Discrete Geometry II
Winter term 2020/21
Prof. Dr. Michael Joswig, Holger Eble

Session 11

Exercise 1 (Reality check: Algebraic Variety)
Let $\mathbb{K} = \mathbb{C}$ be some field, preferably algebraically closed.

a) Recall the definition of an (affine) algebraic variety.

b) Ist the empty set resp. the whole space an algebraic variety?

c) Are (finite-dimensional) linear/affine spaces algebraic varieies?

d) Give an example for an infinite variety.

e) Give an example for an obviously finite, non-empty variety.

Additionally, read about which topology a variety usually is associated with and how finiteness of the variety is expressed... but do that after you'll have finished the other exercises.

Exercise 2 (Recap: Minkowski sum)
Let $f, g \in \mathbb{C}[[x_1, \dots, x_n]]$ be two Laurent polynomials. Show that $N(f \cdot g) = N(f) + N(g)$ holds, where N denotes the Newton polytope.

Notation. Let $K_1, \dots, K_n \subset \mathbb{R}^n$ be convex bodys. Then $\mathrm{vol}_n(K_i)$ denotes the n-dimensional Lebesgue measure of K_i and $V(K_1, \dots, K_r)$ is the mixed volume of $K_1, \dots, K_n$.

Exercise 3 (Recap: mixed volumes, straight forward computations)
Let all $K_i \subset \mathbb{R}^n$ be convex bodies.

a) Recall the definition of the mixed volume $V(K_1, \dots, K_n)$ in terms of the coefficients of the polynomial (?) $\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_n K_n)$.

b) Let $K_1, K_2 \in \mathcal{K}^2$. Give a formula that relates $\mathrm{vol}_2(K_i)$ to the 2-dimensional mixed volume $V(K_1, K_2)$.

Exercise 4 (Easy BKK example)
Consider the following complex polynomials:

$$f(X, Y) := a_1 + a_2 X + a_3 XY + a_4 Y$$
$$g(X, Y) := b_1 + b_2 X^2 Y + b_3 XY^2$$

Assume the coefficients a_i, b_i to be generic, cf. below, and assume that the system has $N < \infty$ solutions in the torus $(\mathbb{C}^*)^2$.

a) Determine N.

b) Explain along the BKK theorem where these N solutions come from.

Exercise 5
Is the subdivision indicated in Exercise 4 a mixed subdivision? If yes:

a) Come up with a labelling and explain why it is a mixed subdivision.

b) Describe the face lattice of the Cayley embedding $C := C(N(f), N(g))$. How do triangulations of C look like? Is the point configuration $\mathrm{vert}(C)$ unimodular, i.e. do all simplices in of $\mathrm{vert}(C)$ have the same volume?

c) Describe the subdivision of the Cayley embedding that corresponds to a). Is it a regular subdivision/triangulation?