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Algebro-Geometric Approach to Nonlinear Integrable Equations

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Preface

In recent years mathematical physicists have developed a new line of investigation in the theory of nonlinear differential equations. It was discovered that there exists a wide class of nonlinear equations which can be solved analytically. The now classical approach used for their integration is the inverse scattering method. Starting in 1974 a periodic version of this method was developed which revealed remarkable relations between the spectral theory of operators, algebraic geometry, the theory of Abelian functions and Riemann surfaces. In this volume we systematically study this subject, mainly the solutions corresponding to the so-called finite-gap initial data of the Korteweg - de Vries equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili and the sine-Gordon equations as well as a few others appearing in various branches of mathematical and theoretical physics.

January 1994 *The Authors*

Dedication

Dedicated to the Leningrad school of mathematical physics
on the occasion of the 60th birthday of L.D. Faddeev

Contents

1. Introduction

In the mid 19th century Jacobi, Abel, Weierstrass and especially Riemann formulated a beautiful theory of algebraic and Abelian functions. In the domain of mathematical physics this theory was first used by K. Neumann, Kowalewski and others. However, after the famous work by Kowalewski on the new integrable case of the rotation of a rigid body around a fixed point in a gravitational field, the theory of Abelian functions was developed mostly without any real connection to mathematical physics. A few remarkable treatise by Drach, Burchnall and Chaundy, and Baker, written between 1919 and 1928, were soon forgotten and their content was rediscovered only around 1975 in connection with the emergence of the theory of the so-called finite-gap solutions of nonlinear equations of the type of the Korteweg - de Vries equation (KdV). This theory was developed by several groups in the USSR and the USA, namely, Novikov, Dubrovin and Krichever in Moscow, Matveev and Its in Leningrad, Lax, McKean, van Moerbeke and M. Kac in New York, and Marchenko, Kotlyarov and Kozel in Kharkov. This theory includes an effective solution of the inverse spectral problems for linear operators with periodic and almost periodic coefficients and results in explicit formulas for eigenfunctions of such operators as well as solutions of the KdV-like equations expressed by in terms of multi-dimensional theta functions. Moreover, it allows one to obtain non-trivial results in the algebraic geometry of Riemann surfaces.

In this volume we have considered a numerous specific applications including the theory of the classical top, the description of the Peierls-Fröhlich problem, which is one of the important problems of solid state physics as well as the investigation and numerical realization of the finite-gap theta function formulas based on the use of automorphic functions.

1.1 The History of the Search for Periodic Solutions of the Korteweg - de Vries Equation

The Korteweg-de Vries equation (KdV)

$$4u_t = 6uu_x + u_{xxx} \tag{1.1}$$

was derived at the end of the 19th century to describe the motion of solitary waves on shallow water. The simplest solutions of (1.1) have the form $u(x-vt)$.

To find them it is sufficient to solve the classical problem of inversion of the elliptic integral. That is, by setting $u = u(x - vt)$, we find an ordinary differential equation for u:

$$-4vu_x = 6uu_x + u_{xxx} \quad . \tag{1.2}$$

After an elementary integration we have

$$-4vu = 3u^2 + u_{xx} + C \quad .$$

Multiplication by $2u_x$ and one more integration yield

$$-4vu^2 = 2u^3 + u_x^2 + 2Cu + C_1$$

or

$$u_x = \sqrt{-2u^3 - 4vu^2 - 2Cu - C_1}$$

and hence,

$$x(u) = \int_{\infty}^{-u} \frac{d\tau}{\sqrt{2\tau^3 - 4v\tau^2 + 2C\tau + C_1}} + C_2 \quad . \tag{1.3}$$

To find u as a function of x is the classical inversion problem for the elliptic integral. Its general solution is a double-periodic meromorphic function $u = -2\wp(x) + \gamma$ of the complex variable x, the so-called Weierstrass elliptic function, and γ is a constant. Both depend on the parameters C, C_1, C_2.

In 1967 *Gardner*, *Green*, *Kruskal* and *Miura* [1.3] discovered a remarkable method of integrating the KdV equation for fast decreasing initial data which is known as the inverse scattering method. This method is closely related to the spectral theory of the Schrödinger operator L with the potential $u(x,t)$. The starting point for this relation, known as the Lax representation for the KdV equation, was discovered by *Lax* [1.4] in 1968:

$$\begin{aligned} &\frac{\partial L}{\partial t} = [L, A], \qquad L = -\frac{d^2}{dx^2} - u(x,t) \quad , \\ &A = \frac{d^3}{dx^3} + \frac{3}{4}\left(u\frac{d}{dx} + \frac{d}{dx}u\right) \quad . \end{aligned} \tag{1.4}$$

Using the Lax representation the eigenvalues of the operator L may be shown to be independent of t. In the case of fast decreasing data, the Lax representation allows us to deduce a simple evolution rule for the scattering data of the operator L, i.e., for the reflection coefficients and normalizing factors [1.5]. The now well-known procedure of reconstructing a potential from its scattering data (developed earlier by Gelfand, Levitan, Marchenko, Faddeev and others) allows us to transform the evolution of the scattering data into the solution of the KdV equation.

In the case of the initial potential $u(x,0)$, corresponding to a reflection coefficient equal to zero (the last condition is invariant with respect to KdV dynamics) the inverse problem can be solved exactly and leads to an important class of solutions of the KdV, to its N-soliton solutions. They have the form

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2}{\partial x^2}\ln\det M \quad , \\ M_{ij} &= \delta_{ij} + \frac{2\sqrt{P_iP_j}}{P_i+P_j}\cdot\exp\left(\frac{\xi_i+\xi_j}{2}\right) \quad , \\ \xi_i &= P_ix - P_i^3t + \xi_{i0},\ i,j=1,\dots,N,\ P_i>0,\ P_i \neq P_j \quad . \end{aligned} \tag{1.5}$$

In (1.5) N is the number of negative eigenvalues λ_i of L, $P_i = 2\sqrt{-\lambda_i}$ and the ξ_{i0} are arbitrary real constants. For a more complete treatment of such solutions including the interaction of N solitary waves, see [1.5]. An elementary purely algebraic approach to the study of the multisoliton solutions interacting with an arbitrary background solution is developed in [1.6].

The immediate application of the inverse spectral method to the periodic initial data was not possible, because, for a long time, there were no effective methods for solving the inverse spectral problem for linear operators with periodic coefficients. It is well-known that the spectrum of the Schrödinger operator with a periodic potential has a zone-structure, i.e., it consists of a sequence of segments $[E_{2k+1}, E_{2k+2}]$, $k=0,1,\dots$, of absolutely continuous components of the spectrum separated by gaps (lacunas)

$$(-\infty, E_1),\ (E_2,E_3),\ \dots\ ,\ (E_{2k},\ E_{2k+1}),\ \dots\ .$$

The length of the kth gap tends to zero as $k\to\infty$ for all continuous periodic potentials. It can happen that the number of lacunas in the spectrum of the Schrödinger operator is finite. The corresponding potentials are called finite-gap potentials.

The role of finite-gap potentials for the analysis of the KdV equation was independently (and from different view-points) discovered by *Novikov*, *Matveev*, *Lax* and *Marchenko* [1.7-11]. It occurred that all finite-gap potentials can be described as solutions of the stationary higher KdV equations (Novikov's equations [1.7]). Then, Matveev noticed that in a paper by *Akhiezer* [1.12] the description of some special class of finite-gap potentials was reduced to the Jacobi inversion problem on the two-sheeted Riemann surface X of an algebraic curve

$$\mu^2 = \prod_{j=1}^{2g+1}(\lambda - E_j) \quad , \tag{1.6}$$

where the $E_j's$ are the boundaries of the nondegenerate gaps. This inverse problem was solved by Akhiezer for $g=1$, i.e., for the case of an elliptic curve. An important observation of Akhiezer was that the pair of Bloch solutions $\psi_{1,2}(x,\lambda)$, $\psi_{1,2}(0,\lambda)=1$ for the finite-gap potentials under consideration may be interpreted as elements of the analytic function $\psi(x, P)$, $P \in X$, with g poles P_j which do not depend on x. In the neighborhood of the point $\lambda = \infty$ this function has the asymptotic behavior

$$\psi(x, P) \cong \exp(\mathrm{i}x\sqrt{\lambda}), \quad \lambda = \pi(P) \quad ,$$

and $\pi(P)$ is the canonical projection of the point $P \in X$ onto the complex plane. These two conditions – pole structure and asymptotics at infinity – define ψ (normalized by $\psi(0, P) = 1$) uniquely.

Analysis of Akhiezer's paper, performed independently by *Dubrovin* [1.13, 14] and *Its* and *Matveev* [1.15, 16], showed that it is possible to generalize Akhiezer's approach, and to solve the related Jacobi inversion problem explicitly. Historically, the main point in this analysis is the study of the spectral problem with zero boundary conditions. Let us denote by λ_i the eigenvalues of the Sturm-Liouville problem with the periodic potential $u(x) = u(x+T)$ and zero boundary conditions, lying in nondegenerate gaps. By $\lambda_i(\tau)$ we denote the eigenvalues of the problem

$$\frac{d^2y}{dx^2} + u(x+\tau)y = -\lambda y, \qquad y\big|_{x=0} = y\big|_{x=T} = 0$$

also lying in nondegenerate gaps:

$$\overline{E_1 \quad E_2} \quad \dot{\lambda}_1 \quad \overline{E_3 \quad E_4} \quad \ldots \quad \overline{E_{2g-1} \quad E_{2g}} \quad \dot{\lambda}_g \quad \overline{E_{2g+1} \quad \infty}$$

The functions $\lambda_i(\tau)$ are periodic with period T and are related to the potential $u(\tau)$ by the following trace formula:

$$u(\tau) = 2\sum_{i=1}^{g} \lambda_i(\tau) - \sum_{j=1}^{2g+1} E_j \quad . \tag{1.7}$$

The quantities $\lambda_i(\tau)$ may be found from the solution of the Jacobi inversion problem. The final expression for this potential $u(\tau)$ with a continuous spectrum is of the form

$$\begin{aligned} u(x) &= 2\frac{d^2}{dx^2} \ln \theta(\boldsymbol{V}x + \boldsymbol{D}) + c \quad , \\ \theta(\boldsymbol{p}) &= \sum_{\boldsymbol{k} \in \mathbb{Z}^g} \exp\{\frac{1}{2}\langle B\boldsymbol{k}, \boldsymbol{k}\rangle + \langle \boldsymbol{p}, \boldsymbol{k}\rangle\}, \quad \boldsymbol{p} \in \mathbb{C}^g \end{aligned} \tag{1.8}$$

first found in [1.15, 16]. B is the period matrix of the Riemann surface X of the curve (1.6) and E_j are the boundaries of the continuous spectrum of the corresponding Schrödinger operator. All objects entering (1.8) are described by simple explicit formulas, which are presented in Chap. 3. Independently, the reduction to the Jacobi inversion problem in the spectral analysis of the finite-gap Hill's equation has been found by *McKean* and *van Moerbeke* [1.17]. Later an infinite-gap case was investigated by *McKean* and *Trubowitz* [1.20].

For the potential to be a solution of the KdV equation, $\boldsymbol{D}(t)$ must be a linear function of t

$$\boldsymbol{D}(t) = \boldsymbol{D}(0) + \boldsymbol{W}\, t \quad ,$$

where the vector $\boldsymbol{W}$ is completely determined by the boundaries of the spectrum, see *Dubrovin* [1.14]. Naturally, these solutions were called the finite-gap solutions of the KdV equation.

Solutions of the Schrödinger equation with the potential of the form (1.8) may be constructed explicitly. The associated formula

$$\psi(x,P) = \frac{\theta\left(\int_{\infty}^{P}\omega + \boldsymbol{U}x + \boldsymbol{D}\right)\cdot\theta(\boldsymbol{D})}{\theta\left(\int_{\infty}^{P}\omega + \boldsymbol{D}\right)\cdot\theta(\boldsymbol{U}x + \boldsymbol{D})}\exp(\Omega_1(P)x) \quad , \tag{1.9}$$

$$P = (\mu, \lambda) \quad ,$$

where $\Omega_1(P)$ is an Abelian integral of the second kind, to be defined in Chap. 3, was found by *Its* and applied in [1.18] to solve the KdV equation. For the subsequent development of the theory this formula proved to be very important. Moreover after the papers [1.21-23] by *Krichever* the ψ-function itself has become the main tool of the theory. We now wish to end our historical remarks related to the first steps of the theory created around 1975 (see also [1.1, 2]) and turn to the contents of this volume.

1.2 Outline of the Content of This Book

Chapter 2 contains some preliminaries related to the theory of Riemann surfaces, theta functions, Abelian integrals, and automorphic functions. It is written in a self-contained manner to enable the reader, possessing a minimal knowledge of elementary complex analysis, to understand the contents of the next chapters and to use these tools in further investigations. For further details the textbooks by *Farkas* and *Kra* [1.24], *Fay* [1.25] and *Mumford* [1.26] could be consulted.

Chapter 3 describes the application of the analysis of in Chapter 2 to the integration of the Korteweg-de Vries and Kadomtsev-Petviashvili

$$\pm 3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x \tag{1.10}$$

equations. Historically (1.1), (1.10) were the most important examples of applications of the methods of algebraic geometry in the 1970's. In our exposition we do not follow the historical development. First, following the idea of *Krichever* [1.21-23], we derive the linear partial differential equations for the so-called Baker-Akhiezer functions. Next we use the fact that the conditions of compatibility of these linear partial differential equations lead to nonlinear evolution equations for their coefficients. Explicit expressions of the Baker-Akhiezer functions in terms of the Riemann theta functions, generalizing (1.9), enable one to compute also the coefficients of associated linear partial differential equations (PDE) and, hence, to solve the related nonlinear evolution equation. In such a way, we get quickly a multi-parametric family of solutions of the Kadomtsev-Petviashvili equation generated by an arbitrary compact Riemann surface (a result first obtained by Krichever). Specifying the surface and the parameters of the construction, we arrive at the solutions of the Korteweg-de Vries equation and Boussinesq equation expressed in terms of the multi-dimensional Riemann theta functions. Generally speaking, the obtained solutions are complex-valued and have pole singularities at some points. In order to make the solutions real-valued and non-singular, we have to impose some complementary restrictions on the parameters of the construction found by *Dubrovin* and *Natanzon* [1.27] for the Kadomtsev-Petviashvili equation. This is realized for both ($\pm$) possibilities of (1.10), usually referred to as KP2 and KP1 equations. In the next part of Chapter 3 we discuss the spectral properties of the solutions of the KdV equation. More exactly, we discuss the spectral properties of the Schrödinger operator, where the potential is given by the theta function solution of the KdV. We establish that the real, non-singular, x-periodic solution $u(x,t)$ of the KdV equation, generated by the hyperelliptic curve (1.6), being considered as a potential function for the Schrödinger operator $L = -\partial_x^2 - u(x,t)$ in $L_2(-\infty,\infty)$, leads to a very specific spectral structure. Namely a number of energy gaps in the continuous spectrum of L is finite, despite the fact that for generic periodic potentials we have an infinite number of energy gaps. In the same section we also prove that the inverse statement is true. All finite-gap periodic potentials may be represented in the form of the second derivative of the logarithm of the Riemann theta function. The matrix B generating this theta function is a matrix of the B-periods of the hyperelliptic Riemann surface X defined by (1.6). In (1.6) g is the number of the energy gaps coinciding with the genus of the Riemann surface; E_j are the boundaries of the continuous spectrum. We show that the so-called Bloch solutions of the finite-gap periodic Schrödinger operator have the same analytic properties as the projections of the Baker-Akhiezer function (1.9) on the upper and lower sheets of the Riemann surface X. Sometimes the g-gap periodic potential may be expressed in terms of elliptic functions. This rather non-trivial phenomenon is discussed in generality in Chapter 7. In Chapter 3 we prove only that the Lamé potential $u(x) = -g(g+1)\cdot\wp(x)$, where $\wp(x)$ is a Weierstrass elliptic function, creates exactly g gaps in the continuous spectrum of operator L. Following a derivation due to Hermite, we show that the associated Bloch solutions are expressed in terms of elliptic functions. We include the almost-forgotten remarkable contribution of Drach in which he was the first to isolate the "integrable" Sturm-Liouville equations with the following property: there exists a fundamental system of solutions $\psi_{1,2}$ such that a product $\psi_1(x,\lambda)\psi_2(x,\lambda)$ is a polynomial of degree g in λ. This property is in particular necessary and sufficient for distinguishing the g-gap periodic potentials from the periodic potentials of the general position, as was shown by *Its* and *Matveev* [1.15, 16]. Various steps of the work of Drach were rediscovered around 1974 by Dubrovin, Gelfand and Dikii, and by Its and

Matveev in connection with analysis of finite-gap solutions of the KdV equation 55 years later. The solution of the inverse spectral problem for finite-gap periodic potentials using [1.15, 16] completes Chapter 3.

The fourth chapter extends the methods of algebro-geometric integration on the nonlinear evolution equations with matrix Lax pairs. Namely, the nonlinear Schrödinger equation (NS equation)

$$iy_t + y_{xx} - 2\sigma|y|^2 y = 0, \quad \sigma = \pm 1 \quad , \tag{1.11}$$

and the sine-Gordon equation (SG equation)

$$v_{xt} = -4 \sin v \tag{1.12}$$

are analyzed thoroughly. Actually, the same methods are successfully applied to the numerous nonlinear systems having very different physical origin. But the NS and SG equations seem to be the most important equations with regard to physical applications and are at the same time the simplest from a technical viewpoint. Particularly, we can say that the NS equation has the same degree of universality as the KdV equation both from a mathematical and physical viewpoint. In this connection it seems to be quite natural that it was the second (after the KdV equation) evolution equation discovered to be integrable by the inverse scattering approach, as it was first shown by *Zakharov* and *Shabat* [1.28] in 1971. The inverse scattering program for the SG equation was realized two years later by *Ablowitz, Kaup, Newell, Segur* [1.29], *Faddeev, Zakharov, Takhtajan* [1.30] and *Takhtajan* [1.31]. The algebro-geometric integration of the NS and SG equations was realized by *Its*, *Kozel* and *Kotlyarov* [1.19, 32] in the spirit of the same spectral approach as applied to the KdV equation. At the same time *Dubrovin* [1.13] considered general matrix $U - V$ pairs and associated linear systems in an algebro-geometric framework. Here we present the modern version of the algebro-geometric solution of (1.11,12) based on the notion of the vector-valued Baker-Akhiezer function introduced by *Krichever* [1.22].

Most attention in this chapter is concentrated on the special problems related to the matrix nature of the associated *U-V* systems. The first of these problems is the necessity to take into account the reduction restrictions on the Baker-Akhiezer functions isolating *U-V* pairs under consideration from the related gauge classes. Associated reduction restrictions may be divided into two different types: involutions accompanied by holomorphic transformations of the spectral variable λ and antiinvolutions $\lambda \to \overline{\lambda}$. The account of the reductions of the first type leads to the appearance of algebraic curves with non-trivial involutions and some coverings over algebraic curves. Reductions of the second type – antiinvolutions – are basically the same as for the scalar case when we try to isolate smooth and real solutions of the KdV or the KP equations. The main problem here is to transfer the reduction restrictions on the parameters specifying the finite-gap solution, given by a non-special divisor $\mathcal{D}$ on the algebraic curve X. In the KdV case and in the NS case with $\sigma = 1$ this problem is trivial by virtue of the possibility of identifying the points of the divisor $\mathcal{D}$ with the eigenvalues of some self-adjoint Sturm-Liouville problem. The absence of such a possibility in the general case, in particular, in the NS case with $\sigma = -1$, and in the sine-Gordon case, considerably changes the situation. In the work of *Kozel* and *Kotlyarov* [1.32] it was realized that associated conditions in terms of the divisor $\mathcal{D}$ for the NS case with $\sigma = -1$ and for the sine-Gordon case take the form of complicated nonlinear systems of algebraic equations. Later Cherednik presented an algebro-geometric interpretation of the Kozel-Kotlyarov conditions exhibiting the same level of effectiveness. A qualitative jump in the solution of the (real) problem was given by *Forest* and *McLaughlin* [1.33], *Dubrovin* and *Natanzon* [1.35], *Belokolos* and *Enol'ski* [1.34] (SG case) and by *Dubrovin* and *Novikov* [1.36] (NS case), who realized that it is more convenient to isolate the real solutions in terms of $\boldsymbol{A}(\mathcal{D})$ – the image of $\mathcal{D}$ on $\boldsymbol{J}(X)$ – the Jacobian of the associated Riemann surface X. It is important to mention that $\boldsymbol{A}(\mathcal{D})$ appears in the theta function representations of the finite-gap solutions in a universal and natural way. In terms of $\boldsymbol{A}(\mathcal{D})$ the problem transforms to a classical question of algebraic geometry: the description of real and imaginary components of the Jacobian in a real algebraic curve. However, it is necessary to mention that the detailed study of $\boldsymbol{A}(\mathcal{D})$ does not solve all problems in the theory of real finite-gap solutions. In particular, in the Hamiltonian aspect of the theory, the structure of the divisor $\mathcal{D}$ itself is of major importance. That is why, despite the fact that the reality problem is solved on the level of the explicit formulas for the finite-gap solutions, its Hamiltonian aspect is still interesting and not yet completely understood.

The nonlinear systems studied in Chap. 4 are the simplest from the point of view of algebraic geometry. We hope that their detailed study will lead the reader quickly and naturally to the understanding of the main tools and tricks of the method, which are used later for a study of more complicated integrable matrix systems arising in Chap. 6. The Hamiltonian aspect of the theory of finite-gap solutions of the NS and SG equations does not appear in our discussion. In particular, the dynamics of the zeros of the Baker-Akhiezer function and the problem of the action-angle variables on the variety of the real finite-gap solutions are not described. The interested reader may find corresponding results in the article by *Novikov* [1.37].

Finite-gap solutions describe the nonlinear interaction of several modes. Despite the existence of an explicit theta function representation of these solutions, an appropriate approach in using them in numerical computations was developed only recently, see Chap. 5. All main physical characteristics of the finite-gap solutions (wave numbers, phase velocities, amplitudes of the interacting modes) are defined by the compact Riemann surface X. Such a parametrization is complicated and seems to be rather ineffective for the investigation of the solutions and for numerical computations. Probably this fact is one of the principle obstacles in practical applications of the finite-gap solutions.

Several papers deal with the problem of the effective construction of the theta function solutions. *Dubrovin* and *Novikov* [1.38] suggested "an algebro-geometric effectivization". The papers by *Nakamura* and *Boyd* [1.39-41] are devoted to a "physical effectivization". All these papers are based on the substitution of the theta-function formulas for the solutions into a nonlinear equation. Here the spectral origin of the parameters is "forgotten" and they are determined directly from the equation. The 2-phase solutions of the KP equation were investigated by *Segur* and *Finkel* [1.42] in the framework of the algebro-geometrical effectivization method. One should note that the substitution technique is applicable only for the construction of 2-phase solutions for $1+1$ integrable nonlinear evolution equations (and for the construction of 3-phase solutions of $2+1$ KP-like equations). Let us also mention that a portrait of the 2-phase solutions of the KdV equation appeared in the book of *Mumford* [1.26], and that the periodic solutions of the KdV equation were studied numerically by *Osborne* and *Segré* [1.43], exploring the Dubrovin-Drach equations for the Dirichlet eigenvalues for arbitrary genus.

The theta functional genus 2 solutions of the KP equation were compared by *Hammack*, *Scheffner* and *Segur* [1.44] with experimental gravitation water waves generated in a basin. The experimental waves were found to be described by them with reasonable accuracy.

A universal (in relation to the number of interacting phases and to the type of the nonlinear equation) approach to the effectivization problem, suggested by *Bobenko* [1.45-48], is presented in Chap. 5. It is based on the Schottky uniformization theory of the Riemann surfaces. The advantage of this approach lies in the fact that all the principal ingredients of the formulas for the finite-gap solutions (holomorphic differentials, period matrix, vectors of B-periods, etc.) are expressed in terms of the uniformization parameters with the help of the Poincaré theta series. In this way all the physically important real solutions are effectively described. The corresponding plots can be found in Chap.5 and in [1.47].

Chapter 6 deals with applications of theta functions to some problems of classical mechanics. First applications of the Jacobi theta functions to the classical top date back to Euler in the 18th century. Euler solved the equations of motion of a rigid body around its center of mass. The multi-dimensional theta functions were first applied by C. Neumann in 1859 while solving the equations of motion of a particle constrained on a sphere under the action of a quadratic potential. The most famous mechanical system of this kind is the Kowalewski top [1.49], which was in the focus of interest in the 19th century.

Despite the discovery of numerous examples of finite dimensional systems integrable in terms of multi-dimensional theta functions, there was at that time no general approach to solving the equations of motion of these systems. Each time the success in integration was based on finding a rather non-trivial change of variables leading to a Jacobi inversion problem. After Kowalewski the most important results in this direction were obtained by *Kötter* [1.50 - 52].

In Chap. 6 we solve the equations of motion of classical tops with the help of finite-gap integration theory. Such an application of the modern theory gives the possibility to obtain important new results even for the classical tops [1.53, 54, 55]. Some of the formulas for the solutions presented in here are simpler than the classical ones.

The theory of finite-gap integration for finite dimensional systems is based on the representation of the equation of motion in Lax form

$$\frac{d}{dt}L(\lambda) + [L(\lambda), A(\lambda)] = 0 \quad .$$

When the Lax representation is found, all the machinery of finite-gap integration theory may be used. At the same time the construction of the Lax representation for the concrete system is a transcendental problem, solved, however, now for all famous tops studied in classical papers. The recent results in this direction are various Lax pairs for the Kowalewski top. One of them found by *Reymann* and *Semenov-Tian-Shanski* [1.79] is used in Chap. 6 for the integration of the equations of motion.

It should be mentioned that there now exists a direct approach to solving integrable systems with two degrees of freedom. This approach was devised by *Adler* and *van Moerbeke* [1.56]. It is based on the study of singularities of the solutions and goes back to fundamental ideas of the *Kowalewski* paper [1.49], inspired by a letter of *Weierstrass* [1.57], who first proposed the idea of finding the integrable cases by analyzing the singularities of the solutions. With the help of the direct approach, the geometry of algebraic curves and Abelian tori arising in various problems of classical mechanics, was investigated in [1.58-62], where important isomorphisms of several tops are found. The advantage of the direct approach is that it starts directly from the equations of motion without an a priori knowledge of the Lax representation. However, the theta function formulas were not derived in this way.

As in Chaps. 3 and 4 the main instrument used in constructing solutions is the Baker-Akhiezer function. The essential difference with the corresponding results of Chaps. 3 and 4 is that in Chap. 6 we obtain the general solution of the problem instead of a particular family of solutions. It is interesting that in some cases the Baker-Akhiezer function itself appears to be very useful for the description of dynamics. The motion in the laboratory frame is described in this way (Sect. 6.9).

It is worth mentioning also that the Lax representations of all classical systems considered in Chap. 6 have non-trivial reduction groups. This in turn implies that the associated spectral curves represent complicated coverings and that the Baker-Akhiezer function has specific analytic properties. As a final result we have a more complicated integration procedure.

Chapter 7 describes the solution of the following problem: what are the restrictions on the Riemann surfaces which allow reduction of the multi-dimensional Riemann theta functions to the theta functions of the lower genera? For the first time it was pointed by *Belokolos* and *Enol'skii* [1.34], that the solution of this problem is closely related to the study of the reduction of the Abelian integrals to elliptic integrals. The possibility of expressing the solutions of high genus by means of one-dimensional theta functions is important for many reasons. First of all, such solutions are the simplest possible from the point of view of numerical calculations. Another point is that such solutions are closely related to different finite dimensional dynamical systems and the description of different physical phenomena. The particular significance of the obtained solutions lies in the fact that they may satisfy some complementary restrictions of physical importance. In particular, 3-gap generalizations of the Bianchi-Lamb ansatz presented in this chapter satisfy the periodicity requirement with respect to the space variables. The content of this chapter is based mainly on works by *Belokolos* and *Enol'skii* [1.63, 64], *Babich, Bobenko* and *Matveev* [1.65, 66] partially reviewed in [1.67].

Following *Igusa* [1.68] the chapter starts with a formulation of the Poincaré theorem on complete reducibility, which is a further development of the Weierstrass reduction theory of multi-dimensional theta functions to Jacobi theta functions. The finite-gap solution of an algebro-geometrically integrable equation is reducible to Abelian functions of lower genera if the associated algebraic curve is a covering over an algebraic curve of a lower genus. The derived solution turns out to be quasi-periodic. To obtain periodic solutions (elliptic solitons [1.69]) one has to impose an additional commensurability condition on the components of the winding vectors.

It is shown that in the case of hyperelliptic curves all reduction conditions can be expressed by demanding some of the theta-constants to vanish. Special varieties in moduli space – such as Humbert varieties (see, e.g., *van der Geer* [1.70]) are considered.

Elliptic potentials of the Schrödinger equation are considered in this chapter and the necessary and sufficient conditions for the elimination of such potentials from the general theta functional formula for finite-gap potentials are given, some examples of the elliptic potentials [1.69, 71] are derived deductively and all characteristics of the corresponding coverings are computed.

There are numerous applications of the finite-gap integration theory in physics and mathematics. One of them, in condensed matter physics as suggested by *Belokolos* [1.76, 77] is dealt with in Chap. 8. An exact description is presented for the Peierls state, which is a collective bound state of electrons and phonons in a one-dimensional conductor. In particular it is proved rigorously that in the Peierls state the phonons produce a finite-gap potential for the electrons. It is shown also that the Peierls state is a lattice of solitons (polarons) at low densities of electrons and a charge density wave at high densities of electrons. Therefore two basic concepts of solid state physics merge here. The rigorous theory of Fröhlich conductivity due to the uniform motion of the Peierls state is also developed. Thus we have one more exactly solvable one-dimensional many-body problem similar to that considered by Bethe, Lieb and Wu, Wiegmann and some others.

The material for this book has been prepared over a long period of time. Originally it was supposed to be the work of authors based in Leningrad and Kiev. The basic part was indeed prepared during the "Leningrad-Kiev" period, when the contact between the authors was rather close. But in the end the collaboration was international in character. Now the USSR does not exist any more, Leningrad changed its name back to the historical one, St. Petersburg, and now the authors reside not just in different cities but even in different countries. To make it easier for the reader to contact *the* competent author with questions or suggestions, we now give a list indicating who feels responsible for which chapter:

Naturally, Chapter 1 has been delt with by the complete team of authors.
Chapter 2 falls into the responsibility of Bobenko, Enol'ski and Its.
Chapter 3 resulted from the joint efforts of Bobenko, Its and Matveev.
Chapter 4 was prepared by Its and Matveev.
Chapters 5 and 6 are due to Bobenko.
Chapter 7 was written by Belokolos and Enol'lski.
Chapter 8 emerged from the work of Belokolos.

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2. Riemann Surfaces and Theta Functions

In this chapter we present some useful information on Riemann surfaces and theta functions. A detailed account of the points discussed below is given by *Hurwitz* and *Courant* [2.1], *Forster* [2.2], *Farkas* and *Kra* [2.3], *Krazer* [2.4], *Krazer* and *Wirtinger* [2.5], *Igusa* [2.6], *Fay* [2.7], *Mumford* [2.8], *Griffith* and *Harris* [2.9].

2.1. Riemann Surfaces

A Riemann surface X is a connected two-dimensional topological manifold with a complex-analytic structure on it. The latter implies that for each point $P \in X$ there is a homeomorphism $\varphi : U \to V$ of some neighborhood $U \ni P$ onto an open set $V \subset \mathbb{C}$, and it is defined so that any two such homeomorphisms φ, $\tilde{\varphi}$ with $U \cap \tilde{U} \neq \emptyset$ are holomorphically compatible, i.e., the mapping $\varphi \circ \tilde{\varphi}^{-1} : \tilde{\varphi}(U \cap \tilde{U}) \to \varphi(U \cap \tilde{U})$, called a transition function, is holomorphic. In what follows, the homeomorphism φ will be referred to as a local parameter. Any set $\{\varphi_i\}$ of holomorphically compatible local parameters such that the appropriate neighborhoods $\{U_i\}$ cover the entire manifold X is called a complex atlas of the Riemann surface X. The union of the atlases that correspond to the same complex-analytic structure on the manifold X, i.e., to the same Riemann surface X, is again an atlas. This property is violated if the atlases making up a union belong to different complex-analytic structures or, which is equivalent, to different, yet topologically identical Riemann surfaces. The number of nonequivalent complex structures on a given two-dimensional manifold, or, equivalently, the number of conformally nonequivalent Riemann surfaces topologically isomorphic to the same two-dimensional manifold is directly linked to the fundamental topological characteristic of orientable two-dimensional manifolds – the genus. We discuss this point in greater detail below.

The simplest examples of Riemann surfaces are any open subset of a complex plane $\mathbb{C}$, the complex plane $\mathbb{C}$ itself and an extended complex plane $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ with complex-analytic structures naturally introduced on them. Non-trivial examples of Riemann surfaces are provided by nonsingular algebraic curves, i.e., sets of points in $\mathbb{C}^2$ defined by equations such as

$$\mathcal{P}(\mu, \lambda) = 0, \qquad (\mu, \lambda) \in \mathbb{C}^2 \quad , \tag{2.1.1}$$

where $\mathcal{P}$ is the polynomial in its arguments that satisfies the condition

$$\text{grad}\ \mathcal{P}\Big|_{\mathcal{P}=0} = \left(\frac{\partial\mathcal{P}}{\partial\mu}, \frac{\partial\mathcal{P}}{\partial\lambda}\right)\Big|_{\mathcal{P}(\mu,\lambda)=0} \neq 0 \ .$$

Furthermore, the curves (2.1.1) essentially exhaust all compact Riemann surfaces. A complex-analytic structure is introduced on the algebraic curve (2.1.1) as follows: the variable λ is taken to be a local parameter in the neighborhoods of the points where $\partial\mathcal{P}/\partial\mu \neq 0$, and the variable μ is a local parameter in the neighborhoods of the points where $\partial\mathcal{P}/\partial\lambda \neq 0$. The holomorphic compatibility of the local parameters introduced results from a complex-analytic version of the implicit function theorem [2.1]. In the important special case of a hyperelliptic curve [1]

$$\mu^2 = \prod_{j=1}^{N}(\lambda - E_j),\ N \in \mathbb{N},\ E_j \in \mathbb{C},\ E_j \neq E_k, \quad j,k = 1,\dots,N, \qquad (2.1.2)$$

the choice of local parameters can be additionally specified. Namely, in the neighborhoods of the points (μ_0, λ_0) with $\lambda_0 \neq E_j\, \forall j$, the local parametrization is defined by the homeomorphism $(\mu, \lambda) \to \lambda$; in the neighborhood of each point $(0, E_j)$ by the homeomorphism

$$(\mu, \lambda) \quad \to \quad \sqrt{\lambda - E_j} \ .$$

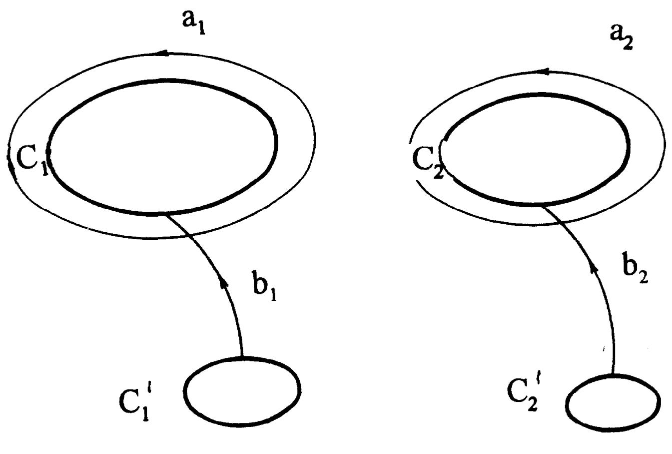

Fig. 2.1. Riemann surface of genus two realized by the action of the Schottky group on the 4-connected region in $\mathbb{CP}^1$

We now consider another fundamental example of a Riemann surface. Let Δ be a region of an extended complex plane, with a group G of holomorphic transformations acting discontinuously on it. The discontinuity of the action of the group implies that for $\forall P \in \Delta$ there exists a small neighborhood $U \ni P$ such that $gU \cap U = \emptyset,\ \forall g \in G,\ g \neq I$. In this case we can introduce an equivalence

[1] When $N = 3$ or 4, the curve (2.1.1) is called elliptic.

relation between the points Δ : $P \sim P' \Leftrightarrow \exists g \in G$, $P' = gP$ and consider the quotient space Δ/G. This is a Riemann surface with a complex structure introduced as follows: Let $\{U_n\}$ denote a system of neighborhoods covering Δ such that $\forall g \in G$, $gU_n \cap U_n = \emptyset$. The appropriate neighborhoods on the quotient space are $\pi(U_n)$, where $\pi : \Delta \to \Delta/G$ is the natural projection associating each point $P \in \Delta$ to its equivalence class. The local parameters are defined as $\varphi_n = (\pi\, |_{U_n})^{-1}$. Next we note that $\pi(U_n) \cap \pi(U_m) \neq \emptyset \Leftrightarrow \exists g_{nm} \in G$:

$$U_n \cap g_{nm} U_m \neq \emptyset.$$

This yields the holomorphic compatibility of the appropriate local parameters φ_n and φ_m, i.e.,

$$\varphi_m \circ \varphi_n^{-1} = g_{nm}^{-1}\Big|_{U_n \cap g_{nm} U_m}.$$

Let F be a $2N$-connected region in $\mathbb{CP}^1$, bounded by closed nonintersecting Jordan curves C_1, C_1', $\cdots$, C_N, C_N' (Fig. 2.1). The linear fractional transformation $\sigma_n z = (\alpha_n z + \beta_n)/(\gamma_n z + \delta_n)$ maps the exterior of C_n into the interior of C_n'. A system of generators $\sigma_1, \cdots, \sigma_N$ induces a Schottky group G, i.e., a group free of relations. The limiting set of the group, $\Lambda(G)$, is a closure of the set of fixed points of elements of G. Its complement $\Omega(G) = \mathbb{CP}^1 \setminus \Lambda(G)$ is the domain of discontinuity of the group, i.e., the domain on which the group acts discontinuously. Therefore, $\Omega(G)/G$ is a Riemann surface.

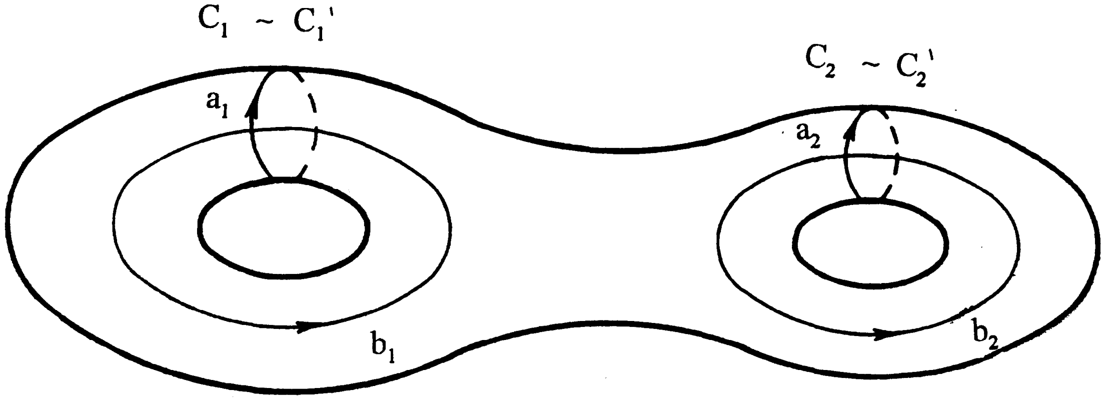

Fig. 2.2. Riemann surface of genus two realized as a Riemann sphere with handles

Let F be a fundamental domain of G, i.e., a domain that contains exactly one point of each orbit of G. This proposition implies that (1) $F \cap gF = \emptyset, \forall g \in G$, $g \neq 1$ (2) $\cup_{g \in G} gF = \Omega(G)$. The Riemann surface $\Omega(G)/G$ may therefore be regarded as the fundamental domain F with identified boundaries $C_n \sim C_n'$. Pasting pairwise $2N$ curves C_n with C_n' on a Riemann sphere, we see that the resulting Riemann surface is compact and equivalent to a sphere with N handles (Fig. 2.2). The number of handles g is called the genus of the Riemann surface. The following general theorem holds:

Theorem 2.1. Any compact Riemann surface is topologically equivalent to a sphere with a finite number of handles.

For any Riemann surface X the notion of a function being holomorphic on some open subset $U \subseteq X$ is well-defined. More specifically, the mapping $f : U \to \mathbb{C}$ is said to be holomorphic if the local representation of the mapping f, i.e., the mapping $f \circ \varphi_j^{-1} : \varphi_j(U_j \cap U) \to \mathbb{C}$, is a holomorphic function in the usual sense for any local parameter φ_j with $U_j \cap U \neq \emptyset$. Thus, a function which is defined on the hyperelliptic curve (2.1.2) and holomorphic in a neighborhood of the point (μ_0, λ_0) with $\lambda_0 = E_j$ can be represented by a convergent Taylor series in integral powers of the variable $(\lambda - E_j)^{1/2}$.

The theory of holomorphic and, more generally, meromorphic functions (an exact definition will be given below) are the main objects of study in the theory of Riemann surfaces. The content of the following analysis is largely dependent on whether the Riemann surface under study is a compact topological variety or not. We shall henceforth assume the Riemann surfaces under consideration to be compact, unless otherwise stated. As far as algebraic curves are concerned, we assume them to be compactified by joining points at infinity. Thus, for the hyperelliptic curve (2.1.2) there are two points at infinity when $N = 2g + 2$, and one such point when $N = 2g + 1$. In the first case, infinitely remote points will be denoted by the symbols ∞^+ and ∞^-, and in the second case by the symbol ∞. The points $\infty^\pm$ are distinguished by the conditions

$$P \equiv (\mu, \lambda) \to \infty^\pm \Leftrightarrow \lambda \to \infty,\ \mu \sim \pm\lambda^{g+1} \quad ,$$

and the local parameter in the neighborhoods of both points is given by the homeomorphism

$$(\mu, \lambda) \to \lambda^{-1} \quad .$$

Similarly, the point ∞ is distinguished by the condition

$$P \equiv (\mu, \lambda) \to \infty \Leftrightarrow \lambda \to \infty,\ \mu \sim \lambda^{g+1/2}$$

and the local parameter in its neighborhood is $\lambda^{-1/2}$.

A cycle on a Riemann surface is an oriented closed curve (not necessarily consisting of a connected component). The cycles can be added and subtracted (by changing orientation), so that the structure of an Abelian group may be introduced on them. Let us define the following equivalence relation: the cycle is equivalent to zero, if it is a boundary of a domain on a Riemann surface (with orientation taken into account). Two cycles are equivalent if their difference equals zero. The above Abelian group factorized under this equivalence relation is said to be the first homology group of the Riemann surface X and is denoted by $H_1(X, \mathbb{Z})$.

We assign to every intersection point of the cycles γ_1 and γ_2 the number 1 if the intersection is as in Fig. 2.3, and -1 if the intersection is as in Fig. 2.4. The

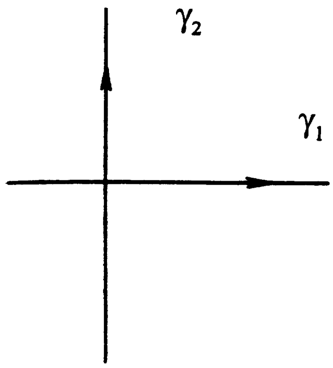

Fig. 2.3. Cycle intersection index $\gamma_1 \circ \gamma_2 = 1$

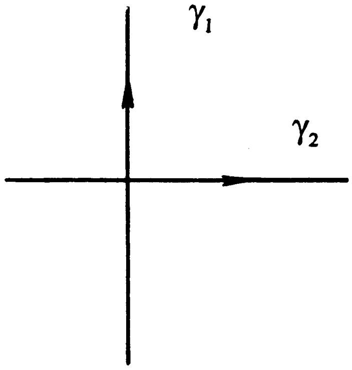

Fig. 2.4. Cycle intersection index $\gamma_1 \circ \gamma_2 = -1$

cycle intersection index $\gamma_1 \circ \gamma_2 = -\gamma_2 \circ \gamma_1$ is the sum of these numbers taken over all intersection points. The notion of an intersection index is extended to the elements of $H_1(X, \mathbb{Z})$.

A basis in $H_1(X, \mathbb{Z})$ is a set of cycles $\gamma_1, \ldots, \gamma_{2g}$ such that any cycle γ, except a zero one, can be represented as a non-trivial integral combination of them. A canonical basis of the cycles of a Riemann surface X of genus g or a canonical basis in $H_1(X, \mathbb{Z})$ is a basis of $a_1, b_1, \ldots, a_g, b_g$ cycles such that $a_n \circ a_m = 0, \quad b_n \circ b_m = 0, \quad a_n \circ b_m = \delta_{nm}$. It is obvious from Fig. 2.2 that such a basis always exists (and its choice is not unique). The canonical bases of cycles for the Riemann surfaces exemplified in the present section are given in Figs. 2.1, 5. In Fig. 2.5, the parts of the cycles that lie on the upper sheet are indicated by solid lines, and those on the lower sheet by broken lines. The cycles b_n in Fig. 2.1 run from the point $z_n \in C_n$ to its equivalent $\sigma_n z_n \in C'_n$. Note that it follows from Fig. 2.5 that a hyperelliptic Riemann surface has genus g if the number N in (2.1.2) is equal to $N = 2g + 1$ or $N = 2g + 2$. We conclude by noting that the examples of Riemann surfaces discussed in this section are of the most general character.

Theorem 2.2. Any compact Riemann surface can be represented as an algebraic curve.

Theorem 2.3. Any compact Riemann surface can be represented as Δ/G, where Δ is a domain in $\mathbb{CP}^1$ and G is a discontinuous group of conformal transformations.

The latter representation is said to be the uniformization of a Riemann surface. A Riemann surface can be uniformized in different ways, depending on the choice of Δ and G.

Both of the above theorems were proved within the context of the Riemann surface uniformization theory developed in the 19th century. The following problem was used as a starting point: Suppose that we have an algebraic curve $\mathcal{P}(\mu, \lambda) = 0$ which defines multi-valued functions $\lambda(\mu)$, $\mu(\lambda)$ via the appropriate

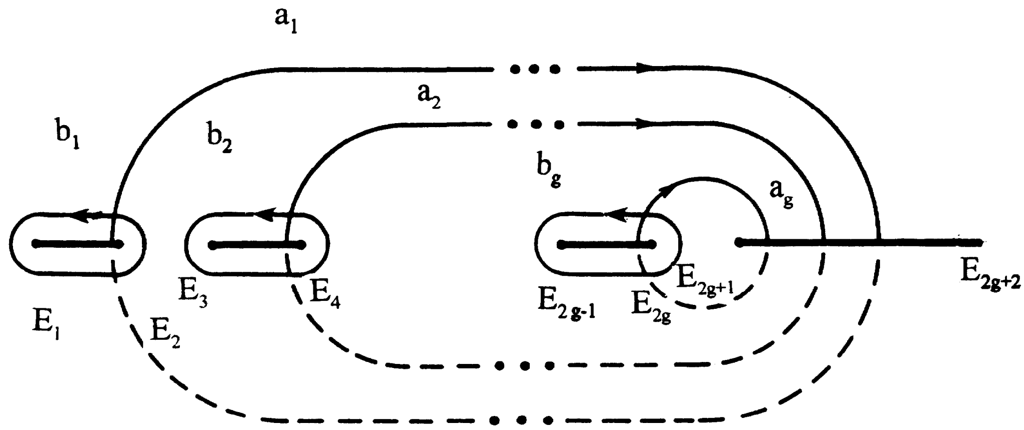

Fig. 2.5. Canonical bases of cycles for the Riemann surface of genus g. The parts of the cycles that lie on the upper sheet are indicated by solid lines and those on the lower sheet by broken lines

mappings $(\mu, \lambda) \to \lambda$, $(\mu, \lambda) \to \mu$. A new variable z must be found such that $\lambda(z)$ and $\mu(z)$ are single-valued functions of z and $\mathcal{P}(\mu(z), \lambda(z)) = 0$ is fulfilled identically. The problem was solved in terms of the discontinuous groups of conformal transformations of a complex plane, which were invented by Poincaré. For a detailed account of the results produced by this beautiful theory we refer the reader to the Appendix and [2.10]. In Sect. 2.3, we shall discuss the simplest non-trivial example of uniformization that served as a starting point in developing the theory.

2.2 Coverings

A natural generalization of the concept of a holomorphic function on a Riemann surface is the notion of a holomorphic mapping of one Riemann surface into another. Namely, the mapping f of a Riemann surface $\tilde{X}$ into a Riemann surface X is said to be holomorphic if any function $\varphi \circ f \circ \tilde{\varphi}^{-1}$ (local notation of the mapping f), where $\tilde{\varphi}, \varphi$ are arbitrary local parameters of the corresponding Riemann surfaces, is holomorphic in the usual sense. Non-constant holomorphic mappings of Riemann surfaces are holomorphic coverings. Let $f : \tilde{X} \to X$ be a holomorphic covering. We give its basic general properties:

(a) f is a surjective map
(b) every point $\tilde{P} \in \tilde{X}$ can be associated with the positive integer $n \geq 1$, which can be characterized as follows: for any neighborhood V of the point P there are neighborhoods $\tilde{U} \subset \tilde{V}$ of the point $\tilde{P}$ and U of the point $f(P)$ such that for any point $P \in U$, $P \neq f(\tilde{P})$ the set $f^{-1}(P) \cap \tilde{U}$ consists precisely of n elements. The number n is the multiplicity with which the mapping f takes the value $f(\tilde{P})$ at the point $\tilde{P}$, and it is denoted by $n(f, \tilde{P})$. The number of points at which $n > 1$ is finite, and they are called branch points of the covering f. The number $\nu = n(f, \tilde{P}) - 1$ for the branch point $\tilde{P}$ is referred

to as a branch number of f at $\tilde{P}$. If there are no branch points, the covering is said to be unramified.

(c) For any point $P \in X$, the set $f^{-1}(P)$ consists of a finite set of elements; moreover, the number $N = \sum_{\tilde{P}\in f^{-1}(P)} n(f, \tilde{P})$ is independent of the choice of the point P and is called the number of sheets of the covering f.

In the situation when there is a holomorphic covering $f : \tilde{X} \to X$, the following alternative terminology is often used: the surface $\tilde{X}$ is called a cover, the surface X is a base of a cover, and the mapping f a projection.

The consequence of the property (b) is that, for every point $\tilde{P} \in \tilde{X}$ which is not a branch point of the covering f, there exists a neighborhood $\tilde{U}$ such that the restriction $f\,|_{\tilde{U}}$ is a homeomorphism. From this and property (c) it follows that when the covering $f : \tilde{X} \to X$ is unramified, every point $P \in X$ has a neighborhood U such that

$$f^{-1}(U) = \bigcup_{j=1}^{m} \tilde{U}_j \quad ,$$

where $\tilde{U}_j \cap \tilde{U}_k = \emptyset$ and all restrictions $f\,|_{\tilde{U}_j}$ are homeomorphisms.

The notion of a holomorphic covering can be extended to the case when a covering surface is not compact. In that case the property (b) remains valid, the number N can take infinite values, and the property (a) should be included in the definition of a covering. In the unramified case, the definition of a covering also involves the property given at the end of the preceding paragraph.

There is an expression that makes it possible to calculate the genus of a cover, in terms of the genus of the base. Let $\tilde{P}_1, \ldots, \tilde{P}_k$ be branch points of an N-sheeted covering $\tilde{X} \to X$, and $\nu_1, \ldots, \nu_k$ branch numbers for these points, and let g be the genus of X. Then the genus $\tilde{g}$ of $\tilde{X}$ is given by the Riemann-Hurwitz formula

$$\tilde{g} = N(g-1) + 1 + \frac{1}{2}\sum_{i=1}^{k} \nu_i \quad . \tag{2.2.1}$$

An example of a holomorphic covering is the mapping

$$\pi(P) = \lambda, \quad P = (\mu, \lambda)$$

of the hyperelliptic Riemann surface (2.1.2) onto $\mathbb{CP}^1$. This is a ramified two-sheeted covering. The points $Q_j = (0, E_j)$ are its second-order branch points. The procedure of "crosswise pasting" two copies of the complex plane $\mathbb{C}$ with a system of cuts enabling one to extract a single-valued branch $\sqrt{\prod(\lambda - E_j)}$ is commonly used to interpret geometrically the covering π (see Fig. 2.5). The construction is a classical Riemann surface of the multivalued function $\mu(\lambda)$.

Among unramified holomorphic coverings there are special universal coverings, i.e., unramified holomorphic coverings for which a covering surface is simply-connected. The important point in Riemann surface theory is that for any compact Riemann surface X there is a canonical technique of constructing a simply-connected, generally noncompact Riemann surface X_0 that covers X holomorphically. The surface X_0 is called a universal covering of X. The canonical technique of constructing X_0 is as follows: we chose a point $P_0 \in X$, and for another arbitrary point $P \in X$ we denote by $\pi(P_0, P)$ a set of homotopical classes of curves γ_{PP_0} on X that begin at P_0 and end at P. Note that the curves γ_{PP_0} and $\tilde{\gamma}_{PP_0}$ are homotopic if the closed curve $\gamma_{PP_0}\tilde{\gamma}^{-1}_{PP_0}$ can be contracted in X to a point. Then the universal covering X_0 is defined as the set $\{(P, \gamma) : P \in X, \gamma \in \pi(P_0, P)\}$, and the projection $f_0 : X_0 \to X$ is given by the equality $f_0(P, \gamma) = P$.

Finally, we define another important class of coverings. Suppose that a group of holomorphic homeomorphisms (conformal automorphisms) $G : X \to X$ acts on the Riemann surface X. We note that for surfaces X of genus $g \geq 2$ the order of the full group G of automorphism of X is finite and can be estimated by the Hurwitz formula ([2.3]):

$$\operatorname{ord} G \leq 84(g-1) \quad . \tag{2.2.2}$$

If G is a finite group of automorphisms of X, then the quotient X/G is a Riemann surface and the covering is said to be normal. The complex structure X/G is introduced by the condition that the projection $\pi : X \to X/G$, which associates each point $P \in X$ with its equivalence class, is holomorphic. In the neighborhoods of the points which are not fixed for any transformation $g \in G$, the local parameter is defined similarly to the quotient Δ/G in Sect. 2.1. If $U_n \subset X$ are such neighborhoods that $U_n \cap gU_n = \emptyset$, $\forall g \in G$, then $\pi(U_n)$ are neighborhoods on the quotient, and

$$\Phi_n = \varphi_n \circ \left(\pi\,|_{U_n}\right)^{-1} : \pi(U_n) \to \varphi_n(U_n)$$

are local parameters. The neighborhoods $\pi(U_n)$ and $\pi(U_m)$ intersect only if there is $g_{nm} \in G$ such that $U_n \cap g_{nm}U_m \neq \emptyset$. The transition functions of this intersection are holomorphic

$$\Phi_m \circ \Phi_n^{-1} = \varphi_m \circ g_{nm}^{-1} \circ \varphi_n^{-1} \quad .$$

Let P_0 be a fixed point of some elements $g \in G$ (the set of such element is called the stabilizer of P_0, $G_{P_0} = g \in G : gP_0 = P_0$), and let U be a neighborhood of P_0 which is invariant with respect to G_{P_0}. The local parameter in the neighborhood $\pi(U)$, which is $\operatorname{ord} G_{P_0}$-sheetedly covered by U, is defined by the product of the values of the local parameter φ at all equivalent points lying in U:

$$\Phi = \prod_{g \in G_{P_0}} \varphi \circ g \circ \left(\pi\,|_U\right)^{-1}, \ \varphi(P_0) = 0 \quad .$$

Although the mapping $(\pi\,|_U)^{-1}$ is $\operatorname{ord} G_{P_0}$-valued, Φ is a homeomorphism of $\pi(U)$ into some neighborhood of a complex plane. In the local parameters chosen, all the transformations $g \in G_{P_0}$ have the form $\varphi \circ g \circ \varphi^{-1} = zh_g(z)$, $h_g(0) \neq 0$, and the projection

$$\Phi \circ \pi \circ \varphi^{-1} = z^{\operatorname{ord} G_{P_0}} \prod_{g \in G_{P_0}} h_g(z)$$

describes the $\operatorname{ord} G_{P_0}$-sheeted holomorphic covering $U \to U/G_{P_0}$, so that P_0 is a branch point of the order $\operatorname{ord} G_{P_0}$ of the covering $X \to X/G$.

2.3 Elliptic Curves

The equation

$$\mu^2 = 4\lambda^3 - g_2\lambda - g_3 = 4(\lambda - e_1)(\lambda - e_2)(\lambda - e_3) \quad , \tag{2.3.1}$$

where g_2, g_3, e_i, $e_1 + e_2 + e_3 = 0$ are some constants, defines an elliptic curve. Any Riemann surface X of genus 1 can be reduced to the form (2.3.1).

We now fix some point $P_0 \in X$ and consider the mapping given by the integral

$$P \to \int_{P_0}^{P} \frac{d\lambda}{\mu} \quad . \tag{2.3.2}$$

First of all we note that this mapping is holomorphic, because the differential $d\lambda/\mu$ is holomorphic at all points of X. Indeed, in the neighborhood of $\lambda = e_i$ the local parameter is equal to $\varphi = \sqrt{\lambda - e_i}$ and $d\lambda/\mu = \text{const}\, d\varphi$; in the neighborhood of $\lambda = \infty$, $\varphi = 1/\sqrt{\lambda}$, we have $d\lambda/\mu \sim -d\varphi$ and the holomorphicity is obvious at all the other points.

Since it depends on the path of integration, the mapping (2.3.2) of the Riemann surface X onto the complex plane is multivalued. Every point $P \in X$ is mapped into an infinite number of points of a complex plane that differ from each other by $2\omega n + 2\omega' m$, $n, m \in \mathbb{Z}$, where

$$2\omega = \int_a \frac{d\lambda}{\mu}, \quad 2\omega' = \int_b \frac{d\lambda}{\mu} \quad . \tag{2.3.3}$$

We define a group G of shifts $z \to z + 2\omega n + 2\omega' m$. Its fundamental domain F is the parallelogram represented in Fig. 2.6 (in the next section we show in particular that ω, ω' and $\operatorname{Im}(\omega'/\omega)$ are always different from zero). Thus, (2.3.2) is the holomorphic mapping between Riemann surfaces of genus 1, $X \to \mathbb{C}/G$. Furthermore the derivative of this mapping is always different from zero. This means that (2.3.2) defines an unramified cover. Obviously, this cover is one-sheeted, see (2.3.3). Consequently, the mapping (2.3.2) is a biholomorphic map of the universal covering of X onto the whole complex plane, X and $\mathbb{C}/G$ are the same Riemann surface, consequently, the functions on X may be represented by the functions on $\mathbb{C}$ which are invariant under the action of the group G, i.e.,

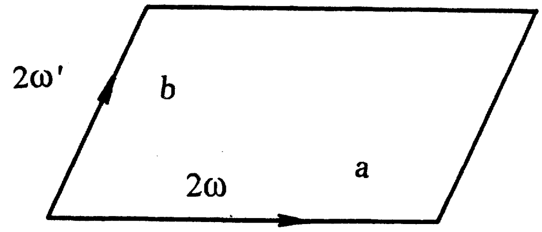

Fig. 2.6. Fundamental domain for elliptic curve

they are doubly periodic functions. Meromorphic doubly periodic functions are said to be elliptic.

The simplest elliptic function is the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{n,m}{}' \left\{ \frac{1}{(z - 2n\omega - 2m\omega')^2} - \frac{1}{(2n\omega + 2m\omega')^2} \right\} \quad . \tag{2.3.4}$$

The prime for the summation sign implies that the summation is taken over all pairs of integers n and m which do not vanish simultaneously. It is easy to show that the series (2.3.4) converges and defines a doubly periodic function with a singularity at $z = 0$ such as $\wp(z) = z^{-2} + O(z^2)$. $\wp(z)$ satisfies the differential equation

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3 \quad , \tag{2.3.5}$$

where

$$\begin{aligned} g_2 &= 60 \sum_{n,m}{}' \frac{1}{(2n\omega + 2m\omega')^4} \quad , \\ g_3 &= 140 \sum_{n,m}{}' \frac{1}{(2n\omega + 2m\omega')^6} \quad . \end{aligned} \tag{2.3.6}$$

Comparing (2.3.4) and (2.3.5), we see that the representation

$$(\mu, \lambda) = (\wp'(z), \wp(z))$$

uniformizes an elliptic curve, with ∞ corresponding to the point $z = 0$ (we set $P_0 = \infty$). Formulas (2.3.6) determine the connection between g_2, g_3 and ω, ω'.

