Topology Friday, 29/10/2021
Winter term 2021/2022
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Exercise Sheet 2

Due-date: Monday, 8/11/2021, *before* the lecture starts.

For this sheet, let $(X, \mathcal{T})$ be a topological space with underlying set X and some topology $\mathcal{T}$ on X. For a given subset $U \subset X$ we denote by $\mathcal{T}_{|U}$ the subspace topology on U induced by $\mathcal{T}$. Here and elsewhere, we'll simply say that U is a subspace of X without explicitly mentioning the subspace topology.

Definition. A topological space is said to be *connected* if the whole space and the empty set are the only subsets, which are both open and closed.

Exercise 1 **5 Points**
We define the following relation $\sim$ on X. For $x, y \in X$ we have $x \sim y$ if and only if there is a connected subspace $(U, \mathcal{T}_{|U})$ of X containing both x and y.

a) Show that $\sim$ is an equivalence relation. We'll refer to the members of the underlying partition of X as *connected components*. Thus, the space X is connected precisely if it has a single connected component.

b) According to a), every connected subspace $U \subset X$ lies in a unique connected component C_U. Show that C_U is the inclusion-wise maximal connected subspace of X containing U. Explain why we're allowed to take $U = \{x\}$ for any $x \in X$.

c) Show that connected components are closed.

d) Determine the number of connected components of the following space X:

$$X := \{0\} \times [-1, 1] \ \cup \ \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0 \right\} \subset \mathbb{R}^2 \ .$$

Exercise 2 **5 Points**
Let X be a set. Show that giving a system of neighborhoods for each $x \in X$ is the same as specifying a topology on X, i.e. show that

a) systems of neighborhoods yield a set system $\mathcal{T} \subset \mathcal{P}(X)$, which satisfies the axioms of a topology on X and that

b) a topology on X yields a system of neighborhoods for every $x \in X$, which satisfy the corresponding four axioms from the lecture.

Further, show that the operations described in a) and b) are inverse to each other.

Exercise 3 **5 Points**

Let p be a fixed prime number. The field $\mathbb{Q}$ is the fraction field of the integers, i.e. every rational number $x \in \mathbb{Q} \setminus \{0\}$ has a unique representation as $x = p^{\nu_p(x)} ab^{-1}$ with unique $\nu_p(x) \in \mathbb{Z}$, $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, both a and b not having p in their prime decomposition. We call

$$|x|_p := \begin{cases} p^{-\nu_p(x)} \ , & \text{if } x \neq 0 \\ 0 \ , & \text{if } x = 0 \end{cases}$$

the *p-adic absolute value* on $\mathbb{Q}$.

a) Show that

   i) we have $|x|_p = 0$ precisely if $x = 0$ holds, that

   ii) the map $|\cdot|_p \colon \mathbb{Q} \setminus \{0\} \to \mathbb{R}_{>0}$ is a group homomorphism and that

   iii) the *strict triangle inequality* $|x+y|_p \leq \max(|x|_p, |y_p|)$ holds for every $x, y \in \mathbb{Q}$.

b) Show that the p-adic absolute value defines a metric on $\mathbb{Q}$, whose induced topology is called the *p-adic topology* on $\mathbb{Q}$ and which will be fixed for now. Further, show that the map $|\cdot|_p$ from a)ii) is continous and that any point of some open metric ball $B_r(x)$ serves as its center. What happens when two open metric balls intersect?

c) For any $x \in \mathbb{Q}$, give a set-theoretic description of the connected component it belongs to.

d) Determine the set (and its cardinality) of continous functions from the real numbers to the rationals equipped with the p-adic topology.

Exercise 4 **3 Points**

Let X be any set and let P be a partition of X.

a) Describe the inclusion-wise smallest topology $\mathcal{T}_P$ on X, such that the members of P are open.

b) Determine the connected components of the space $(X, \mathcal{T}_P)$.

c) Is $(X, \mathcal{T}_P)$ *metrizable*, i.e. does there exist a metric on X, such that the metric topology coincides with $\mathcal{T}_P$?

Exercise 5 **2 Points**

Let $G = G(V, E)$ be a finite graph where all edges have weight 1. Let $d \colon V \times V \to \mathbb{N}$ be a distance function that records shortest paths, i.e. $d(v, w)$ is the length of the shortest path from v to w.

a) Show that d is a metric.

b) Describe the topology induced by d.