Topology
Winter term 2021/2022
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Exercise Session Sheet 2

Again, let X and Y be topological spaces.

Definition. A function $f\colon X \to Y$ is *continous* if for any $x \in X$ and for any neighborhood U of $f(x)$ its inverse image $f^{-1}(U)$ is a neighborhood of x. (ε-δ definition)

Exercise 1
Show that a function $f\colon X \to Y$ is continous if $f^{-1}(O)$ is open for any open set $O \subset Y$.

Exercise 2
Let $A \subset X$ be a subspace. Show that the inclusion map $\mathbf{1}_{X_{|A}} : A \to X$ is continous.

Exercise 3 Let $A \subset X$.

a) Show that $x \in \overline{A}$ holds precisely if every neighborhood of x touches A.

b) show that x lies in the frontier of A precisely if every neighborhood of x touches both A and $X \setminus A$.

Exercise 4 Let $\mathcal{F} \subset \mathcal{P}(X)$ given by complements of finite subsets of X (plus , of course).

a) Show that $\mathcal{F}$ is a topology on X, which is called the *finite-complement-topology.*

b) For some subset $A \subset X$, determine the limit points of A.

Theorem. Let $f\colon X \to Y$ be a function. The following are equivalent:

a) The function f is continous.

b) Given a base $\mathcal{B}$ of Y, $f^{-1}(O)$ is open for any $O \in \mathcal{B}$.

c) We have $f(\overline{A}) \subset \overline{f(A)}$ for any subset $A \subset X$.

d) We have $\overline{f^{-1}(B)} \subset f^{-1}(B)$ for any subset $B \subset Y$.

e) The preimage $f^{-1}(A)$ is closed for any closed set $A \subset Y$.

Exercise 5
Prove a) $\Rightarrow$ b), c) $\Rightarrow$ d) and e) $\Rightarrow$ a) of the above Theorem.

Exercise 6
Let $Y \subset X$ be a subspace. Show that for any given subset $A \subset Y$ the following proudces the same set:

a) Take the closure of A in Y.

b) Intersect the closure of A in X with Y.

Exercise 7 Show that the following are equivalent:

a) The space X is Hausdorff.

b) For every $x \in X$ we have $\{x\} = \bigcap\{\overline{U} \colon U \text{ is a neighborhood of } x\}$.

Exercise 8
Show that if X has a countable base, then X contains a countable dense subset.