Discrete Geometry II
Winter term 2020/21
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Session 9

Exercise 1 (**General note on subdivisions of products.**)
Let $\mathbf{A} = (p_j)_{j\in J} \subset \mathbb{R}^{d_A}$ and $\mathbf{B} = (q_k)_{k\in K} \subset \mathbb{R}^{d_B}$ two point configurations. Then $\mathbf{A} \times \mathbf{B}$ is a pointconfiguration in $\mathbb{R}^{d_A+d_B}$ with point $(p_i \times q_k)_{j\in J,k\in K}$. Given any subdivisions $\mathcal{A}$ of $\mathbf{A}$ and $\mathcal{B}$ of $\mathbf{B}$, there is an induced subdivision $\mathcal{A} \times \mathcal{B}$ of the product $\mathbf{A} \times \mathbf{B}$.

a) How does $\mathcal{A} \times \mathcal{B}$ look like?

b) How do triangulations behave passing to the product?

c) Do you know the number of (symmetry classes of) triangulations of the 3-cube C_3? Is C_3 *unimodular*, do all its full-dimensional simplices have the same dimension? Can you draw the dual graph for some distinct triangulation?

d) What do you know about/expect for the number of triangulations of the n-cube?

Exercise 2 (**Triangulations of prism(Δ_n) revisited.**)
As in the lecture, let $\Delta_n \subset \mathbb{R}^n$ be the standard simplex of dimension $n-1$. The prism over Δ_n, defined as $\operatorname{prism}(\Delta_n) := \Delta_n \times \Delta_2$ has $2\cdot n$ vertices $a_1,\dots,a_n,b_1,\dots b_n$, $n+2$ facets: the top facet (a_i), the bottom facet (b_i) and n *vertical* facets, all affinely isomorphic to $\Delta_{n-1} \times \Delta_2$.

a) Recall how simplices σ of $\operatorname{prism}(\Delta_n)$ look like. How many *vertical facets* does σ, i.e. facets contained in some vertical facet of $\operatorname{prism}(\Delta_n)$ have?

b) Let $\mathcal{T}$ be a triangulation of $\operatorname{prism}(\Delta_n)$ and let $\sigma,\tau \in \mathcal{T}$ be an adjacent pair of (maximal) simplices. Describe the geoemtric situation in terms of *non-vertical facets* of σ and τ. Assume first that σ contains the top (b_i).

c) Let $\sigma = (i_1,\dots,i_n) \in \mathcal{S}_\setminus$ be a permutation. Then the following n simplices form a triangulation of $\operatorname{prism}(\Delta_n)$:

$$\mathcal{T}_\sigma = \{a_{i_1},\dots,a_{i_k},b_{i_k},\dots,b_{i_n} : k=1,\dots,n\}$$

d) Let $\mathcal{T}$ be a triangulation of $\operatorname{prism}(\Delta_n)$. Then $\mathcal{T} = \mathcal{T}_\sigma$ for some $\sigma \in \mathcal{S}_n$.

e) Determine the number of triangulations of $\operatorname{prism}(\Delta_n)$.

f) Are all triangulations of $\operatorname{prism}(\Delta_n)$ regular?

Exercise 3 Show that $\operatorname{prism}(\Delta_n)$ is unimodular.

Let $\mathbf{A} = (a_j)_{j \in J} \subset \mathbb{R}^d$ be a point configuration and let $\mathcal{T}$ be a triangulation of $\mathbf{A}$. The vector

$$\Phi_{\mathbf{A}}(\mathcal{T}) = \sum_{j \in J} \sum_{C \in \mathcal{T}: j \in C} \mathrm{vol}(C) e_j \in \mathbb{R}^J$$

is called *GKZ(Gelfand-Kapranov-Zelevinsky)*-vector of $\mathcal{T}$. The polytope

$$\Sigma\text{-poly}(\mathbf{A}) := \mathrm{conv}\,\{\Phi_{\mathbf{A}}(\mathcal{T}) \colon \mathcal{T} \text{ is triangulation of } \mathbf{A}\} \subset \mathbb{R}^{|J|}$$

is called *secondary polytope* of $\mathbf{A}$ and is of dimension $|J| - d - 1$. Its vertices correspond to the regular subdivisions of $\mathbf{A}$!

Exercise 4 Let $\mathbf{A}_n := \mathrm{vert}(\mathrm{prism}(\Delta_n))$.

a) What is Σ-poly$(\mathbf{A}_3)$? What do its edges correspond to?

b) Show that Σ-poly$(\mathbf{A}_n)$ is *affinely* (not only combinatorially) isomorphic to the $(n-1)$-dimensional permutahedron on n letters.

We now turn our attention to the general product of simplices $\Delta_n \times \Delta_m$.

Exercise 5 There is a $(n \times m)$-grid representation and a complete bipartite graph $K_{n,m}$ representation of $\Delta_n \times \Delta_m$.

a) Explain these.

b) How do faces/facets look like in those representations?

c) How many dots do simplices of $\Delta_n \times \Delta_m$ have in the grid representation and how do they look like? See Exercise 6.

d) How does the triangulation

$$\{a_1, b_1, b_2, b_3\}, \{a_1, a_2, b_2, b_3\}, \{a_1, a_2, a_3, b_3\}$$

of $\mathrm{prism}(\Delta_n)$ look like in the grid representation?

e) Can you already come up with a specific triangulation for $\Delta_3 \times \Delta_3$ or even $\Delta_n \times \Delta_m$?

Exercise 6 A subset of the vertices of $\Delta_n \times \Delta_m$ is

a) affinely independent if and only if the corresponding subgraph of the bipartite graph $K_{n,m}$ is a forest, i.e. doesn't have any cycles.

b) a simplex if and only if the corresponding subgraph of the bipartite graph $K_{n,m}$ is a spanning tree.

c) affinely spanning if and only if the corresponding subgraph of the bipartite graph $K_{n,m}$ is a connected and spanning.

d) ... formulate this on your own.

Exercise 7 The point configuration $\mathrm{vert}(\Delta_n \times \Delta_m)$ is unimodular.