Path-based multi-commodity flows in large networks

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Why path-based flow models?

Flow models in practice have complicated side constraints

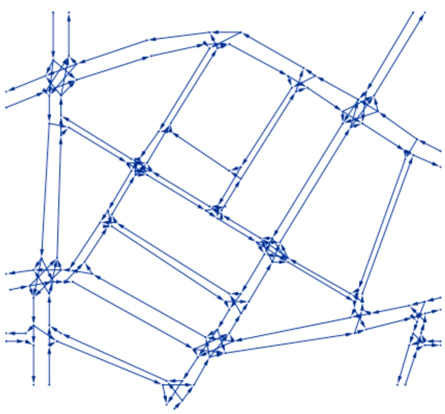
- individual path constraints
  - avoid turns and low level streets in traffic routes
  - restrict the length of routes (no long detours)
  - only way to describe flows over time
- global resource constraints
  - share the network as a common resource
  - path packing problems
- complicated evaluation functions
  - depends on paths (road pricing/track assignment)
  - non-linear in standard flow variables

Overview

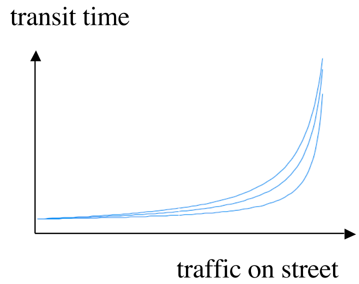
- An application: Traffic routing
- Project goals and working program
  - Static model for traffic flows
  - Fast basic algorithms for large networks
  - Algorithm engineering and software development
  - Dynamic models for traffic flows

Example: Traffic route guidance

a project with DaimlerChrysler



Modeling congestion



Streets have capacities ... ... and traffic-dependent transit times

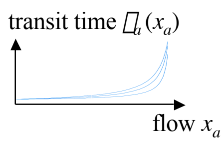
Minimize total travel time

The underlying mathematical model

Static non-linear (fractional) multi-commodity flow model
Models rush hour traffic

- Street network (digraph) $G = (V, A)$

  Arcs $a \in A$ have a
  - capacity $u_a \geq 0$
  - geographic length $l_a \geq 0$
  - link delay function $\tau_a(x_a)$



- Demand: k node pairs (s_i, t_i) with demand $b_i \geq 0$
- Route the demand subject to the capacities such that

  total travel time $\sum_{a \in A} x_a \cdot \tau_a(x_a) \rightarrow \min$

Selfish routing leads to user equilibrium

Users optimize independently and selfish $\Rightarrow$

- reach a Nash equilibrium (user equilibrium), i.e. nobody can improve his route just by himself

$\tau_p(x)$ = transit time of flow x along path p

> x is in Nash equilibrium iff $\tau_p(x) < \tau_q(x^\varepsilon)$ for all paths $p, q \in P$

transfer ε flow units from path p to path q

User equilibrium is not optimal and subject to unwanted effects

Traffic management leads to system optimum

Centralized assignment of routes $\Rightarrow$

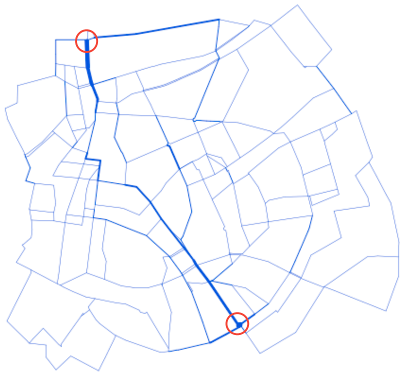
- reach the system optimum i.e. total travel time $\sum_{a \in A} x_a \cdot \tau_a(x_a)$ is minimum
- Can be done in arc flow model, so standard methods available
- Beats user equilibrium — Roughgarden & Tardos [STOC 2000]

But:

- Individual routes may be far too long

Centralized assignment of routes

Centralized assignment of routes



system optimum

restrict length of routes

System optimum under length restriction

Requires path-oriented flow formulation

- P_i set of paths from s_i to t_i that are "not too long"
  i.e., l (path) $\leq (1+\varepsilon) \cdot l$ (shortest path from s_i to t_i)
- $P = P_1 \cup \ldots \cup P_k$
- May represent flow x as
  - vector x^A of arc flow values or
  - vector x^P of path flow values
- $\tau_p(x)$ = transit time of flow x along path $p \in P$
- assign routes from restricted path set P such that
  - all demands are satisfied
  - arc capacities are respected
  - total travel time $\sum_{a \in A} x_a \cdot \tau_a(x_a) = \sum_{p \in P} x_p \cdot \tau_p(x) \rightarrow \min$

