Discrete Geometry II
Winter term 2020/21
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Session 7

Notation.
Here, $\Delta_d \subset \mathbb{R}^d$ denotes the standard simplex. Given a polytope $P \subset \mathbb{R}^d$ we often consider the associated cone $K(P) \subset \mathbb{R}^{d+1}$ whose rays are the rays that pass through the vertices of P, lifted to level one in $\mathbb{R}^{d+1}$.

Exercise 1 (reality check) True or false?

a) The graph of a simplicial polytope determines its combinatorics.

b) The graph of a simple polytope determines its combinatorics.

c) The graph of a polytope determines its dimension.

d) Let P be a d-dimensional polytope and let $a, b \in \Gamma(P)$ be two nodes, $a \neq b$. Then there are d vertex-independent paths connecting a to b.

e) Every acyclic orientation has at least one sink, i.e. a node with out-degree 0. Where did we use acyclic orientations?

f) Balinski's Theorem is a simple consequence of the simplex method.

Exercise 2 Let $P \subset \mathbb{R}^d$ be a lattice polytope. Recall what the h^*-vector $h^*(P)$ is and what its entries count in case P is a simplex. Compute the h^*-vector for the standard simplex.

Exercise 3 Let $Q \subset \mathbb{R}^d$ be a lattice simplex and $P \subset \mathbb{R}^{d+1}$ a pyramid over Q with apex e_{d+1}. Show that $\mathrm{Ehr}_P(z) = \frac{1}{1-z}\mathrm{Ehr}_Q(z)$.

Exercise 4 Describe how one can obtain the statement from Exercise 3 for any lattice polytope, i.e. describe our method to process triangulations for the Ehrhart polynomial. *Note:* There are more methods for counting lattice points in a *fan* (= nice collection of cones) correctly, e.g. passing to half-open cones.

Exercise 5 Let Q and P be two lattice polytopes with $Q \subset P$. Then we have $h^*_j(Q) \leq h^*_j(P)$ for all $j = 0, \ldots, \dim(P)$. Formulate a theorem (= your homework) about triangulations such that the monotonicity statement follows directly from the solution of Exercise 4.

Exercise 6 Recall the polynomial $L_{\Delta_d}(t)$, its leading coefficient and its degree.

Exercise 7 Count the lattice points of $\mathrm{int}(k\Delta_d)$. How does this relate to the Ehrhart-Macdonald reciprocity theorem?

Exercise 8 Check out the `jupyter` notebook on regular subdivision, cf. my homepage.