Topology Friday, 10/12/2021
Winter term 2021/2022
Prof. Dr. Michael Joswig, Holger Eble

Exercise Sheet 8

Due-date: Monday, 3/1/2022, *before* the lecture starts.

Exercise 1 **4 Points**
Let Y be any topological space.

a) Let $f, g\colon \mathbb{S}^1 \to Y$ be two homotopic maps. Show that the spaces $\mathbb{D}^2 \cup_f Y$ and $\mathbb{D}^2 \cup_g Y$ are homotopy equivalent.

b) Let $Y = \mathbb{S}^1$, $f(z) = z^2$ and $g(z) = z^3$. Are the spaces $\mathbb{D}^2 \cup_f \mathbb{S}^1$ and $\mathbb{D}^2 \cup_g \mathbb{S}^1$ homotopy equivalent?

Note: For the fact that X *is glued to* Y *along a map* $f\colon X \supset A \to Y$ one finds both notations $X \cup_f Y$ and $Y \cup_f X$ in the literature. This seems to be a matter of taste since the context is clear and we use $X \cup_f Y$ here.

Exercise 2 **4 Points**
In this exercise we show that the *dunce hat* is contractible.

a) Show that the following two loops α and β in $\mathbb{S}^1$ are homotopic relative to $\{0, 1\}$:

$$\alpha(s) := \begin{cases} e^{4\pi i s}, & \text{for } 0 \leq s \leq \frac{1}{2} \\ e^{4\pi i(2s-1)}, & \text{for } \frac{1}{2} \leq s \leq \frac{3}{4} \\ e^{8\pi i(1-s)}, & \text{for } \frac{3}{4} \leq s \leq 1 \end{cases},$$

$$\beta(s) := e^{2\pi i s}, \text{ for } 0 \leq s \leq 1 .$$

b) Give a formal definition of the *dunce hat* from this week's Wednesday lecture.

c) Show that the dunce hat is contractible.

Definition. A map $f\colon X \to Y$ between topological spaces X and Y is said to be *null homotopic* if it is homotopic to the constant map $c_y\colon X \to Y$ for some $y \in Y$.

Exercise 3 **6 Points**
Let X and Y be topological spaces and let CX denote the cone over X

a) Show that a map $f\colon X \to Y$ is null homotopic if and only if it extends to a map $CX \to Y$.

b) Let $f\colon X \to \mathbb{S}^n$ be a map which is not surjective. Show that f is null homotopic.

c) Show that if X deformation retracts to a point $x \in X$, then for each neighborhood U of x there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is null homotopic.

d) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for $r \in [0,1] \cap \mathbb{Q}$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

Exercise 4 **4 Points**

As usual, let I denote the unit interval $[0,1] \subset \mathbb{R}$.

a) Give a formal, set-theoretic description of the equivalence relation $\sim_M$ on I^2 such that the quotient $I^2/{\sim_M}$ is (homeomorphic to) the Möbius strip.

b) Consider the following examples of a circle C embedded in some surface S:

   A) $S = \mathbb{S}^1 \times \mathbb{S}^1$; $C = \{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 : x = y\}$.

   B) $S = I^2/{\sim_M}$; $C = q(\partial I^2)$, where q is the quotient map $q\colon I^2 \to S$.

   For both A) and B), draw $\iota\colon C \hookrightarrow S$, choose a base point in C, describe generators g_i for the fundamental groups of C and S (i.e. classes of honest loops) and write down the induced homomorphism ι_* of fundamental groups in terms of the generators g_i.

Exercise 5 **2 Points**

Show that the circle is a deformation retract of the Möbius strip, cf. Exercise 4.