The optimization model

min $(\tau^A(\Lambda x^P))^T \Lambda x^P$

Λ = arc-path incidence matrix,
dimension $|A| \times |P|$ $\quad x^A = \Lambda x^P$

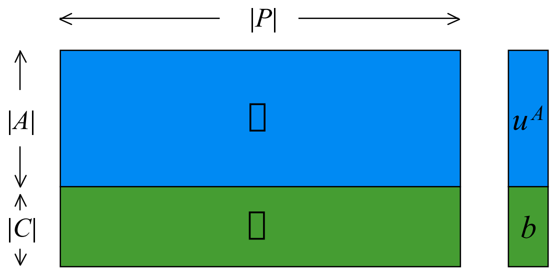
s.t. $\Gamma x^P = b$

$\Lambda x^P \leq u^A$

$x^P \geq 0$

$p \in P$

Γ = commodity-path incidence matrix
dimension $|C| \times |P|$, $C = \{1,\ldots,k\}$

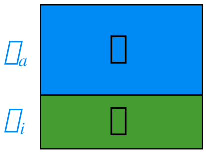


convex objective
linear constraints

The linearized problem

Assume that transit times τ^A = const

- Linear program with a huge number of variables x_p, $p \in P$
- Use revised simplex algorithm with column generation
  - solve LP only with few path variables x_p
    $\Rightarrow$ dual variable values π_a, σ_i
  - optimality condition for whole LP is



$$\sum_{a \in p} (\tau_a - \pi_a) \geq \sigma_i \quad \text{for all } i \in C,\ p \in P_i$$

$\Rightarrow$ constrained shortest path problem
find shortest path from s_i to t_i w.r.t. arc lengths $\tau_a - \pi_a$
such that l (path) $\leq (1+\varepsilon) \cdot l$ (shortest path from s_i to t_i)

The constrained shortest path problem

$$\begin{aligned} \min\ & \tau(p) \\ \text{s.t.}\ & l(p) \leq L \\ & p \text{ is an } s-t \text{ path} \end{aligned}$$

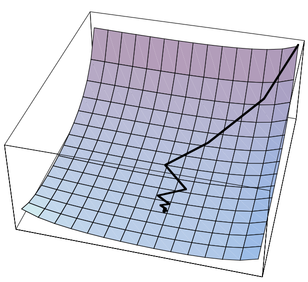
- weakly NP-hard, there are full approximation schemes (of no practical use) [Warburton 1987]
- Branch & bound and Lagrangean relaxation [Beasley & Christofides1989]
- Labeling algorithm (Dijkstra-like) [Aneja, Aggarwal & Nair 1983]
- LP-guided combinatorial algorithms [Mehlhorn & Ziegelmann 2000]

Main steps in the solution method

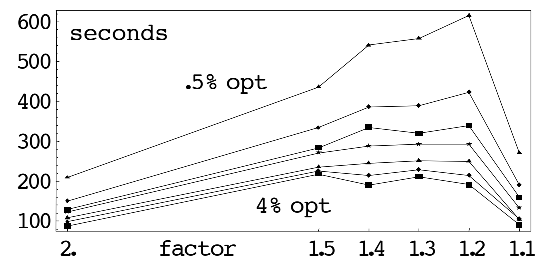
- gradient method (Frank-Wolfe)
  - simplex algorithm
    - algorithm for constrained shortest paths
      - shortest path algorithm, e.g. Dijkstra

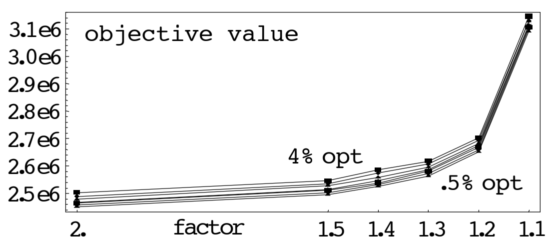
General behavior

Instance REAL
|A| = 4040
|C| = 3166
Ultrasparc 333 MHz



50-200 iterations





Project goals and working program

- **Static model for traffic flows**
  Evaluate quality of solution (compare to user equilibrium)
  Accelerate existing static algorithms (relax capacities, improve subalgorithms);
- **Fast basic algorithms for large networks**
  Design fast basic algorithms, in particular shortest path algorithms (acceleration methods, hierarchical, approximative)
- **Algorithm engineering and software development**
  Reuse of existing software, concepts for combining different libraries, data structures for flow related tasks
- **[Dynamic models for traffic flows]**
  Traffic is modeled as dynamic flow in a directed graph; travel times of arcs are time- and flow-dependent