2.4 Functions, Differentials and Integrals on Riemann Surfaces

2.4.1 General Properties of Functions on Riemann Surfaces

Meromorphic functions, i.e., non-constant holomorphic mappings $f : X \to \mathbb{CP}^1$, constitute a meaningful object of analysis on Riemann surfaces. The local notation

$$f(z) = f \circ \varphi^{-1}(z)$$

of a meromorphic function f in any local parameters φ is a meromorphic function of the variable $z \in \varphi(U)$ in the usual sense. The general properties of holomorphic coverings (Sect. 2.2) imply that the meromorphic function f takes every value $c \in \mathbb{CP}^1$ the same finite number of times (with the multiplicity taken into account). The points $P_0 \in f^{-1}(\infty)$ are said to be poles of the function f. In the neighborhood of any $P_0 \in X$, the meromorphic function f can be represented as a convergent Laurent series:

$$\sum_{j=-N}^{\infty} c_j(z - z_0)^j, \quad z \equiv \varphi(P), \quad z_0 = \varphi(P_0), \tag{2.4.1}$$

where φ is a local parameter, the number $N > -\infty$ and does not depend on a specific choice of φ.

Remark 2.4. In that which follows we shall often use the variable z to locally describe functions and differentials without stating specifically that $z = \varphi(P)$ and $z_0 = \varphi(P_0)$.

The point P_0 is an N-multiple pole (zero) of the function f if and only if $N < 0\,(N > 0)$ in the representation (2.4.1).

Concluding we would like to describe the general properties of meromorphic functions on Riemann surfaces by recalling:

Proposition 2.5. Any two functions f and g, meromorphic on the Riemann surface X, are connected by an algebraic relation,

$$Q(f, g) = 0,$$

where Q is a polynomial of its arguments. Theorem 2.2 given in Sect. 2.1 is a corollary of this proposition.

Corollary 2.6. The following three assertions are equivalent:

(a) the Riemann surface X is hyperelliptic, i.e., it can be given by (2.1.2).
(b) there is a meromorphic function on X that defines the 2-sheeted covering of the sphere $\mathbb{CP}^1$.

(c) there is a function on X that has its unique singularity – a second-order pole – at some point P_0.

Without dwelling at length into the proofs for Proposition 2.5 and Corollary 2.6, we note that the projection

$$\pi : (\mu, \lambda) \to \lambda$$

may be taken as the function appearing in item (b), when the hyperelliptic curve X is given by (2.2). To simplify the notation, this projection will sometimes be identified with the variable itself. For the same reason the expressions

$$1/(\lambda - E_i)$$

or, for odd N, λ itself may denote the function that satisfies condition (c).

2.4.2 Abelian Differentials

In addition to the notion of a function on a Riemann surface, we introduce the notion of an Abelian differential. An Abelian differential on the Riemann surface X is a meromorphic 1-form ω, given on X. This implies that we can write ω locally as $f(z)dz$, where $f(z)$ is a meromorphic function of z in its domain.

For any Abelian differential, the notion of a pole and that of a zero are defined correctly, along with the notions of multiplicities and that of a residue:

$$\operatorname{res}(\omega; P_0) = c_{-1}, \quad \omega(P) = \sum c_j (z - z_0)^j \, dz \quad .$$

Remark 2.7. A generalization of the notion of an Abelian differential is the notion of a differential of weight (p, q), $p \in \mathbb{Q}$, $q \in \mathbb{Q}$ as an object that can be represented in any local parameter in the form

$$f(z, \bar{z})(dz)^p (d\bar{z})^q \quad .$$

Abelian differentials are usually divided into three kinds: holomorphic differentials (first kind), meromorphic differentials with residues equal to zero at all singular points (second kind), and meromorphic differentials of the general form (third kind). For differentials of the third kind the relation holds that

$$\sum_{\substack{\text{over all} \\ \text{singular points}}} \operatorname{res}(\omega) = 0 \ . \tag{2.4.2}$$

The question of whether Abelian differentials exist for an arbitrary Riemann surface is answered by the following classical theorem due to Riemann:

Theorem 2.8. Let X be a Riemann surface of genus g. Then

(a) The dimension of the space of differentials holomorphic on X is equal to g.

(b) For any finite set of points P_j, $P_j \in X$, there is an Abelian differential which is holomorphic on $X \setminus \{P_j\}$ and has, at the points P_j, the poles with arbitrary preassigned principal parts that satisfy only the condition (2.4.2).

Remark 2.9. The principal part $\hat{\omega}_p$ of a differential ω at the point P_0 is defined in a natural way after the local parameter has been fixed:

$$\omega = \sum_{j=-N}^{\infty} c_j(z - z_0)^j dz, \quad \hat{\omega}_p = \sum_{j=-N}^{-1} c_j(z - z_0)^j dz \quad .$$

In the case of hyperelliptic curve (2.1.2), the Abelian differentials that appear in the propositions of Theorem 2.8 can easily be constructed within the format of explicit formulas. Thus, taking the choice of local parameters into account, it is easy to see that the basis in the space of Abelian differentials on the Riemann surface (2.1.2) is formed by

$$\overset{\circ}{\omega}_j = \lambda^{g-j} d\lambda/\mu, \; j = 1, \ldots, g \quad , \tag{2.4.3}$$

where g is the surface genus equal to $N/2 - 1$ for even N, and to $(N-1)/2$ for odd N. This illustrates item (a) of Theorem 2.8. In Sect. 2.7 we will present some constructions which can be regarded as an illustration of item (b) of Theorem 2.8.

For any function f, given on a Riemann surface X, the formula

$$df(P) = \left[\frac{df(z)}{dz}\right] dz, \; z = \varphi(P), \; f(z) = f \circ \varphi^{-1}(z) \quad ,$$

where φ is an arbitrary local parameter, defines an Abelian differential. The inverse question, i.e., the question of a primitive function for an arbitrary Abelian differential leads to a new object – an Abelian integral.

Any Abelian differential ω on the Riemann surface X satisfies the closure condition,

$$d\omega = 0 \; .$$

Therefore, locally, a primitive function for the differential ω always exists and can be defined by the formula

$$\Omega(P) = \int_{P_0}^{P} \omega \tag{2.4.4}$$

for any simply-connected domain on X that involves (in the case of third-kind differentials) no singularities of the differential ω. Formula (2.4.4) considered on the whole surface X, with its genus being non-trivial, defines, in general, a multivalued function called an Abelian integral. The division of Abelian differentials into the three kinds can be naturally extended to Abelian integrals. Locally, Abelian integrals of the first kind are holomorphic functions, Abelian integrals of the second kind are meromorphic functions, and Abelian integrals of the third kind have logarithmic singularities:

$$\omega = \left(\cdots + \frac{c_{-1}}{z} + \cdots\right) dz \Rightarrow \Omega = \cdots + c_{-1} \ln z + \cdots$$

The Abelian differential ω can be restored from the Abelian integral Ω using the obvious equation $d\Omega = \omega$.

2.4.3 Periods of Abelian Differentials

We fix the choice of a basis for a- and b-cycles on X. The specific features of any multivalued Abelian integral Ω of the first or second kind are completely described by its A- and B-periods or cyclic periods

$$A_j = \int_{a_j} d\Omega,\ B_j = \int_{b_j} d\Omega,\ j = 1, \ldots, g \quad , \tag{2.4.5}$$

which are also called A- and B-periods of the differential $d\Omega$. At every point $P \in X$, the Abelian integral Ω takes an infinite set of values, any two of which differ from each other by a quantity independent of P, and is expressed as

$$\int_\gamma d\Omega = \sum_j (m_j B_j + n_j A_j) \ ,$$

where γ is some cycle on X, for which the decomposition

$$\gamma = \sum_j (m_j b_j + n_j a_j)$$

with respect to basis cycles is valid. Thus, we can speak about the value taken by an Abelian integral at the point $P \in X$ modulo its period. For example,

$$\Omega(P_0) = 0 \quad \text{(modulo the periods)} \quad .$$

If Ω is an Abelian integral of the third kind, it is necessary, in order for the character of its multivaluedness to be completely described, to supplement the set of cyclic periods (2.4.5) with a set of polar periods,

$$c_j = \int_{\gamma_j} d\Omega,\ j = 1, \ldots, n \quad , \tag{2.4.6}$$

where γ_j is the cycle homological to zero and containing in itself the point P_j, i.e., the j-th pole of the differential $d\Omega$. Obviously,

$$c_j = 2\pi i \operatorname{res}(d\Omega, P_j)$$

and, by virtue of (2.4.2)

$$\sum_{\substack{\text{over all} \\ \text{singular points}}} c_j = 0 \quad . \tag{2.4.7}$$

We let $\overset{\circ}{X}$ denote the surface obtained by removing all a- and b-cycles from X. Let a_j^+, a_j^- and b_j^+, b_j^- be the left and the right edges of the appropriate cuts. The manifold $\overset{\circ}{X}$ is a manifold with a boundary:

$$\partial \overset{\circ}{X} = \sum_{j=1}^{g} (a_j^+ + b_j^+ - a_j^- - b_j^-) \quad . \tag{2.4.8}$$

Any Abelian integral $\Omega(P)$ of the first or second kind is single-valued on $\overset{\circ}{X}$. It is sufficient to require that the integration path in (2.4.4) should not intersect a- and b-cycles. At the boundary of the surface $\overset{\circ}{X}$, the branch $\Omega(P)$ satisfies the boundary conditions:

$$\begin{aligned} \Omega(P)\Big|_{a_j^+} - \Omega(P)\Big|_{a_j^-} &= -B_j \quad , \\ \Omega(P)\Big|_{b_j^+} - \Omega(P)\Big|_{b_j^-} &= A_j \quad , \end{aligned} \tag{2.4.9}$$

which allow it to be continued to a single-valued function on the universal covering of X.

To distinguish a single-valued branch on a third-kind Abelian integral, it is necessary to draw additional cuts γ_j on the surface X that run from P_0 to P_j, where P_j are the singular points of the Abelian integral under consideration. Equations (2.4.6) are then complemented with the equalities

$$\Omega(P)\Big|_{\gamma_j^+} - \Omega(P)\Big|_{\gamma_j^-} = c_j \quad . \tag{2.4.10}$$

For any pair of Abelian integrals Ω and Ω', we have the following equality from (2.4.8,9)

$$\int_{\partial \overset{\circ}{X}} \Omega' d\Omega = \sum_{k=1}^{g} (A'_k B_k - A_k B'_k) \quad . \tag{2.4.11}$$

Applying the residue theorem to its left-hand side, we obtain the Riemann bilinear relations for the periods of Abelian integrals:

$$\text{(i)} \quad \sum_{k=1}^{g} (A'_k B_k - A_k B'_k) = 0 \quad , \tag{2.4.12}$$

when Ω and Ω' are integrals of the first kind;

$$\text{(ii)} \quad \sum_{k=1}^{g}(A'_k B_k - A_k B'_k) = 2\pi \mathrm{i} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \Omega'(z)|_{z=z_0} , \tag{2.4.13}$$

when Ω' is an integral of the first kind, Ω is an integral of the second kind with a single pole at P_0, and the local parameter in the neighborhood of P_0 is chosen so that

$$d\Omega = [(z-z_0)^{-n} + O(1)]dz , \; n > 1 ;$$

$$\begin{aligned} &\text{(iii)} \quad \sum_{k=1}^{g}(A'_k B_k - A_k B'_k) = 2\pi \mathrm{i} \sum_{j=1}^{n} c_j \Omega'(P_j) \\ &\equiv 2\pi \mathrm{i} \sum_{j=1}^{n} c_j \int_{P_0}^{P_j} d\Omega' \quad , \end{aligned} \tag{2.4.14}$$

when Ω' is an integral of the first kind, Ω is an integral of the third with no more than logarithmic singularities at the points P_j, $j = 1, \dots, n > 1$ and $c_j = \operatorname{res}(d\Omega, P_j)$.

Remark 2.10. The integration contours in (2.4.14) are chosen to be disjoint from the basic a- and b-cycles. In other words in the right hand side of (2.4.14) at all points P_j we take values of the same single-valued branch of the integral Ω'.

The Riemann bilinear relations also involve the formula

$$2\mathrm{i} S_\Omega(\overset{\circ}{X}) \equiv \int_{\partial \overset{\circ}{X}} \bar{\Omega}\, d\Omega = \sum_{k=1}^{g}(-A_k \bar{B}_k + \bar{A}_k B_k) \quad , \tag{2.4.15}$$

where Ω is an Abelian integral of the first kind, and $S_\Omega(\overset{\circ}{X})$ is the area of $\overset{\circ}{X}$ under a conformal mapping given by the integral Ω. From (2.4.15) it follows that the following inequality

$$\mathrm{i} \sum_{k=1}^{g}(A_k \bar{B}_k - \bar{A}_k B_k) > 0 \tag{2.4.16}$$

is valid for cyclic periods of any non-constant Abelian integral of the first kind.

Simple consequences of the Riemann bilinear relations (2.4.16) are the following propositions that complement the main existence theorem for Abelian differentials (Theorem 2.8) with natural uniqueness conditions.

Proposition 2.11. There is no non-zero Abelian differential of the first kind whose A- or B-periods are zero, or all the periods of which are purely imaginary or purely real.

Proposition 2.12. Any Abelian differential of the second or third kind with zero A-periods (B-periods) or with all purely imaginary (purely real) cyclic periods is uniquely defined by its principal parts at the singular points.

Remark 2.13. Ordinarily, when normalizing Abelian differentials (integrals) of the second or third kind, we choose the condition that all the A-periods be zero. Such Abelian differentials (integrals) are said to be normalized.

2.4.4 Period Matrix. Jacobian Variety

Proposition 2.11 enables one to introduce for any Riemann surface X the basis of holomorphic differentials $\{\omega_j\}_{j=1}^{g}$, normalized by the condition

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}, \quad j,k = 1,\ldots,g \quad , \tag{2.4.17}$$

which is dual to the basis of a- and b-cycles. In the hyperelliptic case (2.1.2), the following explicit formula is valid for the differentials:

$$\begin{aligned} \omega_j &= \sum_{k=1}^{g} c_{jk} \overset{\circ}{\omega}_R = \sum_{k=1}^{g} c_{jk} \frac{\lambda^{g-k} d\lambda}{\mu}, \quad j = 1,\ldots,g \quad , \\ c_{jk} &= 2\pi i (A^{-1})_{jk} \quad , \end{aligned} \tag{2.4.18}$$

where the matrix A is defined by the equality

$$A_{jk} = \int_{a_k} \overset{\circ}{\omega}_j, \quad j,k = 1,\ldots,g \quad .$$

The nondegeneracy of the matrix A is a consequence of Proposition 2.11.

An important characteristic of any Riemann surface X is the matrix of B-periods of the basis $\{\omega_j\}$,

$$B_{jk} = \int_{b_j} \omega_k, \quad j,k = 1,\ldots,g \quad .$$

The Riemann bilinear relations (2.4.12) and (2.4.16) allow us to conclude that the matrix B is symmetric, i.e.,

$$B_{kj} = B_{jk}, \quad \forall j \neq k = 1,\ldots,g \tag{2.4.19}$$

and its real part is negatively defined

$$\operatorname{Re} B < 0 \quad . \tag{2.4.20}$$

Under a change of basis of a- and b-cycles,

$$(a_j, b_j) \to (a'_j, b'_j),$$

the matrix B transforms according to the following rule:

$$B \to B' = 2\pi\mathrm{i}(aB + 2\pi\mathrm{i}b)(cB + 2\pi\mathrm{i}d)^{-1} \quad , \tag{2.4.21}$$

where the integral $g \times g$-matrices a, b, c and d are defined from the decompositions

$$\begin{aligned} a_j' &= \sum_{k=1}^{g} d_{jk}a_k + \sum_{k=1}^{g} c_{jk}b_k \quad , \\ b_j' &= \sum_{k=1}^{g} b_{jk}a_k + \sum_{k=1}^{g} a_{jk}b_k \quad . \end{aligned} \qquad j = 1, \dots, g \tag{2.4.22}$$

According to the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

they define an element of the group $\mathrm{Tr}(2g, \mathbb{Z})$ (in the German literature you find instead of the Tr usually an Sp). We denote by $S_g/\mathrm{Tr}(2g, \mathbb{Z}) = \mathcal{U}_g$ the quotient of the space of all $g \times g$-matrices that satisfy (2.4.19), (2.4.20) by the action (2.4.21) of the group $\mathrm{Sp}(2g, \mathbb{Z})$. The set $\mathcal{U}_g$ is called a modular Siegel variety. In the case of small genera, $g = 1, 2$ and 3, the varieties $\mathcal{U}_g$ coincide with the manifold of the moduli M_g of algebraic curves, i.e., the set of classes of conformally equivalent Riemann surfaces of genus g. For $g \geq 4$, we only have the inclusion

$$M_g \subset \mathcal{U}_g \tag{2.4.23}$$

This follows, for example, from a comparison of the dimensions:

$$\begin{aligned} &\dim M_g = 3g - 3, \; g > 1 \\ &\dim \mathcal{U}_g = g(g+1)/2 \quad . \end{aligned}$$

An effective description of the inclusion (2.4.23), or, equivalently, determining the conditions that distinguish from the whole set of symmetric $g \times g$-matrices with a negative real part those matrices which correspond to Riemann surfaces is the subject of the classical Schottky problem. Much progress has recently been made in solving this problem. This has been stimulated in part by the development of the algebro-geometric methods in the theory of nonlinear equations that we discuss in the present book.

Let B be a period matrix of the surface X. We use it to define a lattice Λ in $\mathbb{C}^g$,

$$\Lambda = \{2\pi\mathrm{i}\boldsymbol{N} + B\boldsymbol{M}, \; \boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^g\} \equiv \Pi\mathbb{Z}^{2g}, \quad \Pi = (2\pi\mathrm{i}I; B)$$

and introduce in $\mathbb{C}^g$ the equivalence relation

$$\mathbf{Z} \sim \mathbf{Z}' \Leftrightarrow \mathbf{Z} - \mathbf{Z}' \in \Lambda \quad .$$

The quotient of $\mathbb{C}^g$ by this equivalence relation is a g-dimensional complex torus and called a Jacobian variety, or a Jacobian $J(X)$ of the curve X,

$$J(X) = \mathbb{C}^g / \Lambda \quad .$$

This notion makes it possible to consider correctly and globally Abelian integrals of the first kind on X:

$$\int_{P_0}^{P} \omega_j, \; j = 1, \ldots, g \quad .$$

More specifically, the formula

$$X \ni P \to \int_{P_0}^{P} \omega \equiv \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g \right) \in J(X) \tag{2.4.24}$$

correctly defines the Abelian mapping with basis point P_0 of the Riemann surface X into its Jacobian variety $J(X)$. The Abelian mapping is usually denoted by

$$\int_{P_0}^{P} \omega = A(P) \; .$$

2.4.5 Divisors

The Abelian mapping (2.4.24) can easily be extended from the points of X to more general objects called divisors. A divisor $\mathcal{D}$ on the Riemann surface X is the formal sum of a finite number of its points with integral coefficients

$$\mathcal{D} = \sum_i n_i P_i, \; P_i \in X, \quad n_i \in \mathbb{Z} \quad . \tag{2.4.25}$$

The number $\sum_i n_i$ is referred to as the degree of the divisor $\mathcal{D}$, $\deg \mathcal{D}$. The Abelian mapping of the divisor $\mathcal{D}$ into the Jacobian $J(X)$ is defined by the formula

$$A(\mathcal{D}) = \sum_i n_i \int_{P_0}^{P_i} \omega \equiv \left(\sum_i n_i \int_{P_0}^{P_i} \omega_1, \ldots, \sum_i n_i \int_{P_0}^{P_i} \omega_g \right) \in J(x) \quad .$$

Another classical problem in the theory of functions on Riemann surfaces is the Jacobi inversion problem, i.e., the problem of constructing a mapping inverse to the Abelian mapping. The solution of this problem and the role of theta functions in finding it will be discussed in Sect. 2.5.

The set of all divisors on the Riemann surface X forms an Abelian group $\mathrm{Div}(X)$ with respect to the naturally defined operation of addition. In $\mathrm{Div}(X)$, we introduce in a natural way the notion of a positive divisor

$$\mathcal{D} \geq 0 \Leftrightarrow \mathcal{D} = \sum_i n_i P_i, \quad n_i \geq 0 \, \forall i \quad ,$$

and a partial ordering of divisors

$$\mathcal{D} \leq \mathcal{D}' \Leftrightarrow \mathcal{D}' - \mathcal{D} \geq 0 \quad .$$

The divisor of a function f, given on X, is defined as the sum (2.4.25) where P_i is a zero or a pole of f, and n_i is an appropriate multiplicity. Moreover, it is assumed that $n_i > 0$, if P_i is a zero, and $n_i < 0$, if P_i is a pole. The divisor of f is denoted by (f). The divisor (ω) of the Abelian differential ω is defined in a similar way. For any function f and the Abelian differential ω, the following equalities are valid:

$$\deg(f) = 0, \quad \deg(\omega) = 2g - 2 \quad . \tag{2.4.26}$$

We note also that the divisor (f) of any function f can be decomposed as a difference of two positive divisors: the divisor of zeros of the function f and the divisor of its poles. The same is true for the divisors of Abelian differentials.

The divisor $\mathcal{D} \in \mathrm{Div}(X)$ is said to be a principal divisor if there is a function f on X such that $\mathcal{D} = (f)$. By virtue of (2.4.26), all principal divisors have a zero degree. Moreover, the following holds:

Theorem 2.14. (Abel's theorem). Let $\mathcal{D} \in \mathrm{Div}(X)$ and $\deg \mathcal{D} = 0$. Then the divisor $\mathcal{D}$ is principal if and only if

$$A(\mathcal{D}) = 0 \ .$$

In other words, the sets of points $(P_1, \ldots, P_N)$ and $(Q_1, \ldots, Q_N)$ are sets of the zeros and poles of some function meromorphic on X if and only if for any Abelian differential ω of the first kind

$$\sum_{i=1}^{N} \int_{Q_i}^{P_i} \omega = 0 \quad \text{(modulo the periods).}$$

Two divisors $\mathcal{D}$, $\mathcal{D}' \in \mathrm{Div}(X)$ are said to be equivalent if their difference is a principal divisor. All divisors of Abelian differentials are equivalent, because the quotient of any two Abelian differentials is a meromorphic function. The divisors of Abelian differentials are said to form a canonical class, and the divisors themselves are called canonical ones. We note that by virtue of Abel's theorem, any two equivalent divisors are mapped into the same point of the variety $J(X)$. In particular, all canonical divisors have the same image under Abelian mapping.

The function f (Abelian differential ω) is said to be divisible by the divisor $\mathcal{D}$ if $(f) \geq \mathcal{D}((\omega) \geq \mathcal{D})$. We denote by $F_{\mathcal{D}}$ a linear space of functions meromorphic on X and divisible by the divisor $\mathcal{D}$. Similarly, we introduce a space $d\Omega_{\mathcal{D}}$ for Abelian differentials. The most important result of the classical theory of functions on Riemann surfaces is the

Theorem 2.15. (Riemann-Roch Theorem). Let $\mathcal{D} \in \mathrm{Div}(X)$, where X is a Riemann surface of genus g. Then

$$\dim F_{-\mathcal{D}} - \dim d\Omega_{\mathcal{D}} = 1 - g + \deg \mathcal{D} \quad . \tag{2.4.27}$$

The relation (2.4.27) implies that when the genus of the Riemann surface X is non-trivial, the analysis on it is essentially different from the standard ($g = 0$) complex analysis. In particular, there are very rigid restrictions on the location of the singularities of meromorphic functions. Thus, the Riemann-Roch theorem implies that any set of $k \leq g$ points which is in general position on a Riemann surface of genus g cannot be a set of poles for a meromorphic function. Indeed, let

$$\mathcal{D} = \sum_{j=1} P_j, \; k \leq g, \quad P_j \in X, \quad P_j \neq P_k \quad . \tag{2.4.28}$$

Then, the space $d\Omega_{\mathcal{D}}$ can be identified with the space of solutions to the following linear system:

$$\sum_{i=1}^{g} c_i \omega_i(P_j) = 0, \quad j = 1, \ldots, k \quad , \tag{2.4.29}$$

where $\{\omega_i\}$ is an arbitrary, but fixed basis of holomorphic differentials. In the general position,

$$\operatorname{rank} \left\{ \omega_i(P_j) \right\}_{\substack{i=1,\ldots,g \\ j=1,\ldots,k}} = k \quad .$$

Therefore, $\dim \Omega_{\mathcal{D}} = g - k$ and, in virtue of (2.4.27),

$$\dim F_{-\mathcal{D}} = 1 \quad ,$$

i.e., the only function divisible by the divisor $-\mathcal{D}$ is a constant. The last statement amounts to the fact that there is no meromorphic function for which the divisor $\mathcal{D}$ (or any part of it) is a divisor of the poles.

The arguments in the preceding passage allow us to introduce the notion of speciality and non-speciality for positive divisors of degree $\leq g$. The divisor $\mathcal{D}$, $\mathcal{D} > 0$, $\deg \mathcal{D} \leq g$ is said to be special (non-special), if $\dim F_{-\mathcal{D}} > 1$ ($\dim F_{-\mathcal{D}} = 1$). We have thus clarified that the non-special divisors are divisors of the general position. In the case of huperelliptic curve (2.1.2), the notion of a "general position", or a "non-speciality", can easily be specified by considering the explicit form of appropriate holomorphic differentials. Namely, simple reasoning related to the Vandermond determinant leads us to the conclusion that the non-speciality condition for the divisor (2.4.28) is given by the inequalities

$$\pi(P_j) \neq \pi(P_l), \quad j \neq l \quad , \tag{2.4.30}$$

which, obviously, can always be achieved by a "little stirring".

Remark 2.16. Turning again to the system (2.4.29), we can also introduce the notion of speciality and non-speciality for positive divisors of degree $> g$: the divisor $\mathcal{D}$, $\mathcal{D} > 0$, deg $\mathcal{D} > g$ is special (non-special), if dim $F_{-\mathcal{D}} > 1 - g +$ deg $\mathcal{D}$ (dim $F_{-\mathcal{D}} = 1 - g +$ deg $\mathcal{D}$). In what follows, however, we shall use the terminology in point only for divisors of degree $\leq g$ (unless otherwise stated).

Among special divisors we distinguish divisors such as kP_0, $k \leq g$, $P_0 \in X$. The point P_0 involved here is called a Weierstrass point of the surface X. The number n of Weierstrass points on the surface X of genus $g > 2$ satisfies the Hurwitz inequality

$$2g + 2 \leq n \leq (g-1)g(g+1) \quad ,$$

where the equality $n = 2g + 2$ is attained if and only if X is a hyperelliptic surface. When X is realized canonically by (2.1.2), the Weierstrass points are evidently all branch points, i.e., the points $Q_i = (0, E_i)$, and the points at infinity for odd N.

Here we conclude our review of the general properties of functions and differentials on Riemann surfaces. In Sects. 2.7, 8 we will again return to this theory, more precisely, to its aspects related to the explicit construction of functions with prescribed singularities on Riemann surfaces. It is this constructive aspect of the theory that underlies the apparatus developed in the subsequent chapters and enables the construction and study of the class of algebro-geometrical solutions to nonlinear evolution equations.

2.5 Abelian Functions and Theta Functions

The algebraic-geometrical solutions of completely integrable nonlinear equations and integrable finite-dimensional dynamical systems are expressed by Abelian functions which, in turn, are constructed using theta functions. In this section we give the initial information on Abelian tori and functions on them.

Let $V \in \mathbb{C}^g$ be a complex vector space of dimension g, and Λ a discrete lattice of maximal rank $2g$. The complex torus $T = V/\Lambda$ is called an Abelian torus if there is an Abelian function (i.e., a meromorphic function in g complex variables that has $2g$ independent periods) with period lattice Λ.

Theorem 2.17. (Riemann's Condition.) Let $\Pi = (E, F)$ be $(g \times 2g)$-matrix that defines the lattice Λ, ($\Lambda = EN + FM$, $N, M \in \mathbb{Z}^g$; E, F are $(g \times g)$-matrices). Then the torus V/Λ is Abelian if and only if there is an integral skew-symmetric $(2g \times 2g)$-matrix Q such that

$$\Pi Q^{-1} \Pi^T = 0, \quad \mathrm{Im}\,(\Pi Q^{-1} \bar{\Pi}^T) > 0 \quad . \tag{2.5.1}$$

The period matrix Π is dependent on the choice of a basis both in Λ and on V. When the lattice basis changes, the matrix Π is multiplied from the right by a unimodular $(2g \times 2g)$-matrix C, $\Pi \to \Pi C$. When the basis changes in V, it is multiplied on the left by a complex nondegenerate $(g \times g)$-matrix D, $\Pi \to D\Pi$.

Theorem 2.18. There is such a basis of Λ that

$$Q = Q_\delta = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \quad \Delta_\delta = \operatorname{diag}(\delta_1, \dots, \delta_g), \quad \delta_i \in \mathbb{Z} \tag{2.5.2}$$

with the divisibility $\delta_1 \mid \delta_2 \cdots \mid \delta_g$ taking place.

Let $\Pi = (E, F)$ in this basis. We denote the columns of the matrix E by E_α, $\alpha = 1, \dots, g$. Then, choosing a new basis $e_\alpha = E_\alpha/(2\pi i \delta_\alpha)$ in V, i.e., performing a transformation with the matrix $D = \Delta_\delta E^{-1} 2\pi i$, we have the period matrix

$$\Pi = (2\pi i \Delta_\delta, B) \quad . \tag{2.5.3}$$

This period matrix is referred to as a normalized one.

Theorem 2.19. There are bases in Λ and in V such that $\Pi = (2\pi \mathrm{i} \Delta_\delta, B)$.

The Riemann conditions (2.5.1) and (2.5.2) are rewritten for a matrix of that kind as follows:

$$B = B^T, \ \operatorname{Re} B < 0 \quad . \tag{2.5.4}$$

The numbers δ_i are called elementary divisors of polarization of the torus T. The polarization is said to be principal if $\delta_i = 1$ for all i.

The Abelian torus T^*, defined by the lattice

$$\Pi^* = (\Delta_\delta^{-1}, \Delta_\delta^{-1} B \Delta_\delta^{-1}) \quad , \tag{2.5.5}$$

is dual to an Abelian torus with the lattice (2.5.3) [2.11].

Abelian functions are constructed by means of theta functions. If B satisfies (2.5.4) then a theta function is defined by its multi-dimensional Fourier series

$$\theta(z; B) = \sum_{m \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle m, Bm\rangle + \langle m, z\rangle\right\} \quad , \tag{2.5.6}$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product. Every point $e \in \mathbb{C}^{2g}$ is written uniquely as

$$e = (\varepsilon', \varepsilon'') \begin{pmatrix} 2\pi \mathrm{i} I \\ B \end{pmatrix}, \quad \varepsilon', \varepsilon'' \in \mathbf{R}^g,$$

where

$$[\varepsilon] = \begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix} = \begin{bmatrix} \varepsilon_1', \dots, \varepsilon_g' \\ \varepsilon_1'', \dots, \varepsilon_g'' \end{bmatrix}$$

is the characteristic of e. We shall denote it as $[e]$. For every such characteristic we define on $\mathbb{C}^g \times \mathcal{U}_g$ a theta function with the characteristic $\theta[\varepsilon](z; B)$, using the formula

$$\begin{aligned}
\theta[\varepsilon](z;B) = {} & \exp\left\{\frac{1}{8}\langle\varepsilon', B\varepsilon'\rangle + \frac{1}{2}\langle z + \pi\mathrm{i}\varepsilon'', \varepsilon'\rangle\right\} \\
& \times \theta(z + \pi\mathrm{i}\varepsilon'' + \frac{1}{2}\varepsilon' B; B) \\
= {} & \sum_{m \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle(m + \varepsilon'/2), B(m + \varepsilon'/2)\rangle\right. \\
& \left. + \langle(m + \varepsilon'/2), (z + \pi\mathrm{i}\varepsilon'')\rangle\right\} .
\end{aligned} \tag{2.5.7}$$

When the dependence on B is obvious we shall sometimes use the abbreviation $\theta(z)$. For the characteristic $[\delta]$, δ', $\delta'' \in \mathbb{Z}^g$, from (2.5.7) it follows that

$$\theta[\varepsilon + 2\delta](z;B) = \begin{pmatrix}\varepsilon\\ \delta\end{pmatrix}\theta[\varepsilon](z;B), \quad \theta[\varepsilon](-z;B) = |\varepsilon|\theta[\varepsilon](z;B) \quad , \tag{2.5.8}$$

where $\begin{pmatrix}\varepsilon\\ \delta\end{pmatrix} = \exp\{\pi\mathrm{i}\langle\varepsilon', \delta''\rangle\}$ and $|\varepsilon| = \begin{pmatrix}\varepsilon\\ \varepsilon\end{pmatrix}$. Thus, there are with accuracy to the sign, 2^{2g} theta functions when the characteristics $[\varepsilon]$, ε', $\varepsilon'' \in \mathbb{Z}^g$, are integral. The theta function $\theta[\varepsilon](z; B)$ is even if the characteristic $[\varepsilon]$ is even, i.e., $|\varepsilon| = 1$, and odd if the characteristic $[\varepsilon]$ is odd, i.e., $|\varepsilon| = -1$. Among 2^{2g} theta functions, $2^{g-1}(2^g + 1)$ are even and $2^{g-1}(2^g - 1)$ are odd.

The characteristics $[\varepsilon]$, for which ε_i', ε_j'' or $1, i, j = 1, \ldots, g$ are called half-period characteristics. We shall henceforth be concerned only with characteristics of this kind, unless otherwise stated. The characteristics form a group with respect to summation, and, for brevity, we write the summation of the characteristics multiplicatively $[\varepsilon_1] + \cdots + [\varepsilon_k] = [\varepsilon_1 \cdots \varepsilon_k]$. For Jacobi theta functions (i.e., for $g = 1$) there are 4 characteristics, with one of them being odd $\begin{bmatrix}1\\1\end{bmatrix}$ and the other three even: $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$. For reference, we give the comparison of notations [2.10].

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(z;B) = -\vartheta_1\left(\frac{z}{2\pi\mathrm{i}}\,\middle|\,\frac{B}{2\pi\mathrm{i}}\right), \quad \theta\begin{bmatrix}0\\0\end{bmatrix}(z;B) = \vartheta_3\left(\frac{z}{2\pi\mathrm{i}}\,\middle|\,\frac{B}{2\pi\mathrm{i}}\right),$$
$$\theta\begin{bmatrix}0\\1\end{bmatrix}(z;B) = \vartheta_4\left(\frac{z}{2\pi\mathrm{i}}\,\middle|\,\frac{B}{2\pi\mathrm{i}}\right), \quad \theta\begin{bmatrix}1\\0\end{bmatrix}(z;B) = \vartheta_2\left(\frac{z}{2\pi\mathrm{i}}\,\middle|\,\frac{B}{2\pi\mathrm{i}}\right).$$

In another notation which we shall also use the fourth Jacobi theta function ϑ_4 is denoted by ϑ_0. The functions

$$\theta[\varepsilon] = \theta[\varepsilon](0;B),$$
$$\theta_{i_1\cdots i_n}[\varepsilon] = (\partial^n/\partial z_{i_1}\cdots\partial z_{i_n})\,\theta[\varepsilon](z;B)\big|_{z=0}$$

non-identical to zero on $\mathcal{U}_g$ will be called theta constants. The even characteristic $[\varepsilon]$ will be called a non-singular (singular) one, if $\operatorname{grad}_z \theta[\varepsilon](z; B)\big|_{z=0} \neq 0\ (= 0)$. When $g = 1, 2$, all the characteristics of half-periods are non-singular.

The theta function (2.5.7) has the periodicity property

$$\begin{aligned} &\theta(z + 2\pi e_k; B) = \theta(z; B), \\ &\theta(z + f_k; B) = \exp\left(-\frac{1}{2}B_{kk} - z_k\right)\theta(z; B), \quad k = 1, \dots, g\ , \end{aligned} \tag{2.5.9}$$

where $e_1, \dots, e_g$ are basis vectors in the space $\mathbb{C}^g$ with coordinates $(e_k)_j = \delta_{kj}$, $k, j = 1, \dots, g$ and $f_1, \dots, f_g$, are vectors with coordinates $f_k = Be_k$. For the integral vectors n', $n'' \in \mathbb{Z}^g$ the equalities of (2.5.9) are generalized to the formula

$$\begin{aligned} &\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(z + 2\pi i n' + Bn''; B) = \exp\Big\{ -\frac{1}{2}\langle Bn'', n''\rangle \\ &- \langle z, n''\rangle + \pi i(\langle \varepsilon', n'\rangle - \langle \varepsilon'', n''\rangle)\Big\}\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(z; B) \quad . \end{aligned} \tag{2.5.10}$$

Equality (2.5.10) is called the transformation property of a theta function.

Using the theta functions (2.5.6), we construct Abelian functions with respect to $\Lambda = 2\pi i\mathbb{Z}^g \oplus B\mathbb{Z}^g$; namely, the function

$$f(z) = \frac{\prod_{i=1}^{n} \theta(z + a_i; B)}{\prod_{i=1}^{n} \theta(z + b_i; B)}\ , \tag{2.5.11}$$

where the vectors a_i, $b_i \in \mathbb{C}^g$ are such that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ (modulo $2\pi i\mathbb{Z}^g$) is meromorphic on $T = \mathbb{C}^g/\Lambda$. This is because the denominator is not identical to zero and the condition on a_i and b_i provides, by virtue of the transformation properties (2.5.10), that the function (2.5.11) is invariant under a shift by an arbitrary vector of Λ.

In addition to the transformation property (2.5.10), the theta function has the modular property as a function on $\mathbb{C}^g \times \mathcal{U}_g$. To formulate this we denote by $\mathrm{Tr}(2g, \mathbb{R})$, where $\mathbb{R}$ is the set of real numbers, a group of symplectic matrices of degree $2g$ with coefficients in $\mathbb{R}$, whose elements are written as $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d being square matrices of degree g with coefficients in $\mathbb{R}$ that satisfy the conditions $\sigma^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The following formula transforming the theta function (2.5.7) holds:

$$\begin{aligned} &\theta[\sigma \cdot \varepsilon]\left(2\pi i((cB + 2\pi i d)^{-1})^T z;\ \sigma \cdot B\right) = k \det(cB + 2\pi i d)^{1/2} \\ &\times \exp\Big\{\frac{1}{2}\sum_{i \le j} \left[z_i z_j \times (\partial/\partial B_{ij}) \log \det(cB + 2\pi i d)\right]\Big\}\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(z; B), \end{aligned} \tag{2.5.12}$$

where

$$\sigma \cdot B = 2\pi\mathrm{i}(aB + 2\pi\mathrm{i}b)(cB + 2\pi\mathrm{i}d)^{-1} \quad , \tag{2.5.13}$$

$$\sigma \cdot (\varepsilon', \varepsilon'') = (\varepsilon', \varepsilon'') \begin{pmatrix} d^T & -b^T \\ -c^T & a^T \end{pmatrix} + \operatorname{diag}(cd^T, ab^T) \quad , \tag{2.5.14}$$

k is a constant independent of z, B and the symbol "diag" means that diagonal elements only should be taken away from the matrices (cd^T), (ab^T). Formula (2.5.12) is referred to as the modular property of a theta function.

A modular form relative to $\mathrm{Tr}(2g, \mathbb{R})$ is a holomorphic function f on $\mathcal{U}_g$ that satisfies the equation

$$f(\sigma \cdot B) = \det(cB + d)^n f(B) \quad , \tag{2.5.15}$$

where the number n is called a weight. These forms can be constructed using theta constants [2.11].

2.6 Addition Theorems for Theta Functions

In this section we present two addition theorems for theta functions and some of their corollaries. The first theorem is a Riemann theta formula that "dominates in the labyrinth of quadratic biquadratic relations on theta functions" [2.12].

Theorem 2.20. (Riemann Theta Formula.) Let A_0 be some group of characteristics of order 2^{g-m} and $B_0 = \{[\beta] \big| |\alpha|\,|\beta|\,|\alpha\beta| = 1,\ [\alpha] \in A_0\}$ a conjugate group of characteristics of order 2^{g+m}. Then, for all characteristics $[\zeta], [\eta]$ and the half-period characteristics $[\rho], [\sigma]$ the following equality holds:

$$\begin{aligned} 2^m \sum_{[\alpha] \in A_0} & |\zeta\eta\alpha| \begin{pmatrix} \rho\sigma \\ \eta\alpha \end{pmatrix} \theta[\eta\alpha\rho\sigma](\boldsymbol{u} + \boldsymbol{w}; B)\, \theta[\eta\alpha\rho](\boldsymbol{u} - \boldsymbol{w}; B) \\ & \times \theta[\rho\alpha\sigma](\boldsymbol{v} + \boldsymbol{z}; B)\, \theta[\eta\alpha](\boldsymbol{v} - \boldsymbol{z}; B) \\ = |\zeta\eta| & \sum_{[\beta] \in B_0} |\eta\zeta\beta| \begin{pmatrix} \rho\sigma \\ \zeta\beta \end{pmatrix} \theta[\zeta\beta\rho\sigma](\boldsymbol{u} + \boldsymbol{z}; B) \\ & \times \theta[\zeta\beta\rho](\boldsymbol{u} - \boldsymbol{z}; B)\, \theta[\zeta\rho\sigma](\boldsymbol{v} + \boldsymbol{w}; B)\, \theta[\zeta\beta](\boldsymbol{v} - \boldsymbol{w}; B) \quad , \end{aligned} \tag{2.6.1}$$

where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z} \in \mathbb{C}^g$. In particular, when $m = g$,

$$\begin{aligned} & 2^g \theta[\eta\rho\sigma](\boldsymbol{u} + \boldsymbol{w}; B)\, \theta[\eta\rho](\boldsymbol{u} - \boldsymbol{w}; B)\, \theta[\eta\sigma](\boldsymbol{v} + \boldsymbol{z}; B)\, \theta[\eta](\boldsymbol{v} + \boldsymbol{z}; B) \\ & = \sum_{[\varepsilon]} |\eta\varepsilon| \begin{pmatrix} \rho\sigma \\ \eta\varepsilon \end{pmatrix} \theta[\varepsilon\rho\sigma](\boldsymbol{u} + \boldsymbol{z}; B)\, \theta[\varepsilon\rho](\boldsymbol{u} - \boldsymbol{z}; B) \\ & \quad \times \theta[\varepsilon\sigma](\boldsymbol{v} + \boldsymbol{w}; B)\, \theta[\varepsilon](\boldsymbol{v} - \boldsymbol{w}; B) \quad . \end{aligned} \tag{2.6.2}$$

The Riemann theta formula yields many relations between theta functions and theta constants. For example, we underline here the remarkable Jacobi formula,

$$\theta_1\begin{bmatrix}1\\1\end{bmatrix} = (i/2)\,\theta\begin{bmatrix}0\\0\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}\theta\begin{bmatrix}1\\0\end{bmatrix} \quad , \tag{2.6.3}$$

for which "only isolated generalizations have been found and it remains a tantalizing and beautiful result, but not well understood" [Ref. 2.8., vol. 1, p. 66]. One of the generalizations is the Rosenhain formulas for $g = 2$,

$$D(\delta_1, \delta_2)(B) = \pm\frac{1}{4}\,\theta[\varepsilon_1]\theta[\varepsilon_2]\theta[\varepsilon_3]\theta[\varepsilon_4] \quad , \tag{2.6.4}$$

where $[\varepsilon_i] = [\delta_1\delta_2\delta_{i+2}]$, $i = 1,2,3,4$, $|\delta_j| = -1$, $j = 1,\dots,6$ and where

$$D([\delta_1],\dots,[\delta_g])(B) := \det\left(\partial(\theta[\delta_1],\dots,\theta[\delta_g])/\partial(z_1,\dots,z_g)\big|_{z=0}\right) .$$

If B is a B-matrix of a hyperelliptic curve of genus $g \le 5$, there is the following generalization of formula (2.6.3):

$$D([\delta_1],\dots,[\delta_g])(B) = \pm\left(\frac{1}{2i}\right)^g \theta[\varepsilon_1]\theta[\varepsilon_2]\cdots\theta[\varepsilon_{g+2}] \quad , \tag{2.6.5}$$

where the characteristics $[\delta_1],\dots,[\delta_g]$, $[\varepsilon_1],\dots,[\varepsilon_{g+2}]$ form a special fundamental system. In particular this implies that the characteristics $[\delta_1],\dots,[\delta_g]$ are odd and the characteristics $[\varepsilon_1],\dots,[\varepsilon_{g+2}]$ are even. All characteristics are essentially independent, i.e., for any $1 \le i_1 < \dots < i_{2k} < g$ where $k < 0$, the comparison $[\delta_{i_1}\delta_{i_2}\dots\delta_{i_{2k}} \not\equiv [0]$ holds, and azygetic, i.e., for any $1 \le i < j < k \le g$ the equality $|\delta_i|\,|\delta_j|\,|\delta_l|\,|\delta_i\delta_j\delta_k| = -1$ holds. The other generalizations of formula (2.6.3) and some new results are discussed by *Fay* [2.12] and *Igusa* [2.11].

The other addition theorem that we use is the Koizumi formula [2.13]

$$\begin{aligned}
\prod_{i=0}^{N-1}\theta[\varepsilon^{(i)}](z^{(i)};B) &= \sum_{p}\theta\left[\rho^{(0)} + \begin{pmatrix}2p\\0\end{pmatrix}\right](y^{(0)};NB)\\
&\times \sum_{p_i,\, i=N-1,\dots,2}\theta\begin{bmatrix}\rho^{(1)\prime}+2p-2p_{N-1}\\ \rho^{(1)\prime\prime}\end{bmatrix}\left(y^{(1)};N(N-1)B\right)\\
&\times\theta\begin{bmatrix}\rho^{(2)\prime}+2p_{N-1}-2p_{N-2}\\ \rho^{(2)\prime\prime}\end{bmatrix}\left(y^{(2)};(N-1)(N-2)B\right)\\
&\times\theta\begin{bmatrix}\rho^{(N-2)\prime}+2p_3-2p_2\\ \rho^{(N-2)\prime\prime}\end{bmatrix}\left(y^{(N-2)};6B\right)\\
&\times\theta\begin{bmatrix}\rho^{(N-1)\prime}+2p_2\\ \rho^{(N-1)\prime\prime}\end{bmatrix}\left(y^{(N-1)};2B\right) \quad ,
\end{aligned} \tag{2.6.6}$$

where p, p_i are complete sets of the representatives $\mathbb{Z}^g/N$, $\mathbb{Z}^g/(N-i)$, $i = N-1,\dots 2$.

$$\left(y_j^{(0)},\dots,y_j^{(N-1)}\right)=\left(z_j^{(0)},\dots,z_j^{(N-1)}\right)T,$$
$$\left(\rho_j^{(0)\prime},\dots,\rho_j^{(N-1)\prime}\right)=\left(\varepsilon_j^{(0)\prime},\dots,\varepsilon_j^{(N-1)\prime}\right)T^*,$$
$$\left(\rho_j^{(0)\prime\prime},\dots,\rho_j^{(N-1)\prime\prime}\right)=\left(\varepsilon_j^{(0)\prime\prime},\dots,\varepsilon_j^{(N-1)\prime\prime}\right)T,$$

and the matrices T and $T^*=(T^{-1})^T$ are equal to

$$T=\begin{pmatrix} 1 & -(N-1) & 1 & \cdots & 0 & 0\\ 1 & 1 & -(N-2) & \cdots & 0 & 0\\ 1 & 1 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & \cdots & -2 & 0\\ 1 & 1 & 1 & \cdots & 1 & -1\\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

$$T^*=\begin{pmatrix} N^{-1} & -N^{-1} & 0 & \cdots & 0 & 0\\ N^{-1} & N^{-1}(N-1)^{-1} & -(N-1)^{-1} & \cdots & 0 & 0\\ N^{-1} & N^{-1}(N-1)^{-1} & (N-1)^{-1}(N-2)^{-1} & \cdot & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ N^{-1} & N^{-1}(N-1)^{-1} & (N-1)^{-1}(N-2)^{-1} & \cdots & -3^{-1} & 0\\ N^{-1} & N^{-1}(N-1)^{-1} & (N-1)^{-1}(N-2)^{-1} & \cdots & 6^{-1} & -2^{-1}\\ N^{-1} & N^{-1}(N-1)^{-1} & (N-1)^{-1}(N-2)^{-1} & \cdots & 6^{-1} & 2^{-1} \end{pmatrix}.$$

For $N=2$, (2.6.1) is the well-known second-order addition theorem

$$\begin{aligned}&\theta[\varepsilon^{(0)}](z^{(0)};B)\,\theta[\varepsilon^{(1)}](z^{(1)};B)\\ &=\sum_p \theta\begin{bmatrix}\frac{1}{2}(\varepsilon^{(0)\prime}+\varepsilon^{(1)\prime})+2p\\ \varepsilon^{(0)\prime\prime}+\varepsilon^{(1)\prime\prime}\end{bmatrix}(z^{(0)}+z^{(1)};2B)\\ &\quad\times\theta\begin{bmatrix}\frac{1}{2}(\varepsilon^{(0)\prime}-\varepsilon^{(1)\prime})+2p\\ \varepsilon^{(0)\prime\prime}-\varepsilon^{(1)\prime\prime}\end{bmatrix}(z^{(0)}-z^{(1)};2B)\ .\end{aligned}\tag{2.6.7}$$

In what follows we shall use a special case of the formula (2.6.7):

$$\begin{aligned}&\theta\begin{bmatrix}\alpha\\ \beta+\gamma\end{bmatrix}(z^{(1)}+z^{(2)};B)\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z^{(1)}-z^{(2)};B)\\ &=\sum_j(-1)^{\langle\beta,\delta\rangle}\theta\begin{bmatrix}\delta\\ \gamma\end{bmatrix}(2z^{(1)};2B)\,\theta\begin{bmatrix}\alpha+\delta\\ \gamma\end{bmatrix}(2z^{(2)};2B)\end{aligned}\tag{2.6.8}$$

as well as the inversion of the formula

$$\begin{aligned}&\theta\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}(2z^{(1)};2B)\,\theta\begin{bmatrix}\beta\\ \gamma\end{bmatrix}(2z^{(2)};2B)=2^{-g}\sum_\varepsilon(-1)^{\langle\alpha,\varepsilon\rangle}\\ &\times\theta\begin{bmatrix}\alpha+\beta\\ \gamma+\varepsilon\end{bmatrix}(z^{(1)}+z^{(2)};B)\,\theta\begin{bmatrix}\alpha-\beta\\ \varepsilon\end{bmatrix}(z^{(1)}-z^{(2)};B).\end{aligned}\tag{2.6.9}$$

We mentioned here only the two addition theorems that are referred to in our further discussion. Let us note that the theory of theta functions employs other remarkable addition theorems such as the Frobenius addition theorem for hyperelliptic theta functions [2.8], the trisecant Fay formulas for the Riemann theta functions [2.7] and their generalizations, "irrational" Schottky-Frobenius addition theorems [2.14], etc.

2.7 Construction of Functions and Differentials on a Riemann Surface Using Theta Functions

In this section we consider an algebraic curve X of genus g and describe two classical methods of constructing on it meromorphic functions and meromorphic Abelian differentials, as well as periodic functions with essential singularities. The first method involves giving the divisors of zeros and poles of a meromorphic function, and the second is concerned with giving the principal parts of the poles. Both methods are based on using the fundamental Riemann theorem on theta divisor.

Theorem 2.21. (Riemann Theorem.) Let the curve X be equipped with a canonical basis $(\mathbf{a}, \mathbf{b}) \in H_1(X, \mathbb{Z})$ and $\boldsymbol{K} = (K_1, \ldots, K_g)$ be a vector of Riemann constants,

$$K_j = \frac{2\pi i + Bjj}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \left(\int_{a_l} \omega_l(P) \int_{P_0}^{P} \omega_j \right), \quad j = 1, \ldots, g, \tag{2.7.1}$$

that depends on the point P_0 and the basis $(\mathbf{a}, \mathbf{b})$. Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_g) \in J(X)$ be a vector such that the Riemann theta function, defined by $F(P) = \theta\left(\int_{P_0}^{P} \omega - \boldsymbol{\zeta} - \boldsymbol{K};\ B\right)$, does not vanish identically on X. Then:

(i) The function $F(P)$ has on X exactly g zeros $P_1, \ldots, P_g$ that give a solution to the Jacobi inversion problem

$$\sum_{k=1}^{g} \int_{P_0}^{P_k} \omega_j = \zeta_j, \quad j = 1, \ldots, g \quad ; \tag{2.7.2}$$

(ii) The divisor $\mathcal{D} = P_1 + \cdots + P_g$ is nonspecial;

(iii) The points $P_1, \ldots, P_g$ are defined from (2.7.2) uniquely up to a permutation.

Corollary 2.22. For the nonspecial divisor $\mathcal{D} = P_1 + \cdots + P_g$, the function $F(P) = \theta\left(\int_{P_0}^{P} \omega - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K};\ B\right)$ has on X exactly g zeros $P = P_1, \ldots, P_g$.

To construct a meromorphic function f on X by its divisor, it would be instrumental to have on X a holomorphic function that vanishes if and only if

$P = Q$, $P, Q \in X$ and thus generalizes the function $x - y$ on $\mathbb{C} \setminus \{\infty\}$. Since there is no such function on a Riemann surface of genus $g > 0$ we replace it by introducing on X Fay's prime-form $E(P, Q)$, i.e., a $(-1/2, -1/2)$-differential form holomorphic on $X \times X$ [2.7, 8] (Sect. 2.4):

$$E(P,Q) = \frac{\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \left(\int_Q^P \omega;\ B \right)}{h_\delta(P)\, h_\delta(Q)} , \tag{2.7.3}$$

where $[\delta]$ is a non-singular odd characteristic and $h_\delta^2(P) = \sum_{i=1}^{i=g} \theta_i[\delta]\omega_i(P)$. The Riemann theorem on theta function zeros yields the basic property of the form $E(P, Q)$; namely, it vanishes if and only if $P = Q$; moreover, the zero is of first order. Formula (2.7.3) immediately gives the other properties $E(P, Q) = -E(Q, P)$; if we choose in the neighborhood of $P, Q \in X$ such a local coordinate z that $h_\delta^2(P) = dz(P)$, $h_\delta^2(Q) = dz(Q)$ then

$$E(P,Q) = \frac{z(P) - z(Q)}{\sqrt{dz(P)\, dz(Q)}} \left\{ 1 + o\left[(z(P) - z(Q))^2 \right] \right\} , \tag{2.7.4}$$

and, finally $E(P, Q)$ is invariant under by-passing of a-cycles and is multiplied by $\exp\left(-(1/2)B_{ii} \mp \int_Q^P \omega_i \right)$ when the b_i-cycle is by-passed by the coordinate P (upper sign) and by the coordinate Q (lower sign) in the exponent.

We use the prime-form to construct meromorphic functions and the main differentials on X (e.g., [2.7, 8]).

(a) Let $\mathcal{D} = \sum_{i=1}^n P_i - \sum_{j=1}^n Q_j$ be a divisor of a meromorphic function f, i.e., a divisor that satisfies the condition of the Abelian theorem (Sect. 2.4) Then

$$f(P) = \prod_{i=1}^n \frac{E(P, P_i)}{E(P, Q_i)} \tag{2.7.5}$$

is a single-valued meromorphic function with a divisor of zeros, $\sum P_i$ and a divisor of poles, $\sum Q_i$.

(b) Let us construct differentials of the third kind $d\Omega_{P_0 - Q_0}(P)$ that have zero a-periods, a simple pole P_0 with residue 1 and a simple pole Q_0 with residue -1. We set

$$d\Omega_{P_0 - Q_0}(P) = d_P \ln \frac{E(P, P_0)}{E(P, Q_0)} . \tag{2.7.6}$$

Indeed, $\int_{a_i} d\Omega_{P_0 - Q_0}(P) = 0$, $i = 1, \dots, g$ and we have on local coordinates

$$\begin{aligned} d\Omega_{P_0 - Q_0}(P) = {} & d_P \ln (z(P) - z(P_0)) \\ & - d_P \ln (z(P) - z(Q_0)) + \text{ holomorphic differential} , \end{aligned}$$

with this yielding the statement about residues.

(c) Let us construct a differential of the second kind $d\Omega_Q(P)$ with zero a-periods and the only second-order pole at the point Q. For this purpose we consider the 2-form

$$d\Omega(P,Q) = d_P\, d_Q \ln E(P,Q) \quad . \tag{2.7.7}$$

This form has zero a-periods and is decomposed, in local coordinates, into a series by the formula

$$d\Omega(P,Q) = \frac{dz(P)\, dz(Q)}{\big(z(P) - z(Q)\big)^2} + O(1)\, dz(P)\, dz(Q) \quad . \tag{2.7.8}$$

The second-kind differential that we look for is defined as

$$d\Omega_Q(P) = \frac{d\Omega(P,Q)}{dQ} \quad .$$

To derive finite-gap solutions of completely integrable equations, we need to describe meromorphic functions and functions with essential singularities in terms of the principal parts of the poles. The above method of constructing meromorphic functions does not serve this purpose, because it involves the condition of the Abel theorem. So, using the Riemann theorem on the zeros, we describe another technique. First, we construct on X a meromorphic function with $g + n$ poles, where $n \geq 1$.

Proposition 2.23. Let $\mathcal{D}$ be a non-special divisor of degree g and $\mathcal{D}'$ an arbitrary positive divisor of degree n. A meromorphic function $\Psi(P)$ on X with $g + n$ poles in $\mathcal{D} + \mathcal{D}'$ is defined by the formula

$$\Psi(P) = \frac{\theta\left(\int_{P_0}^{P} \omega + \boldsymbol{W} - \boldsymbol{D};\ B\right)}{\theta\left(\int_{P_0}^{P} \omega - \boldsymbol{D};\ B\right)} \exp \Omega(P) \quad , \tag{2.7.9}$$

where $\Omega(P) = \int_{P_0}^{P} d\Omega$, $d\Omega$ is a normalized Abelian differential of the third kind with poles in $\mathcal{D}'$ (with residues -1), the vector $\boldsymbol{W} = (W_1, \ldots, W_g) = (\int_{b_1} d\Omega, \ldots, \int_{b_g} d\Omega)$ is a vector of b-periods, the vector $\boldsymbol{D} = \boldsymbol{A}(\mathcal{D}) + \boldsymbol{K}$, where $\boldsymbol{A}(\mathcal{D})$ is an Abelian mapping and $\boldsymbol{K}$ is a vector of Riemann constants; the integration path in the integrals $\Omega(P)$ and $\int_{P_0}^{P} \omega$ is chosen to be the same.

Proof. From Corollary 2.22 and (2.7.9) it follows that the function (2.7.9) really has poles only in $\mathcal{D} + \mathcal{D}'$. Therefore, we have to show that the function (2.7.9) does not change when P goes around the arbitrary cycle $\gamma \in H_1(X, \mathbb{Z})$. We denote by $\mathcal{M}_\gamma$ a monodromy operator that corresponds to going around the cycle $\gamma = \sum_{i=1}^{i=g}(N_i a_i + M_i b_i) = \langle \boldsymbol{N}, \boldsymbol{a} \rangle + \langle \boldsymbol{M}, \boldsymbol{b} \rangle$, where $\boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^g$ are the coefficients of decomposition of γ in the basis $(\boldsymbol{a}, \boldsymbol{b})$ in $H_1(X, \mathbb{Z})$. Then, in virtue of the transformation property of the theta function (2.5.10), we have

$$\mathcal{M}_\gamma[\Psi(P)] = \frac{\theta\left(\int_{P_0}^P \omega + \boldsymbol{W} - \boldsymbol{D} + B\boldsymbol{M} + 2\pi i \boldsymbol{N};\ B\right)}{\theta\left(\int_{P_0}^P \omega - \boldsymbol{D} + B\boldsymbol{M} + 2\pi i \boldsymbol{N};\ B\right)} \times \exp\left(\Omega(P) + \langle \boldsymbol{W}, \boldsymbol{M}\rangle\right) = \Psi(P)$$

which completes the proof.

We now construct on X functions with essential singularities. These functions are the generalization of the exponential function exp z on $\mathbb{C}$ which is analytic in $\mathbb{C}$ and has an essential singularity at $z = \infty$. Contrary to this if $g \geq 1$, then in addition to essential singularities, the functions have poles as well.

We fix on X an arbitrary, but finite number of points $Q_1, \ldots, Q_n$ and define local parameters z_j so that $z_j(Q_j) = \infty$. We associate every point Q_j with an arbitrary polynomial $q_j(z_j)$. Next, let $P_1 + \cdots + P_g$ be an arbitrary positive divisor on $X \backslash \{Q_1, \ldots, Q_n\}$ of degree g. We denote by $L(\mathcal{D};\ Q_1, \ldots, Q_n,\ q_1, \ldots, q_n)$ a linear space of functions $\Psi(P)$ on X and satisfying the conditions

(1) The function $\Psi(P)$ is meromorphic on $X \setminus \{Q_1, \ldots, Q_n\}$ and has poles only at the points of the divisor $\mathcal{D} = P_1 + \cdots + P_g$;
(2) In the neighborhood of every point Q_j, $j = 1, \ldots, n$ the following estimate holds:

$$\Psi(P) \exp\{-q_j[z_j(P)]\} = O(1) \quad . \tag{2.7.10}$$

Theorem 2.24. The space $L(\mathcal{D};\ Q_1, \ldots, Q_n,\ q_1, \ldots, q_n)$ is one-dimensional for the non-special divisor $\mathcal{D} \in X$ and the polynomials q_j, $j = 1, \ldots, n$ with sufficiently small coefficients. Its basis is described explicitly by

$$\Psi_0(P) = \frac{\theta\left(\int_{P_0}^P \omega + \boldsymbol{V} - \boldsymbol{D};\ B\right)}{\theta\left(\int_{P_0}^P \omega - \boldsymbol{D};\ B\right)} \exp \Omega(P) \quad , \tag{2.7.11}$$

where $\Omega(P)$ is a normalized Abelian integral of the second kind with poles at the points $Q_1, \ldots, Q_n$, the principal parts of which coincide with the polynomials $q_j(z_j)$, $j = 1, \ldots, n$, V is a vector of the b-periods of the integrals of $\Omega(P)$, i.e.,

$$V_j = \int_{b_j} d\Omega(P), \quad j = 1, \ldots, g \quad , \tag{2.7.12}$$

$\boldsymbol{D} = \boldsymbol{A}(\mathcal{D}) + \boldsymbol{K}$, where $\boldsymbol{A}(\mathcal{D})$ is an Abelian transformation and $\boldsymbol{K}$ is a vector of Riemann constants, and the integration path in the integrals

$$\Omega(P) = \int_{P_0}^P d\Omega(P) \quad \text{and} \quad \int_{P_0}^P \omega$$

is chosen to be the same.

