Topology Friday, 3/12/2021
Winter term 2021/2022
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Exercise Sheet 7

Due-date: Monday, 13/12/2021, *before* the lecture starts.

Recap. Let X be a topological space. The *cone* CX over X is the quotient space $X \times [0,1]/\sim$ where $\sim$ identifies $X \times \{1\}$ to one point, i.e. $CX = X \times [0,1]/(X \times \{1\})$.

Exercise 1 **5 Points**
Let X be a compact Hausdorff space. Show that CX is homeomorphic to the one-point compactification of $X \times [0,1)$. If $A \subset X$ is closed, show that X/A is homeomorphic to the one-point compactification of $X \setminus A$.

Exercise 2 **3 Points**
Let X be a path-connected space. Show that the following assertions are equivalent:

A) Every map $\mathbb{S}^1 \to X$ is homotopic to the constant map.

B) Every map $\mathbb{S}^1 \to X$ extends to a map $\mathbb{D}^2 \to X$.

C) We have $\pi_1(X) = \{1\}$.

Exercise 3 (Leftover from Exercise Session 6) **2 Points**
For a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, let $\overline{f}\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1), z \mapsto f(z)/f(1)$ be the base point preserving variant of f. Show that two homotopic maps $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ induce the same map on fundamental groups $\overline{f}_* = \overline{g}_*\colon \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1)$. Recall why in this case f and g have the same degree, cf. Exercise Session 6.

Definition. Let Y and X be topological spaces. A map $p\colon Y \to X$ is called *covering map* or *cover* (of X) if every point $x \in X$ has an open neighborhood V for which $p^{-1}(V)$ decomposes into a disjoint union of open sets $U_i \subset Y$ such that each restriction of p to U_i is a homomorphism onto V. In particular, a covering map is always surjective.
A cover $p\colon Y \to X$ is an instance of a *space over* X. A *map* from a space $p_1\colon Y_1 \to X$ over X to a second space $p_2\colon Y_2 \to X$ over X is a map $f\colon Y_1 \to Y_2$ such that $p_2 \circ f = p_1$ holds.

Example. Let $I \neq \emptyset$ be a discrete space. The projection $p_X\colon X \times I \to X$ onto the first coordinate is called *trivial cover* of X.

Exercise 4 **2 Points**
Let $n \in \mathbb{N}$. Show that $t_n\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}, x \mapsto x + n$ is a cover of the quotient space $\mathbb{R}/\mathbb{Z}$. Can you draw it?

Exercise 5 **4 Points**

Show that a map $p\colon Y \to X$ is a cover of X precisely if every point $x \in X$ has an open neighborhood V such that the restriction of p to $p^{-1}(V)$ is homeomorphic as space over V to some trivial cover $V \times I_V$ of V. Further, show that if X is connected, then the discrete spaces I_V are all homeomorphic to the same space I. In this case, the cardinality of I is called the *number of sheets of the covering p.*

Exercise 6 **3 Points**

Let $p\colon Y \to X$ be a cover of X. Choose a base point $x \in X$ and some point $y \in p^{-1}(x)$ in the fibre of x. Further, let $F\colon [0,1] \times [0,1] \to X$ be a map with $F(0,t) = F(1,t) = x$ for all $t \in [0,1]$. Show that there is a unique lift $\hat{F}$ of F, i.e. there is a unique map $\hat{F}\colon [0,1] \times [0,1] \to Y$ which fulfills $p \circ \hat{F} = F$ and $\hat{F}(0,t) = y$ for all $t \in [0,1]$.
Hint: Exercise 4 and some Theorem from the lecture.

Exercise 7 **1 Point**

Let $p\colon Y \to X$ be a cover of X and let $p(y) = x$ for some point $y \in Y$. Show that $\pi_1(Y, y)$ is isomorphic to a subgroup H of $\pi_1(X, x)$.
Extra point: In case X is connected, show that the cardinality of the set $\pi_1(X, x)/H$ is equal to the number of sheets of p, cf. Exercise 5.