Discrete Geometry II
Winter term 2020/21
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Session 6

Exercise 1
Show that $\sum_{t\geq 0}\binom{t+d}{d}z^t = \frac{1}{(1-z)^{d+1}}$. This is the base case for Ex. 5 on Sheet 5.

Exercise 2
Let $\Delta \subset \mathbb{R}^d$ be an arbitrary simplex. Write the Erhart series $\mathrm{Erh}_\Delta(z)$ of Δ in terms of the generating function of Δ and consider the representation $\mathrm{Erh}_\Delta(z) = \frac{g(z)}{(1-z)^{d+1}}$. What can you say about the coefficients of g? How does this relate to Exercise 1?

Exercise 3
Let $\Delta = \mathrm{conv}\{v_1, \dots, v_{d+1}\} \subset \mathbb{R}^d$ be a (full-dimensional) lattice simplex. Let K be the cone over $\tilde{\Delta}$ as before. In order to pass to the topological (relative) interior, we define

$$\mathring{\pi}_K := \left\{ \sum_{i=1}^{d+1} \lambda_i (v_i, 1) \colon \lambda_i \in \mathbb{Z}, 0 < \lambda_i \leq 1 \right\} \subset \mathring{K} := \mathrm{int}(K) \ .$$

a) Describe $\mathring{K}$.

b) Every lattice point $v \in \mathring{K}$ can uniquely be written as $v = x + \sum_{i=1}^{d+1} \lambda_i (v_i, 1)$ where $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $x \in \mathring{\pi}_K \cap \mathbb{Z}^{d+1}$.

c) Show that $L_{\mathring{\Delta}}(n) = \sum_{i=1}^{k+1} \partial_i \binom{n-i+k}{k}$ where ∂_i denotes the number of lattice points of $\mathring{\pi}_K$ with level, i.e. last coordinate, equal to i.

d) Assume now Δ to be unimodular, i.e. of minimal volume. Compute its volume. In order to show that $\mathring{\pi}_K$ contains exactly one lattice point consult the `www` and find out which property the the cone spanned by $(v_1, 1), \dots, (v_{d+1}, 1)$ has.

   What is its *level*?

Exercise 4
Let P be a d-dimensional lattice polytope. A unimodular (lattice) triangulation of P is a triangulation $\mathcal{T}$ of P where all $\sigma \in \mathcal{T}$ have the minimal volume.

a) Does every lattice polytope have a unimodular triangulation?

Now, assume that P has a unimodular triangulation $\mathcal{T}$ with f-vector $f(\mathcal{T}) = (f_0, \dots, f_d)$.

b) Let $\sigma \in \mathcal{T}$ with dimension k. Show that $L_{\mathring{\sigma}}(n) = \binom{n-1}{k}$.

c) Give an intuitive argument why the follwing formula holds:

$$L_P(n) = \sum_{\Delta \text{ face of } \mathcal{T}} L_{\mathring{\Delta}}(n)$$

d) Show that $L_P(n) = \sum_{k=0}^{d} \binom{n-1}{k} f_k$.

Show that

$$L_P(n) = \#|nP \cap \mathbb{Z}^d| = \sum_{k=0}^{d} \binom{n-1}{k} f_k$$

Exercise 5

Let P be $3\Delta_2 \subset \mathbb{R}^2$, the 2-dimensional standard simplex dilated by the factor 3. Compute the Ehrhart polynomial $L_P(z)$ using Exercise 4). Check your result with Exercise 6 of Session 5.