Proof. The function (2.7.11) has, by Corollary 2.2, poles exactly at the points of the divisor $\mathcal{D}$ and, by construction, essential singularities at the points

$Q_1, \dots, Q_n$. We show that the function (2.7.11) is single-valued. This is equivalent to the $\Psi_0(P)$ being invariant when the point P goes around an arbitrary cycle $\gamma \in H_1(X, \mathbb{Z})$. Let $\gamma = \langle \boldsymbol{N}, \boldsymbol{a} \rangle + \langle \boldsymbol{M}, \boldsymbol{b} \rangle = \sum_{i=1}^{g} (N_i a_i + M_i b_i)$, where $N_i, M_i \in \mathbb{Z}$, $i, j = 1, \dots, g$ and $(\boldsymbol{a}, \boldsymbol{b})$ is the basis in $H_1(X, \mathbb{Z})$. We denote by $\mathcal{M}_\gamma$ a monodromy operator that corresponds to the cycle γ being traversed. Using the transformation property of the theta function (2.5.10) we have

$$\mathcal{M}_\gamma[\Psi_0(P)] = \frac{\theta\left(\int_{P_0}^{P} \omega + \boldsymbol{V} - \boldsymbol{D} + B\boldsymbol{M} + 2\pi \mathrm{i} \boldsymbol{N};\ B\right)}{\theta\left(\int_{P_0}^{P} \omega - \boldsymbol{D} + B\boldsymbol{M} + 2\pi \mathrm{i} \boldsymbol{N};\ B\right)} \times \exp\left(\Omega(P) + \langle \boldsymbol{V}, \boldsymbol{M} \rangle\right) = \Psi_0(P).$$

Next, let Ψ be an arbitrary element of the space L. Then, the ratio Ψ/Ψ_0 is a rational function L with a divisor of poles that coincides with the divisor $\mathcal{D}' = P_1' + \dots + P_g'$ or zeros of the function $\Psi_0(P)$, for which, by virtue of (2.7.11) the following comparison is valid:

$$\boldsymbol{A}(\mathcal{D}') - \boldsymbol{A}(\mathcal{D}) = \boldsymbol{V} \quad . \tag{2.7.13}$$

For sufficiently small vectors $\boldsymbol{V}$ (i.e., for sufficiently small coefficients of the polynomials q), the theta function in the numerator of (2.7.11) does not vanish identically. Consequently, its pole divisor $\mathcal{D}'$ is non-special, so that Ψ/Ψ_0 is a constant (Sect. 2.4).

Remark 2.25. Non-special divisors are divisors of the general position; therefore, in Theorem 2.23 we can replace the requirement that the coefficients of the polynomials q_j be small with the requirement that the "polynomials be of general form".

Corollary 2.26. Let $\mathcal{D}$ be a non-special divisor and $q_i, i = 1, \dots, n$ are polynomials of the general form. Then, if for $\Psi \in L(\mathcal{D}; Q_1, \dots, Q_n, q_1, \dots, q_n)$ the symbol $O(1)$ is replaced by $o(1)$, at least in one of the estimates (2.7.10), $\Psi(P) = 0$.

Throughout this section we treated the functions $f(P)$ depending on $P \in X$. Finally, we consider an arbitrary meromorphic function f on X and a set of g points of general position (non-special divisor) $\mathcal{D} = P_1 + \dots + P_g$. We define the symmetric function $S_i(f; \mathcal{D})$, $i = 1, \dots, g$, as

$$S_1(f; \mathcal{D}) = \sum_{i=1}^{g} f(P_i), \dots, \ S_g(f; \mathcal{D}) = \prod_{i=1}^{g} f(P_i) \quad .$$

Since the Jacobi inversion problem is solvable for all divisors $\mathcal{D}$, and is uniquely solvable for almost all $\mathcal{D}$, the functions $S_i(f; \mathcal{D})$, $i = 1, \dots, g$ are functions of the points $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_g)$ of the Jacobian $J(X)$, i.e., $2g$-multiple periodic functions with a period lattice $\{2\pi i M + BN\}$. Since these functions are Abelian, they are expressed via theta functions. In particular, the following equality is valid:

$$
\begin{aligned}
S_1(f;\mathcal{D}) = & \frac{1}{2\pi i}\sum_k \int_{a_k} f(P)\omega_k - \\
& - \sum_{f(Q_k)=\infty} \operatorname{res} f(P)\, d\ln\theta\left(\int_{P_0}^{P}\omega - A(\mathcal{D}) - K;\ B\right) .
\end{aligned}
\tag{2.7.14}
$$

2.8 Hyperelliptic Curves

In this section we examine a hyperelliptic curve X of genus g and describe the relevant meromorphic functions on it. Let the hyperelliptic curve X be in the form (2.1.2) and equipped with the basis (a, b) in $H_1(X, \mathbb{Z})$, as indicated in Fig. 2.5. We normalize the basis in the space of holomorphic differentials (2.4.2), $\omega = (\omega_1, \ldots, \omega_g)$, $\omega_i = \sum c_{ij}\lambda^{g-j}d\lambda/\mu$ such that the Riemann matrix $\Pi = (2\pi i I; B)$. We denote by A_j the quantities $A_j = A(Q_j, Q_1)$, $j = 1, \ldots, 2g+2$, where $A(P, Q)$ is an Abelian mapping, and write down the characteristics $[A_j]$ of the points

$$
\begin{aligned}
&[A_1] = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}, \quad [A_2] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} , \\
&[A_3] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}, \quad \ldots , \\
&[A_{2k+1}] = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{matrix}}_{k} & \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} \end{bmatrix} , \\
&[A_{2k+2}] = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{matrix}}_{k} & \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} \end{bmatrix} , \quad \ldots , \\
&[A_{2g+1}] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \quad [A_{2g+2}] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} .
\end{aligned}
\tag{2.8.1}
$$

The calculation of the vector of Riemann constants (2.7.1) is simplified in the present case of a hyperelliptic curve. It results in K being a half-period, with the characteristics given by

$$
[K] = \sum_{|A_i|=-1} [A_i] \pmod 2 , \tag{2.8.2}
$$

which, in the homology basis shown in Fig. 2.5, yields

$$[\boldsymbol{K}] = \begin{bmatrix} g & g-1 & \cdots & 1 \\ g & 1 & \cdots & 1 \end{bmatrix} \quad .$$

We associate each of 4^g half-periods with one of 4^g partitions of the numbers $I = \{1, \ldots, 2g+2\}$ into two groups

$$I = I_m \cup (I/I_m) = \{i_1, \ldots, i_{g+1-2m}\} \cup \{j_1, \ldots, j_{g+1+2m}\} \quad ,$$

where $m = 0, 1, \ldots, [(g+1)/2]$. Each of the partitions is associated with the characteristic

$$[\varepsilon(I_m)] = \left(\left[\boldsymbol{A} \left(\sum_{k=1}^{g+1-2m} Q_{i_k} - (g+1-2m)Q_1 \right) \right] + [\boldsymbol{K}] \right) \ (\mathrm{mod}\ 2). \quad (2.8.3)$$

The number m in (2.8.3) is the multiplicity of the zero of $\theta[\varepsilon](\boldsymbol{z}; B)$ for $\boldsymbol{z} = 0$. For $m = 0$, there are $\binom{2g+1}{g}$ even non-singular characteristics

$$[\varepsilon] = \left(\sum_{k=1}^{g+1} [\boldsymbol{A}_{i_k}] + [\boldsymbol{K}] \right) \quad (\mathrm{mod}\, 2) \quad ,$$

for $m = 1$ there are $\binom{2g+2}{g-1}$ singular characteristics

$$[\delta] = \left(\sum_{k=1}^{g-1} [\boldsymbol{A}_{i_k}] + [\boldsymbol{K}] \right) \quad (\mathrm{mod}\, 2) \quad ,$$

and at last for $m > 1$ there are $\binom{2g+2}{g+1-2m}$ singular characteristics

$$[\eta] = \left(\sum_{k=1}^{g+1-2m} [\boldsymbol{A}_{i_k}] + [\boldsymbol{K}] \right) \quad (\mathrm{mod}\, 2) \quad ,$$

the parity of which is the same as that of the number m and for which the theta function $\theta[\eta](z; B)$ has an mth order zero at $\boldsymbol{z} = 0$.

In particular, for genera $g = 1, 2$ there are no singular characteristics, for genus $g = 3$ there is one even singular characteristic of the vector of Riemann constants $[\boldsymbol{K}]$. We note that for $g > 2$, the characteristic $[\boldsymbol{K}]$ is always singular and the theta function $\theta[\boldsymbol{K}](\boldsymbol{z}; B)$ at $\boldsymbol{z} = 0$ has zero of the order $m = g/2$ if g is even, and $(g+1)/2 = m$ if g is odd. On the other hand, the vanishing of the theta constants to the mth order represents a complete set of the condition that distinguish the B-matrices of hyperelliptic curves in $\mathcal{U}_g$. More specifically, the following holds:

Theorem 2.27. [Ref. 2.3, p. 309]. In order for the matrix B of a curve X to be a B-matrix of a hyperelliptic curve, it is necessary and sufficient that
(a) in the case $g = 4$ or 6 there is one half-period on which the theta function has zeros of order 2 or 3;
(b) in the case of an even genus $g \geq 8$, there are two half-periods on which the theta function has a zero of order $g/2$;
(c) in the case of an odd genus $g \geq 3$, there is one half-period on which the theta function has a zero of order $(g+1)/2$.

We now use the theta functions to construct meromorphic functions on the hyperelliptic curve $X = (\mu, \lambda)$. The simplest of them is the coordinate λ, which is a meromorphic function of the second order. We specify the notation as follows: the curve X is represented as

$$\mu^2 = \lambda(\lambda - 1)\prod_{i=1}^{2g-1}(\lambda - \lambda_i) \quad , \tag{2.8.4}$$

where $\lambda_1, \dots, \lambda_{2g-1}$ are different points in $\mathbb{C}^2 \setminus \{0, 1\}$, with $\pi(Q_1) = 0$, $\pi(Q_2) = 1$, $\pi(Q_{2j+1}) = \lambda_j$, $j = 1, \dots, 2g-1$, $\pi(Q_{2g+2}) = \infty$. We equip the curve (2.8.4) with the basis $(\mathbf{a}, \mathbf{b}) \in H_1(X, \mathbb{Z})$ represented in Fig. 2.5 and consider the function

$$f(P) = c\frac{\theta^2\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 1 & 1 & 0 & \cdots & 0\end{bmatrix}\left(\int_{Q_1}^{P}\omega;\ B\right)}{\theta^2\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\end{bmatrix}\left(\int_{Q_1}^{P}\omega;\ B\right)} \ . \tag{2.8.5}$$

The numerator of the function (2.8.5) is not identically zero. Indeed, e.g., when $P = Q_2$, we use (2.8.1) to find that the numerator is $\theta^2\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 1 & 1 & 0 & \cdots & 0\end{bmatrix}$, where the characteristic $\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 1 & 1 & 0 & \cdots & 0\end{bmatrix}$ is odd and generated by the partition $\{1, 2, 5, 7, \dots, 2g+1\} \cup \{3, 4, 6, \dots, 2g+2\}$ with $m = 0$ [see (2.8.3]. Next, at $P = Q_1$, $f(P)$ has a second-order zero, because the odd characteristic $\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 1 & 1 & 0 & \cdots & 0\end{bmatrix}$, appearing in the numerator, is non-singular, i.e., it corresponds to the partition $I_1 \cup (I \setminus I_1)$ with $I_1 = \{5, \dots, 2g+1\}$, while the even non-singular characteristic $\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\end{bmatrix}$ in the denominator corresponds to the partition $I_0 \cup (I \setminus I_0)$ with $I_0 = \{2g+2, 5, 7, \dots, 2g+1\}$. At $P = Q_{2g+2}$ the function (2.8.5) has a second-order pole, because under this substitution the theta constants appearing in the numerator and denominator are inverse to those considered above, in view of (2.8.1).

We note that the numerator vanishes at g points $Q_1, Q_5, Q_7, \dots, Q_{2g+1}$, and the denominator at g points $Q_{2g+2}, Q_5, Q_7, \dots, Q_{2g+1}$. By the Riemann theorem, these are the only zeros of the numerator and the denominator. We can easily see that $f(Q_k) \neq 0$ and is finite at the ambiguity points Q_k, $k = 5, \ldots, 2g+1$. Thus, the function (2.8.5) has only a second-order zero at $P = Q_1$ and a second-order pole at $P = Q_{2g+2}$. By finding the constant c from the normalization condition $f(Q_2) = 1$, we finally obtain that the coordinate is expressed via theta functions as

$$\lambda = \frac{\theta^2\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \theta^2\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}\left(\int_{Q_1}^{P} \omega;\ B\right)}{\theta^2\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix} \theta^2\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}\left(\int_{Q_1}^{P} \omega;\ B\right)}. \tag{2.8.6}$$

Formula (2.8.6) can be used to derive an expression in even theta constants for the projections $\lambda_i = \pi(Q_{2+i})$ of the branching points at $i = 1, 2, 4, \ldots, 2g-2$. In particular, substituting $P = Q_3$ into (2.8.6) yields

$$\lambda_1 = \frac{\theta^2\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \theta^2\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}{\theta^2\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix} \theta^2\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}} . \tag{2.8.7}$$

We can also obtain expressions in even theta constants for λ_i when $i = 3, 5, \ldots, 2g-1$. To avoid the need for evaluating indeterminacies in the expression for $f(P)$, we must choose appropriate characteristics in (2.8.5).

We note that the possibility of expressing the numbers λ_i via theta functions in different ways indicates the existence of numerous relations between them. Some of these relations will be given below.

Other useful formulas for meromorphic functions can be obtained by calculating the residue for the case of interest in (2.7.12). Let the curve X be given by $\mu^2 = \prod_{i=1}^{2g+1}(\lambda - E_i)$ and $\mathcal{D} = P_1 + \cdots + P_g$ be a nonspecial divisor. As basis point P_0 (lower bound in the Abelian mapping), we choose a point Q_{2g+2}, $\pi(Q_{2g+2}) = \infty$.

We consider the function $f = \lambda$, $f : X \to \mathbb{C}$. According to (2.7.14), we have

$$S_1(\lambda, \mathcal{D}) = c - \operatorname{res} \lambda \, d \ln \theta \left(\int_{Q_{2g+2}}^{P} \omega - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K};\ B \right).$$

To calculate the residue, we take $\lambda = 1/z^2 = \infty$ as a local parameter in the neighborhood of the point Q_{2g+2}. For normalized holomorphic differentials ω and the Abelian transformation we have the decompositions

$$\omega(P) = \left(\boldsymbol{U} + o(z^2)\right) dz, \quad \boldsymbol{A}(P) = \boldsymbol{U} z + o(z^3) \quad , \tag{2.8.8}$$

where the vector $\boldsymbol{U} = (U_1, \ldots, U_g) = -2(c_{11}, \ldots, c_{1g})$. Then

$$d\ln\theta\left(\int_{Q_{2g+2}}^{P}\omega - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K};\ B\right) = dz\left[\partial - \mathbf{U}\ln\theta\,(\boldsymbol{A}(\mathcal{D}) + \boldsymbol{K};\ B)\right.$$
$$\left.+\,\partial_U^2\ln\theta\,(\boldsymbol{A}(\mathcal{D}) + \boldsymbol{K};\ B)\,z + o(z^2)\right]\quad ,$$

where $\partial_U = \sum_{j=1}^{g} U_j\partial/\partial z_j$ is the operator of differentiation in the direction $\boldsymbol{U}$. We finally obtain

$$S_1(\lambda;\mathcal{D}) = -\partial_U^2\ln\theta\left(\left(\int_{Q_{2g+2}}^{P_1} + \cdots + \int_{Q_{2g+2}}^{P_g}\right)\omega + \boldsymbol{K};\ B\right) + C \tag{2.8.9}$$

where the constant C is defined from the formula (2.7.12).

We also consider the function $f = \ln(\lambda - E_k)$, $1 \leq k \leq 2g-1$. In this case we obtain

$$\sqrt{\prod_{j=1}^{g}(\lambda(P_j) - E_k)}$$
$$= h_k\frac{\theta\left(\int_{P_0}^{Q_k}\omega - \left(\int_{P_0}^{P_1} + \cdots + \int_{P_0}^{P_g}\right)\omega - \boldsymbol{K};\ B\right)}{\theta\left(\int_{P_0}^{Q_{2g+2}}\omega - \left(\int_{P_0}^{P_1} + \cdots + \int_{P_0}^{P_g}\right)\omega - \boldsymbol{K};\ B\right)}\ , \tag{2.8.10}$$

where h_k are constants.

We consider a hyperelliptic curve $X = (\mu, \lambda)$ that has no branching at infinity, $\mu^2 = \prod_{k=1}^{2g+2}(\lambda - E_k)$, and let $\mathcal{D} = P_1 + \cdots + P_{g-n+1}$ be a divisor of degree $g-n+1$, $n \leq g$. Furthermore, let $\{i_1, \ldots, i_n\}$ and $\{j_1, \ldots, j_n\}$ be two sets of numbers in $\{1, 2, \ldots, 2g+2\}$. Then, the following formula [Ref. 2.5, p. 776] is valid:

$$\prod_{l=1}^{g-n+1}\sqrt{\prod_{k=1}^{n}\frac{\lambda(P_l) - E_{i_k}}{\lambda(P_l) - E_{j_k}}} = c\frac{\theta\left[\boldsymbol{K} + \sum_{k=1}^{n}\boldsymbol{A}_{i_k}\right](\boldsymbol{A}(\mathcal{D});\ B)}{\theta\left[\boldsymbol{K} + \sum_{k=1}^{n}\boldsymbol{A}_{j_k}\right](\boldsymbol{A}(\mathcal{D});\ B)} \tag{2.8.11}$$

where c is a constant independent of $\mathcal{D}$. Formula (2.8.11) yields for $n = g$

$$\frac{\theta\left[\boldsymbol{K} + \sum_{k=1}^{g}\boldsymbol{A}_{i_k}\right]\left(\int_{P_0}^{P}\omega;\ B\right)}{\theta\left[\boldsymbol{K} + \sum_{k=1}^{g}\boldsymbol{A}_{j_k}\right]\left(\int_{P_0}^{P}\omega;\ B\right)}$$
$$= \frac{\kappa\sqrt{\dfrac{(\lambda - E_{i_1})\cdots(\lambda - E_{i_g})}{(\lambda - E_{j_1})\cdots(\lambda - E_{j_g})}}}{\sqrt[4]{\dfrac{(E_{i_g+1} - E_{i_1})\cdots(E_{i_g+1} - E_{i_g})}{(E_{i_g+1} - E_{j_1})\cdots(E_{i_g+1} - E_{j_g})}}\sqrt[4]{\dfrac{(E_{j_g+1} - E_{i_1})\cdots(E_{j_g+1} - E_{i_g})}{(E_{j_g+1} - E_{j_1})\cdots(E_{j_g+1} - E_{j_g})}}}\ , \tag{2.8.12}$$

and for $n = 1$,

$$\frac{\theta[K + A_j]\left(\left(\int_{P_0}^{P_1} + \cdots + \int_{P_0}^{P_g}\right)\omega;\ B\right)}{\theta[K + A_k]\left(\left(\int_{P_0}^{P_1} + \cdots + \int_{P_0}^{P_g}\right)\omega;\ B\right)} = \kappa\, \frac{\prod_{l=1}^{g} \sqrt{\dfrac{\lambda(P_l) - E_j}{\lambda(P_l) - E_k}}}{\prod_{\substack{m=1\\ m\neq j,k}}^{2g} \sqrt[4]{\dfrac{E_m - E_j}{E_m - E_k}}}, \tag{2.8.13}$$

where κ is a primitive 8th root of unity.

Formula (2.8.11) can also be used to derive the important Thomae formulas [Ref. 2.5, p. 774]:

$$\theta^4[\varepsilon(I_0)] = \det c^{-2} \prod_{\substack{i,j\in I_0\\ i<j}} (E_i - E_j) \prod_{\substack{k,l\in I\setminus I_0\\ k<l}} (E_k - E_l)\ , \tag{2.8.14}$$

$$\begin{aligned} \theta_j^4[\varepsilon(I_1)] ={}& \frac{1}{16} \det c^2 \prod_{\substack{m,n\in I_1\\ m\neq n}} (E_i - E_j) \prod_{\substack{k,l\in I\setminus I_1\\ k\neq l}} (E_k - E_l) \\ &\times \left(\sum_{i=1}^{g} d_{ij} S_{i-1}(I_1)\right)^4\ , \quad j = 1, \ldots, g, \end{aligned} \tag{2.8.15}$$

where $c = \|c_{ij}\|$ and $d = c^{-1}$ are $(g \times g)$-matrices and c_{ij} normalization constants of holomorphic differentials, $S_{i-1}(I_1)$ symmetric functions,

$$S_0(I_1) = 1,\ S_1(I_1) = \sum_{i\in I_1} E_i, \ldots,\ S_{g-1}(I_1) = \prod_{j\in I_1} E_j.$$

We have shown above how the projections E_j of the branch points of the curve X are expressed in terms of theta constants. Let us show now how the normalization constants c_{ij} of holomorphic differentials can be expressed in terms of theta functions.

Proposition 2.28. Let B be such a hyperelliptic point that there is a set $M = ([\delta_1], \ldots, [\delta_g])$ of odd non-singular characteristics satisfying the condition

$$D(M)(B) = \det\left(\partial(\theta[\delta_1], \ldots, \theta[\delta_g])/\partial(z_1, \ldots, z_g)\big|_{z=0}\right) \neq 0\ . \tag{2.8.16}$$

Then, the following formulas are valid:

$$\begin{aligned} c_{ji} ={}& \left(\frac{1}{2} D(M)\right) \det c^{1/2} \sum_{r=1}^{g} \mathcal{D}_{jr} \prod_{\substack{m,n\in I_1^{(r)}\\ m<n}} (E_m - E_n)^{1/4} \\ &\times \prod_{\substack{k,l\in I^{(r)}\setminus I_1^{(r)}\\ k<l}} (E_k - E_l)^{1/4} S_{i-1}(I_1^{(r)})\ , \quad j,k = 1, \ldots, g, \end{aligned} \tag{2.8.17}$$

where D_{jr} are the co-factors of the matrix $\|D(M)(B)\|$, and $I_1^{(r)}$, $I \setminus I_1^{(r)}$, $r = 1, \dots, g$ are the partitions of sets $I = \{1, \dots, 2g+2\} = \{i_1^{(r)}, \dots, i_{g-1}^{(r)}\} \cup \{j_1^{(r)}, \dots, j_{g+3}^{(r)}\}$ that correspond to the characteristics $[\delta_1], \dots, [\delta_g]$.

Proof. Let M be a set of characteristics that satisfy the condition (2.8.16) and $I_1^{(r)}$, $r = 1, \dots, g$ be a set of partitions $I = \{1, \dots, 2g+2\}$ into two groups $I_1^{(r)}$ and $I \setminus I_1^{(r)}$ that correspond to these characteristics. Using (2.8.15), we can obtain the following equality for $[\delta_1]$:

$$\sum_{j=1}^{g} c_{ji}\theta_j[\delta_1] = \frac{1}{2}\det c^{1/2} \prod_{\substack{m,n \in I_1^{(r)} \\ m<n}} (E_m - E_n)^{1/4}$$
$$\times \prod_{\substack{k,l \in I^{(r)} \setminus I_1^{(r)} \\ k<l}} (E_k - E_l)^{1/4} S_{i-1}(I_1^{(1)}) \quad .$$

With k fixed, we write another $g-1$ equation for the characteristics $[\delta_2], \dots, [\delta_g]$. Formulas (2.8.17) are then derived using the Kramer rule.

Remark 2.29. Equation (2.8.17) can be improved by calculating $D(M)(B)$ in even theta functions with the help of a generalized Jacobi formula (2.6.5).

Appendix 2.1 Uniformization of Riemann Surfaces

In this appendix we give the basic points of the theory of uniformization of Riemann surfaces that will be useful in our further discussion (for more details see, for example, [2.1,9].

We study three domains on $\mathbb{CP}^1$ that will be denoted by $\Delta = \{\mathbb{CP}^1, \mathbb{C}, H\}$, where H is the upper half-plane $H = \{z \,|\, \operatorname{Im} z > 0\}$.

Lemma A.1.1. All holomorphic homeomorphisms $g : \Delta \to \Delta$ are of the form

$$
\begin{aligned}
&(a)\ \ \Delta = \mathbb{CP}^1,\ gz = \frac{\alpha z + \beta}{\gamma z + \delta},\ \alpha\delta - \beta\gamma = 1,\ \alpha, \beta, \gamma, \delta \in \mathbb{C}, \\
&(b)\ \ \Delta = \mathbb{C},\ gz = \alpha z + \beta \quad , \\
&(c)\ \ \Delta = H,\ gz = \frac{\alpha z + \beta}{\gamma z + \delta},\ \alpha\delta - \beta\gamma = 1,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}.
\end{aligned}
\tag{2.A.1}
$$

The linear fractional transformations (2.A.1) include

(a) elliptic transformations $|\alpha + \delta| < 2$, $\alpha + \delta \in \mathbb{R}$,

(b) parabolic transformations $|\alpha + \delta| = 2$, $\alpha + \delta \in \mathbb{R}$,

(c) hyperbolic transformations $|\alpha + \delta| > 2$, $\alpha + \delta \in \mathbb{R}$,
(d) all the others are loxodromic, $\alpha + \delta \notin \mathbb{R}$.

Hyperbolic transformations are sometimes classified together with loxodromic ones. Parabolic transformations have one fixed point, the other transformations have two.

Let P_0 be a point of the Riemann surface X. We consider the set of all oriented closed curves (cycles) passing through P_0. A cycle is homotopic to zero, if it can contract to a point, and two cycles are homotopic, if their difference is homotopic to zero. The corresponding group $\pi_1(X, P_0)$ is called a homotopic group. It is easy to see that $\pi_1(X, P_0)$ is isomorphic to $\pi_1(X, P_0')$, so that we can consider $\pi_1(X)$. Just as we choose the homology group basis (Sect. 2.1), we can also choose a basis of $\pi_1(X)$ for a compact Riemann surface X of genus g, with the defining relations as follows:

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \ .$$

The universal covering of X (Sect. 2.2), which we denote by $\widehat{X}$, is itself a Riemann surface, with the complex structure defined by the condition that the projection $\pi : \widehat{X} \to X$ is holomorphic. The action of the group $\pi_1(X)$ on $\widehat{X}$ can be defined as

$$g(P, \gamma) = (P, \gamma \circ g), \ g \in \pi_1(X, P_0) \ .$$

If X is a compact Riemann surface, then it is a discontinuous group of holomorphic transformations that have no fixed points. We denote this group by G_π.

The universal covering of $\hat{X}$ is simply connected and thus conformally equivalent to one of the following domains.

Theorem A.1. (Riemann.) Any simply connected Riemann surface is holomorphically homeomorphic either to $\mathbb{CP}^1$, or to $\mathbb{C}$, or to H.

G_π induces the action on Δ of the discontinuous group of holomorphic transformations that have no fixed points $G = fG_\pi f^{-1}$

$$\begin{array}{ccc} G_\pi \circlearrowleft \hat{X} & \xrightarrow{f} & \Delta \circlearrowleft G \\ \downarrow & & \downarrow \\ X = \hat{X}/G_\pi & & \Delta/G \end{array}$$

Therefore the Riemann surface X is holomorphically equivalent to Δ/G ($X = \Delta/G$).

The groups G for $\mathbb{CP}^1$ and $\mathbb{C}$ can easily be enumerated. In the case of $\mathbb{CP}^1$, this group consists of one identical transformation, and in the case when $\Delta = \mathbb{C}$, it is determined by one or a pair of shifts

$$z \to z + 2\omega, \quad z \to z + 2\omega' \ .$$

Consequently, X in these cases is either $\mathbb{CP}^1$, or a cylinder, or the torus discussed in Sect. 2.3. There are more interesting examples of G for $\Delta = H$. Notice that every element $\alpha \in \pi_1(X)$ generates $A = f\alpha f^{-1} \in G$.

Theorem A.2. Any compact Riemann surface X of genus $g \geq 2$ can be represented as $X = H/G$, where $G: H \to H$ is a group of hyperbolic transformations with generators A_i, B_i, $i = 1, \ldots, g$ that satisfy the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = 1 \quad .$$

There is a generalization of this theorem that enables us to uniformize the Riemann surface X of genus g with n boundaries $X_1, \ldots, X_n$ and m branching points $P_1, \ldots, P_m$ with branch numbers $\nu_1, \ldots, \nu_m$ (infinite ones are included). In this case we add to a and b in $\pi_1(X)$ the generators $c_1, \ldots, c_n$ that describe the circulation along $X_1, \ldots, X_n$ and the $d_1, \ldots, d_m$ that correspond to the circulation around $P_1, \ldots, P_m$, where $d_1^{\nu_1+1} = \cdots = d_m^{\nu_m+1} = 1$. Furthermore, the generators satisfy the relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_n d_1 \cdots d_m = 1 \quad .$$

As before, the appropriate elements A_i, B_i, C_i of the group G are hyperbolic; they have no fixed points in H. Conversely, since d_i has the fixed point P_i, the transformations $D = f\alpha f^{-1}$ are elliptic ($\nu \neq \infty$) and each of them has exactly one fixed point in H. If $\nu = \infty$, the corresponding transformation D is parabolic. The defining relation for this group is as follows:

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} C_1 \cdots C_n D_1 \cdots D_m = 1 \quad ,$$
$$D_1^{\nu_1+1} = \cdots = D_m^{\nu_m+1} = 1 \quad .$$

The groups G, generated as $G: H \to H$, are called Fuchsian groups of the first kind if $n = 0$, and of the second kind if $n \neq 0$.. They have an invariant circle (real axis). The more general groups of transformations with complex coefficients α, β, γ, δ that act on $C\mathbb{P}^1$ are called Klein groups. The important characteristic is the limiting set of the group $\Lambda(G)$, which is a closure of the set of fixed points for the group transformation. Its complement is the discontinuity domain $\Omega(G)$. For Fuchsian groups, the inclusions $\Lambda(G) \subset \mathbb{R}$, $H \subset \Omega(G)$ are valid.

We fix one of the branches of the projection $\pi: \hat{X} \to X$ such that $\pi^{-1}X$ is simply-connected in $\hat{X}$; then $F = f(\pi^{-1}X)$ is a simply-connected domain in H. Moreover,

1. $F \cap gF = \emptyset \; \forall g \in G, \; g \neq I \quad ,$
2. $\bigcup_{g \in G} gF = H \quad .$

F is the fundamental domain of the group G. More generally the fundamental domain of a Klein group is defined as a domain that satisfies the above two conditions where H is to be replaced by $\Omega(G)$. The fundamental domain can be chosen in many ways. This point is treated in more detail in Chapt. 5.

There are Fuchsian groups of the first and second kind. For Fuchsian groups of the first kind $\Lambda(G) = \mathbb{R}$, and F has no segments that lie on the real axis. These groups are used to uniformize Riemann surfaces without boundaries. For a Fuchsian group of the second kind, the domain F has segments that lie on the real axis, and $\Lambda(G) \subset \mathbb{R}$ is a Cantor set. The Fuchsian groups of the second kind are used to uniformize the Riemann surfaces with boundary cycles.

The remarkable Poincaré metric $(dx^2 + dy^2)/y^2$ can be introduced on $H \ni z = x + \mathrm{i}y$. The circles which are orthogonal to the real axis are geodesics of this metric. The Poincaré metric is invariant under the transformations (2.A.1), so that it can be projected onto H/G.

The analytical functions are said to be automorphic if they are invariant under the discontinuous group G of conformal transformations

$$f(gz) = f(z)\ \forall g \in G \quad .$$

The consequence of the uniformization theorem is that the notion of an analytical function on a Riemann surface is equivalent to that of a function automorphic under the action of uniformizing group. The basic tools of constructing automorphic functions are automorphic forms. The analytic function f which obeys the transformation law

$$f(gz) = f(z)(\gamma z + \delta)^n$$

is called an automorphic form of weight $(-n)$. They can be constructed effectively. A Poincaré theta series of the dimension $(-2l)$ is an automorphic form of weight $(-2l)$ defined by

$$\theta(z) = \sum_{g \in G} H(gz)(\gamma z + \delta)^{-2l} \quad ,$$

where $H(z)$ is an analytic function. If $H(z)$ is bounded, the series is absolutely convergent for $l \geq 2$. Theta series of lower dimensions are also convergent for some of the groups G. For example, a theta series of dimension (-2) is convergent for a Fuchsian group of the second kind.

3. Finite-Gap Solutions of the Kadomtsev-Petviashvili and the Korteweg - de Vries Equations

In this chapter we show how the theta function constructions on Riemann surfaces may be applied to solve nonlinear evolution equations of physical interest. The main tool of the theory is the so-called Baker-Akhiezer function, which solves the linear partial differential equations with "finite-gap" coefficients.

3.1 Differential Equations for the Baker-Akhiezer Functions

We have seen above (Sect. 2.7) that the function ψ given by (2.7.11) is – in the presence of only one essential singularity – described up to a multiplicative constant with the following set of parameters:

1) X, a compact Riemann surface of genus g,
2) $P_\infty \in X$, a marked fixed point,
3) a fixed choice of the local parameter $p = k^{-1}$ near the point P_∞ with the property that $k \to \infty$ as $P \to P_\infty$,
4) $Q(k)$, a polynomial,
5) $\mathcal{D} = P_1 + \ldots + P_g$, a non-special divisor on X.

Let us now suppose that the polynomial $Q(k)$ linearly depends on the complementary parameters $x, y, t, \ldots$. We show that in such a case ψ satisfies some linear partial differential equations with respect to these variables.

We first consider the simplest, but simultaneously the most important, example of such a type:

$$Q(k) = kx + k^2y + k^3t \quad .$$

All considerations below will be independent of the actual structure of the polynomial $Q(k)$ and may be easily extended to include the general case.

It will be helpful to list the general analytic properties of $\psi(x, y, t, P)$ as a function defined on X which also depends on x, y, t:

(1) $\psi(x, y, t, P)$ is meromorphic on $X \backslash P_\infty$, and at the point P_∞ it has an essential singularity of the form

$$\psi(x,y,t,P) = [1 + O(k^{-1})]\exp(kx + k^2y + k^3t), \quad P \sim P_\infty . \tag{3.1.1}$$

(2) The divisor of the poles $\mathcal{D}$ is non-special $\mathcal{D} = P_1 + \ldots, + P_g$ ($\mathcal{D}$ does not depend on x, y and t).

As in Sect. 2.7 ψ is uniquely determined by the conditions (1) and (2), and may be explicitly constructed by means of

$$\begin{aligned}\psi(x,y,t,P) =& \frac{\theta\left(\int_{P_\infty}^{P}\omega + \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}\right)\theta(\boldsymbol{D})}{\theta\left(\int_{P_\infty}^{P}\omega + \boldsymbol{D}\right)\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D})} \\ &\times \exp(\Omega_1(P)x + \Omega_2(P)y + \Omega_3(P)t) \quad ,\end{aligned} \tag{3.1.2}$$

where $\Omega_i(P)$ are normalized Abelian integrals of the second kind with singularities at the point P_∞, fixed by the conditions

$$\Omega_i(P) \to k^i + o(1), \quad P \to P_\infty, \quad i = 1,2,3 ,$$

and

$$D_n = -\sum_{k=1}^{g}\int_{P_0}^{P_k}\omega_n - K_n .$$

The b-periods of these integrals are denoted by

$$U_n = \int_{b_n} d\Omega_1, \quad V_n = \int_{b_n} d\Omega_2, \quad W_n \int_{b_n} d\Omega_3 \quad .$$

Formula (3.1.2) possesses the same structure as (2.7.11). To see that (3.1.2) actually coincides with (2.7.11) it is sufficient to remark that the b-periods of the normalized Abelian integral $\Omega(P)$ with the unique singularity at P_∞ given by $\Omega \to Q(k)$, $P \to P_\infty$, are equal to

$$\int_{b_n} d\Omega = U_n x + V_n y + W_n t \quad ,$$

because $\Omega = \Omega_1 x + \Omega_2 y + \Omega_3 t$. A complementary theta functional factor with respect to (2.7.11) provides the normalization (3.1.1).

Theorem 3.1. The function $\psi(x,y,t,P)$ satisfies the following system of PDEs:

$$\partial_y\psi = L_2\psi, \quad \partial_t\psi = L_3\psi \quad , \tag{3.1.3}$$

and the operators L_2 and L_3 are given by the formulas

$$L_2 = \partial_x^2 + u, \quad L_3 = \partial_x^3 + v_1\partial_x + v_2 \quad .$$

Here u, v_1, v_2 are coefficients independent of P which are determined by the following conditions:

$$
\begin{aligned}
(\partial_y - L_2)\psi &= O(k^{-1})\exp(kx + k^2 y + k^3 t) \quad , \\
(\partial_t - L_3)\psi &= O(k^{-1})\exp(kx + k^2 y + k^3 t), \quad P \to P_\infty \quad .
\end{aligned}
\tag{3.1.4}
$$

Proof. The function $(\partial_y - L_2)\psi$ possesses the same analytic properties (1) and (2) as the function ψ, except for different asymptotic behaviour at the marked point. But the function ψ is uniquely determined by its analytic properties and consequently we have

$$
(\partial_y - L_2)\psi = 0
$$

by the uniqueness of the Baker-Akhiezer function (see Theorem 2.24 and Corollary 2.26). The second of the equations in (3.1.3) may be proven in the same way.

Substituting the asymptotic series

$$
\psi = \left[1 + \sum_{n=1}^{\infty} \xi_n(x,y,t) k^{-n}\right] \exp(kx + k^2 y + k^3 t) \tag{3.1.5}
$$

into the system (3.1.3) and equating the coefficients of the leading powers of k gives the following expressions for u, v_1, v_2:

$$
u = -2\xi_{1x}, \quad v_1 = -3\xi_{1x}, \quad v_2 = 3(\xi_1\xi_{1x} - \xi_{1xx} - \xi_{2x}) \quad . \tag{3.1.6}
$$

We now turn to the derivation of the expression for $u(x,y,t)$. For this calculation we need the following asymptotic formulas for the Abelian integrals Ω_i:

$$
\begin{aligned}
&\Omega_1 \to k - \frac{c}{k} + O(k^{-2}), \quad \Omega_2 \to k^2 - \frac{c_2}{k} + O(k^{-2}) \quad , \\
&\Omega_3 \to k^3 - \frac{c_3}{k} + O(k^{-2}) \quad .
\end{aligned}
$$

The constant coefficients c, c_2, c_3 are uniquely determined by the choice of the surface X, the point P_∞, and the local parameter k at this point. However, one needs to know only the value of the constant c, since c_2 and c_3 do not contribute anything to the explicit formula for ξ_{1x}.

Comparing (3.1.5) with the explicit representation (3.1.2) for ψ we get

$$
\begin{aligned}
&\xi_1 = \frac{\partial}{\partial p}\left[\log\frac{\theta\left(\int_{P_\infty}^{P}\omega + \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}\right)\theta(\boldsymbol{D})}{\theta\left(\int_{P_\infty}^{P}\omega + \boldsymbol{D}\right)\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D})} - (cx + c_2 y + c_3 t)p\right] \quad , \\
&p = k^{-1} \quad ,
\end{aligned}
$$

where the partial derivative of the log is to be taken at $p = 0$. Hence

$$
\xi_{1x} = \frac{\partial}{\partial x}\frac{\partial}{\partial p}\left[\log\theta\left(\int_{P_\infty}^{P}\omega + \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}\right)\right] - c \quad . \tag{3.1.7}
$$

It follows from (2.4.13) that in a neighborhood of P_∞ Abel's mapping is of the following form:

$$\int_{P_\infty}^{P} \omega_n = -U_n p + O(p^2) \quad .$$

Consequently in (3.1.7) one can replace differentiation with respect to p by differentiation with respect to x. Finally we arrive at the following exact representation for u:

$$u(x,y,t) = 2\partial_x^2 \log\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}) + 2c \quad . \tag{3.1.8}$$

The coefficient v_2 can also be expressed in terms of theta functions, and we omit the related expression.

We remark that (3.1.8) may be rewritten as

$$u = 2\partial_{z_i}\partial_{z_j} U_i U_j \log\theta(\boldsymbol{z}) + 2c, \qquad \boldsymbol{z} = \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D} \quad .$$

Now it follows from the last formula that u, as a function of $\boldsymbol{z} \in \mathbb{C}^g$, is the Abelian function defined on the torus with a periodic lattice generated by the columns of the matrix $\Pi = (I, B)$, $B_{nm} = \int_{b_n} \omega_m$. Therefore, $u(x,y,t)$ represents a restriction of some Abelian function on the direction $\boldsymbol{z} = \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}$. Such functions are often called quasiperiodic. Already at this step we arrive at a valuable result: for some linear equations with coefficients explicitly expressed in terms of multi-dimensional theta functions we have constructed a one-parameter family of solutions also expressed by means of Riemann theta functions. It now becomes clear that the constructions of the previous chapter lead quite naturally to the explicit integration of the linear partial differential equations with their quasiperiodic coefficients.

3.2 Solution of the Kadomtsev-Petviashvili Equation

Our next task is to show that the function constructed above satisfies the Kadomtsev-Petviashvili equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right] \tag{3.2.1}$$

usually referred to in the literature as the KP equation. The KP equation is a natural two-dimensional integrable generalization of the KdV equation. It describes various physical phenomena, in particular, two-dimensional shallow water-wave propagation.

The KP equation may be considered as the compatibility condition for the system (3.1.3). Indeed, imposing on ψ the requirement $\psi_{ty} = \psi_{yt}$ we get

$$(L_{2t} - L_{3y} + L_2 L_3 - L_3 L_2)\psi = 0 \quad .$$

The operator in parentheses is an ordinary differential operator of finite order in the variable x and $\psi(P)$ represents a one-parameter family of functions belonging to its kernel. Since the kernel is always finite-dimensional, the operator in parentheses is identically equal to zero, as are the coefficients of the derivatives with respect to x of the same orders. More explicitly, we have

$$\begin{aligned} 2v_{1x} - 3u_x &= 0 \quad , \\ -v_{1y} + v_{1xx} + 2v_{2x} - 3u_{xx} &= 0 \quad , \\ u_t - v_{2y} + v_{2xx} - u_{xxx} - v_1 u_x &= 0 \quad . \end{aligned} \tag{3.2.2}$$

Let us remark that the first of these identities also follows from (3.1.6), which implies $2v_1 = 3u$. Differentiating the third of these equalities by x, and substituting into it v_{2x} given by the second equality of (3.2.2), we get (3.2.1).

Thus we have the following proposition:

Theorem 3.2. An explicit formula for the solutions of the KP equation is given by

$$u(x, y, t) = 2\partial_x^2 \log\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}) + 2c \quad , \tag{3.2.3}$$

where $\boldsymbol{D} \in \mathbb{C}^g$ is an arbitrary vector. These solutions are quasiperiodic. They are the so-called finite-gap solutions.[1]

The solution (3.2.3) depends on the Riemann surface X, the point $P_\infty \in X$, the vector $\boldsymbol{D} \in \mathbb{C}^g$, and the local parameter k at the point P_∞. The dependence on the local parameter is the most evident. The transformation of the local parameter given by

$$k \to \alpha k + \beta + \gamma k^{-1} + O(k^{-2}) \quad , \tag{3.2.4}$$

where α, β, γ are arbitrary complex numbers ($\alpha \neq 0$), leads to a different family of solutions of the same KP equation. These new solutions are obtained by the following transformations of x, y, t, u which leave the KP equation (3.2.1) invariant:

$$\begin{aligned} &x \to \alpha x + 2\alpha\beta y + (3\alpha\beta^2 + 3\alpha^2\gamma)t \quad , \\ &y \to \alpha^2 y + 3\alpha^2\beta t \quad , \\ &t \to \alpha^3 t \quad , \\ &u \to \alpha^{-2}u - 2\alpha^{-1}\gamma \quad . \end{aligned} \tag{3.2.5}$$

[1] This name has been explained in the Introduction. More details are given in Sect. 3.5.

3.3 Real Non-Singular Solutions of the KP1 and KP2 Equations

For applications in physics it is important to single out real and smooth solutions from the whole variety of exact solutions obtained. There exist two different variants of the Kadomtsev-Petviashvili equation which cannot be reduced to each other by real-valued change of variables. Equation (3.2.1) is usually referred to in the literature as KP2. It is also often called the stable variant of the KP equation. The change of variables

$$x \to \mathrm{i}x, \quad y \to \mathrm{i}y, \quad t \to \mathrm{i}t \tag{3.3.1}$$

transforms the KP2 into the (unstable) KP1 equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{1}{4}(6uu_x - u_{xxx})\right) \quad . \tag{3.3.2}$$

Both KP1 and KP2 equations may be applied to the description of various interesting phenomena in plasma physics and hydrodynamics. In these applications the most interesting solutions from the view point of physics are usually the real non-singular ones.

To describe real and smooth solutions we start with some preliminary results on real Riemann surfaces (only these Riemann surfaces give rise to the real solutions of KP1 and KP2).

A Riemann surface X is called a real Riemann surface if it admits an anti-holomorphic involution (anti-involution for short): $\tau : X \to X$, $\tau^2 = 1$. The fixed points of the anti-involution τ form connected components which are called the real ovals of the involution τ. Let n be the number of ovals $0 \le n \le g+1$. These ovals will be denoted by $X_0, \dots, X_{n-1}$. Two distinct situations are possible

(1) the surface X is of decomposing type, i.e., the real ovals decompose X into two components X_+ and X_-, $X_- = \tau X_+$, $X_\pm = X/\tau$ are the surfaces of the genus $h = (g+1-n)/2$ with n boundary curves, $(g+1-n) \equiv 0 \pmod 2$;
(2) the surface X is of nondecomposing type, i.e., the ovals do not decompose X, and X/τ is a non-orientable surface.

An M-curve is a Riemann surface with the maximal possible number of ovals $n = g+1$.

Now let X be an M-curve. Take the basis of cycles such that $P_\infty \in X_0$; b-cycles are taken to coincide with the ovals X_j: $b_j = X_j$,

$$\tau b_j = b_j, \quad \tau a_j = -a_j, \quad j = 1, \dots, g \quad . \tag{3.3.3}$$

Taking the local parameter k to satisfy the condition $\tau^* k = \overline{k}$ we have

$$2\pi\mathrm{i}\delta_{kl} = \int_{a_k} \omega_l = \int_{\tau a_k} \tau^*\omega_l = -\int_{a_l} \tau^*\omega_l \Rightarrow \omega_l = \overline{\tau^*\omega_l} \quad . \tag{3.3.4}$$

Quite similarly one could check the formulas

$$\overline{\tau^* d\Omega_i} = d\Omega_i, \quad i = 1,2,3 \quad ,$$

taking into account that the RHS and LHS represent differentials of the second kind with the same singularities. The equalities above prove the reality of the matrix B and of the vectors $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$,

$$\begin{aligned} \overline{B}_{kl} &= \int_{b_k} \overline{\omega}_l = \int_{\tau b_k} \tau^* \omega_l = \int_{b_k} \omega_l = B_{kl} \quad , \\ \overline{U}_k &= \int_{b_k} \overline{d\Omega_1} = \int_{\tau b_k} \overline{\tau^* d\Omega_1} = \int_{b_k} d\Omega_1 = U_k \quad . \end{aligned} \tag{3.3.5}$$

Suppose the vector $\boldsymbol{D}$ is also real. We show that the solution of the KP2 constructed above is smooth and real-valued. The reality follows directly from the reality of the B-matrix and the reality of the argument of the theta function in (3.2.3). The singularities of $u(x,y,t)$ are situated at the zeros of $\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D})$, but there are no zeros for $x,y,t \in \mathbb{R}$. Indeed, we have

$$\begin{aligned} &\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}) \\ &= \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left(\frac{1}{2}\langle B\boldsymbol{m},\boldsymbol{m}\rangle + \langle \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D},\boldsymbol{m}\rangle\right) \\ &=1 + \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^g \\ m_1\geq 0}}{}' \exp\left(\frac{1}{2}\langle B\boldsymbol{m},\boldsymbol{m}\rangle\right) \{\exp\langle \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D},\boldsymbol{m}\rangle \\ &\quad + \exp(-\langle \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D},\boldsymbol{m}\rangle)\} \\ &=1 + 2\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^g \\ m_1\geq 0}}{}' \exp\left(\frac{1}{2}\langle B\boldsymbol{m},\boldsymbol{m}\rangle\right) \mathrm{ch}\langle \boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D},\boldsymbol{m}\rangle > 0 \quad . \end{aligned}$$

In the last transformation we have grouped together the terms corresponding to the vectors $\boldsymbol{m}$ and $-\boldsymbol{m}$. The prime means that the summation is taken over all $\mathbb{Z}^g$ except $\boldsymbol{m} = (0,\dots,0)$. It can also be proven that the conditions for the reality of the solutions of the KP2 equation obtained this way are not only sufficient but also necessary.

On the compact Riemann surface of decomposing type we can always fix a basis of cycles [3.1] $a_1, b_1, \dots, a_h, b_h, a_{h+1}, b_{h+1}, \dots, a_{2h}, b_{2h}, a_{2h+1}, b_{2h+1}, \dots, a_{2h+n-1}, b_{2h+n-1}$ in such a way that $a_{2h+k} = X_k$, $k = 1,\dots,n-1$ – are real ovals of X,

$$\begin{aligned} &a_i, b_i \in X_+, \quad \tau a_i = a_{i+h} \quad \tau b_i = -b_{i+h}, \quad i = 1,\dots,h \quad , \\ &\tau a_{2h+j} = a_{2h+j}, \quad \tau b_{2h+j} = -b_{2h+j}, \quad j = 1,\dots,n-1 \quad . \end{aligned} \tag{3.3.6}$$

Theorem 3.3. [3.2]. For smoothness and reality of the solutions (3.2.3) of the KP1 and KP2 equations [for KP1 it is necessary to make the change (3.3.1) in

(3.2.3)] it is necessary and sufficient that for the triple (X, P_∞, k) and the vector $\boldsymbol{D}$ the following conditions are satisfied:

(1) The Riemann surface X admits an antiholomorphic involution $\tau : X \to X$, $\tau^2 = 1$, where $\tau P_\infty = P_\infty$ and $\tau^* k = \overline{k}$.
(2) The set of all fixed ovals of the involution τ decomposes the surface X into two pieces X^+ and X^-.
(3) If $P_\infty \in X_0$ and the basis of cycles is of the form (3.3.6), then the vector $\boldsymbol{D}$ for the KP1 equation is an arbitrary vector of the form

$$\boldsymbol{D} = (\boldsymbol{\xi}, \overline{\boldsymbol{\xi}}, \boldsymbol{\eta})^T \quad \boldsymbol{\xi} \in \mathbb{C}^h, \quad \boldsymbol{\eta} \in \mathbb{R}^{n-1} \quad . \tag{3.3.7}$$

(4) For the KP2 equation there is an additional topological restriction for the surface X: it must be an M-curve. If a basis of cycles of the form (3.3.6) is chosen ($h = 0$, $n = g+1$), then $\boldsymbol{D}$ is an arbitrary vector with purely imaginary coordinates. If the basis of cycles is such that $\tau a_i = -a_i$, $\tau b_i = b_i$, then $\boldsymbol{D}$ is an arbitrary real vector $\boldsymbol{D} \in \mathbb{R}^g$.

Remark 3.4. The bases of cycles (3.3.3) and (3.3.6) are related by the modular transformation (2.4.22,5.12). With the help of this transformation, we find a reformulation of the conditions imposed on the vector $\boldsymbol{D}$ in the new basis (see also Sects. 5.5, 6).

It is necessary to remark that the solutions constructed above make it possible to build the solution of a general periodic problem. More exactly the following proposition as proved in [3.3] holds:

Theorem 3.5. For any $|t| < T_0$, and an arbitrary smooth periodic solution $u(x, y, t)$ of the KP2 equation there exists a sequence $u_N(x, y, t)$ of finite-gap solutions, converging uniformly along with all derivatives to $u(x, y, t)$ on the whole (x, y) plane.

3.4 Reduction to Korteweg - de Vries and Boussinesq Equations

It may happen that one of the Abelian integrals Ω_2, Ω_3 is single-valued, i.e., a meromorphic function on X. In such a case both the a- and b-periods of these integrals are equal to zero.

Let, for example, Ω_2 be a meromorphic function with a pole of second order at the point P_∞ but without other singularities. In such a case (Corollary 2.6) the Riemann surface X turns out to be hyperelliptic, and P_∞ must be one of the branch points of X. In this case $\boldsymbol{V} = \boldsymbol{0}$ and the solution (3.2.3) of the KP equation depends only on the x and t variables. In the case under consideration the dependence of $\psi(x,y,t,P)$ on y becomes purely exponential, i.e.,

$$\psi(x,y,t,P) = e^{\Omega_2(P)y}\varphi(x,t,P) \quad ,$$

and the function ψ satisfies the Schrödinger equation

$$\partial_x^2\varphi(x,t,P) + u(x,t)\varphi(x,t,P) = \Omega_2(P)\varphi(x,t,P) \quad . \tag{3.4.1}$$

It is also evident that in this case $u(x,t)$ is the solution of the KdV equation:

$$4u_t = 6uu_x + u_{xxx} \quad , \tag{3.4.2}$$

$$u(x,t) = 2\partial_x^2 \log\theta(\boldsymbol{U}x + \boldsymbol{W}t + \boldsymbol{D}) + 2c \quad . \tag{3.4.3}$$

Now we discuss these solutions in some detail. Let X be the Riemann surface of the M-curve, realized as a two-sheeted covering of the complex plane:

$$\mu^2 = \prod_{i=1}^{2g+1}(\lambda - E_i), \quad E_i \in \mathbb{R}, \quad E_1 < E_2 < \ldots < E_{2g+1} \quad . \tag{3.4.4}$$

Fix P_∞ to be a point over infinity on X which is a branch point of the projection $X \to X/\pi$, $\pi : (\mu, \lambda) = (-\mu, \lambda)$. Take a local parameter k at infinity to be of the form $k = \mathrm{i}\sqrt{\lambda}$, then $\Omega_2 = -\lambda$. In this case also

$$\Omega_1(P) = \int_{E_1}^{P} d\Omega_1, \quad d\Omega_1 = \mathrm{i}\frac{\lambda^g + c_1\lambda^{g-1} + \cdots + c_g}{2\mu}d\lambda \quad ,$$

and the constants c_j are defined by the normalizations

$$\int_{a_j} d\Omega_1 = 0 \quad .$$

The curve (3.4.4) admits two involutions: π, defined above and also $\tau(\mu,\lambda) = (-\overline{\mu}, \overline{\lambda})$. The latter is antiholomorphic and has $g+1$ fixed ovals over the intervals $[-\infty, E_1]$, $[E_2, E_3]$, …, $[E_{2g}, E_{2g+1}]$. Under the action of τ, the local parameter k and the basis of cycles shown on Fig. 3.1 transform as follows:

$$\begin{aligned} &\tau^* k = \overline{k}, \quad \tau a_n = a_n, \quad \tau b_n = -b_n \quad , \\ &\overline{\tau^*\omega_n} = -\omega_n, \quad \overline{\tau^* d\Omega_i} = d\Omega_i \quad , \\ &\overline{U_n} = -U_n, \quad \overline{W_n} = -W_n \quad . \end{aligned} \tag{3.4.5}$$

As in the KP2 case and the basis (3.3.6), the vector $\boldsymbol{D}$ is purely imaginary with respect to the basis (3.4.5).

We have constructed all real and smooth finite-gap solutions of the KdV equation.

We note in addition that from the exact representation (2.4.18)

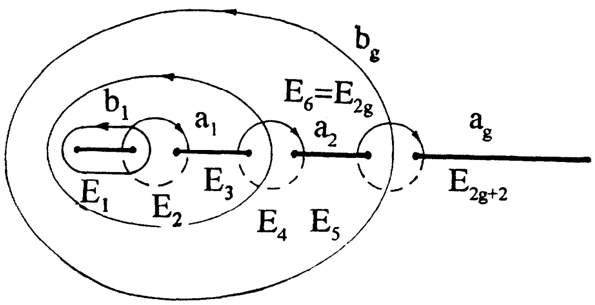

Fig. 3.1. Homology basis for the curve (3.4.4)

$$\omega_n = \sum_{k=1}^{g} \frac{c_{nk}\lambda^{g-k}}{\mu} d\lambda$$

of the normalized holomorphic differentials of the curve (3.4.4), with the help of (2.4.13) we get the following expressions for the coordinates U_n, W_n of the vectors $\boldsymbol{U}$ and $\boldsymbol{W}$:

$$U_n = 2\mathrm{i}c_{n1}, \quad W_n = -2\mathrm{i}\left(c_{n2} + c_{n1}\sum_{k=1}^{2g+1} E_k/2\right) \quad .$$

Quite similarly, if there exists a meromorphic function on X with the same singularity at P_∞ as Ω_3 (i.e., with a third order pole and without other singularities), then W in (3.2.3) vanishes and the dependence of ψ on the variable t becomes purely exponential:

$$\psi(x, y, t, P) = e^{\Omega_3(P)t}\varphi(x, y, P) \; .$$

In this case $u(x, y, t)$ is independent of t and satisfies the nonlinear Boussinesq equation

$$3u_{yy} + \frac{\partial}{\partial x}(6uu_x + u_{xxx}) = 0 \quad .$$

Let us consider a curve X_3,

$$\mu^4 = \prod_{i=1}^{4}(\lambda - E_i) \quad ,$$

representating the simplest example of the so-called trigonal curves. The curve X is called trigonal if there exists a meromorphic function with a third order pole at some point P_0 of X and without other singularities. Such a point must be a Weierstrass point. We recapitulate the general definition of Weierstrass points. If some meromorphic function on X admits a pole at P_0 of order less than or equal to the genus of the surface, and there are no other singularities, the point is called a Weierstrass point. The number N of the Weierstrass points satisfies the Hurwitz inequality

$$2g+2 \le N \le g(g^2-1) \quad ,$$

and $N = 2g+2$ if and only if X is a hyperelliptic curve. In the case of a curve X_3 of genus 3 the number of Weierstrass points may be equal to 16 or 20 and is defined by the choice of the moduli E_i. In particular the branch points $P_i = (0, E_i)$ are Weierstrass points. The local parameters τ_i at the points P_i are defined as follows:

$$\tau_i = \sqrt[4]{\lambda - E_i} \quad .$$

It is evident that the functions

$$f_i = \frac{\prod\limits_{k \neq i} (\lambda - E_k)}{\mu^3}$$

have exactly one pole of third order at the point P_i and no other singularities. All Weierstrass points of X_3 were found in [3.4, 5]. Another remarkable property of the curve X_3, proved in [3.4], is that the associated Riemann theta function splits into a sum of two terms, each term being a product of three one-dimensional theta functions. The B-matrix of the curve X_3 may be explicitly computed with the matrix elements expressed in terms of complete elliptic integrals. In Chap. 7 the general nature of such reductions of multi-dimensional theta functions will be discussed in detail.

3.5 Spectral Properties of the Finite-Gap Solutions

Taking $t = 0$ in (3.4.1) we get the integrable linear Schrödinger equation

$$\left(-\frac{d^2}{dx^2} - u(x)\right) \varphi(x, P) = \lambda \varphi(x, P) \tag{3.5.1}$$

with the potential

$$u(x) = 2\frac{d^2}{dx^2}\log\theta(\boldsymbol{U}x + \boldsymbol{D}) + 2c \quad . \tag{3.5.2}$$

As a function of $z = \mathrm{i}\boldsymbol{U}x \in \mathbb{R}^g$ the potential is a periodic function with a period lattice $2\pi\mathbb{Z}^g$, i.e., a function on the real torus $\mathbb{R}^g/2\pi\mathbb{Z}^g$. Since $\mathrm{i}\boldsymbol{U} \in \mathbb{R}^g$ the function $u(x)$ is obtained by the restriction of this function to a straight line on the torus. Therefore $u(x)$ is a quasiperiodic function.

