Topology
Winter term 2021/2022
Prof. Dr. Michael Joswig, Holger Eble

Exercise Session Sheet 3

On this sheet, let (X, d) be a metric space.

Exercise 1 Show that metric spaces are Hausdorff spaces. Is the finite-complement topology on the natural numbers Hausdorff?

Let now $A \subset X$ be some subset. For $x \in X$ we set $d(x, A) := \inf\{d(x, a) \colon a \in A\}$.

Exercise 2 Show that $x \in \overline{A}$ holds precisely if $d(x, A) = 0$.

Exercise 3 Show that the map

$$\begin{aligned} d(\bullet, A) \colon X &\longrightarrow \mathbb{R} \\ x &\longmapsto d(x, A) \end{aligned}$$

is continous.

Exercise 4 Let A and B be disjoint closed subsets of X. Show that there is a map $f \colon X \to \mathbb{R}$, which takes the value 1 on A, the value -1 and B and values in the open interval $(-1, 1)$ on $X \setminus (A \cup B)$.

Tietze extension theorem. Let $A \subset X$ be a closed subset and let $f \colon A \to \mathbb{R}$ be a map. Then f can be extended to the whole space X, i.e. there is some $g \colon X \to \mathbb{R}$ with $g_A = f$.

Exercise 5 Show that the Tietze extension theorem doesn't hold for arbitrary topological spaces.

Exercise 6 Let Y be *any* set and let $\{f_i \colon i \in \mathbb{N}\}$ be a family of X-valued maps with domain Y, i.e. all f_i map to some fixed metric space. What does (pointwise/uniform) convergence against some $f \colon Y \to X$ mean? Is f continous, given that Y is a topological space?

Exercise 7 (Weierstrass M-test) Let Y be any set and let $\{f_i \colon i \in \mathbb{N}\}$ be a family of real-valued functions on Y. Assume there are non-negative constants M_i such that $|f_i(x)| \leq M_i$ holds for all $x \in Y$ and all $i \in \mathbb{N}$. Further, we assume that the series $\sum_{i=0}^{\infty} M_i$ converges. Then the series $\sum_{i=0}^{\infty} f_i$ converges (absolutely) uniformly on Y.

Exercise 8 Prove Tietzes extension theorem.

Exercise 9 Let $A, B \subset X$ be closed, disjoint subsets of the metric space X. Show that there are disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$ holds.

Exercise 10 Let $A \subset X$ be a closed subset. Show that A is the countable intersection of some open subsets of X.

Exercise 11 Draw two closed and disjoint subsets $A, B \subset \mathbb{R}^2$ such that their distance $d(A, B)$ is zero. Can one of them be compact?