Discrete Geometry II Friday, 20/11/2020
Winter term 2020/21
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Exercise sheet 3

Due-date: Monday, 30/11/2020, via e-mail to eble@math.tu-berlin.de

Exercise 1 **6 Points**
Let $Q := [-1, 1]^2$ and $C := \operatorname{conv}(Q \times 2Q \times \{1\}, 2Q \times Q \times \{-1\})$.

a) Show that

$$C = \{x \in \mathbb{R}^5 \colon -1 \le x_5 \le 1,\ \pm 2x_1 \le 3 - x_5,\ \pm 2x_2 \le 3 - x_5,\ \pm 2x_3 \le 3 + x_5,\ \pm 2x_4 \le 3 + x_5\} .$$

b) Show that C is combinatorially equivalent to $[0, 1]^5$.

c) Show that $\Gamma(C) \cong \Gamma(\pi(C))$ where π deletes the last coordinate.

Exercise 2 **2 Points**
Give an example of an obstract objective function which is not induced by a linear objective funtion.

Exercise 3 **2 Points**
Show that every 3-dimensional cubical polytope has more vertices than facets. Is $f_0 - f_2$ constant for each such polytope?

Exercise 4 **3 Points**
Let $P = [0, 1]^d$ or any other polytope. We consider colorings of vert(P) with two colors red and blue and we call such a coloring a *valid coloring* (for our purposes) if the red vertices can be separated from the blue vertices by some hyperplane $H \subset \mathbb{R}^d$.

a) Count the valid colorings for the d-cube $P = [0, 1]^d$, $d = 1, 2$ and maybe 3.

b) Fix $P = [0, 1]^d$ for some arbitrary dimension d. Show that the valid colorings of $\operatorname{vert}(P)$ can be parametrized via some set of hyperplanes $H_i \subset A(P)$, $i \in I$ finite, where $A(P)$ denotes the space of affine functionals $f \colon \mathbb{R}^d = \operatorname{aff}(P) \to \mathbb{R}$.

c) A finite collection $\mathcal{H}$ of hyperplanes is called *Hyperplane Arrangement.* Define what a chamber of $\mathcal{H}$ is.

Hint: Let $v \in \operatorname{vert}(P)$ and let H_v be the subspace of affine functionals on $\operatorname{aff}(P)$ vanishing on v. Give an explicit description of H_v and determine its dimension. Which property does the (open) halfspace H_v^+ have?

Exercise 5 **3 Points**

Let $X \subset \mathbb{S}^2 = \{x \in \mathbb{R}^3 \colon |x| = 1\}$ be a set of cardinality 6, randomly (uniformly) chosen. Then the set $V := X \cup -X$ has almost surely cardinality 12 and $P := \operatorname{conv}(V)$ is a centrally symmetric 3-polytope which is almost surely simplicial. Prove or disprove: Identifying antipodes on the boundary of P yields almost surely a triangulation of $\mathbb{RP}^2$ with 6 vertices.