The fundamental system of solutions of the Schrödinger equation is obtained by taking the $\varphi(x, P)$ on the upper and lower sheets of the curve X:

$$\varphi_\pm(x, \lambda) = \frac{\theta\left(\int_\infty^{P_\pm} \omega + \boldsymbol{U}x + \boldsymbol{D}\right)\theta(\boldsymbol{D})}{\theta\left(\int_\infty^{P_\pm} \omega + \boldsymbol{D}\right)\theta(\boldsymbol{U}x + \boldsymbol{D})} e^{\Omega_1(P_\pm)x} \quad . \tag{3.5.3}$$

$$P_\pm = (\pm\mu, \lambda), \quad P_+ = \pi P_- \quad .$$

Theorem 3.6. The function $\varphi(x, P)$ has exactly one pole and one zero on each real oval Ov_i of the surface X, lying over the gaps ($Ov_i \to [E_{2i}, E_{2i+1}]$ under the projection $(\mu, \lambda) \to \lambda$).

Proof. The proof will be divided into a number of steps. We start with the following:

Lemma 3.7. For an arbitrary purely imaginary vector $\boldsymbol{D}$ the function $\theta(\int_\infty^P \omega + \boldsymbol{D})$ has exactly g zeros.

The result of the Lemma is a simple consequence of the fact that because $\theta(\boldsymbol{D}) \neq 0$, $\theta(\int_\infty^P \omega + \boldsymbol{D})$ can not be identically equal to zero. Hence it has exactly g-zeros on X. Now to verify that $\theta(\boldsymbol{D}; B) \neq 0$ it will be convenient to apply the modular transformation of theta functions described in Sect. 3.3. We therefore obtain $\theta(\boldsymbol{D}; B) = c\theta(\boldsymbol{D}'; B')\boldsymbol{D} \in \mathbb{R}^g$, $c \neq 0$. The fact $\theta(\boldsymbol{D}'; B') > 0$ was proved in Sect. 3.3. The next step is to prove the following:

Lemma 3.8. The divisor $\mathcal{D} = \sum P_i$ of zeros of $\theta(\int_\infty^P \omega + \boldsymbol{D})$ is invariant with respect to the action of antiholomorphic involution $\tau : \tau\mathcal{D} = \mathcal{D}$.

To prove this result it is sufficient to apply the identity

$$\overline{\theta\left(\int_\infty^P \omega + \boldsymbol{D}\right)} = \theta\left(\int_\infty^P \overline{\omega} + \overline{\boldsymbol{D}}\right) = \theta\left(\int_\infty^{\tau P} \omega + \boldsymbol{D}\right) .$$

It follows that all the points P_i of $\mathcal{D}$ may be divided into pairs P, τP or, alternatively, are situated on the real ovals Ov_i.

The points P_i may be determined from the equality

$$\sum_{i=1}^{g} \int_\infty^{P_i} \omega_n = -D_n - K_n, \quad n = 1, \dots, g \quad , \tag{3.5.4}$$

were $\boldsymbol{K}$ is a vector of Riemann constants which may be constructed in the hyperelliptic case by (see Chap. 2 and also p. 14 of [3.1]):

$$K_n = \frac{1}{2} \sum_{i=1}^{g} B_{ni} + \pi \mathrm{i}(n-2) \quad . \tag{3.5.5}$$

By virtue of $\tau\mathcal{D} = \mathcal{D}$, the unique possibility of obtaining a purely imaginary D_n in (3.5.4) is to choose each of the points P_j to lie on the real ovals Ov_j. This completes the proof of the Theorem.

The fraction in (3.5.3), containing theta functions in its numerator and denominator, is obviously quasiperiodic in the x-variable but the behaviour of the exponential in (3.5.3) depends essentially on the choice of λ.

Lemma 3.9. Choose $P_+(P_-)$ in (3.5.3) to lie on the upper (lower) sheet of the surface X, i.e., on the sheet fixed by the condition $\mathrm{Re}\, k < 0\,(\mathrm{Re}\, k > 0)$, $k = \mathrm{i}\sqrt{\lambda}$ in a neighborhood of infinity. Then

$$1)\quad \lambda \in \mathcal{E}_1 = [E_1, E_2] \cup [E_3, E_4] \cup \ldots \cup [E_{2g+1}, \infty]$$
$$\Rightarrow \mathrm{i}\Omega_1(P) \in \mathbb{R}, \quad P = (\mu, \lambda) \quad .$$
$$2)\quad \lambda \in P\mathbb{C}\backslash\mathcal{E}_1 \Rightarrow \mathrm{Re}\Omega_1(P_+) < 0 \quad ,$$
$$\mathrm{Re}\Omega_1(P_-) > 0 \quad .$$

Proof. The first statement of the Lemma follows from the fact that if $\lambda \in \mathcal{E}_1$, then $\tau P = \pi P$. Indeed, by virtue of the last equality, we have

$$\overline{\Omega_1(P)} = \int_{E_1}^{P} \overline{d\Omega_1} = \int_{E_1}^{P} \tau^* d\Omega_1 =$$
$$\int_{E_1}^{\tau P} d\Omega_1 = \int_{E_1}^{\pi P} d\Omega_1 = -\int_{E_1}^{P} d\Omega_1 = -\Omega_1(P) \quad .$$

This completes the proof of the first statement of the Lemma. For the proof of the second statement of the Lemma it is sufficient to take into account that all b-periods of the integral $\Omega_1(P)$ are purely imaginary. Hence $\mathrm{Re}\,\Omega_1(P_+)$ and $\mathrm{Re}\,\Omega_1(P_-)$ are well-defined harmonic functions on $\mathcal{E}$, $\mathcal{E} = \mathrm{P}\mathbb{C}\backslash\mathcal{E}_1$. The asymptotics $\Omega_1(P) \sim \mathrm{i}\sqrt{\lambda}$, $P \to \infty$ and the choice of the sheets of X imply that for sufficiently large absolute values of λ, $P_\pm = (\pm\mu, \lambda)$, $\mathrm{Re}\Omega_1(P_+) < 0$, $\mathrm{Re}\Omega_1(P_-) > 0$.

Combining the first statement of the Lemma, $\mathrm{Re}\Omega_1(P_\pm) = 0$ for $P_\pm \in \mathcal{E}_1 = \partial\mathcal{E}$, and the maximum principle, the second statement is proven.

From the last Lemma and quasiperiodicity of the fraction involving theta functions in (3.5.3), we conclude that $\mathcal{E}_1$ coincides with the continuous spectrum of the Schrödinger operator (3.5.1), in $L_2(\mathbb{R})$

$$\mathcal{L} = -\frac{d^2}{dx^2} - u(x)$$

with the quasiperiodic potential defined by (3.5.2). Thus the meaning of the name "finite-gap" potential becomes clear.

A knowledge of the explicit solutions of the Schrödinger operator enables one to construct explicitly the spectral matrix of $\mathcal{L}$. Recall the corresponding definitions [3.6]. Let $\theta(x, \lambda), \varphi(x, \lambda)$ be solutions of (3.5.1) fixed by the conditions

$$\varphi'(0, \lambda) = \theta(0, \lambda) = 1, \quad \varphi(0, \lambda) = \theta'(0, \lambda) = 0 \quad .$$

Denote the vector composed from θ and φ by $\boldsymbol{\varphi}$:

$$\boldsymbol{\varphi}(x, \lambda) = \begin{pmatrix} \theta(x, \lambda) \\ \varphi(x, \lambda) \end{pmatrix} \quad .$$

Put the completeness relation in the form

$$\delta(x-y)=\int_{-\infty}^{\infty}\varphi^T(y,\lambda)d\hat{\varrho}(\lambda)\varphi(x,\lambda) \quad ,$$

where the matrix $\hat{\varrho}(\lambda)$ is called a spectral matrix [3.7]:

$$\hat{\varrho}(\lambda)=\begin{pmatrix}\xi(\lambda) & \eta(\lambda)\\ \eta(\lambda) & \zeta(\lambda)\end{pmatrix} \quad .$$

The coefficients are given by

$$\begin{aligned}\xi(\lambda)&=-\frac{1}{\pi}\int_0^\lambda \mathrm{Im}\left[\frac{1}{m_1(\mu+\mathrm{i}o)-m_2(\mu+\mathrm{i}o)}\right]d\mu \quad ,\\ \eta(\lambda)&=\frac{1}{2\pi}\int_0^\lambda \mathrm{Im}\left[\frac{m_1(\mu+\mathrm{i}o)+m_2(\mu+\mathrm{i}o)}{m_1(\mu+\mathrm{i}o)-m_2(\mu+\mathrm{i}o)}\right]d\mu \quad ,\\ \zeta(\lambda)&=-\frac{1}{\pi}\int_0^\lambda \mathrm{Im}\left[\frac{m_2(\mu+\mathrm{i}o)m_1(\mu+\mathrm{i}o)}{m_1(\mu+\mathrm{i}o)-m_2(\mu+\mathrm{i}o)}\right]d\mu \quad ,\end{aligned} \tag{3.5.6}$$

where $m_{1,2}(\lambda)$ are the so-called Weyl-Titchmarsh functions defined by

$$m_{1,2}(\lambda)=\frac{d}{dx}\psi_{1,2}(x,\lambda)_{|_{x=0}} \quad .$$

Here $\psi_{1,2}$ are Weyl solutions of the Schrödinger equation fixed by the conditions

$$\begin{aligned}&\psi_1(0,\lambda)=1, \quad \psi_1(x,\lambda)\in L_2(0,\infty), \quad \mathrm{Im}\,\lambda\neq 0 \quad ,\\ &\psi_2(0,\lambda)=1, \quad \psi_2(x,\lambda)\in L_2(-\infty,0), \quad \mathrm{Im}\,\lambda\neq 0 \quad .\end{aligned}$$

In terms of the spectral matrix, the spectrum of the Schrödinger operator may be described as the support of the measure $d\hat{\varrho}(\lambda)$. Recall also that the spectrum is called absolutely continuous if for all values of λ there exists the locally summable derivative $d\hat{\varrho}/d\lambda$, i.e.,

$$d\hat{\varrho}(\lambda)=\frac{d\hat{\varrho}(\lambda)}{d\lambda}d\lambda \quad .$$

The second statement of Lemma 3.9. proves that Weyl solutions $\psi_{1,2}$ coincide with $\varphi_+(x,\lambda)$ and $\varphi_-(x,\lambda)$ respectively. Hence, for Weyl functions $m_{1,2}(\lambda)$, corresponding to the potential (3.5.2), the following representation holds:

$$m_1(\lambda)=\left.\frac{d\varphi_+(x,\lambda)}{dx}\right|_{x=0}, \quad m_2(\lambda)=\left.\frac{d\varphi_-(x,\lambda)}{dx}\right|_{x=0} \quad . \tag{3.5.7}$$

The formulas (3.5.7) together with (3.5.3) give the exact expressions for the Weyl functions of the Schrödinger equation in terms of theta functions. But it is possible to derive alternative representations for $m_{1,2}$ involving the branch points. Such representations are more natural from the spectral analysis point of view. Recall that, as shown above, the E_j are the boundaries of the continuous spectrum of $\mathcal{L}$. Let the λ_j be the projections of the poles of $\varphi(x, P)$, $P \in X$, on the λ-plane. By virtue of Theorem 3.6, we have exactly g λ's, each of which is situated in the corresponding gap of the continuous spectrum.

$$E_{2j} \leq \lambda_j \leq E_{2j+1}, \quad j = 1, \ldots, g \quad . \tag{3.5.8}$$

The desired expressions for $m_{1,2}$ may be obtained by considering the following combinations of $\varphi_{\pm}(x, \lambda)$ and their derivatives with respect to x at the point $x = 0$:

$$W_{1,2} = \varphi_+(x, \lambda)\varphi'_-(x, \lambda) \mp \varphi'_+(x, \lambda)\varphi_-(x, \lambda)\Big|_{x=0} \quad .$$

W_1 as a function of $P \in X$ has a divisor of poles $\boldsymbol{D} + \pi\boldsymbol{D}$. The zeros of W_1 are placed at the points E_i. These properties, combined with the asymptotics

$$W_1 = -2i\sqrt{\lambda} + \ldots, \quad P \to \infty \quad ,$$

allow one to construct W_1 in a more explicit form

$$W_1(P) = -2\mathrm{i}\frac{\mu}{\prod_{j=1}^{g}(\lambda - \lambda_j)}, \quad P = (\mu, \lambda) \quad . \tag{3.5.9}$$

In contrast with W_1, W_2 is correctly defined as a function of λ and has exactly g simple poles at the points λ_j and a zero at $\lambda = \infty$. The last statement may be checked by substituting the expansions

$$\varphi_{\pm}(x, \lambda) \underset{\lambda \to \infty}{=} \left(1 \pm \frac{\xi_1}{\mathrm{i}\sqrt{\lambda}} + \ldots\right) e^{\pm \mathrm{i}\sqrt{\lambda}x} \tag{3.5.10}$$

into W_2. Hence for W_2 the following formula holds:

$$W_2(\lambda) = \frac{2f(\lambda)}{\prod_{j=1}^{g}(\lambda - \lambda_j)} \quad , \tag{3.5.11}$$

$$f(\lambda) = f_1\lambda^{g-1} + f_2\lambda^{g-2} + \cdots + f_g \quad .$$

At the same time we have the obvious equalities

$$W_1 = m_2 - m_1, \quad W_2 = m_2 + m_1 \quad .$$

Now from (3.5.9) and (3.5.11) we obtain the following representations of the Weyl functions:

$$\begin{aligned} m_1(\lambda) &= \frac{f(\lambda) + \mathrm{i}\mu(\lambda)}{\prod_{j=1}^{g}(\lambda - \lambda_j)} \quad , \\ m_2(\lambda) &= \frac{f(\lambda) - \mathrm{i}\mu(\lambda)}{\prod_{j=1}^{g}(\lambda - \lambda_j)} \quad . \end{aligned} \tag{3.5.12}$$

In (3.5.12) $\mu(\lambda)$ is defined as a single-valued branch of the function

$$\sqrt{\prod_{j=1}^{2g+1}(\lambda - E_j)} \quad ,$$

fixed on $\mathbb{C}\backslash\mathcal{E}$, by the condition: $\mu(\lambda) > 0$ if λ lies on the upper side of the cut $[E_{2g+1}, \infty]$: $\mathrm{Im}\lambda = +0$.

The equalities (3.5.7) enable one to reconstruct the polynomial $f(\lambda)$ from a knowledge of the points λ_j, E_i. We remark that $m_1(\lambda)$ and $m_2(\lambda)$ represent a restriction of the single-valued function $m(P)$ defined on X by the formula

$$m(P) = \frac{f(\lambda) + \mathrm{i}\mu}{\prod_{j=1}^{g}(\lambda - \lambda_j)} = \frac{d}{dx}\varphi(x, P)\bigg|_{x=0} , \quad P = (\mu, \lambda) \quad . \tag{3.5.13}$$

It was proved above that $m(P)$ has exactly g poles $P_k(\mu_k, \lambda_k)$, $k = 1 \dots, g$. This statement does not contradict (3.5.13) if the equalities

$$f(\lambda_k) = +\mathrm{i}\mu_k = \pm\mathrm{i}\mu(\lambda_k) \tag{3.5.14}$$

occur, and the sign $+(-)$ corresponds to the case of P_k lying on the upper (lower) sheet of the surface X. The system (3.5.14) determines the polynomials $f(\lambda)$ uniquely. At the same time, (3.5.14), combined with the inequalities (3.5.8), shows that $f(\lambda)$ is a polynomial with real-valued coefficients.

Formulas (3.5.12) realize the desired representation for the Weyl functions of the operator (3.5.1,2). Now from the link (3.5.6) between Weyl functions and the spectral matrix function

$$\hat{\varrho}(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \xi(\lambda) \end{pmatrix}$$

of the operator $\mathcal{L}$, we deduce the two following consequences:

Lemma 3.10.
(1) $\hat{\varrho}(\lambda)$ is an absolutely continuous function.
(2) the matrix elements of $d\hat{\varrho}/d\lambda$ are equal to

$$\begin{aligned}
\frac{d\xi}{d\lambda} &= \begin{cases} \dfrac{1}{2\pi}\dfrac{\prod_{j=1}^{g}(\lambda - \lambda_j)}{\mu(\lambda)}, & \lambda \in \mathcal{E}_1 \\ \qquad 0, & \lambda \in \mathbb{R}\backslash\mathcal{E}_1 \end{cases} \\
\frac{d\eta}{d\lambda} &= \begin{cases} -\dfrac{1}{2\pi}\dfrac{f(\lambda)}{\mu(\lambda)}, & \lambda \in \mathcal{E}_1 \\ \quad 0, & \lambda \in \mathbb{R}\backslash\mathcal{E}_1 \end{cases} \\
\frac{d\zeta}{d\lambda} &= \begin{cases} \dfrac{1}{2\pi}\dfrac{g(\lambda)}{\mu(\lambda)}, & \lambda \in \mathcal{E}_1 \\ \quad 0, & \lambda \in \mathbb{R}\backslash\mathcal{E}_1 \end{cases}
\end{aligned} \tag{3.5.15}$$

where the function $g(\lambda)$ is defined by the relation

$$g(\lambda) = \frac{f^2(\lambda) + \mu^2(\lambda)}{\prod_{j=1}^{g}(\lambda - \lambda_j)} \quad . \tag{3.5.16}$$

Note that from (3.5.14) it is clear that the right-hand-side of (3.5.16) is a polynomial of degree $g + 1$,

$$g(\lambda) = \lambda^{g+1} + \dots \quad .$$

Equalities (3.5.15, 16) enable us to define more exactly the spectrum of the operator $\mathcal{L}$. The spectrum of $\mathcal{L}$ is absolutely continuous and coincides with $\mathcal{E}_1$. Going further, we can conclude from (3.5.12) that the projections λ_j of the poles of the Baker-Akhiezer functions represent the eigenvalues of the operator $\mathcal{L}$ either on the half-axis $[0, +\infty)$ (the sign "+" in (3.5.14)) or on the half-axis $(-\infty, 0]$ [the sign "-" in (3.5.14)] with the Dirichlet boundary condition at $x = 0$.

Remark 3.11. The projections $\lambda_j(x)$ of the zeros of the Baker-Akhiezer function may also be interpreted in a reasonable way from the spectral point of view. Consider, along with the finite-gap $u(x)$, the shifted potential

$$u_{x_0}(x) = u(x + x_0) \quad .$$

It may be interpreted as a finite-gap potential determined by the same curve X and the divisor $\boldsymbol{D}_{x_0} = \boldsymbol{D} + \boldsymbol{U}x_0$. Associated numbers $\lambda_j^{x_0}$ defined as projections of the zeros of $\theta\left(\int_\infty^P \omega + \boldsymbol{D}_{x_0}\right)$ coincide with $\lambda_j(x)|_{x=x_0}$. Hence $\lambda_j(x_0)$ are the Dirichlet eigenvalues of $\mathcal{L}$ either on the half-axis $[x_0, +\infty)$ (the zero of $\varphi(x, P)$ corresponding to $\lambda_j(x_0)$ is on the upper sheet) or on the half-axis $(-\infty, x_0]$ (if the corresponding zero is on the lower sheet).

Remark 3.12. Consider the zeros $\widetilde{\lambda}_j$, $j = 1, \dots, g+1$ of the polynomial $g(\lambda)$:

$$g(\lambda) = \prod_{j=1}^{g+1}(\lambda - \widetilde{\lambda}_j) \quad . \tag{3.5.17}$$

By virtue of (3.5.12, 16), the points $\widetilde{\lambda}_j$ are the zeros of the Weyl functions $m_1(\lambda)$ or $m_2(\lambda)$. Hence each $\widetilde{\lambda}_j$ is an eigenvalue of the $\mathcal{L}$-operator with the Neumann boundary conditions at the point $x = 0$ on the semi-line $[0, +\infty)$ or on the semi-line $(-\infty, 0]$. The reality of all $\widetilde{\lambda}_j$ is evident. Applying Abel's theorem to the function $m(P)$ (Sect. 2.4.5) we conclude that its zeros Q_k satisfy the equality

$$\sum_{k=1}^{g+1} \int_\infty^{Q_k} \omega = \sum_{i=1}^{g} \int_\infty^{P_i} \omega \quad .$$

The same considerations as in the proof of Theorem 3.6 lead to the following inequalities for $\widetilde{\lambda}_j$:

$$\widetilde{\lambda}_1 \leq E_1, \quad E_{2j} \leq \widetilde{\lambda}_j \leq E_{2j+1}, \quad j = 1, \ldots, g \quad . \tag{3.5.18}$$

The spectral properties of the Schrödinger operator $\mathcal{L}$ established above may be summarized as follows:

Theorem 3.13. Let X be an arbitrary hyperelliptic M-curve

$$\mu^2 = \prod_{i=1}^{2g+1}(\lambda - E_i), \quad E_i \in \mathbb{R}, \quad E_1 < E_2 < \ldots < E_{2g+1} \quad ,$$

$g > 0$, and let $\boldsymbol{D}$ be an arbitrary vector with pure imaginary components. Then

(1) the function $u(x)$, $x \in \mathbb{R}$ defined by (3.5.2) is a real valued, smooth, quasiperiodic function with basic periods $T_j = 2\pi\mathrm{i}/U_j$.
(2) the spectrum of the Schrödinger operator $\mathcal{L} = -d^2/dx^2 - u(x)$ on the whole real line is absolutely continuous and equals $\mathcal{E}_1 = \bigcup_{j=0}^{g-1}[E_{2j+1}, E_{2j+2}] \bigcup [E_{2g+1,\infty}]$.

One of the most remarkable aspects of the theory is that the inverse statement is also true. It can be proven that each real, smooth, quasiperiodic potential $u(x)$ with a finite-gap, absolutely continuous spectrum may be represented in the form of (3.5.2) with an appropriate curve X and divisor $\mathcal{D}$. The reconstruction of X and $\mathcal{D}$ from the spectral data of $\mathcal{L}$ is almost evident. Nevertheless all details of the proof of the last statement are given in Chap. 8. The important particular case of purely periodic potentials is considered in the next section.

Now it is possible to interpret the right-hand-side of (3.4.3) as the solution of the KdV equation with the initial condition $u(x)$, quasiperiodic in x, with the finite-gap (in the sense of the spectral theory) spectrum of the operator $\mathcal{L} = -\partial_x^2 - u(x)$. We remark that the Weyl solutions corresponding to the "KdV-shifted" $\mathcal{L}$-operator $\mathcal{L} = -\partial_x^2 - u(x,t)$ are of the form

$$\begin{aligned} \varphi_\pm(x,\lambda,t) = & \frac{\theta\left(\int_\infty^{P_\pm}\omega + \boldsymbol{U}x + \boldsymbol{W}t + \boldsymbol{D}\right)}{\theta\left(\int_\infty^{P_\pm}\omega + \boldsymbol{W}t + \boldsymbol{D}\right)} \\ & \times \frac{\theta(\boldsymbol{W}t + \boldsymbol{D})}{\theta(\boldsymbol{U}x + \boldsymbol{W}t + \boldsymbol{D})} e^{\Omega_1(P_\pm)x} \quad , \\ & P_\pm = (\pm\mu, \lambda), \quad P_+ = \pi P_- \quad . \end{aligned}$$

From this it is clear that the algebro-geometrical parameters $X, \boldsymbol{D} \in J(X)$ of quasiperiodic, finite-gap potentials depend on t in the following way

$$X = \text{const}, \quad \boldsymbol{D}(t) = \boldsymbol{D}(0) + \boldsymbol{W}t \quad . \tag{3.5.19}$$

The spectral parametrization of the same data by the numbers (λ_j, E_i), yields

$$\frac{d}{dt}E_j = 0, \quad \frac{d\lambda_j}{dt} = \frac{-2\mathrm{i}\left(\sum_{k\neq j}\lambda_k - \frac{1}{2}\sum_{i=1}^{2g+1}E_i\right)\mu(\lambda_j)}{\prod_{k\neq j}(\lambda_j - \lambda_k)} \quad . \tag{3.5.20}$$

To derive (3.5.20) it is sufficient to represent $\boldsymbol{D}(t)$ in the form

$$\boldsymbol{D}(t) = -\sum_{k=1}^{g}\int_{\infty}^{P_k(t)}\omega - \boldsymbol{K}, \quad \pi(P_k(t)) = \lambda_k(t) \quad ,$$

and to differentiate the second of the relations (3.5.19) with respect to t and solve the obtained linear system with respect to $d\lambda_j/dt$.

Remark 3.14. From the relation

$$\boldsymbol{D}(x,t) = \boldsymbol{D}(0,0) + \boldsymbol{U}x + \boldsymbol{W}t$$

it is not difficult to deduce the following autonomous system of differential equations for $\lambda_j(x,t)$ (Dubrovin's equations): (3.5.20) and

$$\frac{\partial\lambda_j}{\partial x} = \frac{-2\mathrm{i}\mu(\lambda_j)}{\prod_{k\neq j}(\lambda_j - \lambda_k)} \quad . \tag{3.5.21}$$

The system (3.5.20, 21) describes the dynamics of the functions λ_j with respect to the shift of the potential and the KdV flow. The system (3.5.20-21) represents the KdV equation in the space of parameters (E_i, λ_j). Let us note that the solution $u(x,t)$ of the KdV equation may be constructed from this data by

$$u(x,t) = 2\sum_{j=1}^{g}\lambda_j(x,t) - \sum_{j=1}^{2g+1}E_j \quad , \tag{3.5.22}$$

which may be easily deduced from the substitution of the expansion (3.5.10) for the functions $\varphi_\pm(x,\lambda)$ in (3.5.9). From the spectral viewpoint, the identity (3.5.22) is the first of the so-called trace formulas. In the following section we shall discuss more thoroughly these relations and their role in the creation of the method of finite-gap integration. With this step we finish the discussion of spectral properties of the finite-gap solutions of the KdV equation. In the following chapters we shall restrict ourselves to consider purely algebraic aspects of the theory for the integrable systems related to the matrix differential operators, taking into account that all the spectral aspects may be treated in complete analogy with the KdV case. Particularly for the NS and SG models all the algebro-geometric ingredients of the theory have the same spectral interpretation with the unique difference that the Schrödinger equation must be replaced by the one-dimensional Dirac system.

3.6 Spectral Properties of the Schrödinger Operator with a Finite-Gap Periodic Potential

In this section we consider some properties of the Schrödinger operator with a real periodic potential. Some of them are well-known (see textbooks [3.6-8]). The main result proven below is the statement inverse to Theorem 3.13 of Sect. 3.5 in the particular case of periodic potentials. We show that each smooth periodic potential, with a finite number of energy gaps, is described by the formula (3.5.2). The associated curve X is defined by the boundaries of the energy gaps, and the divisor $\mathcal{D}$ by the eigenvalues of the Dirichlet problem. In other words we solve explicitly the inverse spectral problem for the Schrödinger operator with the finite-gap periodic potential. Historically, the formula (3.4.3) for exact solutions of the KdV equation was first derived in this manner.

3.6.1 Monodromy Matrix and Bloch Solutions of the Hill Equation

Let the potential $u(x)$ be real and periodic with the period T; $u(x+T) = u(x)$. Introduce a system of fundamental solutions φ, θ of the Schrödinger equation $\mathcal{L}\psi = \lambda\psi$:

$$\varphi(0,\lambda) = 0, \quad \varphi_x(0,\lambda) = 1, \quad \theta(0,\lambda) = 1, \quad \theta_x(0,\lambda) = 0 \quad .$$

The solutions of the system shifted on the period of the potential may be decomposed as follows:

$$\begin{aligned}
&\varphi(x+T,\lambda) = m_{11}\varphi(x,\lambda) + m_{21}\theta(x,\lambda) \\
&\theta(x+T,\lambda) = m_{12}\varphi(x,\lambda) + m_{22}\theta(x,\lambda) \quad , \\
&M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \varphi_x(T,\lambda) & \theta_x(T,\lambda) \\ \varphi(T,\lambda) & \theta(T,\lambda) \end{pmatrix} \quad , \\
&\det M = 1 \quad ,
\end{aligned}$$

where M is called a monodromy matrix. The property $\det M = 1$ follows from the fact that the Wronskian of two solutions of $\mathcal{L}\psi = \lambda\psi$ is independent on x, and hence may be computed at $x = 0$, taking into account the definition of φ and θ.

The fundamental system of the solutions $\psi_{1,2}$, reducing the monodromy matrix to diagonal form, is often called a Bloch (or Floquet) system. Bloch solutions evidently satisfy the condition

$$\psi_{1,2}(x+T,\lambda) = \varrho_{1,2}(\lambda)\psi_{1,2}(x,\lambda) \quad ,$$

where $\varrho_{1,2}$ are the eigenvalues of the monodromy matrix. By virtue of $\det M = 1$ we have $\varrho_1\varrho_2 = 1$. The explicit form of $\varrho_{1,2}$ is obtained immediately by solving $\det(M - \varrho I) = 0$:

$$\varrho_{1,2} = F(\lambda) \pm \sqrt{F^2(\lambda) - 1}, \quad F(\lambda) = \mathrm{Tr}\, M/2 \quad .$$

The map $F \pm \sqrt{F^2 - 1}$ transforms the interior of the interval $[-1, 1]$ to the boundary of the unit circle in the complex plane. The values $F = \pm 1$ correspond to $\varrho_{1,2} = \pm 1$. Hence in the case $|F| \leq 1$ the Bloch solutions are bounded on the whole real axis, and the corresponding values of λ form the continuous spectrum of the Schrödinger operator $\mathcal{L}$. It is well-known that this spectrum represents a sequence of real segments divided by spectrum-free lacunas (or gaps). The length of the gap tends to zero when its number tends to infinity. In general the number of gaps is infinite.

The Bloch solutions may be represented in the form

$$\psi_{1,2} = \theta(x, \lambda) + m_{1,2}\varphi(x, \lambda) \quad ,$$

$$m_{1,2} = \frac{\varphi_x(T, \lambda) - \theta(T, \lambda) \pm 2\sqrt{F^2(\lambda) - 1}}{2\varphi(T, \lambda)} \quad ,$$

where $m_{1,2}$ are the Weyl functions. The Weyl functions are well defined (Sect. 3.5) for arbitrary continuous potentials $u(x)$. In the periodic case they can be determined by the above formulas.

We denote the boundaries of the continuous spectrum of the operator $\mathcal{L}$ by E_i, ordered in a natural way: $E_1 \leq E_2 \leq \ldots \leq E_{2g+1} \leq \ldots$. Hence the spectrum of $\mathcal{L}$ is the union of intervals $[E_1, E_2], \ldots, [E_{2g-1}, E_{2g}], \ldots$. The eigenvalues λ_j of the Dirichlet problem $y(0) = y(T) = 0$, for the Schrödinger equation $\mathcal{L}y = \lambda y$, are situated in the closures of the energy gaps, i.e., in the integrals $[E_2, E_3], \ldots, [E_{2g}, E_{2g+1}], \ldots$.

For the degenerate gaps we have $\lambda_i = E_{2i} = E_{2i+1}$. In the general case the Lyapunov inequalities $E_{2i} \leq \lambda_i \leq E_{2i+1}$ are satisfied. It is important to notice the invariance of the numbers E_i with respect to the shift of the potential $u(x) \to u(x + \tau)$, $\tau \in \mathbb{R}$. The monodromy matrix, as well as the Dirichlet eigenvalues $\lambda_i(\tau)$, are not invariant with respect to the same shift. It is not difficult to prove that by varying τ from 0 to T, we force each λ_i to run twice along the closure of the gap $[E_{2i}, E_{2i+1}]$, and on the ends of the gap $\partial_\tau \lambda_i = 0$.

Later on we need to use the differential equation describing the dependence of the elements of the monodromy matrix M on the shift parameter τ. Since the eigenvalues of the monodromy matrix are invariant with respect to τ, the associated equation may be represented in the form

$$\frac{\partial M}{\partial \tau} = [A, M] \quad , \tag{3.6.1}$$

where the bracket means the matrix commutator. An explicit form of this differential equation may be derived as follows. Let $\varphi(\tau; x, \lambda)$, $\theta(\tau; x, \lambda)$be the solutions of Schrödinger equation with the potential $u(x + \tau)$ fixed by the conditions

$$\varphi(\tau; 0, \lambda) = 0, \quad \varphi_x(\tau; 0, \lambda) = 1, \quad \theta(\tau; 0, \lambda) = 1, \quad \theta_x(\tau; 0, \lambda) = 0 \quad .$$

The solutions $\varphi(0; x+\tau, \lambda)$, $\theta(0; x+\tau, \lambda)$ from another fundamental system for the same equation. The connection between these two systems of solutions is given by the formula (we omit the dependence on λ in notations)

$$\begin{aligned} \theta(\tau; x) &= \varphi_x(0;\tau)\theta(0; x+\tau) - \theta_x(0;\tau)\varphi(0; x+\tau) \quad , \\ \varphi(\tau; x) &= \theta(0;\tau)\varphi(0; x+\tau) - \varphi(0;\tau)\theta(0; x+\tau) \quad . \end{aligned}$$

Differentiation of this formula gives the equalities

$$\begin{aligned} \theta_x(\tau; x) &= \varphi_x(0;\tau)\theta_x(0; x+\tau) - \theta_x(0;\tau)\varphi_x(0; x+\tau) \quad , \\ \varphi_x(\tau; x) &= \theta(0;\tau)\varphi_x(0; x+\tau) - \varphi(0;\tau)\theta_x(0; x+\tau) \quad . \end{aligned}$$

Differentiating the above equalities with respect to τ and putting then $x = T$ one can easily obtain the following system of linear differential equations for the elements of the monodromy matrix $M(\tau)$:

$$\begin{aligned} \partial_\tau\varphi(\tau; T) &= \varphi_x(\tau; T) - \theta(\tau; T) \quad , \\ \partial_\tau\theta(\tau; T) &= \theta_x(\tau; T) + (\lambda + u(\tau))\varphi(\tau; T) \quad , \\ \partial_\tau\varphi_x(\tau; T) &= -\theta_x(\tau; T) + (-u(\tau) - \lambda)\varphi(\tau; T) \quad , \\ \partial_\tau\theta_x(\tau; T) &= (\lambda + u(\tau))(\varphi_x(\tau; T) - \theta(\tau; T)) \quad . \end{aligned} \tag{3.6.2}$$

One may rewrite the right-hand-side of (3.6.2) in commutator form to derive (3.6.1), but we do not need an explicit form of the matrix A.

Subtracting the second equation of the system (3.6.2) from the third and substituting the result into the first equation differentiated with regard to τ, we have

$$\partial_\tau^2\varphi(\tau; T) = -2\theta_x(\tau; T) + 2(-u(\tau) - \lambda)\varphi(\tau; T) \quad .$$

Differentiating the last equation by τ and taking into account the first and the fourth of the equations (3.6.2) we find

$$\partial_\tau^3\varphi(\tau; T) = 4(-u(\tau) - \lambda)\partial_\tau\varphi - 2u'(\tau)\varphi(\tau; T) \quad . \tag{3.6.3}$$

The last equation is the principal result of this section. It is not included in standard textbooks and in the context discussed above was first derived in the work [3.9].

There are many interesting points related to this equation. Thus, *Hermite* observed [3.10] that the product $y_1 y_2$ of an arbitrary pair y_1, y_2 of solutions of the equation

$$-y'' - u(\tau)y = \lambda y$$

satisfies (3.6.3). This simple fact has remarkable consequences in applications to the finite-gap potentials discussed below.

At the end of this section let us write down the identities for the eigenvalues of the Dirichlet problem obtained by substraction of the three first Gelfand-Levitan-Dikii trace formulas [3.11, 12] for shifted and original periodic potential:

$$
\begin{aligned}
\sum_{k=1}^{\infty}(\lambda_k(\tau)-\lambda_k(0)) &= \frac{u(\tau)-u(0)}{2} \quad , \\
\sum_{k=1}^{\infty}(\lambda_k^2(\tau)-\lambda_k^2(0)) &= \frac{u''(0)-u''(\tau)}{4}+\frac{u^2(\tau)-u^2(0)}{2} \quad , \\
\sum_{k=1}^{\infty}(\lambda_k^3(\tau)-\lambda_k^3(0)) &= \frac{u'''(\tau)-u'''(0)}{2} \\
&\quad +\frac{3}{4}\left[u(\tau)u''(\tau)-u(0)u''(0)\right] \\
&\quad +\frac{3}{32}\left[u^{(IV)}(\tau)-u^{(IV)}(0)\right] \\
&\quad +\frac{15}{32}\left[u'^2(\tau)-u'^2(0)\right] \quad .
\end{aligned}
\tag{3.6.4}
$$

3.6.2 Finite-Gap Potentials and the Gelfand-Levitan-Dikii Trace Formulas

We start from the trace formulas of the preceding section. If the potential has exactly g non-degenerate gaps in the continuous spectrum we shall call it a finite-gap potential. For such a potential only g terms in the LHS of the trace formulas (3.1.4) are different from zero. In particular, if all the gaps are degenerated from the first trace formula, we conclude that $u(\tau)=u(0)$. In other words, each smooth periodic 0-gap potential is a constant.

Quite similarly, admitting that there exists only one non-trivial gap, we can eliminate the associated eigenvalue $\lambda_i(\tau)$ from the first two trace formulas and find that the potential $u(\tau)$ satisfies the same nonlinear ordinary differential equation as $2\wp(\tau)+c$, i.e., the differential equation for the Weierstrass elliptic function.

Using higher trace formulas, we can, in principle, derive an ordinary differential equation describing an arbitrary finite-gap potential, but this is not convenient and we do not follow it below.

In addition, we remark that from the first trace formulas it is clear that in the one-gap case the length of the gap is equal to half of the difference between the maximal and minimal values of the potential.

We also mention that by applying the Gelfand-Levitan-Dikii trace formulas to the g-gap potential, one can derive the following important relations:

$$\sum_{i=1}^{g} \lambda_i(\tau) = \frac{u(\tau)}{2} + \frac{1}{2}\sum_{k=1}^{2g+1} E_k \quad ,$$

$$\sum_{i=1}^{g} \lambda_i^2(\tau) = -\frac{u''(\tau)}{4} + \frac{u^2(\tau)}{2} - \frac{1}{2}\sum_{k=1}^{2g+1} E_k^2 \quad ,$$

$$\sum_{i=1}^{g} \lambda_i^3(\tau) = \frac{u^3(\tau)}{2} + \frac{3}{4}u(\tau)u''(\tau) + \frac{15}{32}u'^2(\tau)$$

$$+ \frac{3}{32}u^{(IV)}(\tau) + \frac{1}{2}\sum_{k=1}^{2g+1} E_k^3 \quad ,$$

where E_k are boundaries of nondegenerate gaps.

3.6.3 Criterion for the Periodic Potential to be Finite-Gap

It is well-known that $\varphi(\tau; T), \lambda)$ is an entire function of order 1/2. Its zeros are the eigenvalues $\lambda_i(\tau)$ of the Sturm-Liouville problem. When the number of gaps is finite and equal to g, the function φ admits the following obvious factorization:

$$\varphi(\tau; T, \lambda) = \varphi(0; T, \lambda) \prod_{i=1}^{g} \frac{\lambda - \lambda_i(\tau)}{\lambda - \lambda_i(0)} \quad . \tag{3.6.5}$$

The last formula shows that all dependence of $\varphi(\tau; T, \lambda)$ on τ is concentrated in the polynomial factor

$$S_g(\tau, \lambda) = \prod_{i=1}^{g} (\lambda - \lambda_i(\tau)) \quad . \tag{3.6.6}$$

Hence the existence of the periodic solution of the equation (3.6.3), which is a polynomial of λ of the form (3.6.6), is the necessary condition for the Hill operator to have exactly g gaps in the continuous spectrum.

Next we show that this condition is also sufficient. By virtue of Hermite's lemma mentioned above, the functions $\psi_1\psi_2$, ψ_1^2, ψ_2^2, where the $\psi_{1,2}$ are the Bloch solutions of

$$-y'' - u(\tau)y = \lambda y \quad ,$$

form (for the spectral parameter λ in "general position", i.e., unequal to the boundaries of the spectrum) a fundamental system of solutions of (3.6.3). If λ is not the boundary of a degenerate gap, (3.6.3) has only one periodic solution, which is the product $\psi_1\psi_2$. Since both φ and S_g are periodic in τ the equality

$$\varphi(\tau, \lambda) = S_g(\tau, \lambda)C(\lambda)$$

with some τ-independent function $C(\lambda)$ holds. It means that the number of moving eigenvalues $\lambda_i(\tau)$ is finite and is equal to g, i.e., there are exactly g non-degenerated gaps in the continuous spectrum of the corresponding Hill operator.

Substituting the polynomial S_g into (3.6.3) and equating the coefficients, one can find that the g-gap periodic potential satisfies the ordinary nonlinear differential equation

$$\widetilde{L}(J\widetilde{L})^g 1 = 0, \quad \widetilde{L} = \partial_\tau^3 + 2(u\partial_\tau + \partial_\tau u) \quad ,$$

and J is the operator of indefinite integration $J = \partial_\tau^{-1}$. In the theory of the KdV equation the operator $J\widetilde{L}$ is often called the generating operator.

3.6.4 Charles Hermite and the Lamé Equation

Here we apply the considerations of the previous subsection to the particular case of the Lamé potential $u(\tau) = -N(N+1)\wp(\tau)$. Taking $\wp(\tau) = z$ as a new independent variable, Hermite represented the Lamé equation in so-called algebraic form and remarked that the associated equation (3.6.3) for the product of two solutions of Hill's equation admits a polynomial solution in the z variable. Hermite calculated explicitly the coefficients of that polynomial [3.10]:

$$f_N(\tau, \lambda) = \sum_{z=0}^{N} c_r(\wp(\tau) - e_2)^{N-r} \quad ,$$

$$c_0 = 1, \quad e_1 = \wp(\omega), \quad e_2 = \wp(\omega'), \quad e_3 = \wp(\omega + \omega') \quad .$$

The coefficients c_r are determined by the recurrence relation

$$\begin{aligned}
&4r\left(N - r + 1/2\right)(2N - r + 1)c_r \\
&= (N - r + 1)\{12e_2(N - r)(N - r + 2) - 4e_2(N^2 + N - 3) + 4\lambda\}c_{r-1} \\
&\quad - 2(N - r - 1)(N - r + 2)(e_1 - e_2)(e_2 - e_3)(2N - 2r + 3)c_{r-2} \quad .
\end{aligned}$$

The structure of these formulas shows that c_r are polynomials in λ and hence $f_N(\tau, \lambda)$ is a polynomial in λ of degree N, periodic in the τ solution of (3.6.3).

The criterium proved in the previous subsection shows that the Lamé potential is a periodic potential having exactly N gaps. Since there is not a zero $\lambda = \lambda_0$ of f_N which is independent of τ, all gaps are non-degenerate.

Hermite himself ignored this spectral interpretation of the Lamé potential, but he succeeded in calculating explicitly the Bloch eigenfunctions in the following way: let ψ_1, ψ_2 be a pair of Bloch solutions normalized so that $\psi_1\psi_2 = f_N$ with the Wronskian equal to W. We have

$$\frac{W}{f_N} = \frac{W}{\psi_1\psi_2} = \frac{\psi_{1x}}{\psi_1} - \frac{\psi_{2x}}{\psi_2} = \frac{d}{dx}\log\left(\frac{\psi_1}{\psi_2}\right)$$

and hence

$$\frac{\psi_1}{\psi_2} = \exp\left(\int^x W/f_N \, dx\right) \quad .$$

So we have the following formulas for ψ_1, ψ_2:

$$\psi_1^2 = f_N \frac{\psi_1}{\psi_2} = f_N \exp\left(\int^x W/f_N\, dx\right) \quad ,$$

$$\psi_1 = \sqrt{f_N(x)} \exp\left(W/2 \int^x dx/f_N(x)\right) \quad ,$$

$$\psi_2 = \sqrt{f_N(x)} \exp\left(-W/2 \int^x dx/f_N(x)\right) \quad ,$$

$$\varrho_{1,2}(\lambda) = \exp\left(\pm W/2 \int_0^T dx/f_N(x)\right) \quad .$$

Unfortunately, these beautiful Hermite formulas are not presented in textbooks discussing the Lamé equation. These usually contain only the expressions for the so-called Lamé polynomials, – the solutions corresponding to the boundaries of gaps. The only exception known to the authors is *Akhiezer's* paper [3.13] reproducing Hermite's and Markov's considerations of the Lamé equation.

The Lamé potential is as well of interest in the context of reduction of multidimensional Riemann theta functions to elliptic functions. In Chap. 7 we describe some explicit constructions of the elliptic finite-gap potentials discovered quite recently.

3.6.5 Analytical Properties of the Bloch Functions and the Inverse Spectral Problem for Finite-Gap Potentials

First we show that in the finite-gap case Weyl functions have an extremely simple structure:

$$\begin{aligned} m_{1,2}(\lambda) &= \left(Q(\lambda)/2 \pm i\sqrt{P_{2g+1}(\lambda)}\right) / S_g(0,\lambda) \quad , \\ S_g(\tau,\lambda) &= \prod_{i=1}^{g} (\lambda - \lambda_i(\tau)), \quad Q(\lambda) = \left.\frac{\partial S_g}{\partial \tau}\right|_{\tau=0} , \\ P_{2g+1}(\lambda) &= \prod_{i=1}^{2g+1} (\lambda - E_i) \quad . \end{aligned} \tag{3.6.7}$$

To prove this formula we use the factorization $\varphi(\tau; T, \lambda) = c(\lambda) S_g(\tau, \lambda)$, of φ and the first of the equations in (3.6.2)

$$\varphi_x(\tau; T, \lambda) - \theta(\tau; T, \lambda)|_{\tau=0} = \partial_\tau \varphi(\tau; T, \lambda)|_{\tau=0} = c(\lambda) \partial_\tau S_g(\tau, \lambda)|_{\tau=0} \quad .$$

From the last equality we have ($\tau = 0$)

$$(\varphi_x(\tau; T, \lambda) - \theta(\tau; T, \lambda))/2\varphi(\tau; T, \lambda) = Q(\lambda)/2S_g(0, \lambda) \quad .$$

The zeros of the entire function $F^2(\lambda) - 1$ coincide with the boundaries of the continuous spectrum. The degeneracy of the gap, i.e., the equality $E_{2k} = E_{2k+1}$, corresponds to a double zero of $F^2(\lambda) - 1$; the same λ must be a simple zero of

$\varphi(\tau; T, \lambda)$. From the factorization of these entire functions into infinite products it follows that

$$\frac{F^2(\lambda)-1}{\varphi^2(\lambda)} = qP_{2g+1}(\lambda)/S_g^2(0,\lambda) \quad .$$

The constant q may be determined by substituting the asymptotics for φ, θ, φ_x at infinity [3.9]. Thus we find that $q = -1$. This completes the proof of the representation (3.6.7) for $m_{1,2}(\lambda)$.

From this representation it is clear that the Bloch solutions $\psi_{1,2}(x,\lambda)$ may be considered as two branches of the single-valued function $\psi(x,P)$, defined on the Riemann surface of the square root $\sqrt{P_{2g+1}(\lambda)}$. We consider the analytic properties of this function.

From the representation

$$\psi_{1,2} = \exp\left(\pm\frac{x}{T}\log\left(F(\lambda)+\sqrt{F^2(\lambda)-1}\right)\chi_{1,2}(x,\lambda)\right) \quad ,$$

$$\chi_{1,2}(x,\lambda) = \chi_{1,2}(x+T,\lambda) \quad ,$$

we see that $\psi_1\psi_2$ is a periodic solution of (3.6.3) for $\lambda \neq E_i$. From the condition $\psi_{1,2}(0,\lambda) = 1$ it is evident that for $\lambda \neq E_i$

$$\psi_1\psi_2(x,\lambda) = S_g(x,\lambda)S_g^{-1}(0,\lambda) \quad .$$

The same equality is valid for all λ due to the analyticity of the LHS and RHS for all λ. Consequently the zeros of $\psi(x,P)$ are the points of the Riemann surface $\sqrt{P_{2g+1}(\lambda)}$ situated above $\lambda_i(x)$, and the poles are the points situated above $\lambda_i(0)$. Finally, from the first terms of the "high energy" asymptotics of φ and θ we have

$$\psi(x,P) = e^{i\sqrt{\lambda}x}[1+o(1)], \quad \lambda \to \infty \quad .$$

So far we have verified that the Bloch solutions of the finite-gap periodic potential may be considered as two branches of the function $\psi(x,P)$ taken on the upper and lower sheets of the corresponding Riemann surface, and that $\psi(x,P)$ possesses all the analytical properties of the Baker-Akhiezer function discussed above. Hence $\psi(x,P)$ may be reconstructed by the explicit formula (3.5.3), and the finite-gap potential $u(x)$ itself is given by (3.5.2). It is important to remark that for the purely periodic case, the numbers E_i are restricted to satisfy the transcendental relations

$$U_j = \frac{m_j}{T}, \quad m_j \in \mathbb{Z} \quad , \tag{3.6.8}$$

meaning that the periods $T_j = 2\pi i/U_j$ of the function (3.5.2) (which is generically almost periodic) are commensurable. The entries m_j have the following topological interpretation. When x moves from 0 to T, the point $\lambda_i(x)$ runs m-times along the real oval of the curve X lying above the jth gap $[E_{2j}, E_{2j+1}]$.

It is necessary to notice that there is no criterion which is more effective than (3.6.8) in distinguishing the numbers E_j, corresponding to the purely periodic potentials, from the general situation. Another form of the solution of this problem is given in [3.7], where the boundaries of gaps of purely periodic finite-gap potentials are represented in the form of the Schwartz-Christoffell integrals.

3.6.6 Jules Drach and General Finite-Gap Potentials

In 1919 the French mathematician Jules Drach, studying the cases of the general reduction of the group of rationality of the Sturm-Liouville equation

$$y'' = [\varphi(x) + h]y \quad ,$$

found a large class of integrable cases characterized by the following properties of the solutions of the associated Riccati equation:

$$\varrho' + \varrho^2 = \varphi + h \quad .$$

The Riccati equation admits two solutions $\varrho_{1,2}$ of the form

$$\varrho_{1,2} = \frac{R' \pm \sqrt{\Omega}}{2R}, \quad R = h^n + R_1 h^{n-1} + \ldots \quad ,$$

where Ω is a polynomial in h of order $2n + 1$ with constant coefficients. The function R satisfies the third order linear differential equation

$$R''' - 4R'(\varphi + h) - 2R\varphi' = 0 \quad , \tag{3.6.9}$$

which produces for determination of φ an ordinary differential equation depending on n arbitrary constants $c_1, \ldots, c_n$. Taking into account the existence of the first integral of this equation

$$(R')^2 - 2RR'' + 4R^2(\varphi + h) = \Omega \quad , \tag{3.6.10}$$

let us factorize R

$$R = (h - \omega_1)(h - \omega_2) \cdots (h - \omega_n) \quad .$$

From this factorization we get

$$-\frac{R'}{R} = \frac{\omega_1'}{h - \omega_1} + \cdots + \frac{\omega_n'}{h - \omega_n} \quad . \tag{3.6.11}$$

Now, taking into account that the zeros $h = \omega_i$ are also the roots of $R'^2 - \Omega$ by virtue of (3.6.10) we conclude that at $h = \omega_i$ one of the factors $R' + \sqrt{\Omega}$, $R' - \sqrt{\Omega}$ must be equal to zero. Comparing with (3.6.11), we obtain

$$\frac{\varepsilon_i \omega_i'}{\sqrt{\Omega_i}} = \frac{1}{(\omega_i - \omega_1) \cdots (\omega_i - \omega_n)}, \quad i = 1, \ldots, n$$

and $\Omega_i = \Omega(\omega_i)$, $\varepsilon_i = \pm 1$.

The last result represents just Dubrovin's differential equations, if we identify ω_i with the Dirichlet eigenvalues for the shifted potential. Substituting the polynomial R into (3.6.10), we get

$$\omega_1 + \omega_2 + \cdots + \omega_n = -c_1 + \varphi/2 \quad .$$

The functions ω_i now obviously form a solution of the Jacobi inversion problem. By solving this problem, we can deduce (which was not, however, realized by Drach) representation (3.5.2) for Drach potentials φ which, of course, are nothing but the finite-gap potentials discussed in this chapter. Drach also noted that in a general position these potentials are not periodic but almost periodic. He also observed that the fundamental system of solutions of the associated Sturm-Liouville equation is described by the formula

$$y_{1,2} = \sqrt{R} \exp\left[\pm\sqrt{\Omega}\int \frac{dx}{2R}\right] \quad ,$$

naturally generalizing Hermite's result of the previous subsection. We have reproduced here, almost literally, the principal part of the work by *Drach* [3.14]. It is amazing that this remarkable work containing the constructions rediscovered in connection with the study of the KdV equation by Dubrovin, Its, Matveev, Gelfand, Dikii is referred to very early in the modern literature. Its existence was mentioned in an article by *Ehlers* and *Knörrer* [3.15], devoted to Darboux transformations of the finite-gap potentials.

4. Vector Valued Baker-Akhiezer Functions and Finite-Gap Solutions of the Nonlinear Schrödinger and the Sine-Gordon Equations

4.1 Finite-Gap Solutions of the Nonlinear Schrödinger Equation

The nonlinear Schrödinger (NS) system defines the evolution of two different complex valued functions $y(x,t)$ and $y^*(x,t)$:

$$\begin{aligned} \mathrm{i}y_t + y_{xx} - 2y^*y^2 &= 0 \quad , \\ -\mathrm{i}y_t^* + y_{xx} - 2yy^{*2} &= 0 \quad . \end{aligned} \tag{4.1.1}$$

The NS system reduces to the famous nonlinear Schrödinger equation

$$\mathrm{i}y_t + y_{xx} - 2\sigma|y|^2 y = 0, \quad \sigma = \pm 1 \tag{4.1.2}$$

under the constraint

$$y^* = \sigma\bar{y} \quad . \tag{4.1.3}$$

The Lax representation for the NS equation, first found in 1971 by *Zakharov* and *Shabat* [4.1], may be easily generalized to include the NS system. The most useful form of the Lax representation is the zero-curvature equation

$$U_t(\lambda) - V_x(\lambda) = [V(\lambda), U(\lambda)], \quad \lambda \in \mathbb{C} \quad . \tag{4.1.4}$$

In the case of the NS system, U and V are 2×2 matrices defined by

$$\begin{aligned} U(\lambda) &= -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & \mathrm{i}y \\ -\mathrm{i}y^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \\ V(\lambda) &= 2\lambda U(\lambda) + \begin{pmatrix} -\mathrm{i}yy^* & -y_x \\ -y_x^* & \mathrm{i}yy^* \end{pmatrix} \quad . \end{aligned} \tag{4.1.5}$$

The zero curvature equation is equivalent to the consistency condition of the associated linear system:

$$\begin{aligned} \Psi_x &= U\Psi \quad , \\ \Psi_t &= V\Psi \, , \qquad \Psi = \Psi(\lambda, x, t) \quad . \end{aligned} \tag{4.1.6}$$

The 2×2 matrix valued function $\Psi(\lambda)$ satisfying (4.1.6) plays a crucial role in the construction of algebro-geometric (finite-gap) solutions of the NS system in complete analogy with the scalar (KdV) case.

The first step of our strategy is to find the most general analytical properties of Ψ satisfying (4.1.6) with some matrices U and V having the prescribed form (4.1.5). This may be accomplished by using the following two propositions:

Lemma 4.1. Let $\Psi(\lambda)$ be a $2{\times}2$ matrix function holomorphic in some punctured neighborhood of infinity on the Riemann sphere, smoothly depending on x and t, with the following asymptotic expansion at infinity:

$$\Psi(\lambda, x, t) = \left[I + \sum_{k=1}^{\infty} \Psi_k(x,t)\lambda^{-k} \right] \exp(-\mathrm{i}\lambda x \sigma_3 - 2\mathrm{i}\lambda^2 t \sigma_3)\, C(\lambda) \quad , \tag{4.1.7}$$

where $C(\lambda)$ is some t- and x-independent invertible matrix. Assume also that (4.1.7) allows differentiation by terms with respect to x and t. Then the following asymptotics holds:

$$\begin{aligned} \Psi_x \Psi^{-1} &= U(\lambda) + o(\lambda^{-1}) \quad , \\ \Psi_t \Psi^{-1} &= V(\lambda) + o(\lambda^{-1}) \, , \qquad |\lambda| \to \infty \quad , \end{aligned} \tag{4.1.8}$$

with U and V expressed in terms of the coefficients Ψ_k

$$\begin{aligned} U(\lambda) &= -\mathrm{i}\lambda\sigma_3 + \mathrm{i}[\sigma_3, \Psi_1] \quad , \\ V(\lambda) &= -2\mathrm{i}\sigma_3\lambda^2 + 2\mathrm{i}\lambda[\sigma_3, \Psi_1] + 2\mathrm{i}[\sigma_3, \Psi_2] - 2\mathrm{i}[\sigma_3, \Psi_1]\Psi_1 \quad . \end{aligned} \tag{4.1.9}$$

The proof of Lemma 4.1 is straightforward. It is sufficient to substitute the asymptotic series for Ψ, Ψ_x, Ψ_t into the LHS of (4.1.8) and compute the terms non-vanishing at infinity.

Lemma 4.2. Let $\Psi(\lambda)$, satisfying the conditions of Lemma 4.1, be an exact solution of the system (4.1.6) with the matrix valued polynomials $U(\lambda)$, $V(\lambda)$ defined by (4.1.9). Then the matrices U and V are of the form (4.1.5) with y and y^* proportional to non-diagonal elements of the matrix $\Psi_1(x,t)$

$$y(x,t) = 2(\Psi_1)_{12} \, , \quad y^*(x,t) = 2(\Psi_1)_{21} \quad . \tag{4.1.10}$$

The functions y and y^* form the solution of (4.1.1).

Proof. The first part of the statement of Lemma 4.2 may be proved by checking the identity

$$2\mathrm{i}[\sigma_3, \Psi_2] - 2\mathrm{i}[\sigma_3, \Psi_1]\Psi_1 = \begin{pmatrix} -\mathrm{i}yy^* & -y_x \\ -y_x^* & \mathrm{i}yy^* \end{pmatrix} \quad . \tag{4.1.11}$$

This can be done by substituting (4.1.7) in the first equation of (4.1.6) and equating the coefficients of λ^{-1} on both sides. Thus we get the relation

$$\mathrm{i}[\sigma_3, \Psi_2] - \mathrm{i}[\sigma_3, \Psi_1]\Psi_1 = -\Psi_{1x} \quad .$$

Now it turns out that

$$(2\mathrm{i}[\sigma_3, \Psi_2] - 2\mathrm{i}[\sigma_3, \Psi_1]\Psi_1)_{\mathrm{OD}} = \begin{pmatrix} 0 & -y_x \\ -y_x^* & 0 \end{pmatrix} \quad ,$$

where the subscript OD denotes the off-diagonal part of the matrix on the left-hand-side.

Taking into account the equality

$$(2\mathrm{i}[\sigma_3, \Psi_2] - 2\mathrm{i}[\sigma_3, \Psi_1]\Psi_1)_{\mathrm{D}} = (-2\mathrm{i}[\sigma_3, \Psi_1]\Psi_1)_{\mathrm{D}} = \\ = \begin{pmatrix} -\mathrm{i}yy^* & 0 \\ 0 & \mathrm{i}yy^* \end{pmatrix} \quad ,$$

where the subscript D denotes the diagonal part of the matrix, and adding it with the previous one, we get (4.1.11).

The second part of the statement of Lemma 4.2 may be checked even more simply. Cross differentiation of (4.1.6) gives

$$\{U_t(\lambda) - V_x(\lambda) - [V(\lambda), U(\lambda)]\}\, \Psi(\lambda) = 0 \quad ,$$

for all $\lambda \in \mathbb{C}$.

In a neighborhood of infinity, the matrix Ψ is invertible. This follows from the validity of the expansion (4.1.7). Right multiplication of both sides of the last equation on the matrix Ψ^{-1} shows that the zero curvature equation holds in the neighborhood of infinity. Taking into account the polynomial dependence of U and V on λ, we conclude that the same equation is true for all $\lambda \in \mathbb{C}$, Q.E.D.

Remark 4.3. The constraint (4.1.3) would be realized if we impose on Ψ the complementary restrictions

$$\Psi(\lambda) = \sigma_1 \bar{\Psi}(\bar{\lambda})\, \sigma_1(\lambda)\,, \qquad (\sigma = 1) \quad , \tag{4.1.12}$$

$$\Psi(\lambda) = \sigma_2 \bar{\Psi}(\bar{\lambda})\, \sigma_2(\lambda)\,, \qquad (\sigma = -1) \quad , \tag{4.1.13}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad ,$$

and $\sigma_{1,2}(\lambda)$ are some invertible t- and x-independent 2×2 matrices. We shall discuss this remark more thoroughly in Sect 4.3.

The next step of the finite-gap integration of the NS system is an explicit construction of the function $\Psi(\lambda)$ based on the vector valued Baker-Akhiezer function. Let X be an arbitrary hyperelliptic surface of genus $g \geq 1$ defined by

$$\mu^2 = \prod_{j=1}^{2g+2}(\lambda - E_j) \equiv P_{2g+2}(\lambda) , \qquad E_j \in \mathbb{C},\ E_j \neq E_k,\ j \neq k \quad . \tag{4.1.14}$$

As previously, we use capital letters $P, Q, \ldots,$ to denote the points lying on X, which correspond to different pairs (λ, μ) of complex numbers satisfying (4.1.14). Recall also that the standard projection $\pi : X \to \mathbb{CP}^1$ is defined by

$$\pi(P) = \lambda , \qquad P = (\mu, \lambda) \quad .$$

The projection π defines X as a two-sheeted covering of $\mathbb{CP}^1$. There are exactly two points $\infty^\pm \in X$, with the property $\pi(\infty^\pm) = \infty \in \mathbb{CP}^1$:

$$P \to \infty^\pm \Leftrightarrow \lambda \to \infty , \quad \mu \to \pm\lambda^{g+1} \quad .$$

We now introduce some basic objects, which are related to the curve (4.1.14).

