Topology Friday, 19/11/2021
Winter term 2021/2022
Prof. Dr. Michael Joswig, Holger Eble

Exercise Sheet 5

Due-date: Monday, 29/11/2021, *before* the lecture starts.

Here and elsewhere, *quotient space* is the same as identification space.

Exercise 1 **3 Points**
Let X and Y be topological spaces. Show that X and Y are compact if and only if the product space $X \times Y$ is compact.
Hint: You might use Exercise 8 from Exercise Session 4.

Exercise 2 **4 Points**
Let $X := \mathbb{R}^n \setminus \{\mathbf{0}\}$ be the real euclidean space $\mathbb{R}^n$ with the origin $\mathbf{0}$ removed. We define an equivalence relation $\sim$ on X as follows: For $x, y \in X$ the relation $x \sim y$ holds precisely if x and y lie on a common 1-dimensional linear subspace $L_{x,y}$ of $\mathbb{R}^n$.

a) The quotient map $q\colon X \to X/\sim$ is open, i.e. it maps open sets to open sets.

b) Show that the quotient space $X/\sim$ is compact and Hausdorff.

c) Is $X/\sim$ path-connected? Is it connected?

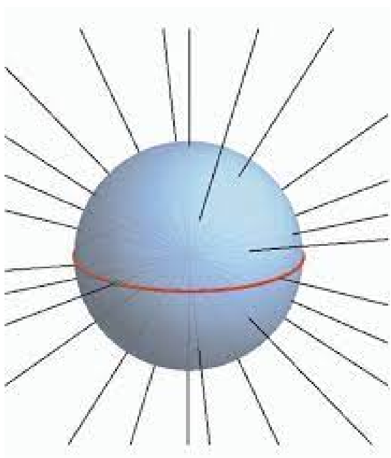

Figure 1: Cartoon of X for $n = 3$, borrowed from https://math.uchicago.edu/∼may.

Exercise 3 **1 Points**
Given an example of a quotient map $q\colon X \to X/\sim$ which is not open.

Exercise 4 **4 Points**

Let X be a set and J an arbitrary index set. For every $j \in J$, let Y_j be a topological space and $f_j\colon X \to Y_j$ a function. Any topology $\mathcal{T}$ on X, which fulfills the following two conditions **A** and **B** is called *initial topology* with respect to the family $(f_j)_{j\in J}$:

A All functions $f_j\colon X \to Y_j$ are continous.

B For every topological space Z and every function $g\colon Z \to X$ we have that g is continous if and only if every composition $f_j \circ g$, $j \in J$, is continous:

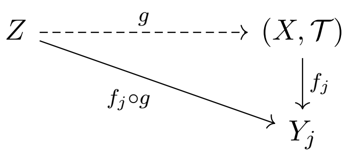

Figure 2: Commutative diagram of the *universal property* **B**.

Prove the following assertions:

a) An initial topology on X always exists.

b) The initial topology $\mathcal{T}$ on X is always unique.

c) For a finite index set J, the cartesian product $X := \prod_{j\in J} Y_j$ and $f_j := p_j$ the projection maps $p_j\colon X \to Y_j$, the initial topology on X coincides with the product topology from the lecture.

d) Now, reconsider an arbitrary index set J and again the cartesian product

$$\prod_{j\in J} Y_j = \left\{ \text{functions } x\colon J \to \bigcup_{j\in J} Y_j \text{ with } x(j) \in Y_j \text{ for all } j \in J \right\}$$

and define the product topology as the initial topology with respect to the projection maps $p_j\colon \prod_{j\in J} Y_j \to Y_j,\ x \mapsto x(j)$ for $j \in J$. Determine a basis for the product topology in this general case.

Exercise 5 **4 Points**

The formulation of the initial topology in Exercise 3 has its roots in category theory. Without delving too deeply into this subject, let us note that in this specific example, and this is true for any statement of this kind, there is a *dual* formulation, which can be obtained by reversing the arrows.

Thus, let X be a set and J an arbitrary index set. For every $j \in J$, let Y_j be a topological space and let $f_j\colon Y_j \to X$ be a function. The statement dual to that of initial topologies yields **the** *final topology* $\mathcal{T}'$ on X, this time via functions $g\colon X \to Z$ for any topological space Z.

0) Make this precise with statements **A'** and **B'** and draw a commutative diagram.

a) Show that a final topology on X always exists.

b) Show that the final $\mathcal{T}'$ topology on X is always unique.

c) Determine a set-theoretic description for $\mathcal{T}'$, i.e. of the form $\mathcal{T}' = \{O \subset X : \ \ldots\}$.

d) Show that the quotient (or identification) topology on any identification space $X/\!\sim$ is the final topology with respect to the projection map $q\colon X \to X/\!\sim$.

Definition. For any index set J and topological spaces X_j, let

$$\coprod_{j\in J} X_j := \bigcup_{j\in J} (X_j \times \{j\})$$

be the *coproduct* of the X_j, equipped with the final topology with respect to the inclusions

$$\begin{aligned} \iota_j\colon X_j &\longrightarrow \coprod_{j\in J} X_j \\ x &\longmapsto (x, j) \ . \end{aligned}$$

The coproduct is simply the union of the X_j forced to be disjoint, with an appropriate topology. (Make one efficient example on your own.)

Exercise 6 **4 Points**

A polyhedron (or *polyhedral surface*) $S \subset \mathbb{R}^n$ as it was introduced in the first lecture can be pictured as a *cell complex* with V vertices, E edges and F faces. For each k-dimensional cell s_k of S, $k \in \{0, 1, 2\}$, create a copy of an appropriate k-dimensional polygon $p_k \subset \mathbb{R}^k$ and collect all these copies p_k in a set $\mathcal{S}$. Thus, the cardinality of $\mathcal{S}$ is $V + E + F$.

a) Describe an equivalence relation $\sim$ on the coproduct $\coprod_{p\in\mathcal{S}} p$, which records the cellular incidences on the surface S, i.e. which tracks the edges of S. This sort of information is called *glueing data*.

b) Let $S' := (\coprod_{p\in\mathcal{S}} p)/\!\sim$ be the corresponding identification space. Show that the abstract space S' is homeomorphic to the polyhedral surface $S \subset \mathbb{R}^n$ equipped with the subspace topology.

c) Can you shrink the set $\mathcal{S}$ and adjust $\sim$ such that the statement of b) still holds?