Measuring route guidance through (un)fairness

Unfairness of a route guidance strategy =

$$\max_{\text{all OD pairs } i} \frac{\text{max transit time } \square_p(x) \text{ of a used path } p \text{ for OD pair } i}{\text{min transit time } \square_p(x) \text{ of a used path } p \text{ for OD pair } i}$$

Unfairness (user equilibrium) = 1

Unfairness (system optimum) may be arbitrarily large

Varying constraint factor

Instance REAL, Gap 0.5%, and geographic distances

					Fairness		
Factor	Value	# Iter.	# Paths	Sec.	geogr.	0-flow	act. flow
∞	2656640.05	214	10533	375	21.35	19.02	15.88
5.00	2657306.23	187	10370	291	4.885	5.881	5.835
2.00	2661115.89	255	10534	224	2	2.867	2.657
1.50	2782771.49	208	8952	376	1.5	1.963	1.927
1.40	2881963.13	203	8536	440	1.4	1.874	2.292
1.30	2988867.60	206	8152	435	1.3	1.782	2.463
1.20	3189798.45	217	7331	426	1.2	1.53	1.762
1.10	4108119.33	105	6051	181	1.1	1.53	2.811
1.05	infeasible						
UE	2909704.41	72	6621	119	10.15	10.41	

Make this approach scalable

- No use of Simplex algorithm
  - Lagrangian relaxation of capacity constraints
  - so far a factor of 2 faster
- Faster computation of constrained shortest paths
  - do standard speed-ups from shortest paths carry over?
- Exploit hierarchy and geography
  - natural in street networks

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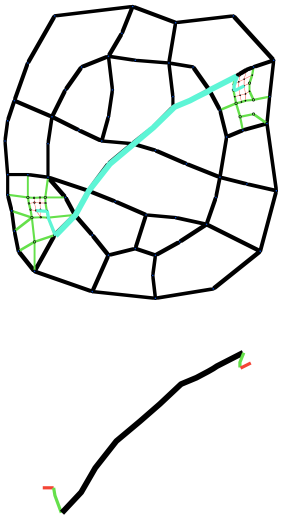
Shortest paths in large networks

- Hierarchy w.r.t. regions and street classes
- Hierarchy w.r.t. graph separators
- Preprocessing
- Acceleration methods

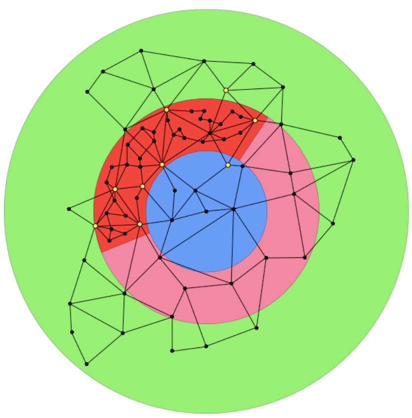
Street-class approach

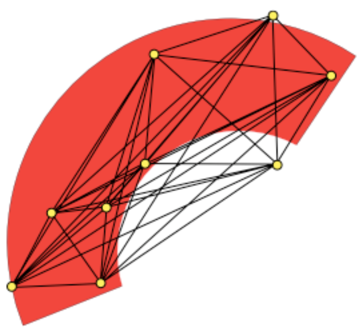


- Decompose the graph into a hierarchy of regions
- Search only for unimodal paths
  - street classes of arcs go first up and then down with the hierarchy
- First results
  - Run time ~ 60% of standard Dijkstra
  - Path lengths ~ 20% longer

