Topology Friday, 26/11/2021
Winter term 2021/2022
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Exercise Sheet 6

Due-date: Monday, 6/12/2021, *before* the lecture starts.

Here and elsewhere, *quotient space* is the same as *identification space*.

Exercise 1 **4 Points**
Choose two formulations of the projective space $\mathbb{RP}^n$ from the lecture and show that $\mathbb{RP}^n$ (in each of the two versions) is homeomorphic to the space $\mathbb{D}^n/\sim$ where $\mathbb{D}^n$ is the closed unit ball $\mathbb{D}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $\sim$ is the equivalence relation on $\mathbb{D}^n$ generated by setting $x \sim y$ if $x = \pm y$ holds on the boundary of $\mathbb{D}^n$, i.e. if $x, y \in \mathbb{S}^{n-1}$ are antipodal.
Attention: This is a statement about homeomorphisms between quotient spaces. You're obliged to use the mapping property of the final topology as we did in the Exercise Session and benefit from induced maps defined on the quotient.

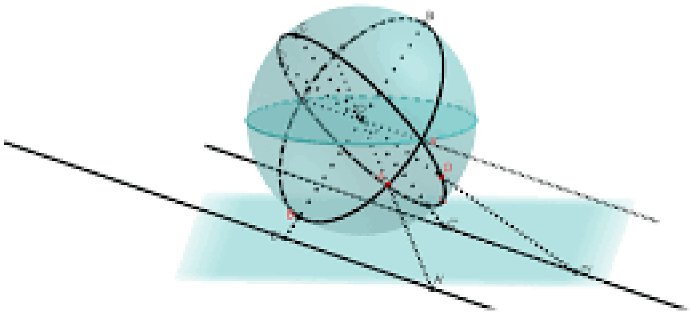


Figure 1: A visualization of $\mathbb{RP}^2$, borrowed from www.geogebra.org. What does it tell you about the circles, the lines and ∞?

Definition. A map $\iota\colon A \to X$ is called *embedding* if ι is injective and the topology on A is the initial topology with respect to the map ι.

Exercise 2 **1 Point**
Show that $\iota\colon A \to X$ is an embedding if and only if ι is a homeomorphism onto its image $\iota(A) \subset X$ equipped with the subspace topology.

Definition. Let X, Y be topological spaces, $A \subset X$ a subspace and $f\colon A \to Y$ a map. Let $\iota_X\colon X \to X+Y$ and $\iota_Y\colon Y \to X+Y$ be the canonical inclusions. Further, we define $\sim_f$ to be the equivalence relation on $X+Y$ *generated* by all relations of the form $\iota_X(a) \sim_f \iota_Y(f(a))$ for $a \in A$. This determines $\sim_f$ on $X+Y$ as:

i) $\iota_Y(y) \sim_f \iota_Y(y')$ if and only if $y = y'$,

ii) $\iota_X(x) \sim_f \iota_Y(y)$ if and only if $x \in A$ and $f(x) = y$,

iii) the symmetric version $(\iota_Y(y) \sim_f \iota_X(x))$ of ii) and

iv) $x \sim_f x'$ if and only if $x = x'$ or $x, x' \in A$ and $f(x) = f(x')$

for all $x, x' \in X$ and $y, y' \in Y$. We set $X \cup_f Y := X + Y/{\sim_f}$ and get composition maps

$$\chi\colon X \xrightarrow{\iota_X} X+Y \xrightarrow{q} X \cup_f Y,$$
$$\iota\colon Y \xrightarrow{\iota_Y} X+Y \xrightarrow{q} X \cup_f Y,$$

where $q\colon X+Y \to X \cup_f Y$ denotes the quotient map. Of course, $X+Y$ and $X \cup_f Y$ are understood to have the final topology for the pair (ι_X, ι_Y) resp. the quotient map q.

Exercise 3 (Reality check) **0 Points**

Show that $\iota_X\colon X \to X+Y$ and $\iota_Y\colon Y \to X+Y$ are embeddings.

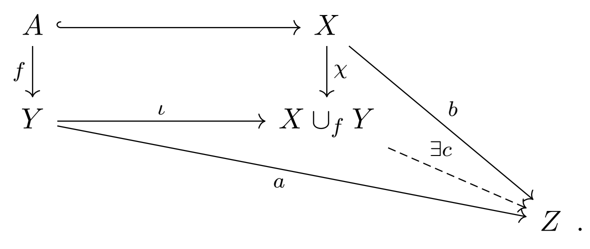
Exercise 4 Let the setup be given as in the above definition. **5 Points**

a) Draw a picture for $X \cup_f Y$. What is the connection to polyhedral surfaces?

b) Show that $\iota\colon Y \to X \cup_f Y$ is an embedding.

c) Is $\chi\colon X \to X \cup_f Y$ an embedding? Prove your conjecture.

d) State and prove the universal mapping property for the space $X \cup_f Y$ and the maps χ, ι and f, which is indicated in the following commutative diagram:



In this situation we say that X is *glued* to Y along the map f.

Exercise 5 **4 Points**

Show that $\mathbb{RP}^2$ is obtained by

a) glueing a 2-disk $\mathbb{D}^2$ to the sphere $\mathbb{S}^1 \subset \mathbb{C}$ along the map $u_2 \colon \mathbb{S}^1 \to \mathbb{S}^1, z \mapsto z^2$.

b) glueing a 2-disk $\mathbb{D}^2$ to the boundary of a Möbius strip $(\mathbb{D}^1 \times \mathbb{D}^1)/\sim$, which is again a sphere $\mathbb{S}^1$. Here, the equivalence relation $\sim$ identifies points $(-1, -x)$ with $(1, x)$ for $x \in [0, 1]$.

Note: A statement similar to b) holds for any n. This yields another reformulation for projective space.

Exercise 6 (Add-on to Exercise 5) **1 Point**

Draw a picture for the situation of Exercise 5a). Of which drawing from the lecture does it remind you? Keep this in mind for the upcomming weeks!

Exercise 7 Let G be a topological group and let $H < G$ be a subgroup. **5 Points**

a) Show that the factor map $q \colon G \to G/H$ is open. As usual, G/H denotes the set of left cosets $G/H := \{gH : g \in G\}$.

b) Show that the closure $\overline{H}$ of H in G is also a subgroup.

c) Let H be a *normal subgroup*, i.e. G/H with the induced group operation is a group or, equivalently, $gHg^{-1} \subset H$ holds for every $g \in G$. Show that $\overline{H}$ is a normal subgroup of G as well.

d) Let U be a neighborhood of the neutral element $e \in G$. Show that there is a neighborhood V of e such that $VV^{-1} \subset U$ holds.

e) In the statement of d), can one replace e with any $g \in G$?