Discrete Geometry II Friday, 06/11/2020
Winter term 2020/21
Prof. Dr. Michael Joswig, Holger Eble

Exercise sheet 1

Due-date: Friday, 13/11/2020, via e-mail to eble@math.tu-berlin.de

All exercises on this sheet cover very basic facts from the course Discrete Geometry I. If you run into trouble for whatever reason, please visit our chat room (`https://riot.polymake.org/#/room/#DGII:polymake.org`) and ask for help if needed.

Polytopes may assumed to be full-dimensional and to contain a chosen point, e.g. $\mathbf{0}$.

Exercise 1 **2 Points**

Let $P \subset \mathbb{R}^n$ be a polyhedron. We saw in the lecture that P can be expressed as

$$P = \operatorname{conv}(V) + \operatorname{cone}(R)$$

for finite sets $V, R \subset \mathbb{R}^n$. (This is an *interior* description of P, sometimes also called $\mathcal{V}$-description of P.)

a) Show that $\operatorname{cone}(R)$ is uniquely determined.

b) Is $\operatorname{conv}(V)$ also unique?

Exercise 2 **2 Points**

Let v be a vertex of a polytope P. Show that v has at least $\dim(P)$ neighbours.

Exercise 3 **2 Points**

Let P be a polytope. Show that the following statements are equivalent:

a) The polytope P is dual to a simplicial polytope.

b) Every vertex figure of P is a simplex.

c) Every vertex has exactly $\dim(P)$ neighbours.

d) Every vertex is contained in exactly $\dim(P)$ facets.

Exercise 4 **2 Points**

Let $C(d,n) \subset \mathbb{R}^d$ be a cyclic polytope with n vertices, $n > d \geq 4$. Show that the graph $G(C(d,n))$ of $C(d,n)$ is the complete graph K_n on n nodes.
Reminder: The graph of a polytope P is an undirected graph having the vertices of P as nodes and the 1-faces of P as edges.

Exercise 5 **2 Points**

Let $Z = \left[p_{(1)}, q_{(1)}\right] + \cdots + \left[p_{(k)}, q_{(k)}\right] \subset \mathbb{R}^n$ be the Minkowski sum of k line segments inside $\mathbb{R}^n$. Give an affine map with domain $[-1,1]^k \subset \mathbb{R}^k$ that maps onto Z.

Exercise 6 **1 Point**

Show that the polytope Z of Exercise 5 is *centrally symmetric*, i.e. it has a center $x_0 \in Z$ such that $x_0 - x \in Z$ holds if and only if $x_0 + x \in Z$ holds.

Exercise 7 **1 Point**

Let $A \in \mathbb{R}^{n,d}$ and $b \in \mathbb{R}^n$. Show that the set of all linear functions $c\colon \mathbb{R}^d \to \mathbb{R}$ such that the linear program

$$\begin{aligned} \max \quad & c \cdot x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

attains its optimum on a face F with $\dim(F) > 0$ has (Lebesgue) measure 0. Thus a randomly picked objective function c almost always yields a single vertex as optimal region of the LP.

Exercise 8 **2 Points**

Describe phase I of the simplex algorithm along an example of your choice.

Exercise 9 **1 Point**

Let P be a polytope. Show that its graph $G(P)$ is connected.

Exercise 10 **1 Point**

Give an example of a connected (undirected) graph $G(V, E)$ such that every $v \in V$ has degree at least d but which is not d-connected, for some $d > 2$.