Topology
Winter term 2021/2022
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Exercise Session Sheet 1

A subset X of some euclidean space $\mathbb{R}^n$ is automatically equiped with the so called *subspace* (or *trace*) *topology*, i.e. $O \subset X$ is open if and only if for every $x \in O$ there is some open Ball $B_x \subset \mathbb{R}^n$ such that $B_x \cap X$ is fully contained in O.

Exercise 1 Show that any subspace $X \subset \mathbb{R}^n$ is *metrizable*, i.e. there exists a metric d on X such that the subspace topology on X is induced by d.

Exercise 2 Describe the subspace topology of $[0,1)$ and $\mathbb{S}^1 \subset \mathbb{R}^2$.

Exercise 3 As usual, we identify $\mathbb{R}^2 \cong \mathbb{C}$ via the homeomorphism $(x,y) \mapsto x + iy$. Consider the function

$$\begin{aligned} \varphi\colon [0,1) &\longrightarrow S^1 \\ t &\longmapsto e^{2\pi i t} \quad . \end{aligned}$$

Is φ continous? Is φ a homeomorphism? Are $[0,1)$ and S^1 homeomorphic?

Exercise 4
In Armstrong's book, for a fixed set X and some given point $x \in X$, a *system of neighborhoods of* x is required to fulfill the following properties:

- The point x lies in each of its neighborhoods.
- The intersection of two neighborhoods of x is a neighborhood of x.
- If N is a neighborhood of x and $N \subset U \subset X$, then U is a neighborhood of x, too.
- Let N be a neighborhood of x. For $N' := \{z \in N \colon N \text{ is a neighborhood of } z\}$ we have that N' is a neighborhood of x.

Elaborate on this in the metric space setting.
Next, how can we pass from systems of neighborhoods to a topology on X?
Important side question: What are neighborhood systems good for?

Exercise 5
Recall from calculus some properties which are respected by homeomorphisms. Can you come up with some property that is not respected? What is a Hausdorff space? What is a discrete space?