1. Let (a_i, b_i) be some canonical basis of oriented cycles in $H_1(X, \mathbb{Z})$, with the standard intersection matrix J

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad ,$$

i.e., the intersection indices of the cycles a_k, b_k are

$$a_i \circ b_j = \delta_{ij} = -b_j \circ a_i , \qquad a_j \circ a_k = b_j \circ b_k = 0 \quad .$$

2. The differentials $\nu_k = \mu^{-1}\lambda^{g-k}d\lambda$, $k = 1, \ldots, g$ form a basis in the space of holomorphic 1-forms defined on X. Their linear combinations ω_j

$$\omega_j = \sum_{k=1}^{g} c_{jk}\nu_k \quad ,$$

satisfying the conditions

$$\oint_{a_k} \omega_j = 2\pi \mathrm{i}\delta_{kj} \quad ,$$

will be referred to as the normalized basis of holomorphic differentials. The normalization condition for the given choice of a_k defines the coefficients c_{jk} uniquely.

3. The matrix of the B-periods of the curve X and the associated theta-function are defined as follows:

$$B_{jk} = \oint_{b_k} \omega_j \quad ,$$

$$\theta(\boldsymbol{p}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^g} \exp\{2^{-1}\langle B\boldsymbol{m}, \boldsymbol{m}\rangle + \langle \boldsymbol{p}, \boldsymbol{m}\rangle\} \quad ,$$

$$\langle \boldsymbol{p}, \boldsymbol{q}\rangle = p_1q_1 + \cdots + p_gq_g , \quad \boldsymbol{p} \in \mathbb{C}^g \quad .$$

4. The Abelian integrals $\Omega_1(P)$, $\Omega_2(P)$, $\Omega_3(P)$, $P \in X$ which are fixed by the following conditions:

a) $\oint_{a_i} d\Omega_j = 0, \quad \forall i,j \quad ,$

b) $\Omega_1(P) = \pm(\lambda + O(1)) \,, \quad P \to \infty^{\pm} \quad ,$
$\Omega_2(P) = \pm(2\lambda^2 + O(1)) \,, \quad P \to \infty^{\pm} \quad ,$
$\Omega_3(P) = \pm(\log\lambda + O(1)) \,, \quad P \to \infty^{\pm} \,, \quad \lambda = \pi(P) \quad .$

c) $\Omega_j(P)$ have no singularities at the points different from $\infty^{\pm}$.

5. An arbitrary divisor $\mathcal{D}$ with $\deg\mathcal{D} = g$ of general position, i.e.,

$$\mathcal{D} = \sum_{i=1}^{g} P_j \,, \quad \pi(P_j) \neq E_j \,, \quad j \neq k \Rightarrow \pi(P_j) \neq \pi(P_k) \quad .$$

The vector-valued Baker-Akhiezer function $\psi(P,x,t) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is uniquely determined by two conditions. The first of these conditions describes the analytic structure of ψ on $X/\{\infty^{\pm}\}$:

I. $\psi(P)$ is meromorphic on $X/\{\infty^{\pm}\}$. Its divisor of poles coincides with $\mathcal{D}$.

The second condition describes the asymptotic behavior of ψ at $\infty^{\pm}$, and shows that ψ has essential singularities at $\infty^{\pm}$.

$$\text{II.} \quad \begin{aligned} &\psi(P) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\lambda^{-1})\right] \exp(-\mathrm{i}\lambda x - 2\mathrm{i}\lambda^2 t) \quad , \\ &P \to \infty^- \,, \qquad \lambda = \pi(P) \quad , \\ &\psi(P) = \alpha\lambda \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\lambda^{-1})\right] \exp(\mathrm{i}\lambda x + 2\mathrm{i}\lambda^2 t) \quad , \\ &P \to \infty^+ \,, \qquad \lambda = \pi(P) \,, \qquad \alpha \in \mathbb{C} \quad . \end{aligned} \tag{4.1.15}$$

As in the scalar case (Chap. 3) $\psi(P)$ is uniquely determined by the conditions I and II and may be explicitly constructed by the formulas

$$
\begin{aligned}
\psi_1(P) =& \frac{\theta\left(\int_{\infty^-}^{P} \omega + \mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t - \boldsymbol{D}\right)\theta(\boldsymbol{D})}{\theta\left(\int_{\infty^-}^{P} \omega - \boldsymbol{D}\right)\theta\left(\mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t - \boldsymbol{D}\right)} \\
&\times \exp\left\{\mathrm{i}x\Omega_1(P) + \mathrm{i}t\Omega_2(P) - \frac{\mathrm{i}}{2}Ex + \frac{\mathrm{i}}{2}Nt\right\} \quad , \\
\psi_2(P) =& \alpha\sqrt{\omega_0}\frac{\theta\left(\int_{\infty^-}^{P} \omega + \mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t - \boldsymbol{D} - \boldsymbol{r}\right)\theta(\boldsymbol{D} - \boldsymbol{r})}{\theta\left(\int_{\infty^-}^{P} \omega - \boldsymbol{D}\right)\theta\left(\mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t - \boldsymbol{D}\right)} \\
&\times \exp\left\{\mathrm{i}x\Omega_1(P) + \mathrm{i}t\Omega_2(P) + \frac{\mathrm{i}}{2}Ex - \frac{\mathrm{i}}{2}Nt + \Omega_3\right\} \quad .
\end{aligned}
\tag{4.1.16}
$$

The vector-valued parameters appearing in (4.1.16) are defined as follows:

$$
\begin{aligned}
&\omega = (\omega_1, \ldots, \omega_g) \quad , \\
&\boldsymbol{V} = (V_1, \ldots, V_g) \, , \quad V_j = \int_{b_j} d\Omega_1 \quad , \\
&\boldsymbol{W} = (W_1, \ldots, W_g) \, , \quad W_j = \int_{b_j} d\Omega_2 \quad , \\
&\boldsymbol{r} = \int_{\infty^-}^{\infty^+} \omega \, , \quad \boldsymbol{D} = \sum_{j=1}^{g} \int_{\infty^-}^{P_j} \omega + \boldsymbol{K}, \quad \mathcal{D} = \sum_{j=1}^{g} P_j \quad , \\
&K_j = \pi\mathrm{i} + \frac{1}{2}B_{jj} - \frac{1}{2\pi\mathrm{i}} \sum_{k \neq j} \int_{a_k} \left[\int_{\infty^-}^{P} \omega_j\right] \omega_k(P) \quad .
\end{aligned}
$$

The quantities E, N, and ω_0 are determined by the second terms of the asymptotic expansions of the integrals $\Omega_i(P)$ at the points $\infty^\pm$:

$$
\begin{aligned}
&\Omega_1(P) = \pm(\lambda - E/2 + o(1)) \, , \quad P \to \infty^\pm \quad , \\
&\Omega_2(P) = \pm(2\lambda^2 + N/2 + o(1)) \, , \quad P \to \infty^\pm \quad , \\
&\Omega_3(P) = \pm(\log\lambda - (1/2)\log\omega_0 + o(1)) \, , \quad P \to \infty^\pm \quad .
\end{aligned}
$$

Due to the fact that the divisor $\mathcal{D}$ is in a general position, the vector $\boldsymbol{D} \in J(X)$ is also in a general position. The vector $\boldsymbol{D}$ may be taken almost arbitrarily as a defining element in the construction of ψ instead of $\mathcal{D}$. The term "general position" must then be understood in that sense that the associated Riemann theta function $\theta(\boldsymbol{U}(P) - \boldsymbol{D})$, $P \in X$ does not vanish identically.

The proof of formulas (4.1.16) for $\psi_1(P)$ is exactly the same as for the scalar case (compare with the proof of the formulas (2.7.11) in Chap. 2). The formulas for ψ_2 may be proved by taking into account the relation (2.4.14) between B-periods of the integrals of the third kind and holomorphic differentials. When applied to the integral Ω_3, this relation reads

$$\int_{b_j} d\Omega_3 = -\int_{\infty^-}^{\infty^+} \omega_j = -r_j \quad .$$

This relation makes it possible to verify that $\psi_2(P)$ is also a single-valued function on X. The other steps in the proof are also identical to the considerations performed in the scalar case.

Remark 4.4. Formulas (2.4.13) from Chap. 2 lead immediately to an alternative description of the vectors $\boldsymbol{V}$, $\boldsymbol{W}$ and the constant E:

$$\begin{aligned} &V_j = 2c_{j1} \ , \quad W_j = 4(c_{j2} + c_{j1}c/2) \quad , \\ &E = c - \frac{1}{\pi \mathrm{i}} \sum_{j=1}^{g} \int_{a_j} \lambda\omega_j \ , \quad c = \sum_{j=1}^{2g+2} E_j \quad . \end{aligned} \tag{4.1.17}$$

We fix some simple connected neighborhood U of the point $\lambda = \infty$ on $\mathbb{CP}^1$ which has no branch points. Then for each $\lambda \in U, \pi^{-1}(\lambda)$ contains exactly two points denoted by $P^\pm \in X$ so that $P^\pm \to \infty^\pm$ when $\lambda \to \infty$. For $\lambda \in U$ the matrix function

$$\Psi(\lambda) = \left(\psi(P^+), \psi(P^-)\right) \tag{4.1.18}$$

is correctly defined.

Now, let us check that the $\Psi(\lambda)$ defined by (4.1.18) satisfies all requirements of Lemmas 4.1 and 4.2. By virtue of (4.1.15), $\Psi(\lambda)$ has the following asymptotic behavior at infinity:

$$\Psi(\lambda) = \left(I + \sum_{k=1}^{\infty} \Psi_k \lambda^{-k}\right) \exp\left\{-\mathrm{i}\lambda x\sigma_3 - 2\mathrm{i}\lambda^2 t\sigma_3\right\} \begin{pmatrix} 0 & 1 \\ \alpha\lambda & 0 \end{pmatrix} \quad . \tag{4.1.19}$$

It is possible to differentiate the asymptotic expansion (4.1.19) in any order with respect to the variables x and t by virtue of the analytical structure of the function Ψ, given by the explicit formulas (4.1.16). Furthermore, this expansion is uniformly convergent with respect to x and t in an arbitrary compact domain. Hence, $\Psi(\lambda)$ fulfills all conditions of Lemma 4.1. Now, considering the first two terms of the expansion (4.1.19) we can calculate the matrix polynomials $U(\lambda)$ and $V(\lambda)$ in accordance with (4.1.9). As we verified, Lemma 4.1 holds for the matrix function $\Psi(\lambda)$ defined by (4.1.18). Therefore the estimates

$$\begin{aligned} &\Psi_x\Psi^{-1}(\lambda) = U(\lambda) + O(1/\lambda) \quad , \\ &\Psi_t\Psi^{-1}(\lambda) = V(\lambda) + O(1/\lambda) \ , \quad \lambda \to \infty \end{aligned} \tag{4.1.20}$$

hold for the derivatives of Ψ.

The vector functions $\boldsymbol{f}_{1,2}$

$$\boldsymbol{f}_1(P) = \boldsymbol{\psi}_x(P) - U(\lambda)\boldsymbol{\psi}(P) \quad ,$$
$$\boldsymbol{f}_2(P) = \boldsymbol{\psi}_t(P) - V(\lambda)\boldsymbol{\psi}(P) \,, \quad \lambda = \pi(P)$$

are meromorphic on $X \setminus \{\infty^{\pm}\}$, their divisor of poles equals $\mathcal{D}$. Let us study the asymptotic behavior of $\boldsymbol{f}_{1,2}$ when P tends to $\infty^{\pm}$. We consider first $\boldsymbol{f}_1(P)$. Its asymptotics at ∞^+ clearly coincides with that of the first column of the matrix function $F_1(\lambda) = \Psi_x(\lambda) - U(\lambda)\Psi(\lambda)$. For $F_1(\lambda)$ we get from (4.1.20)

$$F_1(\lambda) = \left[\Psi_x\Psi^{-1}(\lambda) - U(\lambda)\right]\Psi(\lambda) =$$
$$= O(1/\lambda)\begin{pmatrix} 0 & \exp(-i\lambda x - 2i\lambda^2 t) \\ \alpha\lambda\exp(i\lambda x + 2i\lambda^2 t) & 0 \end{pmatrix} \; (\lambda \to \infty) \quad .$$

In other words, we have

$$\boldsymbol{f}_1(P) = O(1)\exp(\mathrm{i}\lambda x + 2\mathrm{i}\lambda^2 t) \,, \quad P \to \infty^+ \quad ,$$
$$\boldsymbol{f}_1(P) = o(1)\exp(-\mathrm{i}\lambda x - 2i\lambda^2 t) \,, \quad P \to \infty^- \quad .$$

Looking similarly at $\Psi_t(\lambda) - V(\lambda)\Psi(\lambda)$ we get the same estimates for $\boldsymbol{f}_2(P)$. So in complete analogy with the scalar case all requirements of Theorem 2.24 are valid for each component of the vector function $\boldsymbol{f}_{1,2}$. Taking into account Corollary 2.26 we conclude that

$$\boldsymbol{f}_1(P) = \boldsymbol{f}_2(P) \equiv 0 \,.$$

The equality $\pi(P^+) = \pi(P^-)$ enables us to rewrite the last equation in the form

$$\begin{aligned} \Psi_x(\lambda) - U(\lambda)\Psi(\lambda) = 0 \quad , \\ \Psi_t(\lambda) - V(\lambda)\Psi(\lambda) = 0 \quad . \end{aligned} \tag{4.1.21}$$

and hence all requirements of Lemma 4.2 are fulfilled.

Remark 4.5. One can find some intrinsic motivation to use the tools of algebraic geometry in the proof of (4.1.21) for Ψ. The background for this motivation lies in the fact that the concept of the non-special divisor gives a simple way to transform the asymptotic estimates similar to (4.1.20) into exact equations of the type (4.1.21). Together with the hypothesis of the possibility of the analytical continuation of the matrix elements of Ψ on the compact Riemann surface, completed by exact formulas involving multi-dimensional theta functions, this allows the construction of exactly solvable linear matrix problems. The source of this is, in principle, absolutely independent from the preliminary study of periodic spectral problems, although it leads quite naturally to the consideration of all objects which historically emerged as a product of studies of the periodic problems for the Hill and Dirac operators.

To complete the solution of the NS system within the algebro-geometric approach, we need to calculate explicitly the matrix coefficient Ψ_1 in (4.1.19) starting from exact formula (4.1.18) for Ψ, and then refer to (4.1.10). Finally we get the following solutions of the NS system:

$$
\begin{aligned}
y(x,t) &= A\frac{\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D}+\boldsymbol{r})}{\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D})}\exp(-\mathrm{i}Ex+\mathrm{i}Nt) \quad , \\
y^*(x,t) &= \frac{4\omega_0}{A}\frac{\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D}-\boldsymbol{r})}{\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D})}\exp(\mathrm{i}Ex-\mathrm{i}Nt) \quad , \\
A &= \frac{2\theta(\boldsymbol{D})}{\alpha\theta(\boldsymbol{D}-\boldsymbol{r})} \quad .
\end{aligned}
\tag{4.1.22}
$$

Equation (4.1.22) describes a family of exact finite-gap solutions of (4.1.1) depending on $3g+3$ complex-valued parameters. These parameters are $2g+2$ branch points $E_j \in X$, a g-dimensional vector $\boldsymbol{D} \in J(X)$, (or a divisor $\mathcal{D}$), and a complex number A, $A \neq 0$.

4.2 Finite-Gap Solutions of the Sine-Gordon Equation

The zero curvature representation of the sine-Gordon equation

$$
v_{xt} = -4\sin v \tag{4.2.1.}
$$

known since 1973 [4.2-4] is of the form (4.1.4) with the matrices $U(\lambda)$ and $V(\lambda)$ defined by the formulas

$$
\begin{aligned}
U(\lambda) &= -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & -\mathrm{i}v_x/2 \\ -\mathrm{i}v_x/2 & 0 \end{pmatrix} \quad , \\
V(\lambda) &= \frac{1}{\lambda}\begin{pmatrix} -\mathrm{i}\cos v & \sin v \\ -\sin v & \mathrm{i}\cos v \end{pmatrix} \quad .
\end{aligned}
\tag{4.2.2}
$$

First we formulate propositions similar to Lemmas 4.1 and 2 thus fixing the analytical properties of the corresponding matrix Ψ-function:

Let Σ_0 be a punctured neighborhood of the point $\lambda = 0$ on $\mathbb{CP}^1$ and Σ_∞ be a punctured neighborhood of the point $\lambda = \infty$; both of them are invariant with respect to the action of the involution $\lambda \to -\lambda$. On the union $\Sigma_0 \cup \Sigma_\infty$ a 2×2 matrix valued Ψ-function is defined. We suppose that Ψ smoothly depends on the complementary parameters x, t ($\Psi(\lambda) \equiv \Psi(\lambda, x, t)$) such that

(1.) The following asymptotic expansion holds:

$$
\begin{aligned}
\Psi(\lambda, x, t) &= \left(I + \sum_{k=1}^{\infty} \Psi_k(x,t)\lambda^{-k} \right) \\
&\qquad \times \exp(-\mathrm{i}x\lambda\sigma_3)\, C(\lambda)\,, \quad \lambda \to \infty \quad , \\
\Psi(\lambda, x, t) &= \Phi_0(x,t) \left(I + \sum_{k=1}^{\infty} \Phi_k(x,t)\lambda^{k} \right) \\
&\qquad \times \exp\left\{ \frac{-\mathrm{i}t}{\lambda}\sigma_3 \right\} D(\lambda)\,, \quad \lambda \to 0 \quad ,
\end{aligned}
\tag{4.2.3}
$$

(2.) We have the following reduction requirement:

$$\Psi(\lambda) = \sigma_1 \Psi(-\lambda)\, \sigma(\lambda) \quad .$$

In (1)-(2) $\sigma(\lambda)$, $C(\lambda)$, $D(\lambda)$ are invertible matrices independent of x and t.

Lemma 4.6. The function Ψ with the properties described above satisfies the following asymptotic estimates:

$$
\begin{aligned}
\Psi_x \Psi^{-1}(\lambda) &= U(\lambda) + O(1/\lambda)\,, \quad \lambda \to \infty \quad , \\
\Psi_t \Psi^{-1}(\lambda) &= V(\lambda) + O(1)\,, \quad \lambda \to 0 \quad .
\end{aligned}
\tag{4.2.4}
$$

with

$$
\begin{aligned}
U(\lambda) &= -\mathrm{i}\lambda\sigma_3 + \mathrm{i}[\sigma_3, \Psi_1] \quad , \\
V(\lambda) &= \frac{1}{\lambda} V_1\,, \quad V_1 = -\mathrm{i}\Phi_0\sigma_3\Phi_0^{-1} \quad ,
\end{aligned}
\tag{4.2.5}
$$

and the identities

$$
\begin{aligned}
U(\lambda) &= \sigma_1 U(-\lambda)\sigma_1 \quad , \\
V(\lambda) &= \sigma_1 V(-\lambda)\sigma_1 \quad .
\end{aligned}
\tag{4.2.6}
$$

Proof. Substituting the expansions (4.2.3) into left-hand-side of (4.2.4) we see that (4.2.4) is true. The identities (4.2.6) are the direct consequences of the condition 2 and of the uniqueness of the asymptotic expansions (4.2.4).

In addition to the conditions of Lemma 4.6, let the function Ψ be a solution of the linear equations

$$
\begin{aligned}
\Psi_x(\lambda) &= U(\lambda)\Psi(\lambda) \quad , \\
\Psi_t(\lambda) &= V(\lambda)\Psi(\lambda) \quad ,
\end{aligned}
\tag{4.2.7}
$$

with matrix coefficients defined by (4.2.5). Then the following Lemma holds:

Lemma 4.7. Under the conditions formulated above the matrices $U(\lambda)$ and $V(\lambda)$ may be represented in the form (4.2.2) with

$$v(x,t) = \arcsin\,(\Psi_1(x,t))_{12}$$

and the (generically complex valued) function v so defined satisfies the sine-Gordon equation (4.2.1).

Proof. Substituting the first of (4.2.3) into the second equation of the system (4.2.7) we get

$$V_1 = \Psi_{1t} \quad .$$

Therefore, introducing the functions $y(x,t) = 2\,(\Psi_1(x,t))_{12}$, $y^*(x,t) = 2\,(\Psi_1(x,t))_{21}$ we get the relations

$$(V_1)_{12} = \frac{y_t}{2}\,, \quad (V_1)_{21} = \frac{y_t^*}{2} \quad . \tag{4.2.8}$$

From formulas (4.2.5) we have the relations

$$\operatorname{Tr} V_1 = 0\,, \quad \det V_1 = 1 \quad ,$$

which together with (4.2.8) enable us to represent the matrix V_1 in the form

$$V_1 = \begin{pmatrix} -\mathrm{i}\sqrt{1+(1/4)y_t y_t^*} & y_t/2 \\ y_t^*/2 & \mathrm{i}\sqrt{1+(1/4)y_t y_t^*} \end{pmatrix}$$

The reduction identities (4.2.6) mean that the equalities $y = -y^*$ are satisfied and hence it is possible to represent the matrices $U(\lambda)$ and $V(\lambda)$ in (4.2.7) in the form

$$\begin{aligned} U(\lambda) &= -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & \mathrm{i}y \\ \mathrm{i}y & 0 \end{pmatrix} \quad , \\ V(\lambda) &= \frac{1}{\lambda}\begin{pmatrix} -\mathrm{i}\cos v & \sin v \\ -\sin v & \mathrm{i}\cos v \end{pmatrix} \quad , \\ v &= \arcsin\,(y_t/2) \quad . \end{aligned} \tag{4.2.9}$$

To complete the proof it is sufficient to notice that the equality $y = -v_x/2$ follows from the zero curvature equation (4.1.4) for the matrices (4.2.9). The proof of the zero curvature equation itself is absolutely analogous to the one of Lemma 4.2.

Remark 4.8. The way to isolate real valued solutions of the sine-Gordon equation (4.2.1) will be discussed in Sect. 4.3. Here we restrict ourselves to the remark that as in the case of NS the reality of v is equivalent to the reduction restriction (4.1.13) which is complementary to (4.2.6).

Now, we turn to the explicit construction of $\Psi(\lambda)$ by the methods of algebraic geometry. The necessity of satisfying the reduction requirement (2.) introduces in the corresponding construction some new aspects with respect to the case of the NS.[1]

Consider the Riemann surface $\overset{\circ}{X}$ of genus $g \geq 1$, generated by the equation

$$\mu^2 = \overset{\circ}{\lambda} \prod_{j=1}^{2g} (\overset{\circ}{\lambda} - E_j) \quad , \tag{4.2.10}$$
$$E_j \in \mathbb{C}\,, \quad E_j \neq E_k\,, \quad j \neq k \quad .$$

The points of $\overset{\circ}{X}$ and associated two-sheeted covering of the $(\overset{\circ}{\lambda})$-plane will be denoted by $\overset{\circ}{P}$ and $\overset{\circ}{\pi}$:

$$\overset{\circ}{P} = (\mu, \overset{\circ}{\lambda}) \quad ,$$
$$\overset{\circ}{\pi} : \overset{\circ}{X} \to \mathbb{CP}^1\,, \quad \overset{\circ}{\pi}(\overset{\circ}{P}) = \overset{\circ}{\lambda} \quad .$$

The points E_j, 0, ∞, i.e., the branch points of the covering $\overset{\circ}{\pi}$ are, as usual, identified with their projections on the $(\overset{\circ}{\lambda})$-plane. Now, fix the basis of a- and b-cycles, to which we have associated a normalized basis of the holomorphic differentials ω_j and Riemann theta function defined by the matrix of the B-periods: $B_{jk} = \int_{b_j} \omega_k$. We construct on $\overset{\circ}{X}$ a cycle $\mathcal{L}$ such that the points $\{0, \infty\}$, and $\{E_j\}$ lie on different parts of $\mathbb{CP}^1$ divided by $\overset{\circ}{\pi}(\mathcal{L})$ and that the equality

$$\mathcal{L} = \sum_{i=1}^{g} \delta_i a_i\,, \quad \delta_i = \pm 1, 0 \tag{4.2.11}$$

in $H_1(\overset{\circ}{X}, \mathbb{Z})$ is satisfied. For different $\mathcal{L}$ see Remarks 4.10, 12.

Equality (4.2.11) also imposes some restrictions on the choice of the cycles a_k, b_k which are nevertheless not very stringent (see Remark 4.3). On the surface $\overset{\circ}{X}$, cut along the cycle $\mathcal{L}$, it is possible to determine (nonuniquely) a single-valued branch of the function $\sqrt{\overset{\circ}{\pi}}$. Fix in some way such a branch and denote it by $\lambda(\overset{\circ}{P})$. Its boundary values on different sides $\mathcal{L}^+$ and $\mathcal{L}^-$ of $\mathcal{L}$ satisfy the equality

$$\lambda\big|_{\mathcal{L}^+} = -\lambda\big|_{\mathcal{L}^-} \,.$$

[1] In our consideration of the S-G case it is possible to modify the initial U-V pair so that the necessity of the indicated reduction is removed [4.5]. But we do not follow this simplest way, because there are many similar situations, particularly the ones treated in this volume in the chapter devoted to tops, where it is impossible to do without such a reduction simply by the choice of appropriate U-V pairs.

Now, let us cut $\overset{\circ}{X}$ along $\mathcal{L}$ and join two copies of the obtained surface with a cut along $\mathcal{L}$ in order to get the Riemann surface X on which the function λ becomes single-valued if it is continued on the second copy of $\overset{\circ}{X}$ with the change of the sign. Now we fix on $\overset{\circ}{X}$ the Abelian integrals of the second kind by the conditions:

(a.) $\int_{a_j} d\Omega_k = 0, \forall j = 1, \ldots, g\,, \quad k = 1, 2$
(b.) Ω_1 has a unique singularity, a pole at ∞, and Ω_2 has a unique pole at the point 0 with the principal parts given by

$$\Omega_1(\overset{\circ}{P}) = \lambda + o(1)\,, \quad \overset{\circ}{P} \to \infty \quad ,$$
$$\Omega_2(\overset{\circ}{P}) = \lambda^{-1} + o(1)\,, \quad \overset{\circ}{P} \to 0 \quad ,$$

where λ is a value of the fixed branch of $\lambda(\overset{\circ}{P})$.

Now, we fix an arbitrary non-special divisor $\mathcal{D}$ with $\deg \mathcal{D} = g$ and define the vector valued Baker-Akhiezer function $\psi(\overset{\circ}{P}) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of the S-G model by the conditions

(a.) $\psi(\overset{\circ}{P})$ is single valued and analytic on $\overset{\circ}{X}$ cut along $\mathcal{L}$,
(b.) in the neighborhoods of the points ∞ and 0, $\psi(\overset{\circ}{P})$ has an essential singularity described by the formulas

$$\psi(\overset{\circ}{P}) = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + o(1)\right] \exp(-ix\sqrt{\overset{\circ}{\lambda}}) \quad ,$$
$$\overset{\circ}{P} \to \infty\,, \quad \overset{\circ}{\lambda} = \overset{\circ}{\pi}(\overset{\circ}{P}) \quad . \tag{4.2.12}$$
$$\psi(\overset{\circ}{P}) = O(1) \exp(-it/\sqrt{\overset{\circ}{\lambda}}) \quad ,$$
$$\overset{\circ}{P} \to 0\,, \quad \overset{\circ}{\lambda} = \overset{\circ}{\pi}(\overset{\circ}{P}) \quad . \tag{4.2.13}$$

(c.) $\psi(\overset{\circ}{P})$ on $\overset{\circ}{X} / \{\mathcal{L}, \infty, 0\}$ is meromorphic with the divisors of poles equal to $\mathcal{D}$.
(d.) The boundary values $\psi^{\pm}$ of ψ on $\mathcal{L}$ are related by the equality

$$\psi^{+} = \sigma_3 \psi^{-} \,, \tag{4.2.14}$$

i.e., the first component of ψ has no jump on $\mathcal{L}$, and the second component changes the sign when its argument crosses $\mathcal{L}$. As usual $\psi(\overset{\circ}{P})$ may be constructed explicitly by means of the theta functions and Abelian integrals:

$$\psi_1(\overset{\circ}{P}) = \frac{\theta\left(\displaystyle\int_{\infty}^{\overset{\circ}{P}} \omega - \mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \boldsymbol{D}\right)}{\theta\left(\displaystyle\int_{\infty}^{\overset{\circ}{P}} \omega - \boldsymbol{D}\right)} \times \frac{\theta(\boldsymbol{D})}{\theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D}\right)} \exp\left\{-\mathrm{i}x\Omega_1(\overset{\circ}{P}) - \mathrm{i}t\Omega_2(\overset{\circ}{P})\right\} ,$$

$$\psi_2(\overset{\circ}{P}) = \frac{\theta\left(\displaystyle\int_{\infty}^{\overset{\circ}{P}} \omega - \mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \boldsymbol{D} - \mathrm{i}\pi\boldsymbol{\Delta}\right)}{\theta\left(\displaystyle\int_{\infty}^{\overset{\circ}{P}} \omega - \boldsymbol{D}\right)} \times \frac{\theta(\boldsymbol{D})}{\theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta}\right)} \exp\left\{-\mathrm{i}x\Omega_1(\overset{\circ}{P}) - \mathrm{i}t\Omega_2(\overset{\circ}{P})\right\} , \tag{4.2.15}$$

where as before

$$V_j = \int_{b_j} d\Omega_1, \quad W_j = \int_{b_j} d\Omega_2, \quad \boldsymbol{D} = \sum_{j=1}^{g} \int_{\infty}^{\overset{\circ}{P}_j} \omega + \boldsymbol{K}, \quad \mathcal{D} = \sum_{j=1}^{g} \overset{\circ}{P}_j \quad ,$$

and the vector $\boldsymbol{\Delta} \in \mathbb{Z}^g$ is defined by

$$\boldsymbol{\Delta} = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{L}} \omega, \quad \boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_g)$$

or, taking (4.2.11) into account by

$$\Delta_j = \delta_j \quad .$$

We need to comment on the second of the formulas (4.2.15) only. To check it we have to verify that the function $\psi(\overset{\circ}{P})$ given by the right-hand-side of (4.2.15) satisfies the conditions (a.) and (d.). The final condition is equivalent to the invariance of ψ_2 with respect to the movement of $\overset{\circ}{P}$ along b-cycles nonintersecting $\mathcal{L}$ and the change of the sign to the opposite value with respect to the movement along b-cycles intersecting $\mathcal{L}$. Denote the monodromy operator corresponding to one-pass along the cycle b_j by $\mathcal{M}_j$. Then from (4.2.15) we get

$$\mathcal{M}_j[\psi_2(\overset{\circ}{P})] = \frac{\theta\left(\int_{\infty}^{\overset{\circ}{P}} \omega - \mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \boldsymbol{D} - \mathrm{i}\pi\boldsymbol{\Delta} + B\boldsymbol{\Delta}_j\right)\theta(\boldsymbol{D})}{\theta\left(\int_{\infty}^{\overset{\circ}{P}} \omega - \boldsymbol{D} + B\boldsymbol{\Delta}_j\right)\theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta}\right)}$$

$$\times \exp\left\{-\mathrm{i}x\Omega_1(\overset{\circ}{P}) - \mathrm{i}t\Omega_2(\overset{\circ}{P}) - \mathrm{i}x\langle \boldsymbol{V}, \boldsymbol{\Delta}_j\rangle - \mathrm{i}t\langle \boldsymbol{W}, \boldsymbol{\Delta}_j\rangle\right\}$$

with

$$\boldsymbol{\Delta}_j = (\Delta_{j_1}, \ldots, \Delta_{j_g}), \quad \Delta_{ji} = \delta_{ji} \quad .$$

Now, from the transformation property (2.5.10) of the theta functions we obtain

$$\mathcal{M}_j[\psi_2(\overset{\circ}{P})] = \psi_2(\overset{\circ}{P})\exp(i\pi\langle\boldsymbol{\Delta}, \boldsymbol{\Delta}_j\rangle) = \psi_2(\overset{\circ}{P})\exp(i\pi\delta_j) \quad .$$

The indicated behavior of $\psi_2(\overset{\circ}{P})$ follows now from the fact that the cycle b_j intersects $\mathcal{L}$ if in (4.2.11) the associated δ_j is different from zero.

From the Riemann bilinear identities (2.4.13) for the periods of Abelian integrals we also have the following relations:

$$V_j = 2c_{j1}, \quad W_j = -2c_{jg}/\sqrt{p_0}, \quad p_0 = \prod_{j=1}^{2g} E_j \tag{4.2.16}$$

and the coefficients c_{jk} are defined by

$$\omega_j = \sum_k c_{jk} \frac{\overset{\circ}{\lambda}{}^{g-k}}{\mu} d\overset{\circ}{\lambda} \quad .$$

It has already been noted that $\psi_1(\overset{\circ}{P})$ is single-valued on $\overset{\circ}{X}$; $\psi_2(\overset{\circ}{P})$ is two-valued. On X both become single-valued and ψ_2 is extended over all X in the same manner as the function λ.

We denote the points of X by P,

$$P = (\overset{\circ}{P}, \lambda) = (\mu, \overset{\circ}{\lambda}, \lambda) \quad ,$$

and the function $\psi(\overset{\circ}{P})$, continued on X by $\psi(P)$. The surface X is of genus $2g - 1$. It may be viewed as a two-sheeted covering of $\overset{\circ}{X}$, or as a two-sheeted covering of the (λ)-plane, or as a four-sheeted covering of the $(\overset{\circ}{\lambda})$-plane. The related covering mappings are given by

$$\pi_1 : X \to \overset{\circ}{X}, \quad \pi_1(\mu, \overset{\circ}{\lambda}, \lambda) = (\mu, \overset{\circ}{\lambda}) \quad ,$$
$$\pi_2 : X \to \mathbb{CP}^1, \quad \pi_2(\mu, \overset{\circ}{\lambda}, \lambda) = \lambda \quad ,$$
$$\pi_3 : X \to \mathbb{CP}^1, \quad \pi_3(\mu, \overset{\circ}{\lambda}, \lambda) = \overset{\circ}{\lambda} \quad .$$

All these coverings except π_1 are ramified. The branch points of π_2 are the points projected to $\pm\sqrt{E_j}$. The branch points of π_3 are the points projected to E_j, 0, and ∞. The surface X may be described by

$$\mu'^2 = \left(\frac{\mu}{\lambda}\right)^2 = \prod_{j=1}^{2g}(\lambda^2 - E_j) \quad . \tag{4.2.17}$$

In contrast to $\overset{\circ}{X}$ on X there exist two infinity points and two zero points, each situated on its copy of $\overset{\circ}{X}$. We shall denote these points by $\infty^\pm$ and $0^\pm$ respectively. In the realization of (4.2.17) the points $\infty^\pm$ are exactly the same as in the previous example:

$$P \to \infty^\pm \Leftrightarrow \lambda \to \infty, \quad \mu' \simeq \pm\lambda^{2g}, \quad \mu = \lambda\mu' \simeq \pm\lambda^{2g+1} \quad .$$

Similarly,

$$P \to 0^\pm \Leftrightarrow \lambda \to 0, \quad \mu' \simeq \pm\sqrt{p_0}, \quad \mu = \lambda\mu' \simeq \pm\lambda\sqrt{p_0} \quad .$$

The function $\psi(P)$ defined on X yields, by virtue of (4.2.12-13), the following asymptotic estimates:

$$\begin{aligned} \psi(P) &= \left[\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} + o(1)\right] \exp(\mp\mathrm{i}x\lambda), \quad P \to \infty^\pm \quad , \\ \psi(P) &= \left[\begin{pmatrix} m \\ \pm n \end{pmatrix} + o(1)\right] \exp(\mp\mathrm{i}t/\lambda), \quad P \to 0^\pm \quad , \end{aligned} \tag{4.2.18}$$

where the quantities m and n may be easily calculated from the exact formulas (4.2.15):

$$\begin{aligned} m(x,t) &= \frac{\theta(\mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta})}{\theta(\mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t + \boldsymbol{D})} \frac{\theta(\boldsymbol{D})}{\theta(\boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta})} \quad , \\ n(x,t) &= \frac{\theta(\mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t + \boldsymbol{D})}{\theta(\mathrm{i}\boldsymbol{V}x + \mathrm{i}\boldsymbol{W}t + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta})} \frac{\theta(\boldsymbol{D})}{\theta(\boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta})} \quad . \end{aligned} \tag{4.2.19}$$

In deriving (4.2.19) we have taken into account that on the first copy of $\overset{\circ}{X}$ forming X we have

$$\int_\infty^0 \omega = \int_{\infty^+}^{0^+} \omega = \pm\frac{1}{2}\int_{\mathcal{L}} \omega = \pm\mathrm{i}\pi\boldsymbol{\Delta} \quad .$$

Now we define the matrix valued function $\Psi(\lambda)$ by

$$\begin{aligned} \Psi(\lambda) &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\psi(P^+), \psi(P^-)) \\ &= \frac{1}{2}\begin{pmatrix} \psi_1(P^+)+\psi_2(P^+) & , & \psi_1(P^-)+\psi_2(P^-) \\ \psi_1(P^+)-\psi_2(P^+) & , & \psi_1(P^-)-\psi_2(P^-) \end{pmatrix} \quad , \end{aligned} \tag{4.2.20}$$

where $\pi_2(P^\pm) = \lambda$, and

$$P^\pm \to \infty^\pm, \quad \lambda \to \infty \quad ,$$
$$P^\pm \to 0^\pm, \quad \lambda \to 0 \quad ,$$

see also (4.1.18). In contrast with the similar object in the NS case the function (4.2.20) is defined on the union of two simple connected domains – the neighborhood of 0 and ∞ on the λ-plane, invariant with respect to the involution $\lambda \to -\lambda$, and not containing the images of the branch points of the covering π_2. Let us show that the constructed function satisfies all conditions of the Lemmas 4.6 and 4.7. In terms of the matrix-valued function $\Psi(\lambda)$ the asymptotic relations (4.2.18) take the form

$$\begin{aligned} \Psi(\lambda) &= \left(I + \sum \Psi_k \lambda^{-k}\right) \exp(-\mathrm{i}x\lambda\sigma_3), \quad \lambda \to \infty \quad , \\ \Psi(\lambda) &= \Phi_0 \left(I + \sum \Phi_k \lambda^{k}\right) \exp(-\mathrm{i}t\sigma_3/\lambda), \quad \lambda \to 0 \quad , \\ \Phi_0 &= \frac{1}{2}\begin{pmatrix} m+n & , & m-n \\ m-n & , & m+n \end{pmatrix} \quad . \end{aligned} \tag{4.2.21}$$

As in the NS-case, the asymptotic series on the right-hand-side of (4.2.21) are uniformly convergent and may infinitely often be differentiated term by term with respect to x and t. So far we have checked the first conditions of Lemma 4.6 for $\Psi(\lambda)$.

On the surface X, the involution

$$P \to P^\tau : \quad (\mu, \overset{\circ}{\lambda}, \lambda) \to (\mu, \overset{\circ}{\lambda}, -\lambda)$$

interchanging the sheets of the covering π_1 is defined. By construction, the function $\psi(P)$ satisfies the relation

$$\psi(P^\tau) = \sigma_3 \psi(P) \quad .$$

In addition $(\infty^\pm)^\tau = \infty^\mp$. Hence, for $\Psi(\lambda)$ we get the relations

$$\begin{aligned} \Psi(-\lambda) &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\psi(P^{-\tau}), \psi(P^{+\tau})) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sigma_3 (\psi(P^-), \psi(P^+)) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sigma_3 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Psi(\lambda)\sigma_1 = \sigma_1 \Psi(\lambda) \sigma_1 \quad . \end{aligned} \tag{4.2.22}$$

Thus $\Psi(\lambda)$ satisfies the second condition of Lemma 4.6 with $\sigma(\lambda) \equiv \sigma_1$.

We now define the rational matrix valued functions $U(\lambda)$ and $V(\lambda)$ by

$$U(\lambda) = -\mathrm{i}\lambda\sigma_3 + \mathrm{i}[\sigma_3, \Psi_1] \quad ,$$
$$V(\lambda) = -\mathrm{i}\lambda^{-1}\Phi_0\sigma_3\Phi_0^{-1} \quad ,$$

so that the estimates

$$\begin{aligned} \Psi_x\Psi^{-1}(\lambda) &= U(\lambda) + O(1/\lambda), \ \lambda \to \infty \quad , \\ \Psi_t\Psi^{-1}(\lambda) &= V(\lambda) + O(1), \ \lambda \to 0 \quad , \end{aligned} \tag{4.2.23}$$

hold. Then we consider the vector functions

$$\begin{aligned} f_1(P) &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(\frac{\partial}{\partial x} - U(\lambda)\right) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \psi(P) \quad , \\ f_2(P) &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(\frac{\partial}{\partial t} - V(\lambda)\right) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \psi(P) \quad , \\ & \lambda = \pi_2(P) \quad . \end{aligned} \tag{4.2.24}$$

The formulas (4.2.24) define $f_{1,2}$ as single-valued vector functions on the surface X. It is easy to show that they are also single-valued on $\overset{\circ}{X}$, i.e., on the factor X/τ. Indeed, the identity (4.2.22) leads directly to the following relations:

$$\Psi_1 = -\sigma_1\Psi_1\sigma_1, \quad \Phi_0 = \sigma_1\Phi_0\sigma_1$$

for the coefficients of the expansions at the points ∞ and 0. Thus we conclude that $U(\lambda)$ and $V(\lambda)$ satisfy the relations

$$U(-\lambda) = \sigma_1 U(\lambda)\sigma_1, \quad V(-\lambda) = \sigma_1 V(\lambda)\sigma_1 \quad .$$

Combined with the relation

$$\sigma_3 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sigma_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad ,$$

these reductions prove the invariance of $f_{1,2}$ with respect to the action of τ and hence a single-valuedness of $f_{1,2}$ as defined on $\overset{\circ}{X}$.

Further consideration reproduces in general the considerations of Sect. 4.1. As functions depending on $\overset{\circ}{P}$, the $f_{1,2}$ are meromorphic and have poles at points which form a non-special divisor $\mathcal{D}$. To understand the asymptotic behavior of $f_{1,2}$ at $\overset{\circ}{P} \to \infty$ or $\overset{\circ}{P} \to 0$ it is sufficient to look at their behavior on the covering, i.e., to study the behavior of $f_{1,2}(P)$ for $P \to \infty^+$ or $P \to 0^+$. This reduces to the calculation of the asymptotics of the first columns of the matrices

$$F_1(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\Psi_x(\lambda) - U(\lambda)\Psi(\lambda))$$

and

$$F_2(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(\Psi_t(\lambda) - V(\lambda)\Psi(\lambda)\right) \quad ,$$

when $\lambda \to \infty$ and $\lambda \to 0$. By virtue of the estimates (4.2.21, 23) and the estimates

$$\Psi_x \Psi^{-1}(\lambda) = O(1) \quad ,$$
$$\Psi_t \Psi^{-1}(\lambda) = O(1/\lambda), \quad \lambda \to \infty \quad ,$$

following from (4.2.21) we get

$$F_{1,2}(\lambda) = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} O(1/\lambda) \begin{pmatrix} \exp(-\mathrm{i}x\lambda) & 0 \\ 0 & \exp(\mathrm{i}x\lambda) \end{pmatrix}, \quad \lambda \to \infty \quad ,$$
$$F_{1,2}(\lambda) = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} O(1) \begin{pmatrix} \exp(-\mathrm{i}t\lambda^{-1}) & 0 \\ 0 & \exp(\mathrm{i}t\lambda^{-1}) \end{pmatrix}, \quad \lambda \to 0 \quad .$$

Now, recalling that at $P \to \infty^+$, $\lambda = +\sqrt{\overset{\circ}{\lambda}}$, we see that the relations

$$\boldsymbol{f}_{1,2}(\overset{\circ}{P}) = \begin{pmatrix} o(1) \\ O(1) \end{pmatrix} \exp\{-\mathrm{i}x\sqrt{\overset{\circ}{\lambda}}\}, \quad \overset{\circ}{P} \to \infty \quad ,$$
$$\boldsymbol{f}_{1,2}(\overset{\circ}{P}) = \begin{pmatrix} O(1) \\ o(1) \end{pmatrix} \exp\{-\mathrm{i}t/\sqrt{\overset{\circ}{\lambda}}\}, \quad \overset{\circ}{P} \to 0 \quad ,$$

hold. The obtained information about the functions $\boldsymbol{f}_{1,2}$ implies

$$\boldsymbol{f}_1 \equiv 0, \quad \boldsymbol{f}_2 \equiv 0 \quad .$$

As in the NS case, we deduce that all conditions of Lemma 4.7 for $\Psi(\lambda)$ are satisfied.

Remark 4.9. The same a priori motivation of the described construction as in the NS system case may be repeated without difficulties.

Now for the calculation of the solution $v(x,t)$ of (4.2.1.) we use (4.2.5) showing that

$$\exp(-\mathrm{i}v) = m/n \quad .$$

From this relation and (4.2.19) we get the following exact representation for the finite-gap solution $v(x,t)$ of the sine-Gordon equation:

$$v(x,t) = 2\mathrm{i} \log \frac{\theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta}\right)}{\theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D}\right)} \tag{4.2.25}$$

Remark 4.10 a. Let us denote by α the path from ∞ to 0 which is one-half of $\mathcal{L}$. We chose $\mathcal{L}$ in such a way that α and $\pi\alpha$, where π is the hyperelliptic involution on $\overset{\circ}{X}$ comprise $\mathcal{L} = \alpha - \pi\alpha$. We could choose the cycle $\mathcal{L}$ in a different way (denote it by $\mathcal{L}\prime$) adding to $\mathcal{L}$ a small cycle around $\overset{\circ}{\lambda}= 0$. The intersection numbers of α with $\mathcal{L}$ and $\mathcal{L}\prime$ are $\langle\alpha, \mathcal{L}\rangle = 0$, $\langle\alpha, \mathcal{L}\prime\rangle = 1$. Replacing $\mathcal{L}$ by $\mathcal{L}\prime$ we change the sign of the local parameter at $\overset{\circ}{\lambda}= 0$. Therefore if $v(x,t)$ was a solution generated by $\mathcal{L}$ then $\mathcal{L}\prime$ determines the solution

$$v(x,-t)+\pi \ .$$

Remark 4.10 b. Formula (4.2.25) describes the solutions of the SG equation in light-cone variables. For the construction of the solution of the SG equation written in laboratory coordinates, i.e.,

$$u_{tt} - u_{xx} + \sin u = 0 \quad , \tag{4.2.26}$$

it is sufficient to put

$$u(x,t) = v\left(\frac{x+t}{4}, \frac{t-x}{4}\right) \quad . \tag{4.2.27}$$

4.3 Reality Conditions. Reduction of the NS System to the Nonlinear Schrödinger Equation

Reality conditions in the finite-gap integration are understood as reduction constraints related to the existence of the antiholomorphic involution of the "spectral" variable $\lambda : \lambda \to \overline{\lambda}$. In the NS and sine-Gordon cases considered below the "reality" conditions impose the constraints (4.1.12) and (4.1.13) on the solutions of the auxiliary linear system.

In contrast to the "constructive" reduction (2) (see Lemma 4.6) we can realize the conditions (4.1.12) and (4.1.13) only by the appropriate choice of the parameters E_j, $\boldsymbol{D}$ in the exact formulas (4.1.22) and (4.2.25)

4.3.1 NS Equation. Reduction $y^* = \overline{y}$

Let $E_j \in \mathbb{R}$ and $E_1 < E_2 < \ldots < E_{2g+1} < E_{2g+2}$. The associated Riemann surface X is real, i.e., X admits the anti-involution $\tau : (\mu, \lambda) \to (\overline{\mu}, \overline{\lambda})$. The basis (a_k, b_k) of $H_1(X)$ may be chosen so that

$$\tau(a_j) = a_j, \quad \tau(b_j) = -b_j \quad . \tag{4.3.1}$$

The last equalities are understood modulo cycles homological to zero.

An example of such a basis is shown in Fig. 4.1. The Riemann surface X is realized as a two-sheeted covering of the complex plane. The segments

$[E_{2l+1}, E_{2l+2}]$ are the transition lines from the upper to the lower sheet of the surface X. The dotted lines denote the parts of the cycles located on the lower sheet of the surface X. The infinity points ∞^+ and ∞^- are placed on the upper and lower sheets of X, respectively.

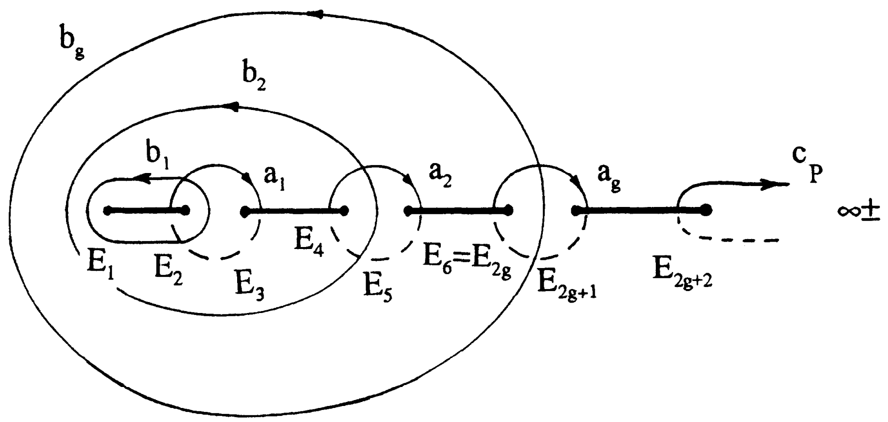

Fig. 4.1. The homology basis for the curve $\mu^2 = \prod_{j=1}^{2g+2}(\lambda - E_j)$ with $E_1 < E_2 < \cdots < E_{2g+1} < E_{2g+2}$.

The individual action τ^* of τ on the holomorphic differentials ν_j is depicted by

$$\tau^* \nu_j = \overline{\nu}_j$$

and hence the matrix A

$$A_{kj} = \int_{a_j} \nu_k$$

satisfies the condition $\operatorname{Im} A = 0$. The coefficients c_{jk} in $\omega_j = \sum_k c_{jk}\nu_k$ form the solution of the system

$$\sum c_{jk} A_{kl} = 2\pi \mathrm{i} \delta_{jl} \quad .$$

From the reality of the matrix A it follows that all c_{jk} must be purely imaginary, i.e., $\operatorname{Re} c_{jk} = 0$. Hence for the normalized differentials ω_j we get

$$\tau^* \omega_j = -\overline{\omega}_j \quad , \tag{4.3.2}$$

and consequently,

$$\overline{V}_j = -V_j , \quad \overline{W}_j = -W_j \quad . \tag{4.3.3}$$

In the case under consideration the matrix of the B-periods is real

$$\overline{B}_{jk} = \int_{b_j} \overline{\omega}_k = -\int_{b_j} \tau^* \omega_k = -\int_{\tau(b_j)} \omega_k = \int_{b_j} \omega_k = B_{jk} \quad . \tag{4.3.4}$$

The integration paths in the definition of the Abelian integrals r_j, ω_j do not intersect the cycles a_j, b_j. Particularly

$$r_j = \int_{\infty^-}^{\infty^+} \omega_j = \lim_{P\to\infty} \int_{c_P} \omega_j \quad ,$$

where the path of integration c_P is shown in Fig. 4.1. c_P is invariant with respect to the action of τ, i.e.,

$$\tau(c_P) = c_P$$

and hence the vector $\boldsymbol{r}$ is purely imaginary:

$$\overline{r}_j = -r_j \quad . \tag{4.3.5}$$

The integrals Ω_j allow the following representations:

$$\Omega_j(P) = \int_{E_{2g+2}}^{P} d\Omega_j \quad ,$$

$$d\Omega_1 = \frac{\lambda^{g+1} - \lambda^g c/2}{\mu}\, d\lambda + \sum_{j=1}^{g} c_j^1 \nu_j \quad ,$$

$$d\Omega_2 = 4\frac{\lambda^{g+2} - \lambda^{g+1}c/2 - d\lambda^g}{\mu}\, d\lambda + \sum_{j=1}^{g} c_j^2 \nu_j \quad ,$$

$$c = \sum_{j=1}^{2g+2} E_j, \quad d = \frac{1}{8}\left[\sum E_j^2 - 2\sum_{j<k} E_j E_k\right] \quad ,$$

$$d\Omega_3 = \frac{\lambda^g}{\mu}\, d\lambda + \sum_{j=1}^{g} c_j^3 \nu_j \quad .$$

The constants c_j^k are determined by the normalization conditions

$$\int_{a_j} d\Omega_k = 0\,, \quad k = 1, 2, 3; \quad j = 1, \ldots, g \quad .$$

Now in complete analogy with the proof of (4.3.2) we can find the transformation law for the differentials $d\Omega_k$ with respect to the action of τ^*:

$$\tau^*(d\Omega_k) = \overline{d\Omega_k} \quad .$$

The constants E, N, and ω_0 may be defined as follows:

$$\begin{aligned} E &= -\lim_{P\to\infty^+}\left[\int_{c_P} d\Omega_1 - 2\lambda\right], \quad \lambda = \pi(P) \quad , \\ N &= \lim_{P\to\infty^+}\left[\int_{c_P} d\Omega_2 - 4\lambda^2\right], \quad \lambda = \pi(P) \quad , \\ \log\omega_0 &= -\lim_{P\to\infty^+}\left[\int_{c_P} d\Omega_3 - 2\log\lambda\right], \quad \lambda = \pi(P) \quad . \end{aligned} \tag{4.3.6}$$

From the relations $\tau(c_P) = c_P$ and $\tau^*(d\Omega_k) = \overline{d\Omega_k}$ derived above it is now clear that the constants E, N, $\log \omega_0$ are real valued:

$$E = \overline{E}\ , \quad N = \overline{N}\ , \quad \log \omega_0 = \overline{\log \omega_0} \ \Leftrightarrow\ \omega_0 > 0 \quad . \tag{4.3.7}$$

For example, by virtue of the definition of E and the relation $\tau^*(P) = P$, we get

$$\begin{aligned}\overline{E} &= -\lim_{P\to\infty^+}\left[\int_{c_P} \overline{d\Omega_1} - 2\lambda\right] = -\lim_{P\to\infty^+}\left[\int_{c_P} \tau^* d\Omega_1 - 2\lambda\right] \\ &= -\lim_{P\to\infty^+}\left[\int_{\tau(c_P)=c_P} d\Omega_1 - 2\lambda\right] = E \quad .\end{aligned}$$

Taking into account (4.3.3-5, 7), the equality

$$y^*(x,t) = \overline{y(x,t)}$$

as applied to (4.1.22) then gives rise to the identity

$$|A|^2 = 4\omega_0 \frac{\theta\big(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \boldsymbol{D} - \boldsymbol{r}\big)\,\theta\big(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \overline{\boldsymbol{D}}\big)}{\theta\big(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \overline{\boldsymbol{D}} - \boldsymbol{r}\big)\,\theta\big(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) - \boldsymbol{D}\big)} \quad .$$

This identity is self-contained only under the condition

$$\overline{\boldsymbol{D}} = \boldsymbol{D} + 2\pi\mathrm{i}\boldsymbol{N} + B\boldsymbol{M}\ , \quad \boldsymbol{N},\, \boldsymbol{M} \in \mathbb{Z}^g \quad . \tag{4.3.8}$$

By virtue of (4.3.8) the representation of $|A|^2$ written above may be transformed to the following form:

$$|A|^2 = 4\omega_0 \exp\langle \boldsymbol{r},\, M\rangle \quad . \tag{4.3.9}$$

Taking into account the relations $\operatorname{Im} B = 0$, $\operatorname{Re} \boldsymbol{r} = 0$ one may insert $\boldsymbol{M} = 0$ into relation (4.3.8).

The resulting reduction constraints ("reality" conditions) in terms of the vector $\boldsymbol{D}$ and the number A take the form

$$\operatorname{Im} \boldsymbol{D} = \pi \boldsymbol{N}, \quad \boldsymbol{N} \in \mathbb{Z}^g \quad , \tag{4.3.10}$$

$$A = 2\sqrt{\omega_0}e^{\mathrm{i}\varphi} \quad . \tag{4.3.11}$$

Remark 4.12. The vector $\boldsymbol{D}$ is defined up to the addition of elements of the lattice $\Lambda = \{2\pi\mathrm{i}\boldsymbol{N} + B\boldsymbol{M},\ \boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^g\}$. As a consequence for the fixed numbers E_j the whole variety of the finite-gap solutions of the "repulsive" NS equation

$$\mathrm{i}y_t + y_{xx} - 2|y|^2 y = 0$$

constructed above splits into 2^g convex components. Each component is fixed by the choice of the vector [2] $\boldsymbol{N} \in \mathbb{Z}^g_{0,1}$ in (4.3.10). The component $\boldsymbol{N} = 0$ is of

[2] $\boldsymbol{N} \in \mathbb{Z}^g_{0,1}$ means that $N_j = 0$ or 1 for all j.

special importance. The related solutions of the NS equation are clearly smooth functions of space and time variables. This follows from the inequality

$$\exp\left\{\frac{1}{2}\langle B\boldsymbol{m},\boldsymbol{m}\rangle + \langle \mathrm{i}(\boldsymbol{V}x+\boldsymbol{W}t)-\boldsymbol{D},\ \boldsymbol{m}\rangle\right\} > 0$$
$$\forall \boldsymbol{m}\in \mathbb{Z}^g\ ,\quad x,t\in\mathbb{R}\quad .$$

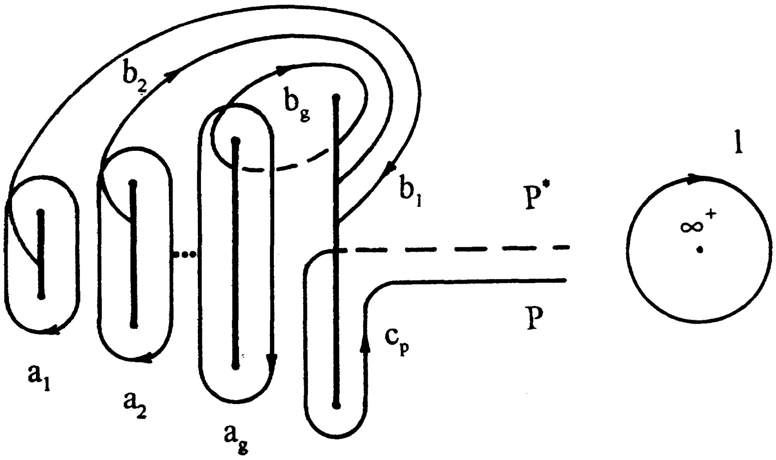

Fig. 4.2. The homology basis for the curve $\mu^2=\prod_{j=0}^{2g+2}(\lambda-E_j)$, $E_{2k+1}=\overline{E_{2k+2}}$, $k=0,\dots,g$.