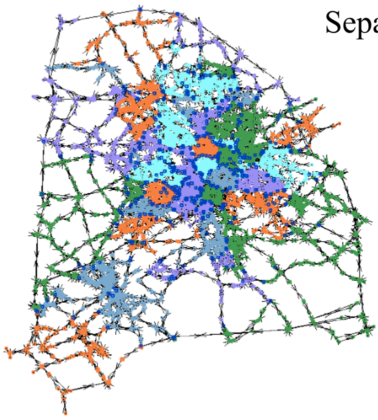
Separator approach

- Find small graph separators that cut the graph into few regions
- Determine distance matrix for separator vertices of each region
- Compute shortest paths using the hierarchy induced by separators





Separator approach



Separators constructed by simple heuristic

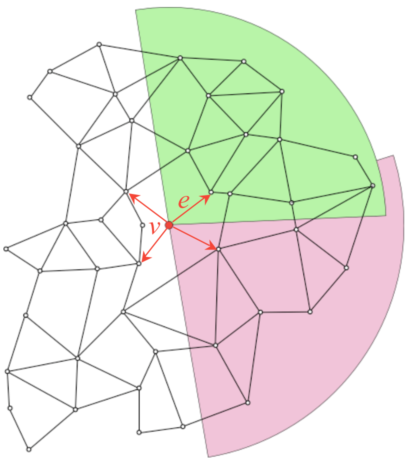
Graph 1:
- 12100 vertices and 19570 arcs
- 603 separator vertices
- 7691 additional arcs
- 54 regions
- 25 sec for preprocessing

Graph 2:
- 3810 vertices and 6027 arcs
- 231 separator vertices
- 2399 additional arcs
- 35 regions
- 2.3 sec for preprocessing

50% improvement on average

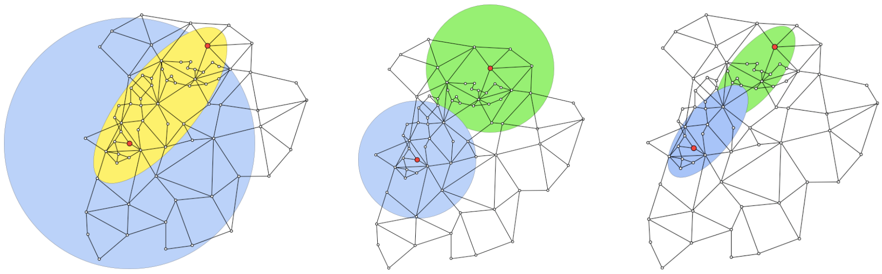
Angle preprocessing

[Brandes, Schulz, Wagner & Wilhalm]



- For every vertex *v* and edge *e* leaving *v*, determine angle containing all vertices having *e* on a shortest path from *v*.
- During algorithm:
  if target vertex not in angle of *e*, then “forget” *e*
- long preprocessing phase
  runtime ~ 40% of normal Dijkstra
- Combination of separator approach and angle preprocessing:
  ~ 25% of normal Dijkstra

Constrained shortest paths: Acceleration



fac	std	bi	bi-do	com
1.50	1490	374	1.54	0.85
1.20	1471	390	9.89	3.47
1.10	1408	403	18.32	9.03
1.05	1435	430	10.38	4.92
1.00	1382	376	1.66	0.91

optimal

fac	bi-do-br	do-br
1.50	1.50	0.86
1.20	1.58	0.86
1.10	1.59	0.86
1.05	4.08	1.72
1.00	1.63	0.88

until first path found

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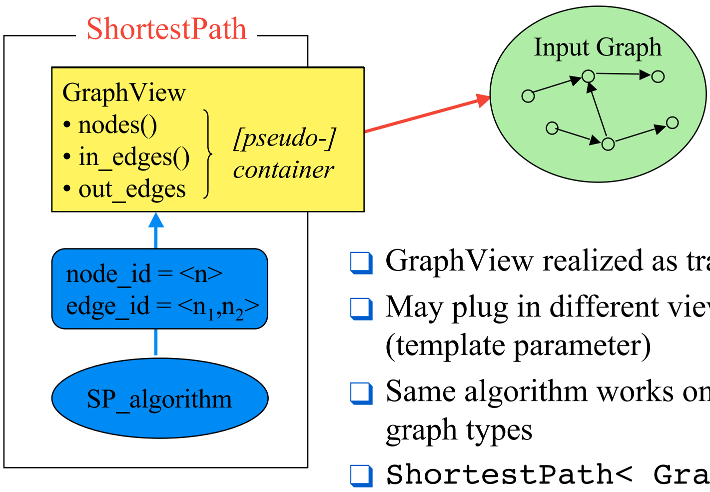
Introduction

