Topology Friday, 4/11/2021
Winter term 2021/2022
Prof. Dr. Michael Joswig, Holger Eble

Exercise Sheet 3

Due-date: Monday, 15/11/2021, *before* the lecture starts.

Let X, Y be topological spaces. Let $\mathcal{T}$ be the underlying topology of X.

Reminder. The space X is called *Hausdorff* if its points can be separated by open sets, i.e. for distinct points $x, y \in X$ there are disjoint open sets $O_x, O_y \subset X$ with $x \in O_x$ and $y \in O_y$.

Exercise 1 **5 Points**

Let a topological space X be given.

a) Show that the following assertions are equivalent:
   i) The space X is Hausdorff.
   ii) For every $x \in X$ we have $\{x\} = \bigcap\{\overline{U} \colon U \text{ is a neighborhood of } x\}$.

b) Show that $\{x\} = \overline{\{x\}}$ holds for all $x \in X$ if X is a Hausdorff space.

c) Give a non-Hausdorff space X such that $\{x\} = \overline{\{x\}}$ holds for all $x \in X$.

d) Let $A \subset X$ be dense and $O \subset X$ be open in X. Show that $O \subset \overline{A \cap O}$ holds.

Exercise 2 **6 Points**

Let $\leq$ be a linear order on X. For $a, b \in X$ we define

$$(a, b] := \{x \in X \colon a < x \leq b\},$$
$$[a, +\infty) := \{x \in X \colon a \leq x\} \quad \text{and so on} \; .$$

Show that the following holds:

a) Let $\mathcal{B} := \{(a, b) \colon a, b \in X \cup \{-\infty, \infty\}\}$. There is a topology $\mathcal{T}$ on X with base $\mathcal{B}$.

b) The topological space $(X, \mathcal{T})$ is a Hausdorff space.

c) Let Z be a topological space, $f \colon Z \longrightarrow X$ be a map and $a \in X$. Show that the set $\{x \in Z \colon f(x) \leq a\}$ is closed.

d) Does the open set (a, b) have any limit point? What is the closure of (a, b)?

e) Let now X be the real numbers and let $\mathcal{B}'$ be the family of all subsets of the form $[a, b)$ for real numbers $a \leq b$. Show that $\mathcal{B}'$ is a base for some topology $\mathcal{T}'$ on X and that all members of $\mathcal{B}'$ are open and closed with respect to this topology. Show that $\mathcal{T}'$, unlike the euclidean topology, doesn't have a countable base.

Reminder. For $A \subset X$, its interior $\mathrm{int}(A) \subset A$ is the biggest open set contained in A. *Reality check:* Show that $\mathrm{int}(A)$ is the union of all open sets of X contained in A.

Exercise 3 Let Z be a subspace of X. **4 Points**

a) Let $A \subset Z$ and denote $\mathrm{int}_Z(A)$ the interior of A in Z. Show that

$$\mathrm{int}(A) =: \mathrm{int}_X(A) \subseteq \mathrm{int}_Z(A)$$

holds and give an example where equality doesn't hold.

b) Let $f\colon X \to Y$ be a function. Assume that X can be written as $X = \bigcup_{i\in\mathbb{N}} A_i$ with $A_i \subset \mathrm{int}(A_{i+1})$ for any i and that all restrictions $f_{|A_i} : A_i \to Y$ are continous. Show that f is continous.

Exercise 4 Still, $(X, \mathcal{T})$ is a topological space. **5 Points**

a) Specify $\mathcal{T}$ such that every real-valued function on X is continous.

b) Given $f\colon X \to Y$, can $\mathcal{T}$ be chosen such that f is always/never continous?

c) Do continous functions map limit points to limit points? Elaborate!

d) Do continous functions map dense sets to dense sets? Elaborate!

e) Do continous functions map open/closed sets to open/closed sets? Elaborate!