Topology Friday, 12/11/2021
Winter term 2021/2022
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Exercise Sheet 4

Due-date: Monday, 22/11/2021, *before* the lecture starts.

Exercise 1 **4 Points**
Let X be a Hausdorff space and let $K \subset X$ be compact.

a) Show that the following holds:

$$K = \bigcap \left\{ \overline{O} \colon O \supset K, O \text{ open} \right\} .$$

b) Let $K' \subset X$ another compact set, disjoint from K. Show that there are disjoint open sets O and O' of X with $K \subset O$ and $K' \subset O'$.

Definition. A space X is called *locally compact* if every point in X has a compact neighborhood.

Exercise 2 **4 Points**
Show that the following spaces are locally compact:

i) any compact space,

ii) any discrete space and

iii) any closed subset of a locally compact space.

Prove that $\mathbb{Q} \subset \mathbb{R}$ is not locally compact. Further, show that local compactness is preserved by homeomorphisms.

Exercise 3 **6 Points**
Let (X, τ_X) be a non-compact topological space. We define a new set $X^+ := X \cup \{\infty\}$ and a topology τ^+ on X^+ by setting

$$\tau^+ := \tau_X \cup \left\{ (X \setminus C) \cup \{\infty\} \colon C \subset X \text{ closed and compact} \right\} .$$

a) Check that τ^+ is a topology on X^+.

b) Show that (X^+, τ^+) is compact and $X \subset X^+$ is dense.

c) Show that X^+ is a Hausdorff space if and only if X is a locally compact Hausdorff space. (A space X is locally compact if every point $x \in X$ has a compact neighbourhood U.)

d) Show that $\mathbb{S}^n$ is realized as the one-point compactification of $\mathbb{R}^n$ via the stereographic projection $\mathbb{S}^n \setminus \{\mathrm{pt}\} \approx \mathbb{R}^n$ (which you know from the first homework sheet).

e) Let X, Y be Hausdorff spaces and let $f\colon X \to Y$ be a map. Show that the following statements are equivalent:

   i) The preimage $f^{-1}(C)$ of every compact set $C \subset Y$ is compact.

   ii) The extension $f^+\colon X^+ \to Y^+$ of f with $f^+(\infty) = \infty$ is continous.

f) What happens when X is compact already?

g) Give an example of two non-homeomorphic spaces X_1 and X_2 such that X_1^+ and X_2^+ are homeomorphic.

Exercise 4 **2 Points**

Find a topological space and a compact subset whose closure is not compact.

Exercise 5 Let X be a metric space. **4 Points**

a) Given a map $f\colon X \to \mathbb{R}^{n+1} \setminus \{0\}$, find a map $g\colon X \to \mathbb{S}^n$, which agrees with f on the set $f^{-1}(\mathbb{S}^n)$.

b) Let $A \subset X$ be a closed subset. Let $f\colon A \to \mathbb{S}^n$ be a map. Show that there is a neighborhood U of A (i.e. of all its points) such that f can be extended over U. *Hint:* Tietze style extension, see Exercise Session 3.