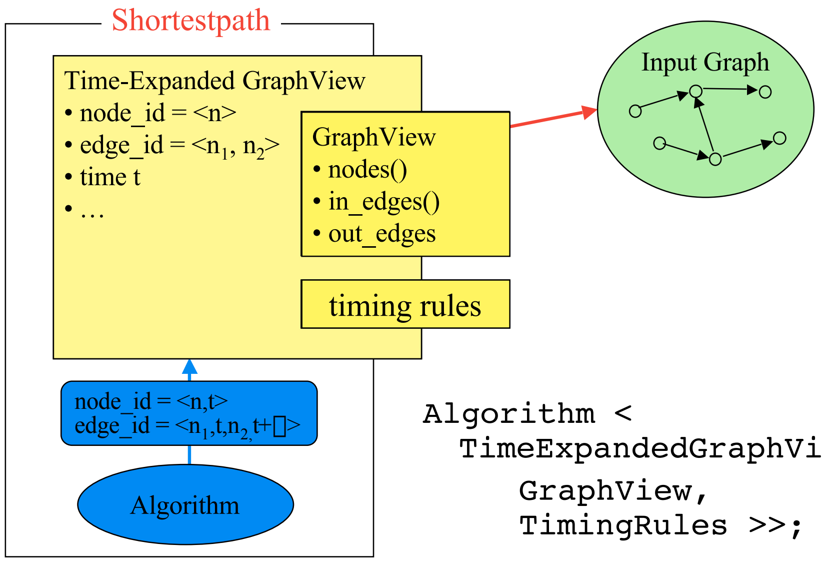
- Generic programming
  - separates data structures and algorithms via abstract requirement specifications
  - uses parametric polymorphism (templates) in C++
- Traits classes
  - provide mappings between types, functions and constants to meet specification/concept requirements
  - determine information about "unknown" types
  - "configure" templates

GraphView



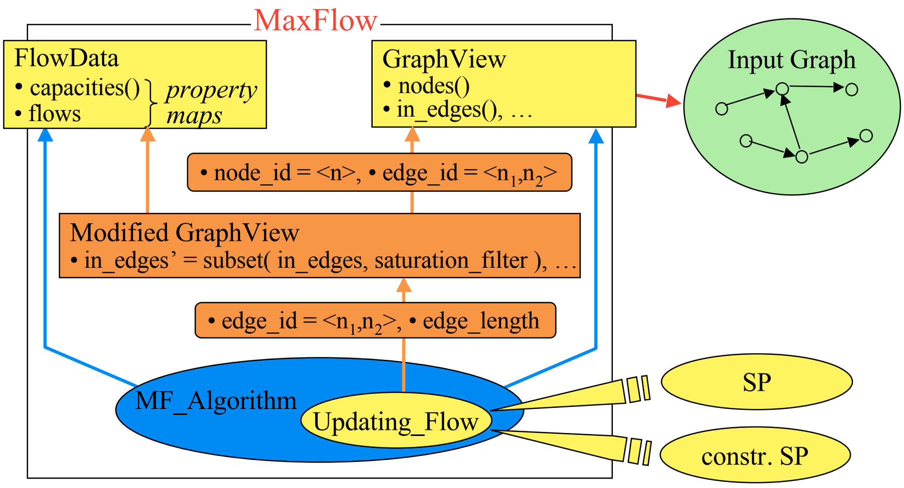
- GraphView realized as traits class
- May plug in different views (template parameter)
- Same algorithm works on different graph types
- `ShortestPath< GraphView >;`

TimeExpanded GraphView




```
Algorithm <
  TimeExpandedGraphView<
      GraphView,
      TimingRules >>;
```

Modified GraphView




```
MaxFlow< GraphView, FlowData,
              Updating_Flow_Builder = SP_Builder >;
```

Algorithm generator

- Multicommodity flow algorithm [Garg & Könemann 1998]: plug in different updating flow algorithms
- Problem: sub-algorithm object needs access to data in main algorithm object, but cannot refer to partially created objects
- Solution: reverse_cast from Instancevariable to main object (e.g. from updating flow module to maxflow algorithm object) (uses pointer offsets in heap allocation table)
- Thus access to data within main object at run time