4.3.2 NS Equation with Attraction. Reduction $y^*=-\overline{y}$

Let X be a hyperelliptic surface defined as above by a polynomial $P_{2g+2}(\lambda)$ with real coefficients with one difference: the zeros of the polynomial $P_{2g+2}(\lambda)$ are now complex valued and ordered as follows:

$$E_{2k+1}=\overline{E_{2k+2}}\ ,\quad k=0,\dots,g\quad .$$

The canonical basis of the cycles a_j, b_j is chosen (see Fig. 4.2.) to satisfy

$$\tau(a_j)=-a_j\ ,\quad \tau(b_j)=b_j+\sum_{k\neq j}a_k\ , \tag{4.3.12}$$

where τ stands for the same involution as above. The replacement for the conditions $y^*=\overline{y}$ by $y^*=-\overline{y}$ leads to a purely imaginary matrix A_{jk}, i.e., the coefficients c_{jk} turn out to be real. Consequently the equalities (4.3.2) and (4.3.3) are replaced by

$$\tau^*\omega_j=\overline{\omega}_j\quad , \tag{4.3.13}$$
$$\overline{V}_j=V_j\quad ,$$
$$\overline{W}_j=W_j\quad . \tag{4.3.14}$$

Next, we show that the structure of the B-matrix is also different from the repulsive case:

$$\overline{B}_{jk} = \int_{b_j} \overline{\omega}_k = \int_{b_j} \tau^* \omega_k = \int_{\tau(b_j)} \omega_k = \int_{b_j} \omega_k + \sum_{l \neq j} \int_{a_l} \omega_k$$
$$= B_{jk} + 2\pi \mathrm{i} \sum_{l \neq j} \delta_{kl} \quad ,$$

i.e.,

$$\overline{B} = B + 2\pi \mathrm{i} B_0 \ , \quad B_0 = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix} \quad . \tag{4.3.15}$$

The behavior of the differentials $d\Omega_j$ with respect to the action of τ^* remains unchanged:

$$\tau^* d\Omega_k = \overline{d\Omega_k} \quad ,$$

as in the repulsive case. The path c_P needed to define the constants E, N, and ω_0 transforms as follows:

$$\tau(c_P) = c_P - \sum_{k=1}^{g} a_k + l$$

This is the reason why (4.3.7) goes now over into

$$\overline{E} = E + \sum_{k=1}^{g} \int_{a_k} d\Omega_1 - \int_l d\Omega_1 = E - \int_l d\Omega_1 \quad ,$$
$$\overline{N} = N - \sum_{k=1}^{g} \int_{a_k} d\Omega_2 + \int_l d\Omega_2 = N + \int_l d\Omega_2 \quad ,$$
$$\overline{\log \omega_0} = \log \omega_0 + \sum_{k=1}^{g} \int_{a_k} d\Omega_3 - \int_l d\Omega_3 = \log \omega_0 - \int_l d\Omega_3 \quad ,$$
$$\int_l d\Omega_{1,2} = 0 \ , \quad \int_l d\Omega_3 = 2\pi \mathrm{i} \quad .$$

Consequently,

$$E = \overline{E} \ , \quad N = \overline{N} \ , \quad \mathrm{Im} \log \omega_0 = \pi \mathrm{i} \Leftrightarrow \omega_0 < 0 \quad . \tag{4.3.16}$$

For the components of $\boldsymbol{r}$ we get instead of (4.3.5)

$$\overline{r}_j = \lim_{P \to \infty^+} \int_{c_P} \overline{\omega}_j = \lim_{P \to \infty^+} \int_{\tau(c_P)} \omega_j = r_j - \sum_k \int_{a_k} \omega_j$$
$$= r_j - 2\pi \mathrm{i} \quad . \tag{4.3.17}$$

Although the behavior of the B-matrix under the action of complex conjugation is more complicated than in the repulsive case, the identity

$$\overline{\theta}(\boldsymbol{p}) = \theta(\overline{\boldsymbol{p}}) \tag{4.3.18}$$

still holds. In our case the relation (4.3.18) may be easily checked using the evident identity

$$\exp\left\{\frac{1}{2}\left\langle 2\pi\mathrm{i}\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}\boldsymbol{m}\,,\,\boldsymbol{m}\right\rangle\right\}$$

$$= \exp\left\{\mathrm{i}\pi\sum_j\left(\sum_{k\neq j} m_k\right)m_j\right\} = \exp\left\{2\pi\mathrm{i}\sum_{k>j} m_k\,m_j\right\} = 1 \quad .$$

In complete analogy with the derivation of the relations (4.3.8-9) from formulas (4.3.13-17) we get

$$|A|^2 \equiv -4\omega_0\frac{\theta\left(\mathrm{i}(\boldsymbol{V}x+\boldsymbol{W}t)-\boldsymbol{D}-\boldsymbol{r}\right)\,\theta\left(\mathrm{i}(\boldsymbol{V}x+\boldsymbol{W}t)+\overline{\boldsymbol{D}}\right)}{\theta\left(i(\boldsymbol{V}x+\boldsymbol{W}t)+\overline{\boldsymbol{D}}-\boldsymbol{r}\right)\,\theta\left(\mathrm{i}(\boldsymbol{V}x+\boldsymbol{W}t)-\boldsymbol{D}\right)}\;, \tag{4.3.19}$$

$$\overline{\boldsymbol{D}} = -\boldsymbol{D} + 2\pi\mathrm{i}\boldsymbol{N} + B\boldsymbol{M}\;,\quad \boldsymbol{N},\boldsymbol{M}\in\mathbb{Z}^g\quad ,$$

$$|A|^2 = -4\omega_0\exp\{-\langle\boldsymbol{r},\,\boldsymbol{M}\rangle\}\quad . \tag{4.3.20}$$

The vectors $\boldsymbol{N}$ and $\boldsymbol{M}$ have to satisfy some complementary restrictions. From (4.3.19) and (4.3.20) we have

$$\boldsymbol{N} = (1/2)B_0\boldsymbol{M}\;,\quad B_0\boldsymbol{M}\in 2\mathbb{Z}^g \Leftrightarrow \sum_{j=1,\,j\neq k} m_j = 2n_k\;,\quad n_k\in\mathbb{Z}\quad ,$$

$$\sum_{j=1}^{g} m_j = 2n_0\;,\quad n_0\in\mathbb{Z}\quad ,$$

and hence

$$\boldsymbol{M} = 2\boldsymbol{M}_0\;,\quad \boldsymbol{M}_0\in\mathbb{Z}^g\quad .$$

This yields

$$\begin{aligned}&\operatorname{Re}\boldsymbol{D} = B_R\boldsymbol{M}_0\;,\quad B_R = \operatorname{Re}B\;,\quad \boldsymbol{M}_0\in\mathbb{Z}^g\quad ,\\ &A = 2\sqrt{-\omega_0}\exp(\mathrm{i}\varphi)\exp\{-\langle\operatorname{Re}\boldsymbol{r},\,\boldsymbol{M}_0\rangle\}\;,\quad \varphi\in\mathbb{R}\quad .\end{aligned} \tag{4.3.21}$$

We can rewrite the last restriction on $\boldsymbol{D}$ in the form

$$\boldsymbol{D} = \boldsymbol{D}_0 + B\boldsymbol{M}_0\;,\quad \operatorname{Re}\boldsymbol{D}_0 = 0\quad .$$

$B\boldsymbol{M}_0$ belongs to the lattice Λ of periods of the theta functions. Therefore the solution $y(x,t)$ parametrized by $\boldsymbol{D}$ is the same as the solution $y_0(x,t)$ obtained by the change $\boldsymbol{D}\to\boldsymbol{D}_0$ (for which $\boldsymbol{M}_0 = 0$), $\varphi\to\varphi_0$,

$$\varphi_0 = \varphi + \pi \sum_{j=1}^{g} m_{0j} \ , \quad \boldsymbol{M}_0 = (m_{01}, \dots, m_{0g}) \quad .$$

So, in the attractive case, the whole variety of the finite-gap solutions with fixed branch points contains one convex component only. This component is eliminated from the solutions of the NS system by demanding

$$\operatorname{Re} \boldsymbol{D} = 0 \ , \quad A = 2\sqrt{-\omega_0} \exp \mathrm{i}\varphi) \quad . \tag{4.3.22}$$

Each of the constructed solutions of (4.1.2) is a smooth function of x and t for $x, t \in \mathbb{R}$. From the exact representation (4.1.23) it is evident that the unique possible singularity of y is the pole

$$y(x,t) \simeq \frac{c}{(x - x_0(t))^n} \ , \quad n \in \mathbb{Z} \ , \quad c \in \mathbb{C} \quad .$$

Substituting the leading term into (4.1.2), we see that c must be equal to zero.

4.3.3 Sine-Gordon Model. Reality Condition $v = \overline{v}$

Let $\overset{\circ}{X}$ in (4.2.25) be a more general curve than in the NS case. Namely, suppose that the branch points may be subdivided into two families:
(a.) E_j, $1 \leq j \leq 2k$, are real and ordered in the following way:

$$E_1 < E_2 < \dots < E_{2k} < 0 \quad ;$$

(b.) E_{2k+j}, $j = 1, \dots, 2(g-k)$, are complex-valued and

$$E_{2k+j} = \overline{E}_{2k+j+1} \quad .$$

The entire number k may be chosen arbitrarily between $k = 0$ and $k = g$. In the case $k = 0\,(k = g)$ there are no branch points of the type (a.) [type (b.)].

On such a surface we can always construct the canonical basis of the cycles (see Fig. 4.3 a for an example of such a basis) satisfying the conditions

$$\begin{aligned} &\tau(a_j) = -a_j \ , \quad j = 1, \dots, g \quad , \\ &\tau(b_j) = b_j \ , \quad j = 1, \dots, k \quad , \\ &\tau(b_j) = b_j - a_j \ , \quad j = k+1, \dots, g \quad . \end{aligned} \tag{4.3.23}$$

[It should be noted that the relations $\tau(\mu, \overset{\circ}{\lambda}) = (-\bar{\mu}, \overline{\overset{\circ}{\lambda}})$ and $\tau^* \sqrt{\overset{\circ}{\lambda}} = -\overline{\sqrt{\overset{\circ}{\lambda}}}$ are valid.] For the bases shown in the figure, the proof of the third of these conditions may be understood from Fig. 4.4.

Starting on (4.3.23) and repeating the calculations, absolutely similar to NS case, we get the following relations for the principal parameters of finite gap solutions (4.2.25):

$$\bar{V}_j = -V_j \ , \quad \bar{W}_j = -W_j \quad . \tag{4.3.24}$$

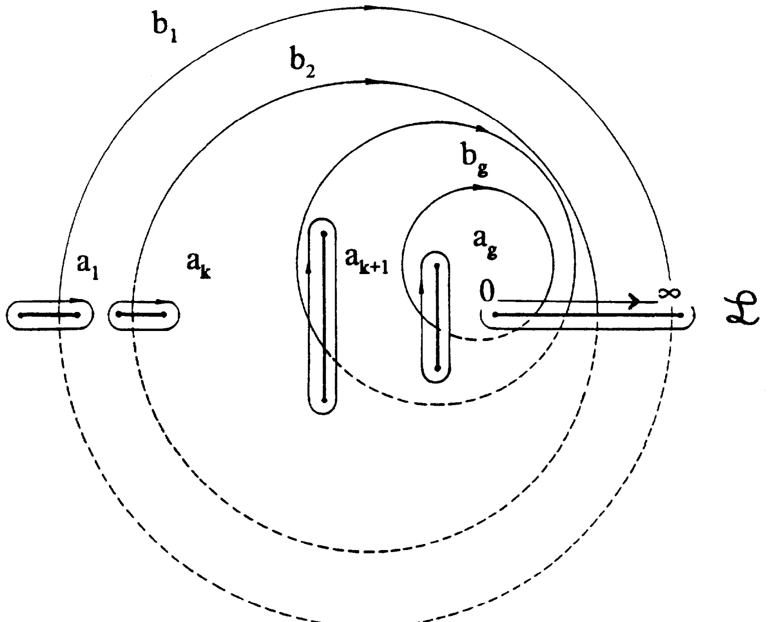

Fig. 4.3a. The homology basis for the curve $\mu^2 = \mathring{\lambda} \prod_{j=0}^{2g} (\mathring{\lambda} - E_j)$, $E_1 < E_2 < \cdots < E_{2k} < 0$, $E_{2k+j} = \overline{E_{2k+j+1}}$, $j = 1, \ldots, 2(g-k)$

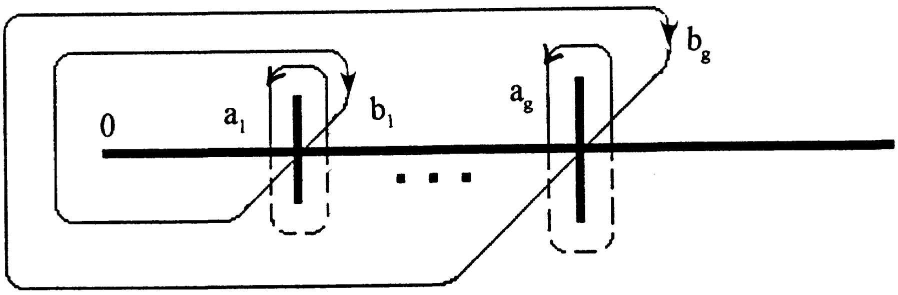

Fig. 4.3b. The choice of cuts and cycles for the small-amplitude regime

$$\begin{aligned} B_{jl} &= \bar{B}_{jl}\,, \quad j \le k \quad , \\ B_{jl} &= \bar{B}_{jl} + 2\pi \mathrm{i}\delta_{jl}\,, \quad j > k \quad . \end{aligned} \tag{4.3.25}$$

Formula (4.3.25) indicates that in the SG case we have to replace (4.3.18) by

$$\begin{aligned} &\bar{\theta}(\boldsymbol{p}) = \theta(\bar{\boldsymbol{p}} + \mathrm{i}\pi \boldsymbol{\Delta}_0) \quad , \\ &\boldsymbol{\Delta}_0 = (0, \ldots, 0, 1, 1, \ldots, 1) \quad . \end{aligned} \tag{4.3.26}$$

Finally, let us choose $\mathcal{L}$ as shown in Fig. 4.3 a, i.e.,

$$\mathcal{L} = \sum_{k=1}^{g} a_k$$

and hence

$$\boldsymbol{\Delta} = (1, 1, \ldots, 1) \quad . \tag{4.3.27}$$

Now we consider the reality conditions for $v(x,t)$. It is evident that the real-valuedness of $v(x,t)$ is equivalent to the following relation:

$$\left| \frac{\theta\,(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D})}{\theta\,(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta})} \right|^2 = 1 \quad . \tag{4.3.28}$$

Using (4.3.23-26), this relation may be rewritten in the form

$$\frac{\theta\,(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D})\ \theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \overline{\boldsymbol{D}} + \mathrm{i}\pi\boldsymbol{\Delta}_0\right)}{\theta\,(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \boldsymbol{D} + \mathrm{i}\pi\boldsymbol{\Delta})\ \theta\left(\mathrm{i}(\boldsymbol{V}x + \boldsymbol{W}t) + \overline{\boldsymbol{D}} + \mathrm{i}\pi\boldsymbol{\Delta}_0 + \mathrm{i}\pi\boldsymbol{\Delta}\right)} = 1 \quad ,$$

and leads to the following restriction on the structure of the vector $\boldsymbol{D}$:

$$\begin{aligned} &\operatorname{Im}\boldsymbol{D} = \frac{\pi}{2}\boldsymbol{\Delta}_1 + \pi\boldsymbol{N}\,, \quad \boldsymbol{N} \in \mathbb{Z}^g \quad , \\ &\boldsymbol{\Delta}_1 = (\underbrace{1,\dots,1}_{k}, 0,\dots,0) \quad . \end{aligned} \tag{4.3.29}$$

The solution (4.2.25) is invariant with respect to a translation of $\boldsymbol{D}$ on the lattice vector $\boldsymbol{\Lambda}$:

$$\boldsymbol{D} \to \boldsymbol{D} + 2\pi\mathrm{i}\boldsymbol{N} + B\boldsymbol{M} \quad . \tag{4.3.30}$$

The imaginary part of the matrix B is by virtue of (4.3.25) of the form

$$\operatorname{Im} B = \pi \begin{array}{l} \left(\begin{array}{cccccc} 0 & & & & & \\ & \ddots & & & 0 & \\ & & 0 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \ddots\\ 0\end{array}}\right\} k \\ \\ \\ \\ \end{array} \end{array} .$$

Hence, using the appropriate transformation (4.3.30), we can always construct a vector $\boldsymbol{N}$ in (4.3.29) in such a way that $N_j = 0$, for all $j \geq k+1$. So the final form of the restrictions on the vector $\boldsymbol{D}$ is:

$$\operatorname{Im}\boldsymbol{D} = \frac{\pi}{2}\boldsymbol{\Delta}_1 + \pi\boldsymbol{N} \quad , \tag{4.3.31}$$

where $\boldsymbol{N} = (\varepsilon_1,\dots,\varepsilon_k, 0,\dots,0)$, $\varepsilon_j = 0, 1$, $j = 1,\dots,k$. Accordingly, with (4.3.31), the whole variety of real finite-gap solutions of the SG equation corresponding to the fixed Riemann surface $\overset{\circ}{X}$ may be subdivided into 2^k convex components. We remark that all constructed solutions are smooth functions of x and t. The last conclusion is an immediate consequence of the equality (4.3.28), and the fact that the theta functions entering in (4.3.28) are entire functions of x and t.

It is rather convenient to describe the small-amplitude regime using a different choice of cuts and cycles, see Fig. 4.3b. The cycles and $\sqrt{\lambda}$ at Fig. 4.3b are transformed according to

$$\tau a_j = a_j, \quad \tau b_j = -b_j + a_j, \quad \tau^*\sqrt{\lambda} = -\overline{\sqrt{\lambda}}\ .$$

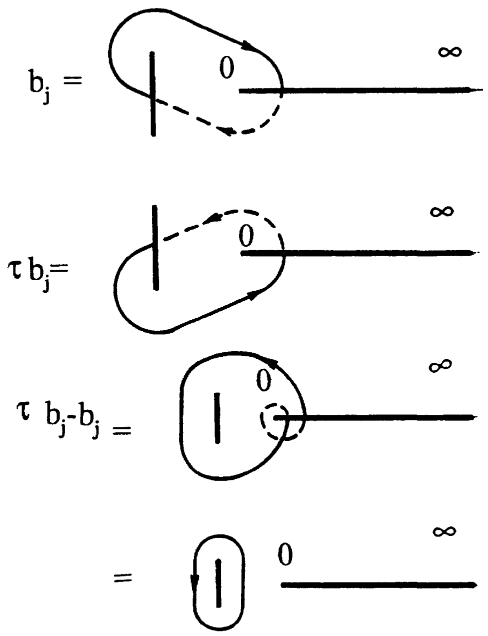

Fig. 4.4. Graphical proof of $\tau b_j - b_j = -a_j$

For the antiholomorphic involution $\pi\tau$ (where π is the hyperelliptic involution) we have the same transformation law of cycles as before (4.3.23):

$$\pi\tau a_j = -a_j, \quad \pi\tau b_j = b_j - a_j \ ,$$

but $\sqrt{\mathring{\lambda}}$ is transformed in a different way:

$$(\pi\tau)^* \sqrt{\mathring{\lambda}} = \overline{\sqrt{\mathring{\lambda}}} \ .$$

Instead of repeating the calculations for this case, we remark, that the transformation

$$\tau \longrightarrow \pi\tau \ , \quad \sqrt{\mathring{\lambda}} \longrightarrow \mathrm{i}\sqrt{\mathring{\lambda}} \ , \tag{4.3.32}$$

yields for the case of Fig. 4.3b the same transformation laws as before in the case of Fig. 4.3a. The change of coordinates with $\sqrt{\mathring{\lambda}}$ in (4.3.32) is equivalent to the complex transformation

$$x \longrightarrow -\mathrm{i}x \ , \quad t \longrightarrow \mathrm{i}t \ , \tag{4.3.33}$$

which preserves the form of (4.2.1). If we make the substitution (4.3.33) in (4.2.25), we see that the condition of real-valuedness (for the solution of the sine-Gordon equation) goes over into

$$\left| \frac{\theta(\boldsymbol{V}x - \boldsymbol{W}t + \boldsymbol{D})}{\theta(\boldsymbol{V}x - \boldsymbol{W}t + \boldsymbol{D} + \pi\mathrm{i}\boldsymbol{\Delta})} \right|^2 = 1 \quad .$$

Using the same method as above and the fact that the theta function is an even function we obtain

$$\boldsymbol{D} + \bar{\boldsymbol{D}} = \pi \mathrm{i}(\boldsymbol{\Delta} - \boldsymbol{\Delta}_0) \quad . \tag{4.3.34}$$

In the case under consideration with $\boldsymbol{\Delta} = \boldsymbol{\Delta}_0$ it yields

$$\mathrm{Re}\ \boldsymbol{D} = 0$$

and

$$v(x,t) = 2\mathrm{i} \log \frac{\theta(\boldsymbol{V}x - \boldsymbol{W}t + \boldsymbol{D} + \pi \mathrm{i}\boldsymbol{\Delta})}{\theta((\boldsymbol{V}x - \boldsymbol{W}t) + \boldsymbol{D})} \quad , \tag{4.3.35}$$

for the solution of the sine-Gordon equation of this kind of spectral curve, i.e., for small-amplitude waves for small cuts in Fig. 4.3 b.

4.4 Degeneracy of Finite-Gap Solutions. Multi-Soliton Solutions

Here we discuss for the NS model the limit structure of the finite-gap solution formula due to the degeneration of the associated algebraic curves, leading to curves with nodes and cusps. The simplest degeneration leads to the construction of so-called multi-soliton solutions in the representation given by Hirota. Next we show that the consideration of more complicated degenerations gives the possibility of finding solutions with interesting and previously unknown properties. An alternative way to describe some of these solutions is the application of dressing procedures, the so-called Darboux transformations to the finite-gap background. Here we do not compare these two approaches although such a comparison is also of some interest because the formulas obtained for the same objects are different. At first we discuss the simplest case when the genus of the curve X degenerates to become equal to zero in the limit case.

In such a limit a finite-gap solution transforms to a multi-soliton. We start here from the general formulas for the NS system. The constraint $y^* = \pm \bar{y}$ will be realized at the last step by taking an appropriate choice for the parameters in a degenerate version of our formulas.

We consider an arbitrary curve X of the form (4.1.14), take some value $\alpha > 0$ and put

$$E_1 = -E_{2g+2} = -\alpha \quad .$$

The basis of cycles is chosen as shown in Fig. 4.5

Now, we consider the limit

$$E_{2k},\ E_{2k+1} \to \lambda_k,\ k = 1, \ldots, g,\ \lambda_k \neq \lambda_j \quad . \tag{4.4.1}$$

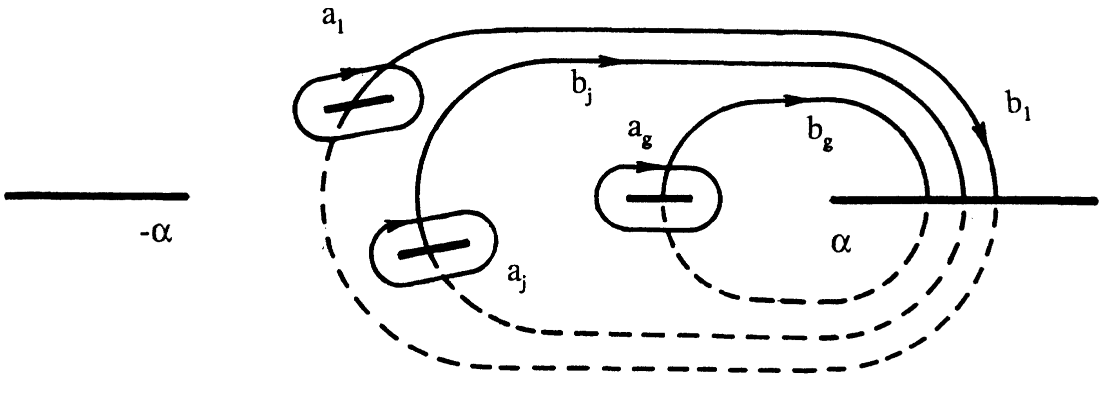

Fig. 4.5. Homology basis of cycles for an arbitrary curve of the form (4.1.14)

For a moment we do not impose any complementary restrictions on the positions of the points λ_k. In the limit (4.4.1) the function $\mu = \sqrt{P_{2g+2}(\lambda)}$ transforms to the function $\mu_0(\lambda)$ defined by

$$\mu_0(\lambda) = \sqrt{\lambda^2 - \alpha^2} \prod_{k=1}^{g} (\lambda - \lambda_k) \quad .$$

The square root in μ_0 is determined by by the condition $\sqrt{\lambda^2 - \alpha^2} > 0$, when λ is lying on the upper side of the cut joining α and $+\infty$ on the upper sheet of the curve X_0 (i.e., the Riemann surface of the function $\sqrt{\lambda^2 - \alpha^2}$). It is necessary to remark that each cycle a_j is placed on the upper sheet of the curve X_0 and surrounds the point λ_j clockwise. The b_j-cycle transforms to a curve starting from the point λ_j on the upper sheet, then crossing the cut $[\alpha, +\infty]$, going to the lower sheet and returning to the point λ_j, following the lower sheet. Now, the limit form of the normalized Abelian differentials ω_j is [3]

$$\omega_j^0(\lambda) = \frac{\varphi_j^0}{\sqrt{\lambda^2 - \alpha^2} \prod_{k=1}^{g} (\lambda - \lambda_k)} d\lambda \quad ,$$
$$\varphi_j^0(\lambda) = c_{j1}^0 \lambda^{g-1} + c_{j2}^0 \lambda^{g-2} + \ldots + c_{jg}^0 \quad .$$

The functions $\varphi_j^0(\lambda)$ may be determined from the normalization conditions:

$$2\pi \mathrm{i} \delta_{kj} = \int_{a_k} \omega_j^0 = -2\pi \mathrm{i} \operatorname{res}(\omega_j^0, \lambda_k) \quad .$$

Thus we get

[3] In the calculations performed in this and the next section it will be convenient to admit traditional inaccuracy, denoting the points of X_0 and their projections on the complex plane by the same character λ.

$$\varphi_j^0(\lambda) = c_{j1}^0 \prod_{l \neq j} (\lambda - \lambda_l), \quad c_{j1}^0 = -\sqrt{\lambda_j^2 - \alpha^2} \equiv -\mathrm{i}\kappa_j \quad . \tag{4.4.2}$$

According to (4.4.2), the differentials ω_j^0 may be cast into the form

$$\omega_j^0(\lambda) = \frac{c_{j1}^0}{\sqrt{\lambda^2 - \alpha^2}(\lambda - \lambda_j)} d\lambda \quad . \tag{4.4.3}$$

Now from (4.4.2) the limiting values of c_{j2}^0 are

$$c_{j2}^0 = -c_{j1}^0 \sum_{l \neq j} \lambda_l = \mathrm{i}\kappa_j \sum_{l \neq j} \lambda_l \quad .$$

Hence, the limits of the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ are given by (compare with (4.1.17))

$$\begin{aligned} &V_j \to V_j^0 = 2c_{j1}^0 = -2\mathrm{i}\kappa_j \quad , \\ &W_j \to W_j^0 = 4\mathrm{i}\left(\kappa_j \sum_{l \neq j} \lambda_l - \kappa_j \sum \lambda_l\right) = -4\mathrm{i}\kappa_j\lambda_j \quad . \end{aligned} \tag{4.4.4}$$

The Abelian integrals Ω_j, $j = 1,2,3$ are (in the limit considered here) equal to

$$\begin{aligned} &\Omega_1^0(\lambda) = \sqrt{\lambda^2 - \alpha^2} \quad , \\ &\Omega_2^0(\lambda) = 2\lambda\sqrt{\lambda^2 - \alpha^2} \quad , \\ &\Omega_3^0(\lambda) = \log\left(\lambda + \sqrt{\lambda^2 - \alpha^2}\right) - \log\alpha \quad . \end{aligned}$$

Now we can easily get the following expansions for $\Omega_i^0(\lambda)$ when λ tends to infinity:

$$\begin{aligned} &\Omega_1^0(\lambda) = \pm\lambda \mp \alpha^2/2\lambda + \dots \quad , \\ &\Omega_2^0(\lambda) = \pm(2\lambda^2 - \alpha^2 + o(1)) \quad , \\ &\Omega_3^0(\lambda) = \pm(\log\lambda - \log\alpha/2 + o(1)) \quad . \end{aligned}$$

From the above we conclude that in the limit (4.4.1) the constants E, N, ω_0 go to

$$E \to E^0 = 0\ , \quad N \to N^0 = -2\alpha^2\ , \quad \omega_0 \to \omega_0^0 = \alpha^2/4 \quad . \tag{4.4.5}$$

Now let us calculate the limit values of the elements of the matrix B. First of all we remark that

$$\begin{aligned} \int_\alpha^{\lambda_0} \omega_j \longrightarrow c_{j1}^0 \int_\alpha^{\lambda_0} \frac{d\lambda}{(\lambda - \lambda_j)\sqrt{\lambda^2 - \alpha^2}} = \\ = -\log \frac{\sqrt{\dfrac{\lambda_j - \alpha}{\lambda_j + \alpha}} - \sqrt{\dfrac{\lambda_0 - \alpha}{\lambda_0 + \alpha}}}{\sqrt{\dfrac{\lambda_j - \alpha}{\lambda_j + \alpha}} + \sqrt{\dfrac{\lambda_0 - \alpha}{\lambda_0 + \alpha}}} \quad . \end{aligned} \tag{4.4.6}$$

The integration path in (4.4.6) is supposed to lie on the upper sheet of the surface X_0 and not to go across the points λ_j. Also the principal branch is taken for the logarithm in (4.4.6) and

$$\sqrt{\frac{\lambda-\alpha}{\lambda+\alpha}} \equiv \frac{\sqrt{\lambda^2-\alpha^2}}{\lambda+\alpha}$$

is used to avoid uncertainties.

Starting from this point we consider λ_j to be ordered according to

$$\operatorname{Re}\lambda_j > \operatorname{Re}\lambda_k\ , \quad j > k \quad .$$

Under this assumption the limit values B^0_{jk} of the matrix elements B_{jk}, $j \neq k$, are given by the formulas:

$$\begin{aligned} B^0_{jk} &= 2\int_{\lambda_j}^{\alpha} \omega^0_k = 2\log\frac{\gamma_k-\gamma_j}{\gamma_k+\gamma_j}\ , \quad j > k \quad , \\ B^0_{jk} &= B^0_{kj} = 2\int_{\lambda_k}^{\alpha} \omega^0_j = 2\log\frac{\gamma_j-\gamma_k}{\gamma_k+\gamma_j}\ , \quad j < k \quad , \\ \gamma_j &= \sqrt{\frac{\lambda_j-\alpha}{\lambda_j+\alpha}} \equiv \frac{\sqrt{\lambda_j^2-\alpha^2}}{\lambda_j+\alpha} \quad . \end{aligned} \tag{4.4.7}$$

In the limit (4.4.1) we easily obtain for the diagonal elements of the matrix B the asymptotic estimate

$$\operatorname{Re} B_{jj} = 2\log|E_{2j+1}-E_{2j}| + O(1) \quad . \tag{4.4.8}$$

We conclude that in the considered limit

$$\operatorname{Re} B_{jj} \to -\infty\ , \quad j = 1,\dots,g \quad . \tag{4.4.9}$$

The final step we have is to study the behavior of the vector $\boldsymbol{r}$. Turning to (4.4.6) we find

$$r_j \to r^0_j = 2\int_{\alpha}^{\infty^+} \omega^0_j = -2\log\frac{\gamma_j-1}{\gamma_j+1} \quad . \tag{4.4.10}$$

We now discuss what happens with the theta functions defining the solutions (4.1.22) in the limit case. To obtain a reasonable answer we need to specify the behavior of the arbitrarily chosen vector $\boldsymbol{D}$. From now on we assume

$$D_j = \frac{1}{2}B_{jj} + 2\eta_j \tag{4.4.11}$$

to hold where the quantities η_j ($j = 1,\dots,g$) are supposed to be chosen arbitrarily but to be invariant with respect to variations of the branch points E_j .[4] Formula

[4] The above form of $\boldsymbol{D}$ arises quite natural from spectral approach considerations.

(4.4.11) enables one to represent the exponents in the exponentials entering in the definition of $\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D}-\varepsilon\boldsymbol{r})$, $\varepsilon=-1,0,1$ in the form

$$\frac{1}{2}\sum B_{jj}m_j(m_j-1)+\sum_{j<k}B_{jk}m_jm_k \\ +\sum_j m_j(\mathrm{i}V_jx+\mathrm{i}W_jt-2\eta_j-\varepsilon r_j) \quad .$$

From (4.4.9) it is clear that only the terms corresponding to the vectors $\boldsymbol{m}$, whose coordinates m_j take the values 0 and 1, may give a non-zero contribution to the series forming the Riemann theta function of the degenerated surface. Taking into account (4.4.4-7,10), we find that in the limit (4.4.1) the Riemann theta function $\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D}-\varepsilon\boldsymbol{r})$ transforms to the finite sum

$$\begin{aligned}\theta_\varepsilon(x,t)=\sum_{m_j=0,1}\exp&\left\{\sum_{j>k}\log\left[\frac{\gamma_j-\gamma_k}{\gamma_j+\gamma_k}\right]^2 m_jm_k\right.\\ &\left.+\sum_j m_j\left(2\kappa_jx+4\kappa_j\lambda_jt+2\varepsilon\log\frac{\gamma_j-1}{\gamma_j+1}-2\eta_j\right)\right\} \quad .\end{aligned} \tag{4.4.12}$$

Hence, the limit form of the solutions of the NS system is given by

$$\begin{aligned}y(x,t)&=A\,\frac{\theta_{-1}(x,t)}{\theta_0(x,t)}\exp(-2\mathrm{i}\alpha^2t)\ ,\\ y^*(x,t)&=\frac{\alpha^2}{A}\,\frac{\theta_1(x,t)}{\theta_0(x,t)}\exp(2\mathrm{i}\alpha^2t)\ .\end{aligned} \tag{4.4.13}$$

The last formulas describe a family of $(2g+2)$-parametric solutions of the NS system (4.1.1), fixed by the choice of the parameters A, α, λ_j, η_j and involving elementary functions only. Varying those parameters one can obtain different types of solutions which have direct physical meaning. We shall consider here two of the most typical examples.

Example 1. Multi-soliton solutions on the constant background of the NS equation (4.1.2) with $\sigma=1$.

Here A, λ_j, η_j are assumed to satisfy the conditions

$$\begin{aligned}&\operatorname{Im}\lambda_j=0\ ,\quad \lambda_j\in(-\alpha,\alpha)\ ,\quad j=1,\dots,g\quad ;\\ &A=\alpha\exp(\mathrm{i}\varphi)\ ,\quad \varphi\in\mathbb{R}\ ,\quad \operatorname{Im}\eta_j=0\quad .\end{aligned} \tag{4.4.14}$$

From (4.4.14) we get

$$\begin{aligned}&\kappa_j=\sqrt{\alpha^2-\lambda_j^2}>0\ ,\quad \gamma_j=\mathrm{i}\frac{\kappa_j}{\lambda_j+\alpha}\ ,\quad \log\left(\frac{\gamma_j-\gamma_k}{\gamma_j+\gamma_k}\right)^2\in\mathbb{R}\quad ,\\ &\log\frac{\gamma_j-1}{\gamma_j+1}=-2\mathrm{i}\arctan\frac{\kappa_j}{\lambda_j+\alpha}+\mathrm{i}\pi\quad .\end{aligned} \tag{4.4.15}$$

Taking into account (4.4.12) and (4.4.15), we find that $\theta_\varepsilon(x,t)$ satisfies the condition

$$\overline{\theta_\varepsilon}(x,t) = \theta_{-\varepsilon}(x,t) \quad ,$$

i.e., under the assumptions (4.4.14), the solutions (4.4.13) satisfy the reduction:

$$y(x,t) = \overline{y^*(x,t)} \quad .$$

By the same assumptions the smoothness conditions will be satisfied, too. So we get a $(2g+2)$-parametric family of smooth solutions of the NS equation in the repulsive case.

Next, we consider the asymptotic properties of the constructed solution $y(x,t)$. Let t be fixed and send $x \to \infty$. Then the leading term in the sum (4.4.12) corresponds to the vector

$$\boldsymbol{m} = (1,1,\ldots,1) \quad .$$

Therefore we get for $y(x,t)$ the following asymptotic behavior for $x \to \infty$:

$$y(x,t) \sim \alpha \exp\left(-2\mathrm{i}\alpha^2 t + \mathrm{i}\varphi + 4\mathrm{i}\sum_j \arctan\frac{\kappa_j}{\lambda_j+\alpha}\right) \quad . \tag{4.4.16}$$

Similarly we get from (4.4.12) an asymptotic formula for $x \to -\infty$:

$$y(x,t) \sim \alpha \exp(-2\mathrm{i}\alpha^2 t + \mathrm{i}\varphi) \,, \quad x \to -\infty \quad . \tag{4.4.17}$$

Now, consider the asymptotic behavior of this solution for large time: $t \to \pm\infty$. Let us assume that for some $j = j_0$ we have

$$x + 2\lambda_{j_0} t = O(1) \,, \quad t \to \pm\infty \quad . \tag{4.4.18}$$

Taking into account our suggestion about the ordering of the λ_j's, we conclude that the leading term for $t \to \infty$ in the sum (4.4.12) is generated by two vectors

$$\boldsymbol{m} = (\underbrace{0,\ldots,0}_{j_0},\ 1,\ 1,\ 1,\ldots,\ 1)$$

and

$$\boldsymbol{m} = (\underbrace{0,\ldots,1}_{j_0},\ 1,\ 1,\ 1,\ldots,\ 1) \quad .$$

Accordingly for $y(x,t)$ we get

$$
\begin{aligned}
y(x,t) &= \frac{1+\exp\Theta_1(x,t)}{1+\exp\Theta_2(x,t)} \\
&\quad \times \alpha\exp\left\{-2\mathrm{i}\alpha^2 t + \mathrm{i}\varphi + 4\mathrm{i}\sum_{j=j_0+1}^{g}\arctan\frac{\kappa_j}{\lambda_j+\alpha}\right\} \\
&= \alpha\frac{1+\exp\Theta_3(x,t)}{1+\exp\Theta_4(x,t)}\exp(-2\mathrm{i}\alpha^2 t + \mathrm{i}\varphi^+) \quad , \\
&\qquad t\to\infty\,, \quad x+\lambda_{j_0}t = O(1) \quad , \\
\Theta_1(x,t) &= \sum_{j=j_0+1}^{g}\log\left(\frac{\gamma_j-\gamma_{j_0}}{\gamma_j+\gamma_{j_0}}\right)^2 + 2\kappa_{j_0}x + 4\kappa_{j_0}\lambda_{j_0}t \\
&\quad + 4\mathrm{i}\arctan\frac{\kappa_j}{\lambda_{j_0}+\alpha} - 2\eta_{j_0} \quad , \\
\Theta_2(x,t) &= \sum_{j=j_0+1}^{g}\log\left(\frac{\gamma_j-\gamma_{j_0}}{\gamma_j+\gamma_{j_0}}\right)^2 + 2\kappa_{j_0}x + 4\kappa_{j_0}\lambda_{j_0}t - 2\eta_{j_0} \quad , \\
\Theta_3(x,t) &= 2\kappa_{j_0}(x-x_{j_0}^+) + 4\kappa_{j_0}\lambda_{j_0}t + 4\mathrm{i}\arctan\frac{\kappa_{j_0}}{\lambda_{j_0}+\alpha} \quad , \\
\Theta_4(x,t) &= 2\kappa_{j_0}(x-x_{j_0}^+) + 4\kappa_{j_0}\lambda_{j_0}t \quad ,
\end{aligned}
\tag{4.4.19}
$$

where

$$
\begin{aligned}
x_{j_0}^+ &= \frac{\eta_{j_0}}{\kappa_{j_0}} + \frac{1}{\kappa_{j_0}}\sum_{j=j_0+1}^{g}\log\left|\frac{\gamma_j+\gamma_{j_0}}{\gamma_j-\gamma_{j_0}}\right| \quad , \\
\varphi^+ &= \varphi + 4\sum_{j=j_0+1}^{g}\arctan\frac{\kappa_j}{\lambda_j+\alpha} \quad .
\end{aligned}
\tag{4.4.20}
$$

Now it is evident that the functional structure of the asymptotics for $t\to-\infty$ must be the same. Only the phase shifts have to be changed

$$
\begin{aligned}
x_{j_0}^+ \to x_{j_0}^- &= \frac{\eta_{j_0}}{\kappa_{j_0}} + \frac{1}{\kappa_{j_0}}\sum_{j=1}^{j_0-1}\log\left|\frac{\gamma_j+\gamma_{j_0}}{\gamma_j-\gamma_{j_0}}\right| \quad , \\
\varphi^+ \to \varphi^- &= \varphi + 4\sum_{j=1}^{j_0-1}\arctan\frac{\kappa_j}{\lambda_j+\alpha} \quad .
\end{aligned}
\tag{4.4.21}
$$

The right-hand-side of the last equality in (4.4.19) describes the simple one-soliton solution or, more precisely, a simple kink, because the non-zero boundary conditions for $x\to\pm\infty$, $\sigma=1$ are satisfied. So the constructed solution describes the interaction of the g-kinks of the nonlinear Schrödinger equation.

The formulas (4.4.20) and (4.4.21) give the associated expressions for the displacement of the center of mass and the phases of the solutions in interaction, respectively. Equations (4.4.16-17) show that the phase shift for the boundary condition corresponding to the multi-soliton case is an additive function in the number of solitons.

Remark 4.13. The multi-soliton solution of the NS equation with $\sigma = -1$ (the attractive case, which is more widely discussed in the literature) may be deduced [4.6] from (4.4.13) as a result of the following choice of the parameters:

$$
\begin{aligned}
&g = 2N, \quad A = \alpha, \quad \lambda_{N+j} = \overline{\lambda_j}, \quad \operatorname{Im}\lambda_j > 0, \quad j = 1,\dots,N \quad , \\
&\eta_j = -\frac{1}{2}\log\alpha - \mathrm{i}\frac{\pi}{4} + c_j, \quad j = 1,\dots,2N \quad , \\
&\overline{c}_{N+j} = c_j, \quad j = 1,\dots,N; \quad \alpha \to 0 \quad .
\end{aligned}
$$

Example 2. Multi-phase modulations of the stationary solution of the NS equation with $\sigma = -1$.

Let us assume that

$$
\begin{aligned}
&g = 2N, \quad A = \alpha, \quad \lambda_{N+j} = -\lambda_j, \quad 0 < \lambda_j < \alpha \quad , \\
&\eta_j = \eta_j^0 + \mathrm{i}d_j + c_j, \quad \eta_{N+j} = \eta_j^0 + \mathrm{i}d_j - c_j \quad , \\
&\eta_j^0 = \frac{1}{2}\sum_{\substack{l=1 \\ l \neq j}}^{N} \log\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l} + \frac{1}{2}\sum_{l=1}^{N}\log\frac{1+\gamma_j\gamma_l}{1-\gamma_j\gamma_l} \quad , \\
&d_j, c_j \in \mathbb{R}, \quad j = 1,\dots,N \quad .
\end{aligned}
\tag{4.4.22}
$$

Proposition 4.14. Under the conditions (4.4.22) the functions y, y^* defined by (4.4.13) satisfy

$$
y^*(\mathrm{i}x,t) = \overline{y(\mathrm{i}x,t)} \tag{4.4.23}
$$

Proof. Under the restrictions imposed on the parameters λ_j, η_j, the exponents $I_\varepsilon(m,x,t)$ of the exponentials entering the definition of the functions $\theta_\varepsilon(x,t)$ may the rewritten as follows: [5]

[5] Here and below Latin indices are taken to run from 1 to N and Greek indices from 1 to $2N$.

$$
\begin{aligned}
I_\varepsilon(\boldsymbol{m} \mid x,t) = &\sum_{j>l} \left(\log \left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l} \right]^2 m_\ell m_j \right. \\
&\left. + \log \left[\frac{\gamma_{N+j} - \gamma_{N+l}}{\gamma_{N+j} + \gamma_{N+l}} \right]^2 m_{N+l} m_{N+j} \right) \\
&+ \sum_{\ell,j} \log \left[\frac{\gamma_{N+j} - \gamma_l}{\gamma_{N+j} + \gamma_l} \right]^2 m_{N+j} m_l \\
&+ \sum_j \big(2(\kappa_j x - \mathrm{i} d_j)(m_j + m_{N+j}) \\
&\qquad + (4\kappa_j \lambda_j t - 2c_j)(m_j - m_{N+j}) \big) \\
&+ 2\varepsilon \sum_j \left(\log \frac{\gamma_j - 1}{\gamma_j + 1} m_j + \log \frac{\gamma_{N+j} - 1}{\gamma_{N+j} + 1} m_{N+j} \right) \\
&- 2 \sum_j \eta_j^0 (m_j + m_{N+j}) \quad .
\end{aligned}
$$

Let the vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ be related by the equalities

$$
m_j = 1 - n_{N+j} \ , \quad m_{N+j} = 1 - n_j \quad .
$$

Then it is evident that

$$
\begin{aligned}
&\overline{I_\varepsilon}((\boldsymbol{m}(\boldsymbol{n}) \mid \mathrm{i}x, t) \\
&= I_\varepsilon(x) + \sum_{j>l} \left(\log \left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l} \right]^2 (1 - n_{N+j})(1 - n_{N+l}) \right. \\
&\qquad \left. + \log \left[\frac{\gamma_{N+j} - \gamma_{N+l}}{\gamma_{N+j} + \gamma_{N+l}} \right]^2 (1 - n_j)(1 - n_l) \right) \\
&\qquad + \sum_{j,l} \log \left[\frac{\gamma_{N+j} - \gamma_l}{\gamma_{N+j} + \gamma_l} \right]^2 (1 - n_j)(1 - n_{N+l}) \\
&\qquad + \sum_j \big(2i(\kappa_j x - d_j)(n_j + n_{N+j}) + (4\kappa_j \lambda_j t - 2c_j)(n_j - n_{N+j}) \big) \\
&\qquad + 2\varepsilon \sum_j \left(n_{N+j} \log \frac{\gamma_j - 1}{\gamma_j + 1} + n_j \log \frac{\gamma_{N+j} - 1}{\gamma_{N+j} + 1} \right) \\
&\qquad + 2 \sum_j \eta_j^0 (n_j + n_{N+j}) \quad ,
\end{aligned}
\tag{4.4.24}
$$

where

$$
I_\varepsilon(x) = -4\mathrm{i} \sum_j (\kappa_j x - d_j) - 2\varepsilon \sum_\nu \log \frac{\gamma_\nu - 1}{\gamma_\nu + 1} - 4 \sum_j \eta_j^0 \quad .
$$

We now remark that under the condition $\lambda_j = -\lambda_{N+j}$, taking into account the formulas

$$\kappa_{N+j} = \kappa_j = \sqrt{\alpha^2 - \lambda_j^2} > 0 \quad ,$$
$$\gamma_\nu = \frac{\sqrt{\lambda_\nu^2 - \alpha^2}}{\lambda_\nu + \alpha} = \mathrm{i}\frac{\kappa_\nu}{\lambda_\nu + \alpha} \quad ,$$

the equalities

$$\begin{gathered}\gamma_{N+j} = -\frac{1}{\gamma_j}, \quad \frac{\gamma_j - 1}{\gamma_j + 1} = -\frac{\gamma_{N+j} + 1}{\gamma_{N+j} - 1} \quad , \\ \frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l} = -\frac{\gamma_{N+j} - \gamma_{N+l}}{\gamma_{N+j} + \gamma_{N+l}}, \quad \frac{\gamma_{N+j} - \gamma_l}{\gamma_{N+j} + \gamma_l} = \frac{\gamma_{N+l} - \gamma_j}{\gamma_{N+l} + \gamma_j}\end{gathered} \tag{4.4.25}$$

hold. They allow to transform (4.4.24) in the following way:

$$\begin{aligned}\overline{I_\varepsilon}(\boldsymbol{m}(\boldsymbol{n}) \,|\, \mathrm{i}x, t) = I_\varepsilon(x) + \sum_{\nu > \mu} \log \left[\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu}\right]^2 \\ + I_{-\varepsilon}(\boldsymbol{n} \,|\, \mathrm{i}x, t) + I(\boldsymbol{n}) + 4 \sum_j \eta_j^0 (n_j + n_{N+j}) \quad ;\end{aligned}$$

$$\begin{aligned}I(\boldsymbol{n}) = &- \sum_{l=1}^{N-1} \sum_{j=l+1}^{N} (n_j + n_{N+j}) \log \left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l}\right]^2 \\ &- \sum_{l=1}^{N-1} (n_l + n_{N+l}) \sum_{j=l+1}^{N} \log \left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l}\right]^2 \\ &- \sum_{j=1}^{N} n_j \sum_{l=1}^{N} \log \left[\frac{\gamma_{N+j} - \gamma_l}{\gamma_{N+j} + \gamma_l}\right]^2 \\ &- \sum_{l=1}^{N} n_{N+l} \sum_{j=1}^{N} \log \left[\frac{\gamma_{N+j} - \gamma_l}{\gamma_{N+j} + \gamma_l}\right]^2 \quad .\end{aligned}$$

Taking again into account (4.4.25), we get

$$
\begin{aligned}
I(\boldsymbol{n}) = &-\sum_{j=2}^{N}(n_j+n_{N+j})\sum_{l=1}^{j-1}\log\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2 \\
&-\sum_{j=1}^{N-1}(n_j+n_{N+j})\sum_{l=j+1}^{N}\log\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2 \\
&-\sum_{j=1}^{N}(n_j+n_{N+j})\sum_{l=1}^{N}\log\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right]^2 \\
= &-\sum_{j=1}^{N}(n_j+n_{N+j})\left[\sum_{l\neq j}\log\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2+\sum_{l=1}^{N}\log\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right]^2\right] \\
= &-4\sum_{j}\eta_j^0(n_j+n_{N+j}) \quad .
\end{aligned}
$$

In other words,

$$
\overline{I_\varepsilon}(\boldsymbol{m}(\boldsymbol{n})\,|\,\mathrm{i}x,t) = I_\varepsilon(x)+\sum_{\nu>\mu}\log\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2 + I_{-\varepsilon}(\boldsymbol{n}\,|\,\mathrm{i}x,t) \quad .
$$

Consequently we have

$$
\begin{aligned}
\overline{\theta_\varepsilon}(\mathrm{i}x,t) &= \sum_{\boldsymbol{m},\, m_\nu=0,1}\exp\overline{I_\varepsilon}(\boldsymbol{m}\,|\,\mathrm{i}x,t) = \sum_{\boldsymbol{n},\, n_\nu=0,1}\exp\overline{I_\varepsilon}(\boldsymbol{m}(\boldsymbol{n})\,|\,\mathrm{i}x,t) \\
&= \exp\left\{I_\varepsilon(x)+\sum_{\nu>\mu}\log\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2\right\}\sum_{\boldsymbol{n},\, n_\nu=0,1}\exp I_{-\varepsilon}(\boldsymbol{n}\,|\,\mathrm{i}x,t) \\
&= E(x)\theta_{-\varepsilon}(\mathrm{i}x,t) \quad ,
\end{aligned}
$$

where, by virtue of the second of the equalities (4.4.25), the function E is independent of ε and has the form

$$
E(x) = \exp\left\{-4\mathrm{i}\sum_j(\kappa_j x-d_j)-4\sum_j\eta_j^0+\sum_{\nu>\mu}\log\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2\right\} \quad .
$$

Now from $\overline{\theta_\varepsilon}(\mathrm{i}x,t) = E(x)\theta_{-\varepsilon}(\mathrm{i}x,t)$ it is seen that the identity (4.4.23) in (4.4.13) is true. This completes the proof of Proposition 4.14.

Taking into account that under the condition (4.4.23) the function

$$
v(x,t) = y^*(\mathrm{i}x,t)
$$

is a smooth [6] solution of the NS equation with $\sigma=-1$, i.e., v satisfies

[6] The smoothness follows from the fact that the substitution of the fraction $c/(x-x_0(t))^n$, x, $x_0 \in \mathbb{R}$, $c \in \mathbb{C}$, $n \in \mathbb{Z}$ into (4.4.26) implies $c=0$, by virtue of the sign of the nonlinear term in (4.4.26).

$$\mathrm{i}v_t + v_{xx} + 2|v|^2 v = 0 \quad . \tag{4.4.26}$$

From Proposition 4.14 we get now:

Theorem 4.15. Let $N \geq 1$, $N \in \mathbb{Z}$. Then for all $\alpha, \varphi \in \mathbb{R}$, $\alpha > 0$ and $\{\lambda_j,\, x_{0j},\, t_{0j}\}$ $j = 1, \ldots, N$, restricted only by the conditions $0 < \lambda_j < \alpha$, $x_{0j},\, t_{0j} \in \mathbb{R}$, there exists a solution of (4.4.26) which depends smoothly on x and t and is of the form

$$v_N(x,t) = \alpha \frac{\theta_1(x,t)}{\theta_0(x,t)} \exp(2\mathrm{i}\alpha^2 t - \mathrm{i}\varphi) \quad , \tag{4.4.27}$$

where

$$\begin{aligned}
\theta_\varepsilon(x,t) &\equiv \theta_\varepsilon(\mathrm{i}x,t) \\
&= \sum_{m_\nu=0,1} \exp\Bigg\{ \sum_{j>l} \log \left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l}\right]^2 (m_j m_l + m_{N+j} m_{N+l}) \\
&\quad + \sum_{j,l} \log \left[\frac{1+\gamma_j\gamma_l}{1-\gamma_j\gamma_l}\right]^2 m_{N+j} m_l \\
&\quad + 2\varepsilon \sum_j \log \left[\frac{\gamma_j - 1}{\gamma_j + 1}\right] (m_j - m_{N+j}) \\
&\quad - 2\sum_j \eta_j^0 (m_j + m_{N+j}) + 2\sum_j \delta_j (t - t_{0j})(m_j - m_{N+j}) \qquad (4.4.28) \\
&\quad + 2\mathrm{i} \sum_j \kappa_j (x - x_{0j})(m_j + m_{N+j}) \Bigg\} \quad ,
\end{aligned}$$

$$\varepsilon = 1, 0$$

$$\kappa_j = \sqrt{\alpha^2 - \lambda_j^2} > 0, \quad \delta_j = 2\lambda_j \sqrt{\alpha^2 - \lambda_j^2}, \quad \gamma_j = i\frac{\kappa_j}{\lambda_j + \alpha}$$

$$\eta_j^0 = \frac{1}{2}\sum_{l=1}^{N} \log \frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l} + \frac{1}{2}\sum_{l=1}^{N} \log \frac{1+\gamma_j\gamma_l}{1-\gamma_j\gamma_l} \quad .$$

Remark 4.16. It can be shown that $|v_N|^2$ may be represented in the form different from (4.4.27), namely

$$|v_N(x,t)|^2 = \frac{\partial^2}{\partial x^2} \log \theta_0(x,t) \quad . \tag{4.4.29}$$

The formulas (4.4.27, 28) describe a $3N + 2$-parametric smooth family of almost periodic (in x) solutions of the NS equation (4.1.2) with $\sigma = -1$. The basic periods T_j are

$$T_j = \frac{\pi}{\kappa_j}, \quad j = 1, \ldots, N \quad .$$

These solutions represent a multi-phase generalization of the simplest (one or two phase) solutions of this type found first in [4.7]. The solutions found in [4.7] describe the weak one-phase or two-phase modulations of the "stationary" wave solution $v_0(x,t)$ [7], i.e.,

$$v_0(x,t) = \alpha \exp(2\mathrm{i}\alpha^2 t - \mathrm{i}\varphi) \quad . \tag{4.4.30}$$

For arbitrary N, the solution $v_N(x,t)$ describes the weak N-phase modulation of (4.4.30). In addition, the existence of the solution $v_N(x,t)$ implies the instability of the stationary solution $v_0(x,t)$ with respect to small multi-phase modulation perturbations. More precisely, the following theorem holds:

Theorem 4.17. Let the choice of the λ_j imply the inequalities

$$\max_j \delta_j < 2 \min_j \delta_j \equiv 2\delta_0 \quad . \tag{4.4.31}$$

Then for $t \to \pm\infty$ the following asymptotics hold:

$$\begin{aligned} v_N(x,t) = \alpha \Big[1 + \sum_{j=1}^{N} A_j^{\pm} \cos 2\kappa_j (x - x_{0j}) \exp(-2\delta_j |t| \mp \mathrm{i}\alpha_j) \\ + O\left(\exp(-4\delta_0 |t|)\right) \Big] \exp\left(2\mathrm{i}\alpha^2 t - \mathrm{i}\varphi + \mathrm{i}\varphi^{\pm}\right) \quad ; \\ A_j^{\pm} = -4\frac{\kappa_j}{\alpha} \exp(\pm 2\delta_j t_{0j}) \prod_{j=1}^{N} \frac{1+\gamma_j\gamma_l}{1-\gamma_j\gamma_l} \prod_{l \neq j}^{N} \frac{\gamma_j + \gamma_l}{\gamma_j - \gamma_l} \quad , \\ \alpha_j = \arctan \frac{\lambda_j}{\kappa_j}, \quad \varphi^{\pm} = \mp 4 \sum_{l=1}^{N} \arctan \frac{\kappa_j}{\lambda_j + \alpha} = \\ = \mp 2 \sum_{l=1}^{N} \arctan \frac{\kappa_l}{\lambda_l} \quad . \end{aligned} \tag{4.4.32}$$

Proof. Consider for definiteness the case $t \to +\infty$, looking at the asymptotic behavior of $v_N(x,t)$ up to terms of order $O(\exp(-4\delta_0 t))$. The associated leading terms in the sums θ_ε correspond to the vectors $\boldsymbol{m}$ with the following coordinates:

---

[7] It has to be mentioned that in the applications to fiber wave-guide optics considered in [4.7], the roles of the space and time variables have to be interchanged.

$$
\begin{aligned}
1 \leftrightarrow & m_1 = m_2 = \ldots = m_N = 1 \quad , \\
& m_{N+1} = m_{N+2} = \ldots = m_{2N} = 0 \quad , \\
\exp(-2\delta_1 t) \leftrightarrow & \begin{cases} m_1 = m_2 = \ldots = m_N = 1 \quad , \\ m_{N+1} = 1,\ m_{N+2} = \ldots = m_{2N} = 0 \quad ; \\ m_1 = 0,\ m_2 = \ldots = m_N = 1 \quad , \\ m_{N+1} = m_{N+2} = \ldots = m_{2N} = 0 \quad . \end{cases} \\
\exp(-2\delta_N t) \leftrightarrow & \begin{cases} m_1 = m_2 = \ldots = m_N = 1 \quad , \\ m_{N+1} = m_{N+2} = \ldots = m_{2N-1} = 0,\ m_{2N} = 1 \quad ; \\ m_1 = m_2 = \ldots = m_{N-1} = 1 \quad , \\ m_N = m_{N+1} = \ldots = m_{2N} = 0 \quad . \end{cases} \\
\exp(-2\delta_j t) \leftrightarrow & \begin{cases} m_1 = m_2 = \ldots = m_N = 1 \quad , \\ m_{N+j} = 1,\ m_{N+1} = \ldots = \widehat{m}_{N+j} = \ldots = m_{2N} = 0 \quad ; \\ m_j = 0,\ m_1 = \ldots = \widehat{m}_j = \ldots = m_N = 1 \quad , \\ m_{N+1} = m_{N+2} = \ldots = m_{2N} = 0 \quad . \end{cases}
\end{aligned}
$$

where $\widehat{m}_j$ means, as usual, that the corresponding term must be omitted. Consequently, the asymptotic behavior of the θ_ε for $t \to \pm\infty$ is given by the formula

$$
\begin{aligned}
\theta_\varepsilon(x,t) = & \exp\Big(2\mathrm{i}\sum_j \kappa_j(x - x_{0j}) + 2\sum_j \delta_j(t - t_{0j}) - 2\sum_j \eta_j^0 \\
& + 2\varepsilon \sum_j \log\frac{\gamma_j - 1}{\gamma_j + 1} + \sum_{j>l}\log\left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l}\right]^2\Big) \\
& \times \Bigg\{1 + \sum_{j=1}^{N}\Bigg[\exp\Bigg(\sum_{l=1}^{N}\log\left[\frac{1 + \gamma_j\gamma_l}{1 - \gamma_j\gamma_l}\right]^2 - 2\varepsilon\log\frac{\gamma_j - 1}{\gamma_j + 1} \\
& - 2\eta_j^0 + 2i\kappa_j(x - x_{0j}) - 2\delta_j(t - t_{0j})\Bigg) \\
& + \exp\Bigg(-\sum_{\substack{l \neq j \\ l=1}}\log\left[\frac{\gamma_j - \gamma_l}{\gamma_j + \gamma_l}\right]^2 - 2\varepsilon\log\frac{\gamma_j - 1}{\gamma_j + 1} + 2\eta_j^0 - \\
& - 2\mathrm{i}\kappa_j(x - x_{0j}) - 2\delta_j(t - t_{0j})\Bigg)\Bigg] + O\left(\exp(-4\delta_0|t|)\right)\Bigg\} \quad .
\end{aligned}
\tag{4.4.33}
$$

Similar calculations for $t \to -\infty$ yield formulas that may be obtained from (4.4.33) by virtue of the transformation

$$\delta_j \to -\delta_j \ , \quad \varepsilon \to -\varepsilon \quad .$$

The behavior of the function $v_N(x,t)$ claimed in the theorem follows now from (4.4.33), taking into account an explicit formula for η_j^0. This completes the proof.

Remark 4.18. Omitting the restriction (4.4.31) imposed on the parameters δ_l the weaker result remains true, i.e.,

$$
\begin{aligned}
v_N(x,t) =& \alpha[1+o(1)]\exp(2\mathrm{i}\alpha^2 t - \mathrm{i}\varphi + \mathrm{i}\varphi^{\pm}) \\
& t \to \pm\infty \quad , \\
\Delta\varphi = \varphi^+ - \varphi^- &= -4\sum_{j=1}^{N} \arctan\frac{\kappa_j}{\lambda_j} \quad .
\end{aligned}
$$

4.5 Partial Degeneracy of the Finite-Gap Formulas. Multi-Phase Modulations of the Cnoidal Wave

In complete analogy to the KdV one-gap solution, i.e., the genus one solution of the NS equation will be referred to as a cnoidal wave solution or simply cnoidal wave. For this solution, the Riemann surface X turns out to be an elliptic curve. Hence, the solution describing NS cnoidal wave may be represented by means of the Jacobi elliptic functions. In particular, for the attractive case, ($\sigma = 1$), we get from the general formula (4.1.22) the following representation of the cnoidal wave solution:

$$
\begin{aligned}
y(x,t) = \frac{1+k}{k}\mathrm{dn}&\left((x-x_0)\frac{1+k}{k};\ \frac{2\sqrt{k}}{1+k}\right) \times \\
&\exp\left(2\mathrm{i}\frac{1+k^2}{k^2}t + \mathrm{i}\varphi\right) \quad , \\
&0 < k < 1\,, \quad \varphi \in \mathbb{R} \quad .
\end{aligned}
\tag{4.5.1}
$$

Of course, (4.5.1) may be obtained also by the direct substitution of the corresponding ansatz into the NS equation.

Different limiting processes, performed in the variety of parameters of the general finite-gap formulas, corresponding to the degeneracy of X into an elliptic curve, lead to solutions describing different types of perturbations of the cnoidal wave (4.5.1). Just as in the preceding section, we can obtain multi-soliton-like solutions on the elliptic background as well as a multi-mode kind of perturbation of the cnoidal wave, described by explicit formulas. In this section we consider only multi-mode perturbations of (4.5.1). The perturbations of the multi-soliton type may be constructed and studied in the same manner.

