Topology Friday, 22/10/2021
Winter term 2021/2022
Prof. Dr. Michael Joswig, Holger Eble

Exercise Sheet 1

Due-date: Monday, 1/11/2021, *before* the lecture starts.

Here, a subset X of some euclidean space $\mathbb{R}^n$ is automatically equiped with the so called *subspace* (or *trace*) *topology*, i.e. $O \subset X$ is open if and only if for every $x \in O$ there is some open Ball $B_x \subset \mathbb{R}^n$ such that $B_x \cap X$ is fully contained in O.


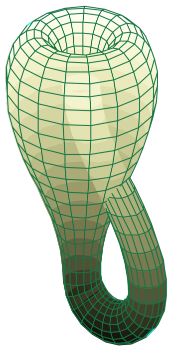

Figure 1: Left: Topology and cartography - pic borrowed from imaginary-exhibition.com. Right: Klein Bottle (somehow) embedded in $\mathbb{R}^3$ necessarily has a self-intersection - pic borrowed from wikipedia.

Exercise 1 **4 Points**

Show that the n-dimensional unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \colon \|x\| = 1\}$ with a singleton point removed, say the n-th standard basis vector e_n, is homeomorphic to $\mathbb{R}^{n-1}$.
Hint: Use the map

$$\begin{aligned} P \colon \mathbb{S}^{n-1} \setminus \{e_n\} &\longrightarrow \mathbb{R}^{n-1} \\ x &\longmapsto \left(\frac{x_1}{1-x_n}, \dots, \frac{x_{n-1}}{1-x_n} \right) \quad . \end{aligned}$$

Show that P is continous and explain in your own words how this map works, starting with $n = 2$. Then provide a continous inverse.

Exercise 2 **4 Points**

Show that the interval $[0, 1]$ is not homeomorphic to $\mathbb{S}^1$.

Exercise 3 **4 Points**

Given $n \in \{2, 0, -2, -4\}$, construct a polyhedron P with $\chi(P) = n$.
Extra point question: Does there exist a polyhedron Q with $\chi(Q) = 1$? Construct it or explain why it does not exist.

Exercise 4 **4 Points**

Pick a suitable pair of the four polyhedra from the lecture (cf. Figure 2 plus the surface of an octahedron) and show that they are homeomorphic.

Exercise 5 **4 Points**

Consider the example from the lecture with Euler characteristic 0, i.e. the prism over a 5-gon with a smaller 5-gon pushed through, c.f. in Figure 2 the central object. Construct a spanning tree T and describe the dual Γ of T, i.e. name interesting properties.

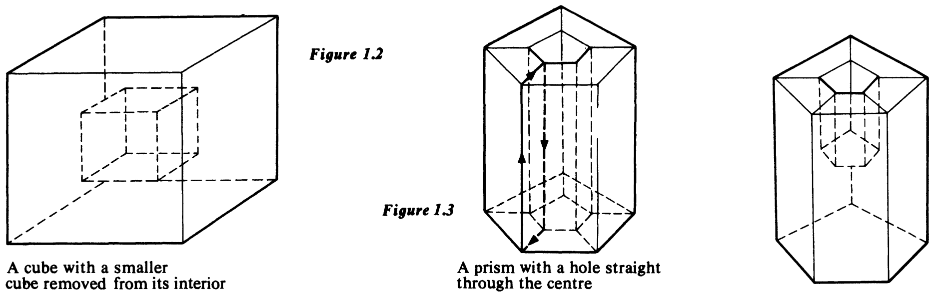

Figure 2: Examples from the first lecture. Source: Armstrong - Basic Topology.