Discrete Geometry II Tuesday, 8/12/2020
Winter term 2020/21
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Exercise sheet 5

Due-date: Thursday, 17/12/2020, via e-mail to eble@math.tu-berlin.de

Exercise 1 **4 Points**
Let m be some natural number and let d be some fixed dimension. Show that the polytope

$$R_d(m) = \operatorname{conv}(\mathbf{0}, e_1, \dots, e_{d-1}, \sum_{i=1}^{d-1} e_i + me_d) \subset \mathbb{R}^d$$

is a d-dimensional simplex with $\#(R_d(m) \cap \mathbb{Z}^d) = d + 1$.
How does this statement relate to Pick's theorem?

Exercise 2 **2 Points**
Let T be an elementary tetrahedron (i.e. T is a lattice simplex in 3-space, without any lattice points other than the four vertices). What are lower and upper bounds for its volume?

Exercise 3 **4 Points**
Let R be an integral domain, i.e. R is a commutative ring such that $ab = 0$ implies $a = 0$ or $b = 0$.

a) Prove that $ab = ac$ implies $b = c$.

The field $\operatorname{Quot}(R)$ is the *quotient field of* R and consists of fractions $\frac{a}{b}$, where $a \in R$ and $R \setminus \{0\}$, with the obivous ring operations and identities. (Think of $\mathbb{Z} \hookrightarrow \operatorname{Quot}(\mathbb{Z}) = \mathbb{Q}$.) The quotient field $\operatorname{Quot}(R)$ is the smallest field containing R. Now let K be a field and consider two cases, $R_1 := K[X]$ and $R_2 = K[[X]]$. In the first case, the elements of the quotient field $K(X) := \operatorname{Quot}(K[X])$ are called *rational functions*. In the second case, the elements of the quotient field $K((X))$ of the formal power series $K[[X]]$ are called *Laurent series*.

b) Prove that $K((X)) = \{\sum_{-k}^{\infty} a_i X^i \colon k \in \mathbb{Z}, a_i \in K \text{ for all } i\}$.

c) Prove that $K(X)$ is a subfield of $K((X))$ by giving an explicit embedding $K(X) \hookrightarrow K((X))$, i.e. explain how a rational funtion can be rewritten as Laurent series.

Exercise 4 **2 Points**

Prove Lemma 1 of today's (Dec 8) exercise session: Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = 0$. For $0 \leq j \leq d$ let

$$f_j(t) := \binom{t+d-j}{d} \in \mathbb{F}[t] \ .$$

Show that the f_j form a basis of the $\mathbb{F}$-vector space $\mathbb{F}[t]_{\leq d}$ of polynomials with degree bounded by d.

Exercise 5 **3 Points**

Prove Lemma 2 of today's (Dec 8) exercise session: Show that for all $j \in \mathbb{Z}_{\geq 0}$ we have

$$\sum_{k \in \mathbb{Z}_{\geq 0}} \binom{k+d-j}{d} z^k = \frac{z^j}{(1-z)^{d+1}} \ .$$

Start with $j = 0$. What does the left hand side represent in this case?

Exercise 6 **1 Points**

Compute the h^* vector of the 3-dimensional cross-polytope, see today's (Dec 8) exercise session.