We fix the number k by the condition $0 < k < 1$ and assume

$$
\begin{aligned}
&g = 2N+1\,, \quad N \geq 1\,, \quad 1 = E_1 < E_2 < \ldots < E_{g+1} = 1/k \quad , \\
&E_{g+2} = -E_1\,, E_{g+3} = -E_2, \ldots, E_{2g+2} = -E_{g+1} \quad ,
\end{aligned}
\tag{4.5.2}
$$

to hold. Now we fix the basis of cycles a_ν, b_ν according to Fig. 4.6.

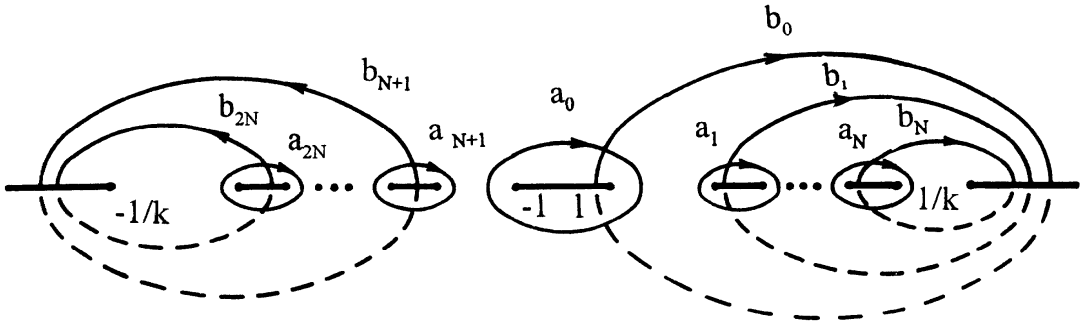

Fig. 4.6. Homology basis of cycles a_ν, b_ν, $0 < \nu < 1$ for the curve (4.5.2)

The cycles a_ν, b_ν are ordered by ν, $\nu = 0, \ldots, g-1 = 2N$. Consider the limit

$$\begin{aligned} &E_{2j},\, E_{2j+1} \to \lambda_j\,,\ j = 1, \ldots, N \quad ; \\ &1 < \lambda_1 < \lambda_2 < \ldots < \lambda_N < 1/k \quad . \end{aligned} \tag{4.5.3}$$

In such a limit

$$E_{2N+2+2j}\,,\ E_{2N+3+2j} \to \lambda_{N+j} = -\lambda_j$$

and the function $\mu = \sqrt{P_{2g+2}(\lambda)}$ tends to the function $\mu_0(\lambda)$:

$$\mu_0(\lambda) = \sqrt{(\lambda^2-1)(\lambda^2-k^{-2})}\prod_{j=1}^{N}(\lambda^2-\lambda_j^2) \quad .$$

The square root in the definition of μ_0 is assumed to be positive for the values of λ lying on the upper bank of the cut $[k^{-1}, \infty)$ on the upper sheet of the associated Riemann surface. We see that, in the limit under consideration, X degenerates into an elliptic curve X_0, defined by the equation

$$\omega^2 = (\lambda^2-1)(\lambda^2-k^{-2}) \quad .$$

Now let us study the limit behavior of the different objects in formula (4.1.22) for the general finite gap solution of the NS system. First, consider the behavior of the Abelian differentials ω_ν:

$$\begin{aligned} &\omega_\nu \to \omega_\nu^0(\lambda) = \frac{k\varphi_\nu^0(\lambda)\,d\lambda}{\prod_{\alpha>0}(\lambda-\lambda_\alpha)\sqrt{(\lambda^2-1)(k^2\lambda^2-1)}} \quad , \\ &\varphi_\nu^0(\lambda) = c_{\nu 1}^0\lambda^{2N} + c_{\nu 2}^0\lambda^{2N-1} + \ldots + c_{\nu g}^0 \quad . \end{aligned}$$

The normalization condition

$$\int_{a_\mu} \omega_\nu = 2\pi \mathrm{i}\delta_{\mu\nu} \quad ,$$

in our case, takes the form

$$\int_{a_0} \omega_\nu^0 = -2k \int_{-1}^{1} \frac{\varphi_\nu^0(\lambda)\, d\lambda}{\prod\limits_{\alpha>0}(\lambda-\lambda_\alpha)\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} = 2\pi\mathrm{i}\,\delta_{0\nu} \quad ,$$

$$\begin{aligned}\int_{a_\mu} \omega_\nu^0 &= -2\pi\mathrm{i}\,\mathrm{res}\,(\omega_\nu^0; \lambda_\mu) \\ &= -\frac{2\pi\mathrm{i}k}{\sqrt{(\lambda_\mu^2-1)(k\lambda_\mu^2-1)}} \frac{\varphi_\nu^0(\lambda_\mu)}{\prod\limits_{\substack{\alpha>0\\ \alpha\neq\mu}}(\lambda_\mu-\lambda_\alpha)} = 2\pi\mathrm{i}\,\delta_{\mu\nu}\,,\ \mu>0 \quad ,\end{aligned} \tag{4.5.4}$$

where in the last equality λ_μ is considered to be lying on the upper sheet of X_0. From (4.5.4) it turns out that

$$\begin{aligned}\varphi_0^0(\lambda) &= c_{01}^0 \prod_{\alpha>0}(\lambda-\lambda_\alpha) \quad , \\ \varphi_\nu^0(\lambda) &= (c_{\nu 1}^0\lambda + d_\nu) \prod_{\substack{\alpha>0\\ \alpha\neq\nu}}(\lambda-\lambda_\alpha)\,, \quad \nu>0 \quad .\end{aligned}$$

By virtue of $\lambda_{N+j} = -\lambda_j$ the constants $c_{\nu 1}^0$, d_ν satisfy the equalities

$$\begin{aligned}&-4kc_{01}^0 \int_0^1 \frac{d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} = 2\pi\mathrm{i} \quad , \\ &\frac{c_{j1}^0\lambda_j + d_j}{\sqrt{(\lambda_j^2-1)(1-k^2\lambda_j^2)}} = \frac{c_{N+j,1}^0\lambda_j - d_{N+j}}{\sqrt{(\lambda_j^2-1)(1-k^2\lambda_j^2)}} = -\frac{\mathrm{i}}{k} \quad , \\ &\int_{-1}^{1} \frac{c_{j1}^0\lambda + d_j}{(\lambda-\lambda_j)\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}\, d\lambda \\ &\quad = \int_{-1}^{1} \frac{c_{N+j,1}^0\lambda + d_{j+N}}{(\lambda+\lambda_j)\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}\, d\lambda = 0 \quad .\end{aligned} \tag{4.5.5}$$

From these equalities we get the important relation

$$c_{N+j,1}^0 = c_{j,1}^0\,, \quad d_{N+j} = -d_j \quad . \tag{4.5.6}$$

Let the quantities γ_j, $K(k)$, M_j, I_j by

$$\begin{aligned}\gamma_j &= \sqrt{(\lambda_j^2-1)(k^{-2}-\lambda_j)} > 0 \quad , \\ K(k) &= \int_0^1 \frac{d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} \quad , \\ M_j &= \int_{-1}^{1} \frac{1}{(\lambda-\lambda_j)} \frac{\lambda\, d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} \quad , \\ I_j &= \int_{-1}^{1} \frac{1}{(\lambda-\lambda_j)} \frac{d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} \quad .\end{aligned}$$

Now (4.5.5) leads to the following representation of $\omega_\nu^0(\lambda)$:

$$\begin{aligned}\omega_0^0(\lambda) &= -\frac{\pi \mathrm{i}}{2K}\frac{d\lambda}{\sqrt{(\lambda^2-1)(k^2\lambda^2-1)}} \quad ,\\ \omega_\nu^0(\lambda) &= \frac{c_{\nu 1}^0\lambda + d_\nu}{(\lambda-\lambda_\nu)}\frac{d\lambda}{\sqrt{(\lambda^2-1)(\lambda^2-k^{-2})}}, \quad \nu > 0 \quad ,\end{aligned} \tag{4.5.7}$$

$$\begin{aligned}&c_{j,1}^0 = \frac{\mathrm{i}}{2K}\gamma_j I_j\,, \quad d_j = -\frac{\mathrm{i}}{2K}\gamma_j M_j \quad ,\\ &c_{N+j,1}^0 = c_{j,1}^0\,, \quad d_{N+j} = -d_j \quad .\end{aligned} \tag{4.5.8}$$

Now we consider the limits of the vectors $\boldsymbol{V}$, $\boldsymbol{W}$ and of constant E. From (4.1.17) and (4.5.8) we obtain

$$\begin{aligned}&\boldsymbol{V} \to \boldsymbol{V}^0\,, \quad V_0^0 = -\frac{\pi\mathrm{i}}{kK}\,, \quad V_j^0 = -2\mathrm{i}\kappa_j = V_{N+j}^0 \quad ,\\ &\kappa_j = \mathrm{i}c_{j1}^0 = -\frac{\gamma_i I_j}{2K} \quad .\end{aligned} \tag{4.5.9}$$

To compute the limiting value $\boldsymbol{W}_0$ of $\boldsymbol{W}$ we recall

$$\begin{aligned}&c_{01}^0\prod_{\alpha>0}(\lambda-\lambda_\alpha) = c_{01}^0\lambda^{2N} - c_{01}^0\lambda^{2N-2}\,\frac{1}{2}\sum\lambda_j^2 + \ldots \quad ,\\ &(c_{\nu 1}^0\lambda + d_\nu)\prod_{\substack{\alpha>0\\ \alpha\neq\nu}}(\lambda-\lambda_\alpha) = c_{\nu 1}^0\lambda^{2N} + (c_{\nu 1}^0\lambda_\nu + d_\nu)\lambda^{2N-1} + \ldots \quad ,\end{aligned}$$

$\nu > 0$, and hence,

$$c_{02}^0 = 0\,, \quad c_{j2}^0 = c_{j1}^0\lambda_j + d_j = -c_{N+j,2}^0 \quad .$$

From the last relation, taking into account (4.1.17), (4.5.8) and the limit formula

$$c = \sum E_j = 0 \quad ,$$

we obtain:

$$\begin{aligned}W_0^0 = 0\,, \quad W_j^0 &= 4c_{j2}^0 == \frac{2\mathrm{i}\gamma_j}{K}(\lambda_j I_j - M_j) = -4\mathrm{i}\gamma_j \quad ,\\ W_{N+j}^0 &= -W_j^0 = 4\mathrm{i}\gamma_j \quad .\end{aligned} \tag{4.5.10}$$

The limit value E^0 of the constant E may be calculated from (4.1.17). The limit form of (4.1.17) is

$$E^0 = c - \frac{2}{\pi\mathrm{i}}\int_{-1}^{1}\lambda\omega_0^0(\lambda) + 2\sum_{\nu>0}\mathrm{res}\,(\lambda\omega_\nu^0(\lambda), \lambda_\nu) \quad ,$$

or, by virtue of the relations

$$c = 0, \quad f(\lambda) = -f(-\lambda), \quad f(\lambda) = \lambda\omega_0^0(\lambda)/d\lambda \quad ,$$

$$E^0 = 2\sum_{\nu>0} \operatorname{res}\left(\lambda\omega_\nu^0(\lambda);\ \lambda_\nu\right) = 2\sum \frac{1}{\mathrm{i}\gamma_j}(\lambda_j^2 c_{j1}^0 + \lambda_j d_j)$$
$$- 2\sum \frac{1}{\mathrm{i}\gamma_j}(\lambda_j^2 c_{N+j,1} - \lambda_j d_{N+j}) \quad .$$

From the last formula, taking into account (4.5.6), we conclude that

$$E^0 = 0 \quad . \tag{4.5.11}$$

Consider now the limits of the Abelian integrals Ω_2, Ω_3 and of the constants N, ω_0. Define the functions Ω_2^0, Ω_3^0 by the formulas

$$\Omega_2^0(\lambda) = 2\sqrt{(\lambda^2 - 1)(\lambda^2 - k^{-2})} \quad ,$$
$$\Omega_3^0(\lambda) = \frac{1}{2}\log\left(\lambda^2 - \frac{1+k^2}{2k^2} + \sqrt{(\lambda^2 - 1)(\lambda^2 - k^{-2})}\right) - \frac{1}{2}\log\frac{1-k^2}{2k^2} \quad .$$

It is not difficult to show that the functions Ω_2^0 and Ω_3^0 satisfy, as functions defined on $\overset{\circ}{X}$, the conditions

$$\text{(a.)} \quad \int_{a_\nu} d\Omega_{2,3} = 0 \quad ,$$
$$\text{(b.)} \quad \Omega_2^0(\lambda) = \pm(2\lambda^2 + \ldots) \quad ,$$
$$\Omega_3^0(\lambda) = \pm(\log\lambda + \ldots) \quad , \quad \lambda \to \infty^\pm \quad .$$

Expanding Ω_2^0 and Ω_3^0 at the neighborhoods of the points $\infty^\pm$ of the curve $\overset{\circ}{X}$, we find the following limit values for the constants N and ω_0:

$$N \to N^0 = -2\frac{k^2+1}{k^2}, \quad \omega_0 \to \omega_0^0 = \frac{1-k^2}{4k^2} \quad . \tag{4.5.12}$$

The Limit Values of the Vector $\boldsymbol{r}$. Let $\boldsymbol{r}_0$ be defined by the formula

$$\boldsymbol{r} \to \mathrm{i}\boldsymbol{r}_0 \quad ,$$

in the limit under consideration.

On the upper sheet of X_0 the following relation holds:

$$\int_{\infty^-}^{\infty^+} \omega_0^0(\lambda) = \frac{1}{2}\left[\int_{\infty^-}^{\infty^+} \omega_0^0(\lambda) + \int_{-\infty^-}^{-\infty^+} \omega_0^0(-\lambda)\right]$$
$$= \frac{1}{2}\left[\int_{\infty^-}^{\infty^+} \omega_0^0(\lambda) + \int_{-\infty^+}^{-\infty^-} \omega_0^0(\lambda)\right]$$
$$= -\frac{1}{2}\int_{a_0} \omega_0^0(\lambda) = -\pi\mathrm{i} \quad ,$$

$$-2\pi i = 2\pi i \quad \operatorname{res}(\omega_j^0; \lambda_j) = \int_{\infty^-}^{\infty^+} \omega_j^0(\lambda) + \int_{-\infty^+}^{-\infty^-} \omega_j^0(\lambda)$$
$$= \int_{\infty^-}^{\infty^+} \omega_j^0(\lambda) + \int_{\infty^+}^{\infty^-} \omega_j^0(-\lambda) = \int_{\infty^-}^{\infty^+} \omega_j^0(\lambda) + \int_{\infty^-}^{\infty^+} \omega_{j+N}^0(\lambda) \quad .$$

In the derivation of these relations we have used the equalities

$$\omega_0^0(-\lambda) = -\omega_0^0(\lambda)\,, \quad \omega_j^0(-\lambda) = -\omega_{N+j}^0(\lambda)\,,\ \lambda > 1/k\,, \tag{4.5.13}$$

where the automorphism $\lambda \to -\lambda$ has to be understood as acting in a same way as on the complex plane $\mathbb{C}$ on each sheet of X_0.

The meaning of the notation $\pm\infty^\pm$ is illustrated in Fig. 4.7.

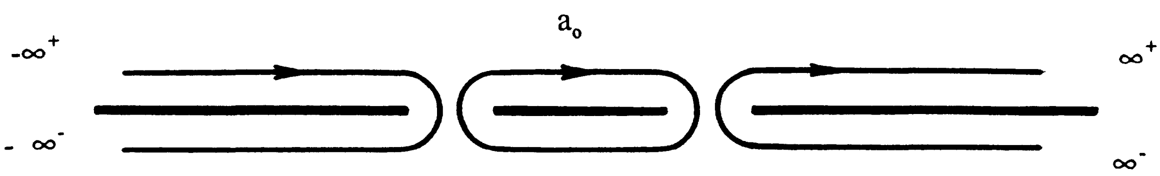

Fig. 4.7. The meaning of the notation $\pm\infty^\pm$ is illustrated

From the above we get the following expression for $\boldsymbol{r}_0$:

$$r_0^0 = -\pi\,, \quad r_j^0 = \frac{\gamma_j}{K} \int_{1/k}^{\infty} \frac{I_j\lambda - M_j}{(\lambda - \lambda_j)\sqrt{(\lambda^2 - 1)(\lambda^2 - k^{-2})}}\, d\lambda$$
$$= -r_{N+j}^0 - 2\pi \quad . \tag{4.5.14}$$

Limit Values of the B-Matrix. As in Sect. 4.4 we deduce that the real part of the diagonal elements of the matrix B has in the limit under consideration the following behavior:

$$\operatorname{Re} B_{\nu\nu} \to -\infty\,, \quad \nu > 0 \quad , \tag{4.5.15}$$

and at the same time B_{00} tends to a finite limit: [8]

$$B_{00} \to B_{00}^0 = -\frac{\pi}{K} \int_1^{1/k} \frac{1}{\sqrt{(\lambda^2 - 1)(1 - k^2\lambda^2)}}\, d\lambda =$$
$$-\frac{\pi K'}{K} \equiv \pi i \tau \quad . \tag{4.5.16}$$

For the elements B_{0j}, B_{jl}, $j \neq l$, and $B_{N+j,l}$ we obtain the limit values:

[8] We use the standard notation [4.8] k, k', K, K' for the main elliptic integrals and their modules.

$$\begin{aligned}
B^0_{0j} = B^0_{j0} &= 2\int_{\lambda_j}^{1/k} \omega^0_0(\lambda) \\
&= -\frac{\pi}{K}\int_{\lambda_j}^{1/k} \frac{1}{\sqrt{(\lambda^2-1)(1-k^2\lambda^2)}}\, d\lambda \equiv 2\pi \mathrm{i}\tau_{0j} \quad , \\
B^0_{jl} = B^0_{lj} &= 2\int_{\lambda_l}^{1/k} \omega^0_j(\lambda) \\
&= \frac{\gamma_j}{K}\int_{\lambda_l}^{1/k} \frac{I_j\lambda - M_j}{(\lambda-\lambda_j)\sqrt{(\lambda^2-1)(k^{-2}-\lambda^2)}}\, d\lambda \equiv 2\tau_{lj}\,,\ j<l \quad ; \\
B^0_{N+j,l} = B^0_{l,N+j} &= 2\int_{\lambda_l}^{1/k} \omega_{N+j}(\lambda) \\
&= \frac{\gamma_j}{K}\int_{\lambda_l}^{1/k} \frac{I_j\lambda + M_j}{(\lambda+\lambda_j)\sqrt{(\lambda^2-1)(k^{-2}-\lambda^2)}}\, d\lambda \equiv 2\tau_{N+j,l} \quad .
\end{aligned} \tag{4.5.17}$$

Taking into account that for $1 < |\lambda| < 1/k$ the relations (4.5.13) must be replaced by

$$\omega^0_0(-\lambda) = \omega^0_0(\lambda)\,, \quad \omega^0_j(-\lambda) = \omega^0_{N+j}(\lambda) \quad .$$

We conclude that the following relations hold:

$$\begin{aligned}
B^0_{N+j,0} = B^0_{0,N+j} &= 2\int_{-\lambda_j}^{-1/k} \omega^0_0(\lambda) \\
&= 2\int_{\lambda_l}^{1/k} \omega^0_0(\lambda) = B_{0j} = 2\pi\mathrm{i}\tau_{0j} \\
B^0_{N+j,N+l} = B^0_{N+l,N+j} &= 2\int_{-\lambda_l}^{-1/k} \omega^0_{N+j}(\lambda) \\
&= 2\int_{\lambda_l}^{1/k} \omega^0_{N+j}(-\lambda) = 2\int_{\lambda_l}^{1/k} \omega^0_j(\lambda) = B^0_{lj} = 2\tau_{lj}\,;\ j<l \quad ,
\end{aligned} \tag{4.5.18}$$

$$\begin{aligned}
\tau_{N+l,j} &= \frac{1}{2}B^0_{N+l,j} = \int_{-\lambda_l}^{-1/k} \omega^0_j(\lambda) = \int_{\lambda_l}^{1/k} \omega^0_{N+j}(\lambda) \\
&= \frac{1}{2}B^0_{l,N+j} = \tau_{N+j,l} \quad .
\end{aligned} \tag{4.5.19}$$

Formulas (4.5.18-19) complete the calculation of the limit values of the different parameters of (4.1.22).

Define D_ν and D_0 as follows:

$$D_\nu = \frac{1}{2}B_{\nu\nu} + 2\eta_\nu\,, \quad \nu > 0\,; \quad D_0 = 2\pi\mathrm{i}\eta_0 \quad ,$$

where η_0, η_ν, $\nu > 0$ are chosen to be arbitrary constants invariant with respect to degeneration $X \to X_0$. The same consideration as in Sect. 4.4 shows that the non-zero counterpart in the series defining $\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D}-\varepsilon\boldsymbol{r})$, $\varepsilon=-1,\,0,\,1$ is generated by terms corresponding to the vectors $\boldsymbol{m}\in\mathbb{Z}^g$ with the property

$$-\infty < m_0 < \infty\,, \quad m_\nu = 0,1\,, \quad \nu \geq 1 \quad .$$

Now from (4.5.9-10, 17-18) we conclude that in the limit (4.5.3) theta series entering in (4.1.22) become transformed into a finite sum of one-dimensional theta functions:

$$\begin{aligned}
&\theta(\mathrm{i}\boldsymbol{V}x+\mathrm{i}\boldsymbol{W}t-\boldsymbol{D}-\varepsilon\boldsymbol{r}) \to \theta_\varepsilon(x,t)\\
&\quad= \sum_{m_\nu=0,1} \vartheta_3\left(-\frac{\mathrm{i}x}{2kK}-\eta_0-\frac{\varepsilon}{2}+\sum_j \tau_{0j}(m_j+m_{N+j})\,\middle|\,\frac{\tau}{2}\right)\\
&\quad\times\exp\Big\{2\sum_{l>j}\tau_{lj}(m_l m_j+m_{N+l,N+j})+2\sum_{l,j}\tau_{N+j,l}m_{N+j}m_l\\
&\quad+2\mathrm{i}\sum_j(-i\kappa_j x)(m_j+m_{N+j})+4\sum_j\gamma_j t(m_j-m_{N+j})\\
&\quad-2\sum_j(\eta_j m_j+\eta_{N+j}m_{N+j})-\mathrm{i}\varepsilon\sum_j r_j^0(m_j-m_{N+j})\Big\}\,,
\end{aligned} \tag{4.5.20}$$

where $\vartheta_3(p\,|\,\tau)$ is the third Jacobi elliptic function

$$\vartheta_3(p\,|\,\tau)=\sum_{m_0=-\infty}^{\infty}\exp\{\pi\mathrm{i}m_0^2\tau+2\pi\mathrm{i}m_0 p\} \quad .$$

As a result we get the following solution of the NS system:

$$\begin{aligned}
y_N(x,t)&=A\,\frac{\vartheta_{-1}(x,t)}{\vartheta_0(x,t)}\exp\left(-2\mathrm{i}\frac{k^2+1}{k^2}t\right)\,,\\
y_N^*(x,t)&=\frac{1-k^2}{Ak^2}\,\frac{\vartheta_1(x,t)}{\vartheta_0(x,t)}\exp\left(2\mathrm{i}\frac{k^2+1}{k^2}t\right)\,,
\end{aligned} \tag{4.5.21}$$

where the functions $\vartheta_\varepsilon(x,t)$ are defined by (4.5.20) and the formulas (4.5.11-12) for E, N and ω_0 are taken into account. The solution (4.5.21) is dependent on N real parameters λ_j, $2N+1$ complex numbers η_0, η_j, η_{N+j} and a complex number A. It describes the interaction of $2N$ "solitons" of the system (4.1.1) on the background of one-gap periodic solution of the same system. More precisely the constructed solution describes some rather special type of soliton interactions, related to the restriction $\lambda_{N+j}=-\lambda_j$, imposed on the values of λ_j. The reason for imposing the last restriction, leading directly to some symmetries in the structure of V_ν^0, W_ν^0, r_ν^0 and $B_{\mu\nu}^0$, is to obtain an elliptic analog of the solutions of the NS equation considered in Sect. 4.4.

Now let us replace x by ix in (4.5.21) and impose the following restrictions on the parameters η_ν and A:

$$|A| = \frac{1}{k}\sqrt{1-k^2}\,, \quad \operatorname{Im}\eta_0 = -\mathrm{i}\sum_j \tau_{0j}\,, \quad \operatorname{Im}\eta_j = \operatorname{Im}\eta_{N+j} \quad ,$$
$$\operatorname{Re}\eta_j + \operatorname{Re}\eta_{N+j} = \sum_{l=j+1}^{N} \tau_{lj} + \sum_{l=1}^{j-1} \tau_{jl} + \sum_{l} \tau_{N+j,l} \quad . \tag{4.5.22}$$

Taking into account that the quantities τ, τ_{0j} are purely imaginary and that the $\tau_{\ell,j}$, $\tau_{N+j,\ell}$, κ_j, γ_j, r_j^0 are real, reproducing all steps of the proof of Proposition 4.14 we get: [9]

Proposition 4.19. Under the conditions (4.5.22) imposed on the parameters in the formulas (4.5.21) the equality

$$\overline{y}_N(\mathrm{i}x, t) = y_N^*(\mathrm{i}x, t)\,, \quad x, t \in \mathbb{R}$$

holds.

Now we get the main theorem of this section:

Theorem 4.20. Let

$$\begin{gathered} N \geq 1\,, \quad N \in \mathbb{Z}; \quad x_0, k, \varphi \in \mathbb{R}\,, \quad 0 < k < 1 \quad ; \\ 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_N < 1/k\,, \quad x_{0j}, t_{0j} \in \mathbb{R}\,, \quad j = 1, \ldots, N \quad . \end{gathered}$$

The equation

$$\mathrm{i}v_t + v_{xx} + 2|v|^2 v = 0 \tag{4.5.23}$$

allows thus for the smooth (with respect to x and t variables) solution defined by the formula

$$v_N(x,t) = k^{-1}\sqrt{1-k^2}\frac{\vartheta_1(x,t)}{\vartheta_0(x,t)} \exp\left\{2\mathrm{i}\frac{k^2+1}{k^2}t + \mathrm{i}\varphi\right\} \quad , \tag{4.5.24}$$

[9] This procedure includes making now reference to (4.5.19) instead of (4.4.25) as previously.

$$
\begin{aligned}
\theta_\varepsilon(x,t) \equiv & \theta_\varepsilon(\mathrm{i}x,t) \\
= & \sum_{\substack{m_\nu=0,1 \\ \nu=1,\dots,2N}} \vartheta_3\Big(\frac{x-x_0}{2kK} - \sum_j \tau_{0j} \\
& + \sum_j \tau_{0j}(m_j+m_{N+j}) + \frac{1-\varepsilon}{2}\Big|\frac{\tau}{2}\Big) \\
& \times \exp\Big\{2\sum_{l>j}\tau_{lj}(m_l m_j + m_{N+l}m_{N+j}) + 2\sum_{lj}\tau_{N+j,\ell}m_{N+j}m_l \\
& + 2\mathrm{i}\sum_j \kappa_j(x-x_{0j})(m_j+m_{N+j}) \\
& + 4\sum_j \gamma_j(t-t_{0j})(m_j-m_{N+j}) \\
& - \sum_j \tau_j(m_j+m_{N+j}) - \mathrm{i}\varepsilon\sum_j r_j^0(m_j-m_{N+j})\Big\}\,, \quad \varepsilon=0,1 \quad ,
\end{aligned} \tag{4.5.25}
$$

where

$$
\begin{aligned}
\tau_{lj} &= \frac{\gamma_j}{2K}\int_{\lambda_l}^{1/k} \frac{I_j\lambda - M_j}{(\lambda-\lambda_j)\sqrt{(\lambda^2-1)(k^{-2}-\lambda^2)}}\,d\lambda\,, \quad l>j \quad , \\
\tau_{N+j,l} &= \frac{\gamma_j}{2K}\int_{\lambda_l}^{1/k} \frac{I_j\lambda + M_j}{(\lambda+\lambda_j)\sqrt{(\lambda^2-1)(k^{-2}-\lambda^2)}}\,d\lambda \quad , \\
I_j &= \int_{-1}^{1} \frac{d\lambda}{(\lambda-\lambda_j)\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} \quad , \\
M_j &= \int_{-1}^{1} \frac{\lambda\,d\lambda}{(\lambda-\lambda_j)\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}} \quad ,
\end{aligned}
$$

and r_j^0 and τ_{0j} are defined as in (4.5.14) and (4.5.17), respectively,

$$
\begin{aligned}
\tau_j &= \sum_{l=j+1}^{N}\tau_{lj} + \sum_{l=1}^{j-1}\tau_{jl} + \sum_l \tau_{N+j,l} \quad , \\
\kappa_j &= -\frac{\gamma_j I_j}{2K}\,, \quad \gamma_j = \sqrt{(\lambda_j^2-1)(k^{-2}-\lambda_j^2)} \quad .
\end{aligned}
$$

Remark 4.21. Performing the change of variable $\lambda \to u$,

$$
u = \int_0^\lambda \frac{1}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}\,d\lambda\,, \quad \lambda = \mathrm{sn}\,(u,k) \quad ,
$$

uniformizing the curve X_0, (compare with Chap. 1), it is possible to rewrite the parameters of (4.5.25) in the following way:

$$
\begin{aligned}
\tau_{0j} &= \tau/2 - \mathrm{i}u_j/2K \quad , \\
\tau_{lj} &= \frac{\pi\mathrm{i}\tau}{2} + \frac{\pi}{2K}(u_l + u_j) - \log\frac{\vartheta_1\big(\mathrm{i}(u_j+u_l)/4K \,|\, \tau/2\big)}{\vartheta_1\big(\mathrm{i}(u_l-u_j)/4K \,|\, \tau/2\big)} \quad , \\
& \qquad\qquad l > j \quad , \\
\tau_{N+j,l} &= \frac{\pi\mathrm{i}\tau}{2} + \frac{\pi}{2K}(u_l + u_j) - \log\frac{\vartheta_2\big(\mathrm{i}(u_j+u_l)/4K \,|\, \tau/2\big)}{\vartheta_2\big(\mathrm{i}(u_j-u_l)/4K \,|\, \tau/2\big)} \quad , \\
r_j^0 &= \arctan\frac{-\mathrm{i}\,\mathrm{sn}\,(\mathrm{i}u_j,k)}{1+\mathrm{dn}\,(\mathrm{i}u_j,k)} \quad (\mathrm{mod}\,2\pi) \quad , \\
\gamma_j &= -\,\frac{1-k^2}{k}\mathrm{i}\frac{\mathrm{sn}\,(\mathrm{i}u_j,k)}{\mathrm{dn}^2(\mathrm{i}u_j,k)} \quad , \\
\kappa_j &= -\,\frac{1-k^2}{k}\mathrm{i}\frac{\mathrm{sn}\,(\mathrm{i}u_j,k)}{\mathrm{cn}\,(\mathrm{i}u_j,k)\,\mathrm{dn}\,(\mathrm{i}u_j,k)} \\
& \quad - \frac{\mathrm{i}}{kK}\Pi(K\,,\,\mathrm{i}u_j + K + \mathrm{i}K') \quad ,
\end{aligned}
\tag{4.5.26}
$$

where $\vartheta_{1,2}(p\,|\,\tau)$ are the Jacobi theta functions and $\Pi(u,a)$ is the canonical elliptic integral of the third kind:

$$
\begin{aligned}
\Pi(u,a) &= \int_0^u \frac{k^2\,\mathrm{sn}\,a\;\mathrm{cn}\,a\;\mathrm{dn}\,a\;\mathrm{sn}^2 u}{1-k^2\mathrm{sn}^2 a\;\mathrm{sn}^2 u}\,du \\
&= \frac{1}{2}\log\frac{\vartheta_0(u-a)/2K\,|\,\tau)}{\vartheta_0(u+a)/2K\,|\,\tau)} + u\left[\frac{d}{du}\log\vartheta_0(u/2K\,|\,\tau)\right]_{u=a} \quad .
\end{aligned}
$$

The free parameters u_j in (4.5.26) satisfy the conditions

$$0 < u_1 < u_2 < \ldots < u_N < K' \quad .$$

and are related to the initial parametrization by the formula

$$\lambda_j = \mathrm{sn}\,(\mathrm{i}u_j + K, k) \quad .$$

The solution (4.5.24) represents a quasi-periodic function of x with the following group of the real periods:

$$T_0 = 2kK\,, \quad T_j = \pi/\kappa_j\,, \quad j = 1,\ldots,N \quad .$$

This solution describes the phenomena of modulation instability of the cnoidal wave solution (4.5.1). Exactly as in Sect. 4.4, the following asymptotic formulas, describing the behavior of (4.5.24) for $t \to \pm\infty$ can be proved:

Theorem 4.22. Let the parameters λ_j be chosen in agreement with the inequalities

$$\max_j \gamma_j < \min_j \gamma_j \equiv 2\gamma_0 \quad . \tag{4.5.27}$$

Then the following asymptotic estimate holds:

$$
\begin{aligned}
v_N(x,t) &= \frac{1+k}{k}\,\mathrm{dn}\left((x-x_0)\frac{1+k}{k}\,,\,\frac{2\sqrt{k}}{1+k}\right)\\
&\times\Big[1+\sum\left(A_j^{\mp}\exp\{-2\mathrm{i}\kappa_j(x-x_0)\}+B_j^{\mp}\exp\{2\mathrm{i}\kappa_j(x-x_0)\}\right)\\
&\times\exp\{\pm4\gamma_j(t-t_{0j})\}+O(\exp\{-8\gamma_0|t|\})\Big]\\
&\times\exp\left(2\mathrm{i}\frac{1+k^2}{k^2}t+\mathrm{i}\varphi+\mathrm{i}\varphi^{\mp}\right)\ ,\quad t\to\mp\infty\quad ,
\end{aligned}
$$

where

$$
\begin{aligned}
A_j^{\mp}\equiv A_j^{\mp}(x/T_0) &= \frac{\vartheta_3\left(\dfrac{x-x_0}{2kK}-\tau_{0j}\,\Big|\,\dfrac{\tau}{2}\right)}{\vartheta_3\left(\dfrac{x-x_0}{2kK}\,\Big|\,\dfrac{\tau}{2}\right)}\exp\{\Delta_j\mp\mathrm{i}r_j^0\}\\
&-\frac{\vartheta_0\left(\dfrac{x-x_0}{2kK}-\tau_{0j}\,\Big|\,\dfrac{\tau}{2}\right)}{\vartheta_0\left(\dfrac{x-x_0}{2kK}\,\Big|\,\dfrac{\tau}{2}\right)}\exp\{\Delta_j\}\quad ,
\end{aligned}
$$

$$
\begin{aligned}
B_j^{\mp}\equiv B_j^{\mp}(x/T_0) &= \frac{\vartheta_3\left(\dfrac{x-x_0}{2kK}+\tau_{0j}\,\Big|\,\dfrac{\tau}{2}\right)}{\vartheta_3\left(\dfrac{x-x_0}{2kK}\,\Big|\,\dfrac{\tau}{2}\right)}\exp\{\Delta_j\mp\mathrm{i}r_j^0\}\\
&-\frac{\vartheta_0\left(\dfrac{x-x_0}{2kK}+\tau_{0j}\,\Big|\,\dfrac{\tau}{2}\right)}{\vartheta_0\left(\dfrac{x-x_0}{2kK}\,\Big|\,\dfrac{\tau}{2}\right)}\exp\{\Delta_j\}\quad ,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_j &= \sum_l\tau_{N+j,l}-\sum_{l=j+1}^{N}\tau_{lj}-\sum_{l=1}^{j-1}\tau_{jl}\quad ,\\
\varphi^{\mp} &= \pm\sum_j r_j^0\quad .
\end{aligned}
$$

Remark 4.23. Omitting, as in Sect. 4.4, the condition (4.5.27) we can see that the rougher result still holds:

$$v_N(x,t) = \frac{1+k}{k}\,\mathrm{dn}\left((x-x_0)\frac{1+k}{k}, \frac{2\sqrt{k}}{1+k}\right)$$
$$\times \exp\left(2\mathrm{i}\frac{1+k^2}{k^2}t + \mathrm{i}\varphi + \mathrm{i}\varphi^{\mp}\right)[1+o(1)]\,, \quad t \to \mp\infty \quad ,$$
$$\Delta\varphi = \varphi^+ - \varphi^- = -2\sum_j \arctan\frac{-\mathrm{i}\,\mathrm{sn}\,(\mathrm{i}u_j, k)}{1+\mathrm{dn}\,(\mathrm{i}u_j, k)} \quad .$$

4.6 Alternative Approach to the Finite-Gap Integration of Matrix Systems

In this section, we follow in essence the scheme presented in [4.5] and describe a construction which enables one to obtain all essential ingredients of the approach represented in Sect. 4.1-2 especially the Riemann surfaces, the Baker-Akhiezer functions and so on, in a purely deductive way. The only restriction appearing for the class of solutions of the NEE for which this construction is formulated is the assumption that there exists a matrix valued rational (in λ) function $L(\lambda)$ satisfying the system

$$\begin{aligned} L_x &= [U, L] \quad , \\ L_t &= [V, L] \quad . \end{aligned} \tag{4.6.1}$$

Let us start with the consideration of the NS system (4.1.1). Without loss of generality, $L(\lambda)$ may be assumed to be a polynomial in λ:

$$L(\lambda) \equiv L(\lambda;\; x,\, t) = \sum_{j=0}^{N} \lambda^{N-j} L_j(x,t) \quad . \tag{4.6.2}$$

$\mathrm{Tr}\, L(\lambda)$, $\det L(\lambda)$ and, more generally, the coefficients of the characteristic polynomial $Q(\mu, \lambda) = \det(L(\lambda) - \mu I)$ are obviously the integrals of motion of the system (4.6.1).

Without loss of generality we may assume that

$$\mathrm{Tr}\, L(\lambda) = 0 \quad . \tag{4.6.3}$$

The general case may be easily reduced to satisfy such a requirement by subtraction of some x and t independent diagonal matrix from L.

Now an equation

$$Q(\lambda,\, \mu) = 0 \tag{4.6.4}$$

determines some algebraic curve X – a spectral curve of $L(\lambda)$. Due to (4.6.3), equation (4.6.4) may be rewritten in the form

$$\mu^2 = -\det L(\lambda) \equiv A^2(\lambda) + B(\lambda)\, C(\lambda) \quad , \tag{4.6.5}$$

where A, B, C are determined by the structure of L:

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} \quad . \tag{4.6.6}$$

From the system (4.6.1) and (4.1.5) we get for the coefficient L_0 in (4.6.2) the relation

$$[\sigma_3, L_0] = 0\,, \quad L_{0x} = L_{0t} = 0 \quad .$$

One of the natural choices of L_0, satisfying the above relation, is

$$L_0 = -\sigma_3 \tag{4.6.7}$$

Consequently in the situation of "general position" (4.6.4) takes the form

$$\mu^2 = \lambda^{2N} + \ldots = \prod_{j=1}^{2N} (\lambda - E_j)\,, \quad E_j \in \mathbb{C}\,, \quad E_j \neq E_k \quad .$$

Assuming $N \equiv g+1$ we reconstruct the Riemann surface X which was a starting point in the construction of the finite-gap solutions of the system (4.1.1). Our next task is to show that the Baker-Akhiezer function $\psi(P, x, t)$, introduced axiomatically in Sect. 4.1, appears quite naturally in the approach of this section as a common eigenfunction of three commuting operators $\partial_x - U(\lambda)$, $\partial_t - V(\lambda)$ and $L(\lambda)$.

Let $\widehat{\Psi}(\lambda, x, t)$ be a matrix solution of (4.1.6) normalized by the condition

$$\widehat{\Psi}(\lambda, 0, 0) = I \quad . \tag{4.6.8}$$

The eigenvectors of the matrix $L(\lambda, x, t)$, corresponding to the eigenvalue μ, $P = (\mu, \lambda) \in X$, normalized by the condition $h_1 \equiv 1$, imposed on its first component, would be denoted by $\boldsymbol{h}(P, x, t)$. The matrix function $\widehat{\Psi}$ defined above is an entire function of λ. The vector function $\boldsymbol{h}(P, x, t)$ is a meromorphic function of P on X. Hence the meromorphic function $\psi(P, x, t)$

$$\psi(P, x, t) = \widehat{\Psi}(\lambda, x, t)\, \boldsymbol{h}(P, 0, 0)$$

is defined correctly on X. $\psi(P, x, t)$ satisfies the following equations:

$$\begin{aligned} \psi_x(P, x, t) &= U(\lambda, x, t)\psi(P, x, t) \\ \psi_t(P, x, t) &= V(\lambda, x, t)\psi(P, x, t) \\ L(\lambda, x, t)\, \psi(P, x, t) &= \mu\psi(P, x, t) \quad . \end{aligned} \tag{4.6.9}$$

We need to comment only on the last of these equations, which may be easily checked by using the relation

$$L(\lambda, x, t)\widehat{\Psi}(\lambda, x, t) = \widehat{\Psi}(\lambda, x, t)\, L(\lambda, 0, 0) \quad ,$$

following directly from the system (4.6.1) and the uniqueness of the solution $\widehat{\Psi}$ of the system (4.1.6) satisfying (4.6.8).

Next we show that $\psi(P, x, t)$ possesses all the properties of the Baker-Akhiezer function discussed in Sect. 4.1. $\boldsymbol{h}(P, x, t)$ may be represented in the form

$$\boldsymbol{h} = \begin{pmatrix} 1 \\ \left(\frac{\mu - A(\lambda,x,t)}{B(\lambda,x,t)}\right) \end{pmatrix} , \quad P = (\mu, \lambda) \quad . \tag{4.6.10}$$

Equation (4.6.5) shows that the zeros $\lambda_j(x,t)$ of the polynomial $B(\lambda)$ satisfy the relation

$$\mu_j^2 = A^2(\lambda_j) \quad .$$

Now it is clear that the poles of $\boldsymbol{h}(P, x, t)$ are located at the points $P_j(x, t)$ defined by

$$\begin{aligned} &P_j(x,t) = \left(\mu_j^-(x,t)\, , \, \lambda_j(x,t)\right) \quad , \\ &B(\lambda;\, x, t)\big|_{\lambda=\lambda_i} = 0 \quad , \\ &\mu_j^-(x,t) = -A(\lambda, x, t)\big|_{\lambda=\lambda_j(x,t)} \quad . \end{aligned}$$

In the general situation, the number of poles coincides with the degree of the polynomial $B(\lambda)$, i.e., with the genus of X and the condition $\pi(P_j) \neq \pi(P_k)$ of nonspeciality of the divisor $\mathcal{D}$ holds (here as in Sect. 4.1 $\pi(P) = \lambda$). So we have shown that the divisor of poles of ψ in the finite part of X is

$$\mathcal{D} = \sum_{j=0}^{g} P_j(0,0) \quad .$$

Let us now turn to the study of the singularities of $\psi(P, x, t)$ at the points $\infty^\pm$. A priori it is evident that these points are the essential singularities of ψ. The precision of the corresponding behavior of ψ may be realized as follows: by virtue of the third of the equations in (4.6.9), we have:

$$\psi(P, x, t) = \varphi(P, x, t)\, \boldsymbol{h}(P, x, t) \quad ,$$

φ being a scalar function depending on $P \in X$. Let us introduce [compare with (4.1.18)] the matrix valued function $\Psi(\lambda, x, t)$:

$$\Psi(\lambda, x, t) = \left(\psi(P^+)\, , \, \psi(P^-)\right) \quad . \tag{4.6.11}$$

Ψ is correctly defined in some neighborhood of infinity on $C\mathbb{P}^1$. By virtue of (4.6.11) we may factorize Ψ as follows:

$$\Psi(\lambda, x, t) = H(\lambda, x, t)\,\Phi(\lambda, x, t) \quad ,$$

where

$$H(\lambda, x, t) = \left(\boldsymbol{h}(P^+, x, t)\,,\ \boldsymbol{h}(P^-, x, t)\right) \quad ,$$

$$\Phi(\lambda, x, t) = \begin{pmatrix} \varphi(P^+, x, t) & 0 \\ 0 & \varphi(P^-, x, t) \end{pmatrix} \equiv \begin{pmatrix} d_1(\lambda, x, t) & 0 \\ 0 & d_2(\lambda, x, t) \end{pmatrix} \quad .$$

From the relations (4.6.5-7), we get for the polynomial $A(\lambda)$

$$\begin{aligned} A(\lambda, x, t) = &- \lambda^{g+1} + \frac{c}{2}\lambda^g \\ &+ \left(\frac{B_0(x,t)\,C_0(x,t)}{2} + \frac{c^2}{8} - \frac{d}{2}\right)\lambda^{g-1} + \ldots \quad , \end{aligned} \tag{4.6.12}$$

where $B_0(x,t)$, $C_0(x,t)$, c and d are determined from the expansions

$$\begin{aligned} &B(\lambda, x, t) = \lambda^g B_0(x,t) + \ldots\,; \quad C(\lambda, x, t) = \lambda^g C_0(x,t) + \ldots \quad , \\ &\mu^2 = \lambda^{2g+2} - c\lambda^{2g+1} + d\lambda^{2g} + \ldots \quad . \end{aligned} \tag{4.6.13}$$

We also have the identities

$$B_0(x,t) = y(x,t)\,, \quad C_0(x,t) = -y^*(x,t) \quad , \tag{4.6.14}$$

which follow easily from (4.6.1). The relations (4.6.10,12-14) lead to the following asymptotic expansion for $H(\lambda, x, t)$:

$$H(\lambda) = \left[I + \frac{1}{\lambda}H_1 + \frac{1}{\lambda^2}H_2 + \ldots\right]\begin{pmatrix} 0 & 1 \\ 2\lambda/y & 0 \end{pmatrix} \quad , \tag{4.6.15}$$

where $(H_1)_{11} = 0$, $(H_1)_{12} = y/2$, $(H_1)_{21} = y^*/2$. The substitution of Ψ, given by (4.6.11), into the system (4.6.9), taking into account (4.6.15), gives [10]

$$\begin{aligned} &\frac{\partial}{\partial x}\log\, d_1 = \mathrm{i}\lambda + \frac{\partial}{\partial x}\log y + O(1/\lambda) \quad , \\ &\frac{\partial}{\partial x}\log\, d_2 = -\mathrm{i}\lambda + O(1/\lambda) \quad , \\ &\frac{\partial}{\partial t}\log\, d_1 = 2\mathrm{i}\lambda^2 + \frac{\partial}{\partial t}\log y + O(1/\lambda) \quad , \\ &\frac{\partial}{\partial t}\log\, d_2 = -2\mathrm{i}\lambda^2 + O(1/\lambda) \quad . \end{aligned}$$

Now, taking into account the identities

$$d_1(\lambda, 0, 0) \equiv d_2(\lambda, 0, 0) \equiv 1 \quad ,$$

following from the definition of the function ψ, we get

[10] Roughly speaking our consideration is an inversion of arguments used in the proof of the Lemmas 4.1-2.

$$\begin{aligned} d_1(\lambda, x, t) &= \frac{y(x,t)}{y(0,0)} \exp\{i\lambda x + 2i\lambda^2 t\} \left(1 + O(1/\lambda)\right) \quad , \\ d_2(\lambda, x, t) &= \exp\{-i\lambda x - 2i\lambda^2 t\} \left(1 + O(1/\lambda)\right) \ , \quad \lambda \to \infty \quad . \end{aligned} \tag{4.6.16}$$

Formulas (4.6.11, 15-16) certify the validity of the following asymptotics for $\Psi(\lambda, x, t)$:

$$\Psi(\lambda, x, t) = \left(I + \sum_{k=1}^{\infty} \Psi_k \lambda^{-\kappa}\right) \exp\left\{-i\lambda x \sigma_3 - 2i\lambda^2 t \sigma_3\right\} \begin{pmatrix} 0 & 1 \\ 2\lambda/y(0,0) & 0 \end{pmatrix}$$

$$\lambda \to \infty \quad .$$

This asymptotics is equivalent (here we use the inverted variant of the considerations in Sect. 4.1 once again) to the asymptotic estimates (4.1.15) for $\psi(P, x, t)$, i.e., the condition (II). By this step the derivation of the properties (I-II) of the Baker-Akhiezer function for the NS model is achieved.

Therefore we have established all the properties of the Baker-Akhiezer function for the NS model.

Before turning to the sine-Gordon case, let us note

$$L(\lambda) = \Psi(\lambda, x, t) \mu \sigma_3 \Psi^{-1}(\lambda, x, t) \quad , \tag{4.6.17}$$

reconstructing the matrix L by means of the known Baker-Akhiezer function.

In the framework of the axiomatic approach of Sect. 4.1, when the matrix Ψ is fixed a priori by (4.1.18), the polynomial structure of right-hand-side of (4.6.17) may be checked by copying the proof of the polynomial structure of $\Psi_x(\lambda, x, t)\Psi^{-1}(\lambda, x, t)$ and $\Psi_t(\lambda, x, t)\Psi^{-1}(\lambda, x, t)$ in Sect. 4.1. In other words the deductive approach of this section and the axiomatic approach of Sect. 4.1 lead to the same variety of the finite-gap solutions of the NS system.

Now let us turn to the sine-Gordon equation. It is convenient to change the gauge of the corresponding U-V pair, performing the gauge transformation

$$\Psi(\lambda) \to \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Psi(\lambda) \quad ,$$

which is equivalent to fixing U and V in the form

$$\begin{aligned} U(\lambda) &= -i\lambda\sigma_1 - i\frac{v_x}{2}\sigma_3 \quad , \\ V(\lambda) &= -\frac{i}{\lambda} \begin{pmatrix} 0 & \exp(-iv) \\ \exp(iv) & 0 \end{pmatrix} \quad . \end{aligned} \tag{4.6.18}$$

The constraints (4.2.6) (reduction restriction) take, in the new gauge, the form

$$\begin{aligned} U(\lambda) &= \sigma_3 U(-\lambda) \sigma_3 \quad , \\ V(\lambda) &= \sigma_3 V(-\lambda) \sigma_3 \quad . \end{aligned} \tag{4.6.19}$$

We retain the old notation for Ψ.

Without loss of generality the matrix $L(\lambda)$ may be chosen in the same traceless polynomial form as for the NS model. From the system (4.6.1), for the highest coefficient L_0, and for the lowest coefficient L_N, we get the relations

$$[\sigma_1, L_0] = 0\,, \quad L_{0x} = L_{0t} = 0\,, \quad \left[L_N\,, \begin{pmatrix} 0 & \exp(-\mathrm{i}v) \\ \exp(\mathrm{i}v) & 0 \end{pmatrix}\right] = 0 \quad .$$

Accordingly, we can assume that L_0 satisfies the relation

$$L_0 = \sigma_1 \quad , \tag{4.6.20}$$

and L_N must be of the form

$$L_N = \begin{pmatrix} 0 & m^2 \\ n^2 & 0 \end{pmatrix}\,, \quad \frac{m}{n} = \exp(-\mathrm{i}v) \quad . \tag{4.6.21}$$

From (4.6.19) we see that both $L(\lambda)$ and $\sigma_3 L(-\lambda)\sigma_3$ are solutions of (4.6.1). The transformation

$$L(\lambda) \to L(\lambda) - \sigma_3 L(-\lambda)\sigma_3 \tag{4.6.22}$$

allows us to suppose that the reduction relation

$$\sigma_3 L(-\lambda)\sigma_3 = -L(\lambda) \tag{4.6.23}$$

holds.

Now from (4.6.20) it follows that the degree of the polynomial $L(\lambda)$ must be even in N,

$$N = 2g$$

Retaining the same notations for the matrix elements of L as used for NS model, we get the defining equation for the spectral curve X

$$(\mu')^2 = -\det\, L(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^{4g} + \ldots \quad . \tag{4.6.24}$$

Due to (4.6.23), the determinant of L satisfies $\det L(\lambda) = \det L(-\lambda)$. Hence the curve X possesses the involution

$$\tau \;:\; (\mu', \lambda) \to (-\mu', -\lambda) \equiv (\mu', \lambda)^\tau$$

and the defining equation (4.6.24) must be of the form

$$(\mu')^2 = \prod_{j=1}^{2g} (\lambda^2 - E_j) \quad .$$

The quotient $X/\tau \equiv \overset{\circ}{X}$ may be identified with the Riemann surface given by the polynomial equation

$$\mu^2 = \overset{\circ}{\lambda} \prod_{j=1}^{2g} (\overset{\circ}{\lambda} - E_j) \quad .$$

The projection $\pi_1 : X \to \overset{\circ}{X}$ is then described by the formula

$$\pi_1(\mu', \lambda) = (\lambda\mu', \lambda^2) \equiv (\mu, \overset{\circ}{\lambda}) \quad .$$

The Riemann surfaces X, $\overset{\circ}{X}$ and the covering $\pi_1 : X \to \overset{\circ}{X}$ are the basic algebro-geometric structures, appearing in all constructions of Sect. 4.2.

Now let us consider the common eigenfunction $\psi(P, x, t)$; $P \in X$ of operators $\partial_x - U$, $\partial_t - V$ and $L(\lambda)$, which is defined in complete analogy with the previous discussion of the NS model:

$$\psi(P, x, t) = \widehat{\Psi}(\lambda, x, t)\, \boldsymbol{h}(P, 0, 0) \quad ,$$

where Ψ and h have the same meaning as in the preceding section. The unique difference is that now $\widehat{\Psi}$ possesses two essential singularities at the zero point of $\mathbb{CP}^1$ and at infinity. Consequently $\psi(P, x, t)$ as a function of P has four essential singularities on X situated at the points $\infty^\pm$ and $0^\pm$. Performing a simple analysis similar to study of the NS model we get the asymptotic estimates

$$\begin{aligned} &\psi(P) = \left[\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} + O\left(\frac{1}{\lambda}\right)\right] \exp\{\mp \mathrm{i}\lambda x\}, \quad P \to \infty^\pm \quad , \\ &\psi(P) = O(1) \exp\{\mp \mathrm{i}t/\lambda\}, \quad P \to 0^\pm \quad , \\ &\lambda = \pi_2(P), \quad \pi_2(\mu', \lambda) \equiv \lambda \quad . \end{aligned} \tag{4.6.25}$$

Equation (4.6.23) shows that the polynomial $A(\lambda)$ is an odd function and $B(\lambda)$ is an even polynomial in λ. Hence the divisor of poles of the function $\psi(P)$ (denoted by $\mathcal{D}$ as above) is of even degree. $\mathcal{D}$ is invariant with respect to the action of τ

$$\mathcal{D}^\tau = \mathcal{D}, \quad \deg \mathcal{D} = 2g \quad . \tag{4.6.26}$$

The vector function ψ satisfies

$$L(\lambda, x, t)\psi(P, x, t) = \mu\psi(P, x, t) \quad .$$

The reduction properties of the matrix $L(\lambda)$ signify that

$$\sigma_3 \psi(P^\tau, x, t) = c(P)\, \psi(P, x, t) \quad . \tag{4.6.27}$$

From the property (4.6.26) we deduce that

$$c(P) = \text{const} \quad . \tag{4.6.28}$$

Asymptotic estimates (4.6.25) and the equalities

$$(\infty^+)^\tau = \infty^-, \quad \pi_2(P^\tau) = -\pi_2(P)$$

enable one to make the identity (4.6.28) more precise

$$c(P) \equiv 1 \quad . \tag{4.6.29}$$

From (4.6.27,29) it is easy to conclude that the first component $\psi_1(P)$ of ψ is a single-valued function on $\overset{\circ}{X}$. The second component $\psi_2(P)$ is two-valued on $\overset{\circ}{X}$.

Let X^+ be a sheet of the covering π_1, containing the points 0^+ and ∞^+. This sheet may be identified with the surface $\overset{\circ}{X}$, cut along the path $\mathcal{L}$ (Sect. 4.2). We assume that some single-valued branch of the multi-valued (on $\overset{\circ}{X}$) function λ is chosen on $\overset{\circ}{X} \setminus \mathcal{L}$, and local parameters at the points $0, \infty \in \overset{\circ}{X}$ are chosen such that

$$\lambda = \sqrt{\overset{\circ}{\lambda}}, \quad \overset{\circ}{P} \equiv (\mu, \overset{\circ}{\lambda}) \to 0, \infty \quad .$$

Let us put

$$\psi(\overset{\circ}{P}) = \psi(P)\Big|_{X+}, \quad \overset{\circ}{P} = \pi_1(P) \quad .$$

In terms of $\psi(\overset{\circ}{P})$ the properties (4.6.25, 27) take the form of (4.2.12-14) respectively. $\psi(P_0)$ itself is single-valued and holomorphic on $\overset{\circ}{X} \setminus \mathcal{L}$. Its divisor of poles $\mathcal{D}^0$ is given by $\mathcal{D}^0 = \pi_1(\mathcal{D})$, $\deg \mathcal{D}^0 = g$, i.e., $\deg \mathcal{D}^0$ coincides with the genus of the curve $\overset{\circ}{X}$. In other words, we have completely reconstructed all axioms a.) - d.) for the Baker-Akhiezer function of the SG model.

Remark 4.24. The matrix $L(\lambda)$, playing a central role in the considerations of this section, is defined as the V-operators for one of the integrable systems, belonging to an infinite hierarchy associated with the given nonlinear integrable equation. Hence the variety of all finite-gap solutions related to the algebraic (hyperelliptic) curves of fixed genus coincides with the variety of the stationary solutions of one of the "higher" analogues of the considered equation.

Remark 4.25. As presented in this section, the version of the finite-gap integration turns out to be more appropriate when applied to finite dimensional integrable systems. The reason is that in the finite dimensional case the zero curvature representation is replaced by the matrix Lax equation, $L_t = [V, L]$. In other words, the differential operator $\partial_x - U(\lambda)$ is replaced by the matrix operator $L(\lambda)$. That is why the introduction of the algebro-geometric context in the problem appears quite naturally; the Riemann surface appears from the very beginning as a spectral curve of the matrix L. The deeper meaning of this lies in the coincidence for finite dimensional case of the variety of finite-gap solutions with the whole variety of the solutions of the system.

5. Uniformization of Riemann Surfaces and Effectivization of Theta Function Formulas

The solutions of the nonlinear integrable equations constructed in the previous two chapters are parametrized by compact Riemann surfaces. This seems to be a rather complicated parametrization if one wants to investigate these solutions or just plot them. Here we show how this problem can be solved using the Schottky uniformization of Riemann surfaces.

The main body of results presented in this chapter was obtained by *Bobenko* [5.1, 2]. Sections 5.7, 9, devoted to the qualitative analysis and the calculation of the finite-gap solutions of the KdV and KP equations follow references [5.3-5].

5.1 The Schottky Uniformization

Let $C_1, C_1', \dots, C_N, C_N'$ be a set of $2N$ mutually disjoint Jordan curves on $\mathbb{C}$, which comprise the boundary of a $2N$-connected domain F (Fig. 5.1).

The linear transformation

$$\frac{\sigma_n z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n}, \quad |\mu_n| < 1, \quad n = 1, \dots, N \tag{5.1.1}$$

transforms the outside of a boundary curve C_n onto the inside of the boundary curve C_n', $\sigma_n C_n = C_n'$. The points A_n and B_n are fixed points of the loxodromic transformation σ_n (Appendix 2.A).

The elements σ of the group PSL$(2,\mathbb{C})$ have the following representation:

$$\begin{aligned} &\sigma z = \frac{\alpha z + \beta}{\gamma z + \delta}, \\ &\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{A - B} \begin{pmatrix} A\sqrt{\mu} - B/\sqrt{\mu} & AB(1/\sqrt{\mu} - \sqrt{\mu}) \\ \sqrt{\mu} - 1/\sqrt{\mu} & A/\sqrt{\mu} - B\sqrt{\mu} \end{pmatrix}. \end{aligned} \tag{5.1.2}$$

The center of the isometric circle is given by

$$-\delta/\gamma = (B\sqrt{\mu} - A/\sqrt{\mu})(\sqrt{\mu} - 1/\sqrt{\mu})^{-1}$$

and its radius equals $|\gamma|^{-1}$.

The transformations $\sigma_1, \dots, \sigma_N$ generate a Schottky group G [5.6]. The fundamental domain of G is F. If all the boundary curves C_n, C_n' are circles, then

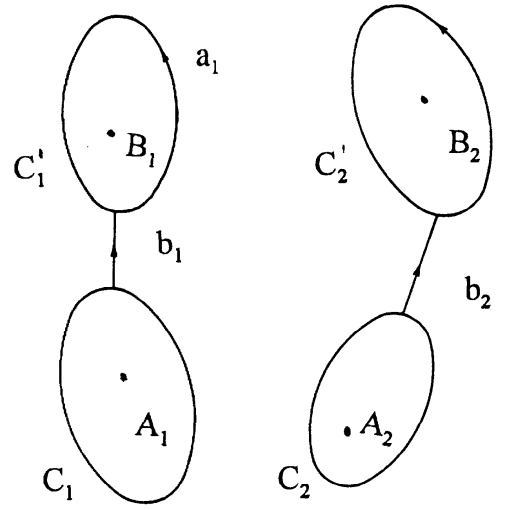

Fig. 5.1. The fundamental domain F. a_n coincides with C'_n positively oriented, b_n runs on F between the points $z_n \in C_n$ and $\sigma_n z_n \in C'_n$, and b-cycles do not mutually intersect

the Schottky group is called classical [5.7]. More generally, Schottky groups can be characterized as those finitely generated, discontinuous groups which are free and purely loxodromic [5.8]. This turns out to be equivalent to the previous definition because any free system of generators of such a group gives rise to a fundamental domain F as constructed above [5.9]. Let $\Omega(G)$ be the set of discontinuity of G, then Ω/G is a compact Riemann surface of genus N.

