Topology Friday, 7/1/2022
Winter term 2021/2022
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Exercise Sheet 9

Due-date: Monday, 17/1/2022, *before* the lecture starts.

Exercise 1 **4 Points**
Let K be a geometric simplicial complex and let $v \in \operatorname{vert}(K)$ be a vertex, i.e. a 0-dimensional simplex, of K. Let $E(K, v)$ be the edge group of K based at v. Show that the group homomorphism $\varphi : E(K, v) \to \pi_1(|K|, v)$ constructed in the Wednesday lecture is injective.

Exercise 2 **2 Points**
Show that the dunce hat (cf. Homework Sheet 8, Exercise 2) is triangulable.

Exercise 3 **4 Points**
Let K be a geometric simplicial complex in $\mathbb{R}^n$. The *cone* CK *on* K is a geometric simplicial complex in $\mathbb{R}^{n+1}$ and was constructed in the Monday lecture: Choose an apex $v \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$ and take the pyramids $\hat{\sigma} := \operatorname{conv}(\sigma, v)$ for $\sigma \in K$ to be the members of CK. Show that this yields indeed a geometric simplicial complex. Further, show that the space $|CK|$ and the cone $C|K| := (|K| \times I)/(|K| \times \{1\})$ over $|K|$ are homeomorphic.

Definition: locally finite complex. Let K be a possibly infinite geometric simplicial complex in $\mathbb{R}^n$. The complex K is called *locally finite* if every vertex of K belongs to only finitely many simplices of K.

Exercise 4 **2 Points**
Show that a geometric simplicial complex K in $\mathbb{R}^n$ is locally finite precisely if its realization $|K| \subset \mathbb{R}^n$ is a locally compact topological space.

For the rest of this exercise sheet let K be an abstract simplicial complex.

Definition: Realization of an abstract simplicial complex (as a *colimit*). As we discussed in the exercise session, the realization $|K|$ can be given by glueing the simplices of K together according to the cell structure of K. To do so, we set $\Delta^F := \{x \in \mathbb{R}^F \colon \sum x = 1, \text{ all } x_k \geq 0\}$ for any finite set F. It is a $(\#F - 1)$-dimensional simplex with $\#F$ vertices. Let now $\sigma \in K$ be a face of K. For any face $\tau \subset \sigma$ of σ there is a canonical embedding

$$\begin{aligned} \iota_\tau^\sigma \colon \Delta^\tau &\longrightarrow \Delta^\sigma \\ x &\longmapsto (\mathbf{1}_\tau(k) \cdot x_k)_k \end{aligned}$$

with the indicator function $\mathbf{1}_\tau$ of τ. A realization of $|K|$ can be given as

$$|K| := \left(\coprod_{\sigma \in K} \Delta^\sigma \right) /\sim \ ,$$

where for $x \in \Delta^\sigma$ and $y \in \Delta^\rho$ we have $x \sim y$ if and only if there is some $z \in \Delta^{\sigma \cap \rho}$ with $\iota^\sigma_{\sigma\cap\rho}(z) = x$ and $\iota^\rho_{\sigma\cap\rho}(z) = y$. Further, for any $\sigma \in K$ we define the map $\chi_\sigma \colon \Delta^\sigma \to |K|$ as the composition of the quotient map with the inclusion into the disjoint union.

Exercise 5 (Reality check) **3 Points**

These statements immediately follow from the universal property of final topologies.

a) Let S be a polyhedral surface (cf. lecture of the first week, *polyhedron* in Armstrong's book) and let $K(S)$ be the underlying abstract simplicial complex. Show that S and $|K(S)|$ are homeomorphic, cf. Homework Sheet 5, Exercise 6.

b) Let K be any abstract simplicial complex with vertex set $V(K) = [n]$. Let $\Delta := \operatorname{conv}\{e_i \colon i \in [n]\} \subset \mathbb{R}^n$ and $\Delta[\sigma] := \operatorname{conv}\{e_i \colon i \in \sigma\} \subset \Delta$ for any non-empty $\sigma \in K$. Let K' be the geometric simplicial complex $K' := \{\Delta[\sigma] \colon \sigma \in K\}$. Show that the realizations of K and K' are canonically homeomorphic.

c) Show that the maps χ_σ are closed embeddings.

Definition: Path-components of K. The *vertices* $V(K)$ of K are precisely those members of K, which are singleton sets. Two vertices $x, y \in V(K)$ lie in the same *path-component* if there is a sequence $z_0, z_1, \ldots, z_n \in V(K)$ of vertices of K with $x = z_0$, $y = z_n$ and $\{z_k, z_{k+1}\} \in K$ for all k. Coming from an equivalence relation, the path-components of K partition the vertex set $V(K)$.
Now, let $C \subset V(K)$ be a path-component of K. We set

$$K_C := \bigcup_{\substack{\sigma \in K \\ \sigma \subset C}} \chi_\sigma(\Delta^\sigma) \subset |K|$$

and we identify the sets K_C as the connected components of $|K|$ in the following exercise.

Exercise 6 **5 Points**

Let K still be an abstract simplicial complex.

a) Show that $|K| = \bigcup K_C$, where C runs through the path-components of K.

b) Every set K_C is path-connected.

c) Every set K_C is clopen (open and closed) in $|K|$. For this, recall which topology is given on $|K|$ by the construction above.

d) We have a partition $\{K_C \colon C \text{ path-component of } K\}$ of $|K|$.

e) The connected components of the realization $|K|$ are open and coincide with the path-components of $|K|$. Further, the connected components of $|K|$ are in bijection with the connected components of the graph $|K^{(1)}| \subset |K|$ of $|K|$, where $K^{(1)}$ only takes into account the faces of K with dimension less or equal to one, i.e. with cardinality less or equal to two.