According to a classical theorem [5.10], any compact Riemann surface X of genus N can be represented in this form. More precisely, let N homologically independent simple disjoint loops $v_1, \dots, v_N$ be chosen on X. Then X, being cut along these loops, is a plane region. It is mapped conformally to the fundamental domain F of the corresponding Schottky group G, v_n being mapped exactly onto the curves C'_n, C_n. Two loop systems $v_1, \dots, v_N$ and $v'_1, \dots, v'_N$ generating the same subgroups in $H_1(X, \mathbb{Z})$ can determine the same group G but with a different choice of generators. A difference of the subgroups leads to a difference of the uniformizing Schottky groups G and G'. One may choose a canonical basis $H_1(X, \mathbb{Z})$ such that a-cycles coincide with the loops $v_n = a_n$. This canonical basis of cycles of Ω/G is illustrated in Fig. 5.1: a_n coincides with C'_n positively oriented, b_n runs on F between the points $z_n \in C_n$ and $\sigma_n z_n \in C'_n$, and b-cycles do not mutually intersect.

Denote by G_n the subgroup of G generated by σ_n. The cosets G/G_n and $G_m \setminus G/G_n$ are the sets of all elements

$$\sigma = \sigma_{i_1}^{j_1} \dots \sigma_{i_k}^{j_k}$$

so that $i_k \neq n$ and for $G_m \setminus G/G_n$ in addition $i_1 \neq m$. The following Lemma is adopted from the classical papers [5.11,12]:

Lemma 5.1. If the series

$$\omega_n = \sum_{\sigma \in G/G_n} \left(\frac{1}{z - \sigma B_n} - \frac{1}{z - \sigma A_n} \right) dz \tag{5.1.3}$$

are absolutely convergent, then they define holomorphic differentials normalized in the basis shown in Fig. 5.1. The period matrix is given by

$$\begin{aligned} B_{nm} &= \sum_{\sigma\in G_m\backslash G/G_n} \log\{B_m, A_m, \sigma B_n, \sigma A_n\},\ m\neq n \quad , \\ B_{nn} &= \log\mu_n + \sum_{\sigma\in G_n\backslash G/G_n,\,\sigma\neq I} \log\{B_n, A_n, \sigma B_n, \sigma A_n\} \quad , \end{aligned} \tag{5.1.4}$$

where the curly brackets indicate the cross-ratio

$$\{z_1, z_2, z_3, z_4\} = (z_1 - z_3)(z_2 - z_4)(z_1 - z_4)^{-1}(z_2 - z_3)^{-1} \quad . \tag{5.1.5}$$

Proof. The series (5.1.3) have no poles in F. The normalization conditions

$$\int_{a_m} \omega_n = 2\pi\mathrm{i}\,\delta_{nm} \tag{5.1.6}$$

are proved by calculating the integrals by residues.

If $\sigma = \sigma_{i_1}^{j_1}\ldots\sigma_{i_k}^{j_k}$, $\sigma\neq I$, then both the points σB_n and σA_n are inside of C'_{i_1} when $j_1>0$, and inside of C_{i_1} when $j_1<0$. For $\sigma = I$ we have B_n inside of C'_n and A_n inside of C_n. That gives the value (5.1.6) of the integral along a_n. It is easy to prove that $\omega_n(\sigma z) = \omega_n(z)$, so ω is a holomorphic differential on $X = \Omega/G$. Using the invariance of the cross-ratio with respect to the linear transformations

$$\{\sigma z_1, \sigma z_2, \sigma z_3, \sigma z_4\} = \{z_1, z_2, z_3, z_4\} \quad ,$$

one can derive (5.1.4) from the definition of the period matrix

$$B_{nm} = \int_z^{\sigma_m z} \omega_n \quad .$$

The finite-gap solutions do not depend on the choice of the canonical basis of $H_1(X,\mathbb{Z})$. Therefore, for the purposes of the finite-gap integration theory, it is sufficient to consider the Schottky group G, determined by some fixed uniformization Ω/G of the Riemann surface X (i.e., by some fixed loops $v_1,\ldots, v_N$). So we need to solve two problems:

1. For a given Riemann surface X does there exist a Schottky uniformization Ω/G (i.e., choice of the loops $v_1,\ldots, v_N$) such that the series (5.1.3) determined by this uniformization are absolutely convergent?
2. Can the set $S = \{A_1, B_1, \mu_1,\ldots, A_N, B_N, \mu_N\}$ of the uniformization parameters be explicitly described?

It is apparently impossible to give a solution to these two problems in the general case. Analogous problems were considered for the Schottky groups independent of the Riemann surface uniformization. Closely connected with the second problem is the question of whether for the arbitrary Riemann surface a uniformization exists with a classical Schottky group. The answer to this question is also unknown. The existence of nonclassical Schottky groups (for an arbitrary system of generators) was proved in [5.7] though no concrete examples are known [5.13].

5.2 Convergence of Poincaré Series

The series (5.1.3) are (-2)-dimensional Poincaré theta series. They can be written in the slightly different form

$$\omega_n = \sum_{\sigma \in G_n \backslash G} \left(\frac{1}{\sigma z - B_n} - \frac{1}{\sigma z - A_n} \right) \sigma' z \, dz, \quad \sigma' z = \frac{1}{(\gamma z + \delta)^2} \quad .$$

For the general Schottky group, they can be absolutely divergent [5.14,15]. However, if a Schottky group is classical and satisfies some restrictions, then (-2)-dimensional theta series are convergent.

Assume that the Schottky group is classical and that $2N-3$ circles $L_1, \ldots, L_{2N-}$ can be fixed on the fundamental domain F so that the following conditions are satisfied:

(i) The circles $L_1, \ldots, L_{2N-3}$, $C_1, C_1', \ldots, C_N, C_N'$ are mutually disjoint.
(ii) The circles $L_1, \ldots, L_{2N-3}$ divide F into $2N-2$ regions $T_1, \ldots, T_{2N-2}$.
(iii) Each T_i has exactly three boundary circles (see Figs.5.2,7).

Let us call these Schottky groups *circle decomposable.*

Lemma 5.2. (The Schottky condition). [5.6, 16]. (-2)-dimensional Poincaré series corresponding to the circle decomposable Schottky groups are absolutely convergent.

In particular, each Schottky group which has an invariant circle is always circle decomposable, and the series are convergent [5.10, 11]. The convergence can also be proved in the case when the circles $C_k, C_k', k = 1, \ldots, N$, are small enough and far enough apart (the corresponding estimates can be found in [5.11, 12]).

Let us pick out one of these regions T_i, $i = 1, \ldots, 2N-2$ (Fig. 5.2).

Consider any two circles of the boundary of T_i. Let R, r be their radii and e be the distance between their centers. So, considering various pairs of circles, we assign three numbers K_i^1, K_i^2, K_i^3 to each T_i,

$$K = \left(\frac{R^2 + r^2 - e^2}{2Rr} \right)^2 - 1 \quad .$$

Set $K = \min(K_1^1, K_1^2, \ldots, K_{2N-2}^2, K_{2N-2}^3)$. The proof [5.6,16] of the Schottky convergence principle shows that the series converges better for the larger K.

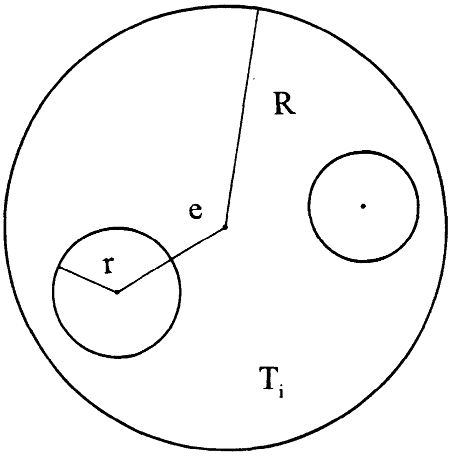

Fig. 5.2. A region T_i having exactly three boundary circles

The speed of convergence can be estimated by the maximal K possible among various decompositions

$$K^* = \max K \quad .$$

The Schottky condition can be slightly generalized. Let us consider a domain $\mathcal{F}$, which is a union of several fundamental domains

$$\mathcal{F} = \bigcup_{\sigma_s \in \{\sigma_1, \ldots, \sigma_k\}} \sigma_s F \quad ,$$

where $\sigma_1, \ldots, \sigma_k \in G$ are certain elements of g. The generalization of the Schottky condition is the following: if $\mathcal{F}$ can be decomposed into regions T each bounded by three circles (i.e., the conditions (i), (ii), (iii) are satisfied for $\mathcal{F}$) then the (-2)-dimensional Poincaré series are absolutely convergent.

The convergence of the Poincaré theta series gives information on a metrical property of the limiting set of the group $\Lambda(G)$ and vice versa. If ν is the minimal dimension for which the $(-\nu)$-dimensional Poincaré series converge absolutely, then the Hausdorff measure of $\Lambda(G)$ is equal to $\nu/2$. So the 1-dimension measure of $\Lambda(G)$ of a Schottky group with divergent (-2)-dimensional Poincaré series is infinite. Corresponding examples of the classical Schottky groups with fundamental domains bounded by isometrical circles are constructed in [5.14,15]. Calculations for various Schottky groups of a critical dimension ν for which the theta series still converge can be found in [5.17]. The class of Schottky groups with convergent (-2)-dimensional Poincaré series was geometrically characterized in [5.29], p. 24.

5.3 Schottky Uniformization of Riemann Surfaces of the Decomposition Type

As was already mentioned, the problems formulated in Sect. 5.1 are rather difficult and their solutions in the general case are unknown. However, in the case of real Riemann surfaces, which is the most important for applications, these two problems can be completely resolved. In this section we give a simple proof of this for real Riemann surfaces of decomposing type, which determine real non-singular solutions of the equations KP1 and KP2 (Sect. 3.3).

Let X be a real Riemann surface of genus N, $\tau: \ X \to X$ an anti-holomorphic involution of decomposing type, having n fixed ovals (real ovals) $X_0, \ldots, X_{n-1}$. The ovals decompose X into two components X_+ and X_-, which are the Riemann surfaces of signature (g, n), i.e., X_+, X_- are homeomorphic to a surface of genus $g = (N + 1 - n)/2$ with n boundary contours.

Lemma 5.3. For any real decomposing Riemann surface X a Schottky uniformization by the Fuchsian group of the second kind exists. The series (5.1.3) corresponding to such an uniformization are absolutely convergent.

Proof. Consider the Fuchsian uniformization H/G of the surface X_+, see App. 2.1, where H is the upper complex half-plane $H = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$. The surface X_+ has a boundary contour. Therefore G is the Fuchsian group of the second kind. It is a purely hyperbolic group with generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, $\gamma_1, \ldots, \gamma_n$, $n \geq 1$, satisfying one restriction

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \gamma_1 \ldots \gamma_n = 1 \ .$$

If we consider the action of G on all $\mathbb{C}$, then the factor $\overline{H}/G$ with $\overline{H} = \{z \in \mathbb{C}, \operatorname{Im} z < 0\}$ is conformally equivalent to X_-. Elements

$$\sigma_i = \alpha_i, \ \sigma_{g+i} = \beta_i, \ \sigma_{g+i} = \gamma_j, \ i = 1, \ldots, g; \ j = 1, \ldots, n-1$$

generate a free, purely hyperbolic group - the Schottky group uniformizing the Riemann surface X. It is well known [5.10] that the (-2)-dimensional Poincaré theta series always converge for the Fuchsian groups of the second kind. In our case the Schottky group has an invariant circle, which is the real axis.

The Fuchsian groups of the second kind are well investigated. In particular the complete description of the set S of generator parameters (Sect. 5.1) can be obtained as described in Appendix 5.1.

5.4 Schottky Uniformization of the M-Curves

As pointed out in Sect. 3.3, the periodic problem for the KP2 equation is solved in terms of the M-curves. In this section we consider in detail various Schottky uniformizations of the M-curve X and the corresponding fundamental domains F of the Schottky groups G.

In this case the surface X_+ (Sect. 5.3) is homeomorphic to a sphere with $N+1$ holes and therefore with $N+1$ boundary contours $X_0, \dots, X_N$ (Fig. 5.3). Let $[L]$ be a system of N mutually non-intersecting curves $L_1, \dots, L_N$ on X_+ such that the surface $X_+[L]$, which is X cut along all the contours L_n, is simple connected. An example of such a system of cuts is given in Fig. 5.3.

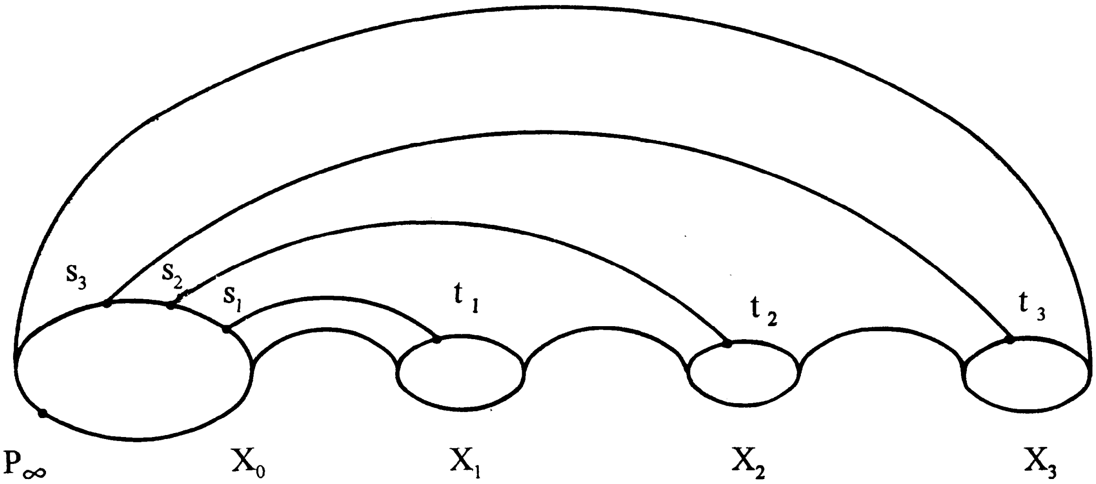

Fig. 5.3. The system of cuts on the Riemann surface

A Fuchsian uniformization of X_+ is H/G. There is a natural projection $\Phi : H \to X_+ = H/G$. Groups equivalent in PSL(2, $\mathbb{R}$) uniformize conformally equivalent Riemann surfaces. It should be recalled that for the KP equation we have a Riemann surface with a marked point P_∞. Let us fix the normalization $\Phi(\infty) = P_\infty$. The inverse mapping $\Phi^{-1} : X_+[L] \to H$ is uniquely determined by two conditions:

(1) $\Phi^{-1}(P\infty) = \infty$,
(2) $\Phi^{-1}(X_+[L])$ is connected.

Then $F = \Phi^{-1}(X_+[L]) \subset H$ is a fundamental domain of the group G. $\Phi^{-1}(L_n)$ consists of a pair of contours C_n, C_n'. They belong to boundary ∂F, and there is a hyperbolic transformation mapping one onto another $\sigma_n C_n = C_n'$ (5.1.1), where

$$A_n,\ B_n,\ \mu_n \in \mathbb{R}, \quad 0 < \mu_n < 1 \quad .$$

The free system of generators of G is $\sigma_1, \dots, \sigma_N$. Every system $[L]$ induces in this way a certain fundamental domain F. The fundamental domain F, corre-

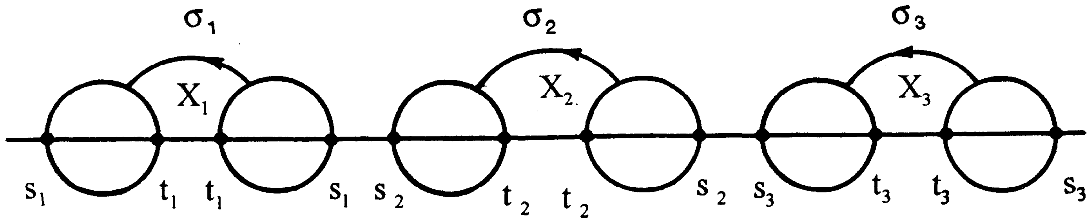

Fig. 5.4. The fundamental domain of the Schottky group

sponding to the cut system $[L]$ of Fig. 5.3 of the Schottky group constructed in Sect. 5.3, is presented in Fig. 5.4.

The Poincaré metric on H (Appendix 2.A) induces a metric on $X_+ = H/G$. Let the curves $L_n(s_n, t_n)$ be mutually non-intersecting and go from one boundary contour X_i to the other boundary contour X_j, $i \neq j$. There is a geodesic $L'_n(s_n, t_n)$ in the Poincaré metric with the same boundary points s_n, t_n. It is evident that the geodesics L'_n are also mutually non-intersecting. $\Phi^{-1}(L'_n)$ are geodesics on H, so that the fundamental domain $\Phi^{-1}(X_+[L'])$ is bounded by circles. As a result we see that the Schottky group G is classical.

The complete description of the set $S = \{A_1, B_1, \mu_1, \ldots, A_N, B_N, \mu_N\}$ can be easily obtained (App. 5.1). The set S depends on the fixed system of generators. We describe S for generators determined by the system of curves $[L]$ shown in Fig. 5.5 and, as a corollary, by the fundamental domain of the Schottky group shown in Fig. 5.6. For these generators S is described in the following final form:

$$\begin{aligned} &B_N < B_{N-1} < \ldots < B_1 < A_1 < \ldots < A_N \quad , \\ &0 < \sqrt{\mu} < 1, \quad i = 1, \ldots, N \quad , \\ &\{B_n, A_n, B_{n+1}, A_{n+1}\} > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}} \right)^2 \quad , \\ &\qquad\qquad n = 1, \ldots, N-1 \quad . \end{aligned} \tag{5.4.1}$$

The proof of this statement is given in App. 5.1. We also mention that the generators $\sigma_1, \sigma_1^{-1}\sigma_2, \ldots, \sigma_{N-1}^{-1}\sigma_N$ induce the fundamental domain of the form shown in Fig. 5.4.

To study small-amplitude waves of the KP2 equation it is more convenient to consider another Schottky uniformization of the M-curve X. The surface X_+ can always be mapped to the upper half-plane with N discs removed and with P_∞ mapped to ∞ [5.10]. Then the group G is described as follows (Fig. 5.7):

$$B_n = \bar{A}_n, \quad \mathrm{Im} A_n > 0, \, 0 < \sqrt{\mu} < 1, \, n = 1, \ldots, N \quad . \tag{5.4.2}$$

In this case C_n and C'_n are the isometric circles of the transformations σ_n and σ_n^{-1}. C_n and $C_{n'}$ are mutually complex conjugated. Since their centers and radii are known (Sect. 5.1), it is easy to write the conditions for the circles to be disjoint:

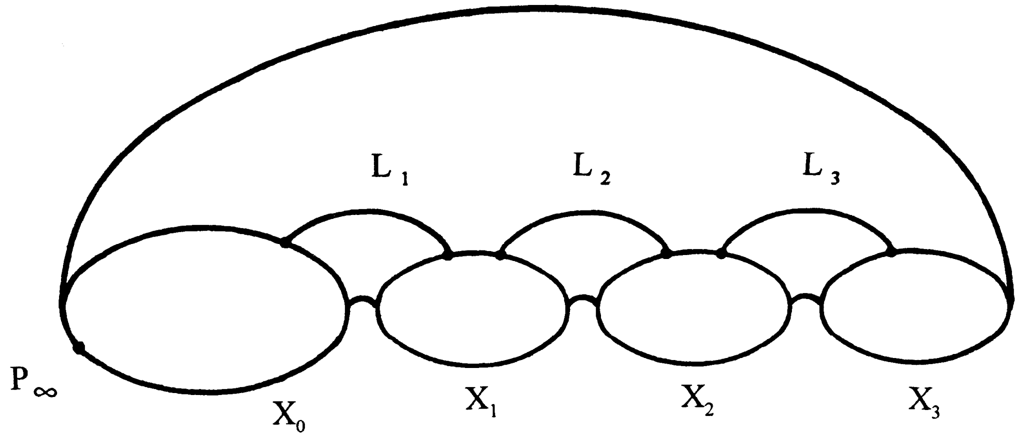

Fig. 5.5. The system of cuts $[L]$

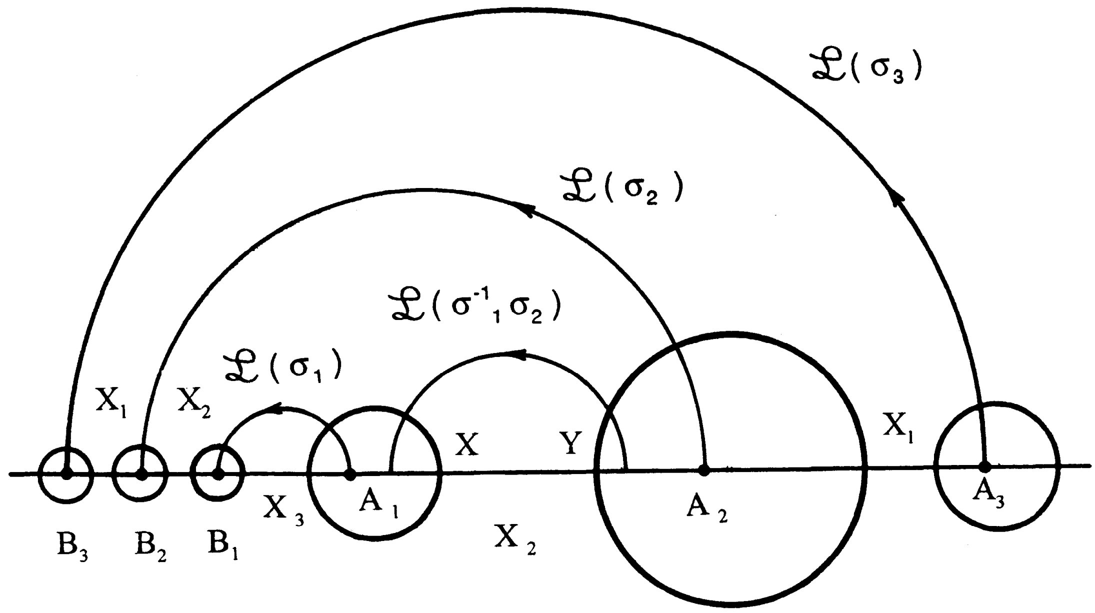

Fig. 5.6. The fundamental domain of the Schottky group

$$\begin{aligned}\left|\frac{\bar{A}_n\sqrt{\mu_n}-A_n/\sqrt{\mu_n}}{\sqrt{\mu_n}-1/\sqrt{\mu_n}}-\frac{\bar{A}_m\sqrt{\mu_m}-A_m/\sqrt{\mu_m}}{\sqrt{\mu_m}-1/\sqrt{\mu_m}}\right|& \\ >-\frac{2\mathrm{Im}\,A_n}{\sqrt{\mu_n}-1/\sqrt{\mu_n}}-\frac{2\mathrm{Im}\,A_m}{\sqrt{\mu_m}-1/\sqrt{\mu_m}}& \ .\end{aligned} \tag{5.4.3}$$

These inequalities together with (5.4.2) determine S. However, for this uniformization the convergence of the series (5.1.3) can be proved not for any point of S but only for the subset of the circle-decomposable groups. In particular, the series always converge when $N = 2$.

We call the two Schottky uniformizations of the M-curves, as described above, I and II (UI and UII). The loops v_n of UII (Fig. 5.7) are chosen uniquely – they are the real ovals of τ without P_∞. In the UI case, v_n can be chosen in many ways. Two examples are presented in Fig. 5.4,6. The most natural is

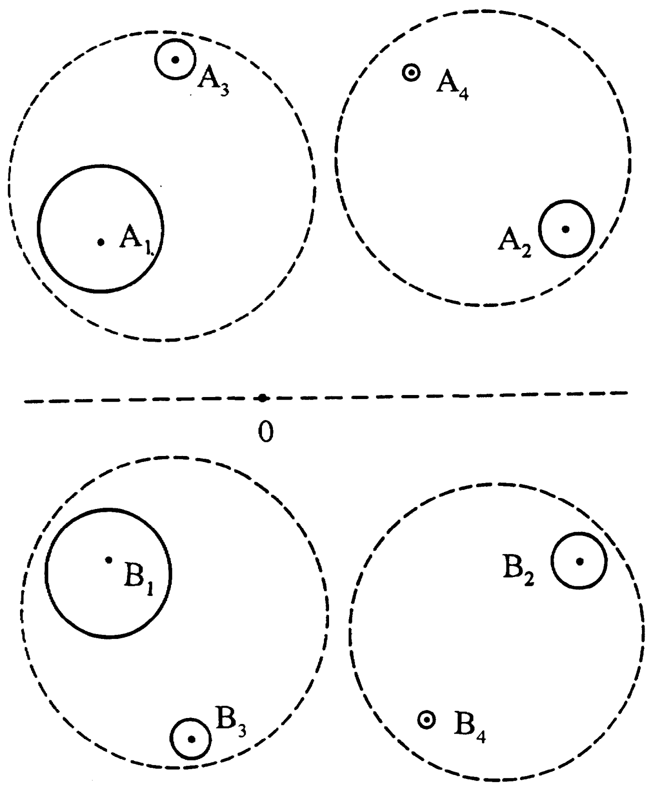

Fig. 5.7. The fundamental domain of a decomposable Schottky group (dotted lines show the circle decomposition). The corresponding Schottky group leads to the solution of the KP2 equation shown in Fig. 5.9

the choice v_n when the value K^* is maximal and the series (5.1.3) are the most rapidly convergent ones.

5.5 Solutions of the KP2 Equation

Let us return to the KP2 equation. The local parameter in the neighborhood of $P_\infty = \infty$ is equal to $k = z^{-1}$. Recall the general formula for the finite-gap solution of the KP equation (Chap. 2)

$$u(x,y,z) = 2\frac{\partial^2}{\partial x^2}\log\theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}) + 2c \quad .$$

The reciprocity law (2.4.13) for the normalized Abelian differentials of the first and second kinds allows us to present the vectors $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ in the form

$$U_n = f_n(0), \quad V_n = \frac{d}{dp}f_n(p)\big|_{p=0}, \quad W_n = \frac{1}{2}\frac{d^2}{dp^2}f_n(p)\big|_{p=0} \quad ,$$

where $p = k^{-1}$ is the local parameter near P_∞, $p(P_\infty) = 0$, and $\omega_n = f_n(p)\,dp$ is the representation of the normalized holomorphic differentials in the neighborhood of P_∞. Then we have from (5.1.3) and (5.5.1)

$$\begin{aligned} U_n &= \sum_{\sigma\in G/G_n} (\sigma A_n - \sigma B_n) \quad , \\ V_n &= \sum_{\sigma\in G/G_n} ((\sigma A_n)^2 - (\sigma B_n)^2) \quad , \\ W_n &= \sum_{\sigma\in G/G_n} ((\sigma A_n)^3 - (\sigma B_n)^3) \quad . \end{aligned} \tag{5.5.2}$$

The Abelian integral of the second kind (Sect. 3.1) is written in the form

$$\Omega_1 = \sum_{\sigma\in G,\sigma\neq I} \left(\sigma z - \frac{\alpha}{\gamma}\right) + z \quad .$$

Then, applying the definition of the constant c, we get

$$c = \sum_{\sigma\in G,\sigma\neq I} \gamma^{-2} \quad . \tag{5.5.3}$$

The KP equation allows the following transformation: if $u(x,y,z)$ is a solution, then

$$\tilde{u}(x,y,t) = u(x - \frac{3}{2}\alpha t, y, t) - \alpha \tag{5.5.4}$$

is also a solution of the KP equation.

For UII the vectors $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{D}$ are purely imaginary. The periodicity condition

$$u(x,y,t) = u(x+2\pi, y, t) = u(x, y+2\pi, t)$$

leads to a restriction of the parameters

$$\mathrm{i}\boldsymbol{U}, \mathrm{i}\boldsymbol{V} \in \mathbb{Z}^N \quad . \tag{5.5.5}$$

Let us fix the solution in the class (5.5.4) normalized by the condition

$$\int_0^{2\pi} u(x,y,t)dx = 0 \quad .$$

As a result we obtain

$$\begin{aligned} u(x,y,t) &= 2\frac{\partial^2}{\partial x^2} \log\theta(\boldsymbol{U}x + \boldsymbol{V}y + \tilde{\boldsymbol{W}}t + \boldsymbol{D}) \quad , \\ \tilde{\boldsymbol{W}} &= \boldsymbol{W} - 3c\boldsymbol{U} \quad . \end{aligned} \tag{5.5.6}$$

In the UI case $\boldsymbol{D} \in \mathbb{R}^n$ is a real vector, and the series are convergent because the statement on the convergence of the series $\sum_{\sigma\in G}(\sigma a - \sigma b)$ is a corollary to the one of the convergence of $\sum_{\sigma\in G,\sigma\neq I} \gamma^{-2}$, [5.10].

So with the help of Theorems 3.2, 3.3 the following theorem is proved.

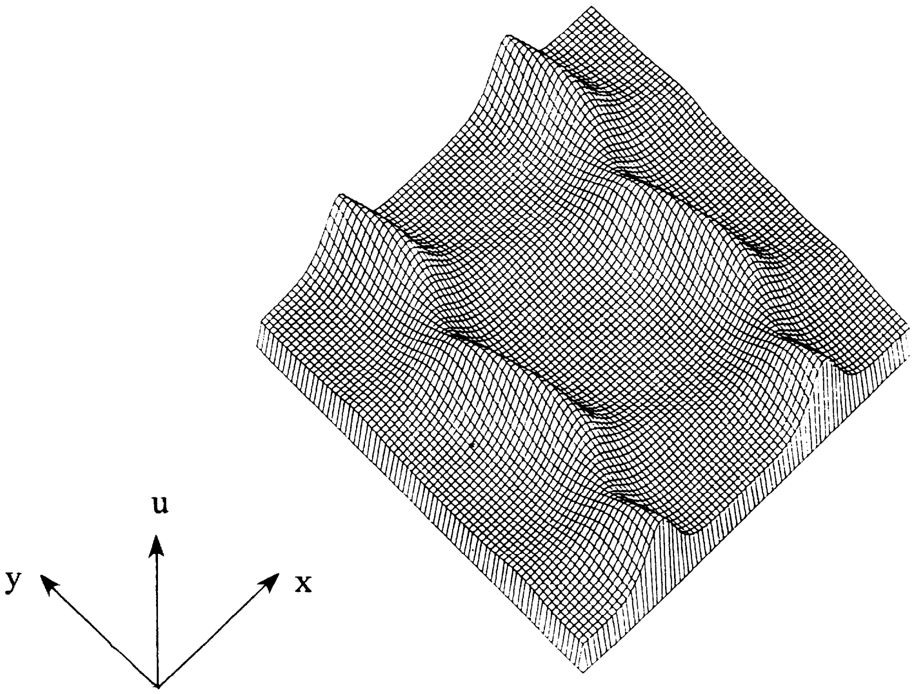

Fig. 5.8 a. The large-amplitude two-phase periodic solution in the soliton regime. The soliton structure is evident. The isometric projection of the surface $u(x, y, 0)$. This surface is represented as a set of lines which are produced by the intersection of the wave surface with planes being parallel to the x and y, respectively

Theorem 5.4. (UI) All real non-singular finite-gap solutions of the KP2 equation are given by the formulas (5.1.4), (5.5.2-3, 6), where the parameters A_n, B_n, μ_n belong to the set (5.4.1) and $D \in \mathbf{R}^N$ is an arbitrary real vector.
(UII) For circle decomposable groups (in particular for any two-phase solution) real non-singular finite-gap solutions of the KP2 equation can be described by the formulas (5.1.4), (5.5.2-3,6), where the parameters A_n, μ_n belong to the set (5.4.2,3) and $\boldsymbol{D}$ is an arbitrary imaginary vector.

When the generators are chosen as in Fig. 5.6, then all the U_n are arranged as follows:

$$0 < U_1 < U_2 < \ldots < U_N \quad .$$

We also remark that the simple periodicity condition (5.5.5) shows that the UII representation is more convenient for isolating the periodic solutions. In this case, A_n, μ_n are the natural convenient parameters of the solution, because, for a given X, UII is unique and we have a one-to-one correspondence between A_n, $\mu_n \in S$ and the solutions of the KP2 equation.

In [5.5], with the help of the formulas described above calculations of the finite-gap solutions of the KP2 equation were carried out. The periodicity conditions could be taken into account in a rather simple way. The calculations were done using the formula (5.5.6) and UII. For a given diagonal of the period matrix

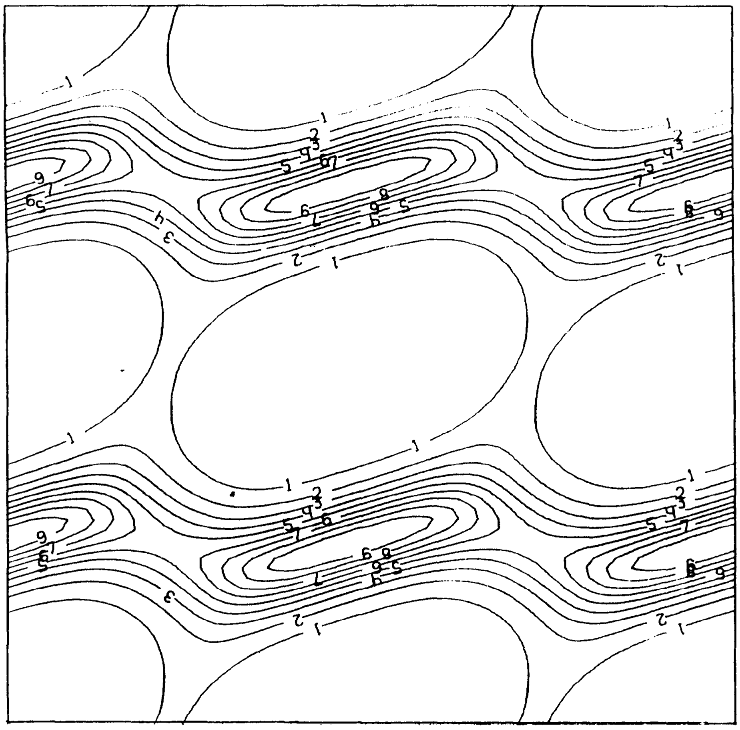

Fig. 5.8 b. The isolines $u(x, y, 0)$ = const, i.e., the contour plot of the same solution. The range of variations of x and y equals two periods of the solution, i.e., $[0, 4\pi]$, $t = 0$

B_{nn} and vectors $\boldsymbol{U}, \boldsymbol{V}$ (amplitudes and wave numbers of harmonics) the parameters A_n, μ_n of UII were calculated. Then the formulas of the present chapter give all the parameters $B, \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$. As a result, computer plots of various finite-gap solutions were constructed in [5.5].

Figures 5.8 a,b show the typical large-amplitude two-phase solution. It has been calculated by UII with the parameters

$$\begin{array}{ll} \boldsymbol{U} = \mathrm{i}\,(1.000,\ 1.000), & \\ \boldsymbol{V} = \mathrm{i}\,(0.000,\ 1.000), & c = -0.217,\ t = 0, \quad x, y \in [0, 4\pi] \\ \boldsymbol{W} = \mathrm{i}\,(-0.171,\ 0.823), & B = -2\pi \begin{pmatrix} 0.400 & 0.225 \\ 0.225 & 0.600 \end{pmatrix} . \\ \boldsymbol{D} = (0.000,\ 0.000). & \end{array}$$

The uniformization parameters for this solution are:

$$A_1 = -0.024 + \mathrm{i}\,0.507\ , \quad \mu_1 = 0.076\ ,$$
$$A_2 = 0.588 + \mathrm{i}\,0.556\ , \quad \mu_1 = 0.014\ ,$$

A more complicated interaction of four phases is presented in Fig. 5.9. The fundamental domain of the corresponding Schottky group is presented in Fig. 5.7. The parameters used are as given below:

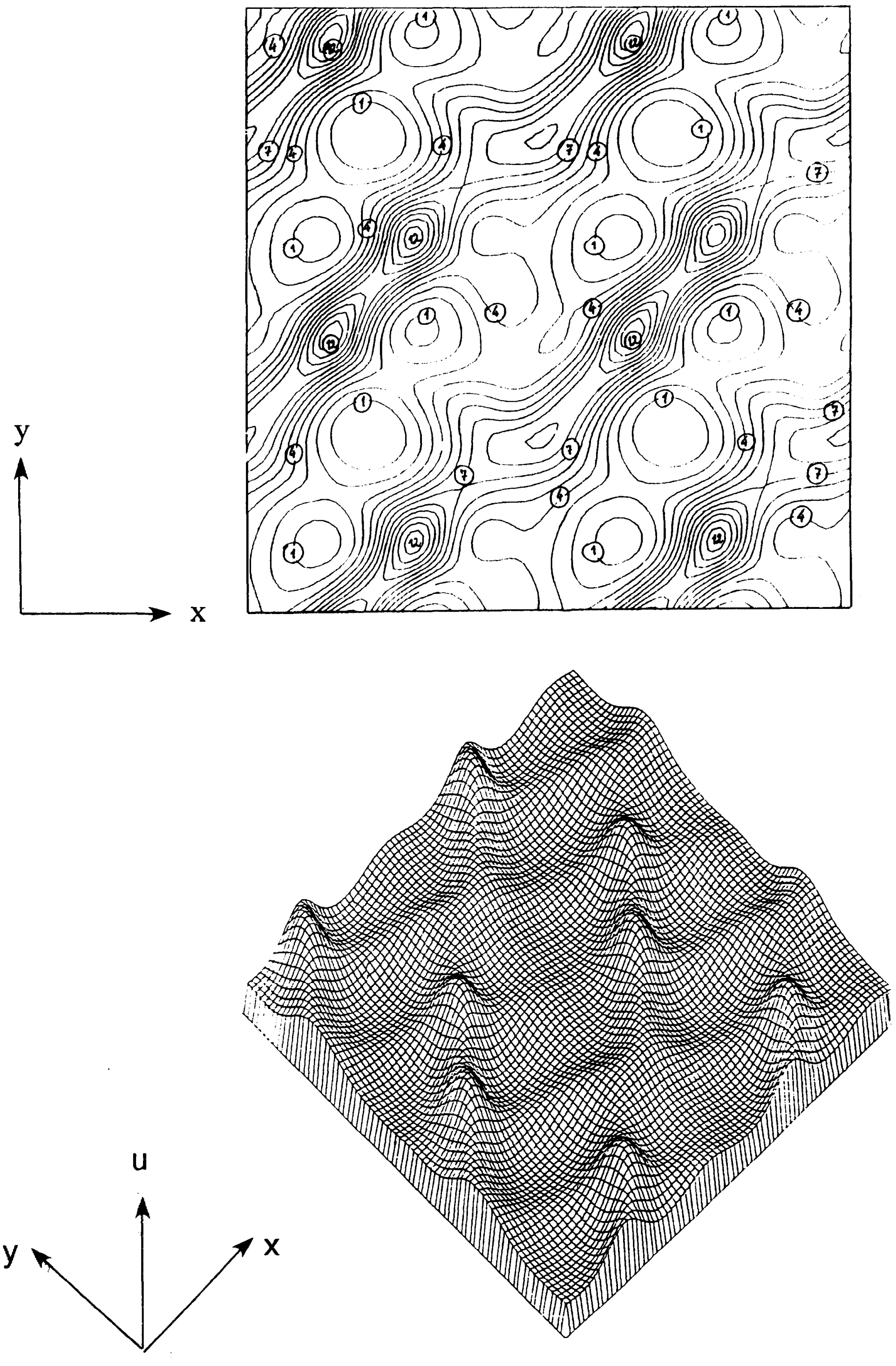

Fig. 5.9. The interaction of four phases

$$B = -2\pi \begin{pmatrix} 0.500 & 0.062 & 0.288 & 0.155 \\ 0.062 & 0.800 & 0.116 & 0.252 \\ 0.288 & 0.116 & 1.100 & 0.325 \\ 0.155 & 0.252 & 0.325 & 1.500 \end{pmatrix} \quad ,$$

$$U = \mathrm{i}\,(1.000,\ 1.000,\ 2.000,\ 2.000) \quad ,$$

$$V = \mathrm{i}\,(-1.000,\ 2.000,\ -1.000,\ 2.000) \quad ,$$

$$W = \mathrm{i}\,(0.480,\ 2.682,\ -2.138,\ -7.930) \quad ,$$

$$D = \mathrm{i}\,(0,\ 0,\ 0,\ 0),\ t = 0,\ c = -0.118,\ x, y \in [0, 4\pi] \quad ,$$

$$A_1 = -0.500 + \mathrm{i}\,0.499, \qquad \mu_1 = 0.0429 \quad ,$$

$$A_2 = 0.970 + \mathrm{i}\,0.515, \qquad \mu_2 = 0.645 \times 10^{-2} \quad ,$$

$$A_3 = -0.253 + \mathrm{i}\,1.107, \qquad \mu_3 = 6.19 \times 10^{-4} \quad ,$$

$$A_4 = 0.479 + \mathrm{i}\,1.055, \qquad \mu_4 = 6.9 \times 10^{-5} \quad .$$

5.6 Solution of the KP1 Equation

The finite-gap solutions of the KP1 equation are determined by a general real Riemann surface of decomposing type, not by the M-curves only (Sect. 3.3). Let X be such a surface, $X_0, \dots, X_{n-1}$ - n fixed ovals of the antiholomorphic involution τ. Then X_+ is a Riemann surface of genus $g = (N + 1 - n)/2$ with n boundary contours $X_0, \dots, X_{n-1}$ (Fig. 5.10).

The arguments quite similar to those of Sect. 5.4 show that a set of non-intersecting geodesics $L_n, n = 1, \dots, N$, such that $X_+[L]$ is simply connected, induces a fundamental domain F of the Fuchsian classical Schottky group G. Let us fix L_n with the ends on X_j. Figure 5.11 presents F constructed by the contour set $[L]$ shown in Fig. 5.10. This is one of the possible choices of F when the boundary circles C_n, C_n' are combined in canonical "fours" and "pairs". Invariant lines of g pairs of generators intersect.

Note that the fundamental domain of the Fuchsian group of the second kind shown in Fig. 5.11 (restriction of the Schottky group to the upper half-plane) differs from the canonical one [5.18]. The latter is determined by the contours L_n, starting from some internal point $P_0 \in X_+$. The set of generators of a canonical fundamental domain is not free (Sect. 5.3).

In general, let C_n, C_n', $n = 1, \dots, N$ be N pairs of mutually disjoint circles orthogonal to the real axis and also such that one is not inside of the others. The order of circles is arbitrary. Every couple C_n, C_n' determines a hyperbolic transformation σ_n. A set $\sigma_1, \dots, \sigma_N$ generates a Schottky group which uniformizes a real decomposing Riemann surface with the number of ovals determined by an arrangement of the circles.

The set S of parameters of the generators $\sigma_1, \dots, \sigma_N$ is also completely described by analyzing invariant lines of G. Corresponding results for generators shown in Fig. 5.11 are obtained in [5.19].

The canonical basis of cycles indicated in Fig. 5.11 is transformed by the anti-involution $\tau z = \bar{z}$:

$$\begin{aligned} &\tau a_k = -a_k, \quad k = 1, \dots, N, \quad , \\ &\tau b_i = b_i - a_{g+i} \quad , \\ &\tau b_{g+i} = b_{g+i} - a_i, \quad i = 1, \dots, g, \; j = 1, \dots, n-1 \quad , \\ &\tau b_{2g+j} = b_{2g+j} \quad . \end{aligned} \tag{5.6.1}$$

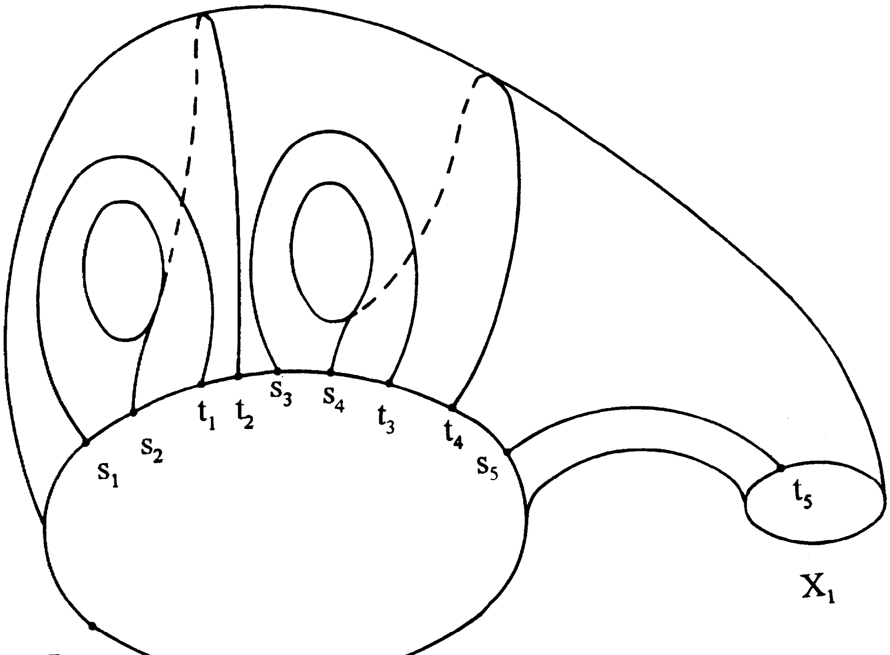

Fig. 5.10. The Riemann surface of genus $g = (N+1-n)/2$ with n boundary contours $X_0, \dots, X_{n-1}$

The basis a'_k, b'_k connected with it by the equalities

$$\begin{aligned} &a'_i = a_i - b_{g+i}, \quad b'_i = b_i \quad , \\ &a'_{g+i} = -b_{g+i}, \quad b'_{g+i} = a_{g+i} - b_i \quad , \\ &a'_{2g+j} = -b_{2g+j}, \quad b'_{2g+j} = a_{2g+j} \end{aligned} \tag{5.6.2}$$

in canonical and anti-involution transforms it by the rule (3.3.6). Exactly this basis is used in Theorem 3.3 for the description of the vector $\boldsymbol{D}$. Using the

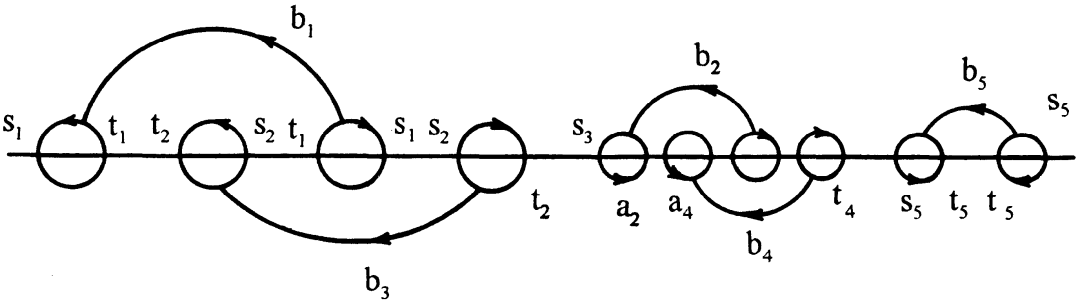

Fig. 5.11. The fundamental domain F constructed by the contour set $[L]$ of Fig. 5.10

modular transformation law (2.5.12) of theta functional argument, one can easily rewrite conditions (3.3.7) on the vector $\boldsymbol{D}$ in the basis indicated in Fig. 5.11. The period matrix in the basis (5.6.1) is equal to

$$B = B_R + \pi \mathrm{i} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{cc} \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} & \begin{array}{c} g \\ g \\ n-1 \end{array} \\ \begin{array}{ccc} g & g & n-1 \end{array} & \end{array},$$

where B_R is real. The theta functional argument in the formula for the solution of the KP1 equation due to Theorem 3.3 is of the following form:

$$z' = (\xi, \overline{\xi}, \eta)^T, \quad \xi \in \mathbb{C}^g, \quad \eta \in \mathbb{R}^{n-1}$$

in the basis a'_k, b'_k. Relation (5.6.2) of two bases can be written in the matrix form (2.4.22)

$$a = \begin{pmatrix} I & 0 & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

$$c = \begin{pmatrix} 0 & -I & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix}, \quad d = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(the dimensions of these matrices are indicated above). Substituting the above formulas in the transformation law (2.5.12) we see that the theta functional argument in the basis (5.6.1) is imaginary. So the following theorem is proved:

Theorem 5.5. All real nonsingular finite-gap solutions of the KP2 equation are given by the formula

$$u(x, y, t) = -2\frac{\partial^2}{\partial x^2} \log \theta(\mathrm{i}(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}y + \boldsymbol{D})) + 2c \quad ,$$

where the constants are determined by the equalities (5.1.4), (5.5.2, 3) and the vector $\boldsymbol{D} \in \mathbb{R}^N$ is arbitrary.

5.7 Multi-Soliton Solutions and Small-Amplitude Waves

From the general N-phase wave solution we arrive, by a limiting procedure, at two kinds of simply described degenerate solutions, namely at the multi-soliton solutions and at small-amplitude waves. Carrying out the limit

$$\mu_n \to 0, \quad n = 1, \dots, N \quad ,$$

the circles C_n and C_n' collapse to the points A_n and B_n respectively. Let us describe this limiting process in detail.

In this degenerate case for all non-identity mappings σ and for arbitrary $a, b \in F$, the equality $\sigma a = \sigma b$ holds. Therefore in the series (5.1.4, 5.5.2, 3) only the terms corresponding to $\sigma = I$ are non-zero

$$\begin{aligned} &\operatorname{Re} B_{nn} \to -\infty,\ B_{nm} \to \log\{B_m, A_m, B_n, A_n\},\ n \neq m,\ c \to 0 \quad , \\ &U_n \to A_n - B_n,\ V_n \to A_n^2 - B_n^2,\ W_n \to A_n^3 - B_n^3 \quad . \end{aligned} \tag{5.7.1}$$

The vector $\boldsymbol{D}$ is an arbitrary parameter, and we can fix its asymptotic behavior as we wish. Let D_n be equal to

$$D_n = -B_{nn}/2 + \eta_n + o(1) \quad , \tag{5.7.2}$$

where η_n are finite constants. Then the argument of the exponential function in the series (5.5.6) is seen to be given by

$$\begin{aligned} \frac{1}{2} \sum_n B_{nn} k_n (k_n - 1) + \sum_{n<m} B_{nm} k_n k_m \\ + \sum_n k_n \left(U_n x + V_n y + W_n t + \eta_n + o(1) \right) \quad . \end{aligned}$$

Since all terms of the series with $k_n \neq 0, 1$ are identically zero, this series is finite. Let $\{0, 1\}^N$ be the set of all N-dimensional vectors with coordinates equal to 0 or 1, then the limit (5.7.1) leads to

$$\theta(\boldsymbol{U}x + \boldsymbol{V}y + \tilde{\boldsymbol{W}}t + \boldsymbol{D}) \to \theta(x, y, t) \quad ,$$

where

$$
\begin{aligned}
\theta(x,y,t) = &\sum_{k\in\{0,1\}^N}\prod_{n<m}\left(\frac{(B_m-B_n)(A_m-A_n)}{(B_m-A_n)(A_m-B_n)}\right)^{k_n k_m} \\
&\times\exp\left(\sum_n k_n[(A_n-B_n)x\right. \\
&\qquad\left.+(A_n^2-B_n^2)y+(A_n^3-B_n^3)t+\eta_n]\right) \quad .
\end{aligned}
\tag{5.7.3}
$$

Finally, the solution of the KP equation is given by

$$
u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(x,y,t) \quad . \tag{5.7.4}
$$

For solution (5.7.4) to be real and non-singular, we have to apply the limiting procedure described above to the UI type solution of the KP2 equation, i.e., choose A_n, B_n to be real. The degeneration (5.7.3) for the UII type solutions of the KP2 and solutions of the KP1 equations does not lead to non-singular real solutions.

To obtain UII type degenerate non-singular real solutions we need to choose a finite parameter $\boldsymbol{D}$ so that the conditions of Theorem 3.3 are satisfied. Then we have a small amplitude limit

$$
\begin{aligned}
&\theta(\boldsymbol{U}x+\boldsymbol{V}y+\tilde{\boldsymbol{W}}t+\boldsymbol{D}) \\
&\quad\to 1+\sum_{n=1}^{N}\sqrt{\mu_n}\left[\exp(U_n x+V_n y+\tilde{W}_n t+D_n)\right. \\
&\qquad\qquad\left.+\exp(-U_n x-V_n y-\tilde{W}_n t-D_n)\right] \quad , \\
&U_n\to \mathrm{i}u_n = A_n-\overline{A}_n,\ V_n\to \mathrm{i}v_n = A_n^2-\overline{A}_n^2 \quad , \\
&\tilde{W}_n\to \mathrm{i}w_n = A_n^3-\overline{A}_n^3 \quad , \\
&B_{nn}\to\log\mu_n,\ D_n=\mathrm{i}d_n,\ c\to 0 \quad , \\
&u(x,y,t)\to -4\sum_n\sqrt{\mu_n}u_n^2\cos(u_n x+v_n y+w_n t+d_n) \quad .
\end{aligned}
\tag{5.7.5}
$$

Clearly (5.7.5) represents a linear superposition of N non-interacting Fourier modes of small amplitude.

As long as μ_n and C_n for UII are small, the solution of the KP2 equation is described by the linear limit (5.7.5). When phase amplitudes and, consequently, the size of the circles C_n, C_n' increase, the phases start to interact, but the solution remains described by the general UII formula. At last, with further amplitude increase, we reach the near-solution regime and the UI description becomes more natural.

5.8 Solutions of the KdV Equation

As already mentioned in Sect. 3.4, if the surface X is hyperelliptic and the point P_∞ is a branch point, then the solution (5.5.6) of the KP equation reduces to the solution of the KdV equation

$$4u_t = 6uu_x + u_{xxx} \quad .$$

All finite-gap solutions of the KdV equation are obtained in this way. Let us describe these solutions, drawing attention to the choice of the basis of cycles (for all details see Chap. 3).

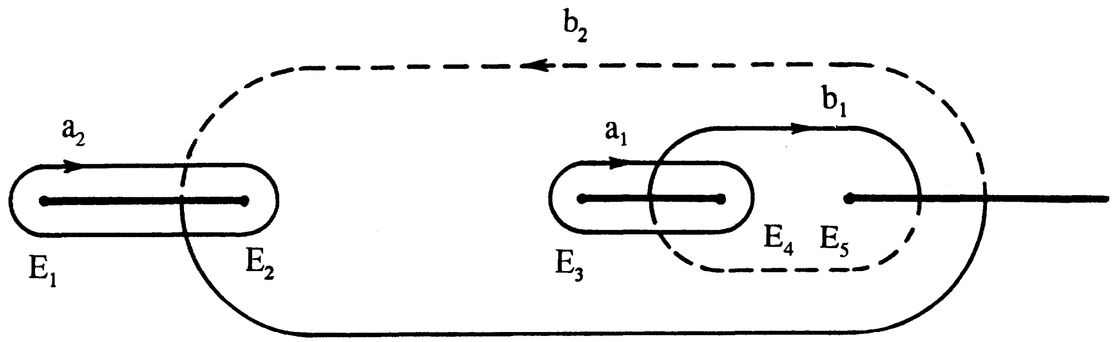

Fig. 5.12. The basis of cycles for an algebraic curve (5.8.1) in the case of genus two

The Riemann surface X of the hyperelliptic curve (Fig. 5.12)

$$\mu^2 = \prod_{i=1}^{2N+1} (\lambda - E_i), \; E_i \in \mathbf{R}, \; E_1 < E_2 < \ldots < E_{2N+1} \tag{5.8.1}$$

admits the hyperelliptic involution $\pi : (\lambda, \mu) \to (\lambda, -\mu)$ as well as the antiholomorphic involutions $\tau : (\lambda, \mu) \to (\overline{\lambda}, -\overline{\mu})$ and $\pi\tau : (\lambda, \mu) \to (\overline{\lambda}, \overline{\mu})$. Since X is an M-curve, both τ and $\pi\tau$ have $N+1$ fixed ovals. Fixed ovals of τ are situated over the gaps $[-\infty, E_1], \ldots, [E_{2N}, E_{2N+1}]$ and fixed ovals of $\pi\tau$ are over the allowed bands of the spectrum $[E_1, E_2], \ldots, [E_{2N+1}, +\infty]$.

Fixing the local parameter $k = i\sqrt{\lambda}$, $p = k^{-1}$ near $P_\infty : \lambda = \infty$, we see that $\tau^* k = \bar{k}$. All real non-singular finite-gap solutions of the KdV equation are described by (3.4.3):

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log \theta(\boldsymbol{U}x + \boldsymbol{V}y + \boldsymbol{W}t + \boldsymbol{D}) + 2c \quad , \tag{5.8.2}$$

where all constants are determined by expressions (5.5.1) (see also Sect. 3.4), and the vector $\boldsymbol{D}$ satisfies certain reality conditions depending on the choice of cycles on X. In particular, for the cycles of Fig. 5.12 we have $\tau a = -a$,

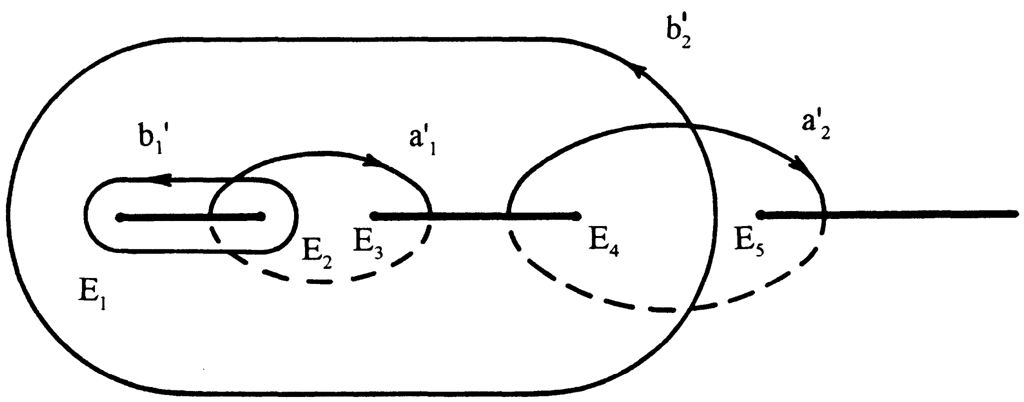

Fig. 5.13. The basis of cycles for an algebraic curve of genus two

$\tau b = b$ and D is an arbitrary real vector $D \in \mathbb{R}^N$. On the contrary, for the cycles of Fig. 5.13 we have $\tau a = a$, $\tau b = -b$ and D is an arbitrary imaginary vector $D \in \mathrm{i}\mathbb{R}^N$. Figure 5.13 may be interpreted in a somewhat different way. Consider the anti-involution $\pi\tau$ instead of τ. It acts on the cycles in quite the same way as τ in the first case: $\pi\tau a = -a$, $\pi\tau b = b$, but $(\pi\tau)^* k = -\overline{k}$.

It is clear that the finite-gap solution itself does not depend on the choice of cycles on the Riemann surface. Thus Figs. 5.12, 13 represent various parametrizations of the same finite-gap solutions. However, as before for the KP2 equation, we use both parametrizations. Finally, note that the transformation of the local parameter $k \to ak + bk^{-1} + o(k^{-2})$ induces the transformation

$$x \to ax + 3a^2bt, \; t \to a^3t, \; u \to a^2u + 2ab \tag{5.8.3}$$

of the solution. With imaginary a, b it transforms a solution in the parametrization of Fig. 5.12 into another solution in the parametrization of Fig. 5.13.

An arbitrary hyperelliptic M-curve can be uniformized in the following way. Let $C_1, \ldots, C_N$ be disjoint circles orthogonal to real axis and lying to the right of zero. The mapping $\pi z = -z$ transforms them to circles $C'_1, \ldots, C'_N$. As above, the pair C_n, C'_n determines a hyperbolic transformation σ_n. The fixed points of σ_n and σ_n^{-1} are A_n and $B_n = -A_n$. The isometrical circles of σ_n and σ_n^{-1} are C_n and C'_n respectively. The center of C_n is situated at the point $A_n(1+\mu_n)/(1-\mu_n)$ and its radius equals $2A_n\sqrt{\mu_n}/(1-\mu_n)$. In this case the Schottky group G is a subgroup of the group with generators α_n and π: $\alpha_n^2 = 1$, $\sigma_n = \pi\alpha_n$. The intersection points of the circles with real axis, as well as 0 and ∞, are the fixed points of the hyperelliptic involution $\pi z = -z$. The antiholomorphic involution is given by $\tau z = \overline{z}$.

As was mentioned in Sect. 5.1 the Schottky uniformization depends on the choice of the loops $v_1, \ldots, v_N$ on the Riemann surface. Below we consider two different Schottky uniformizations UI and UII of the curve (5.8.1), determined by Fig. 5.12 and Fig. 5.13 respectively (we always put $a_n = v_n$). The Schottky

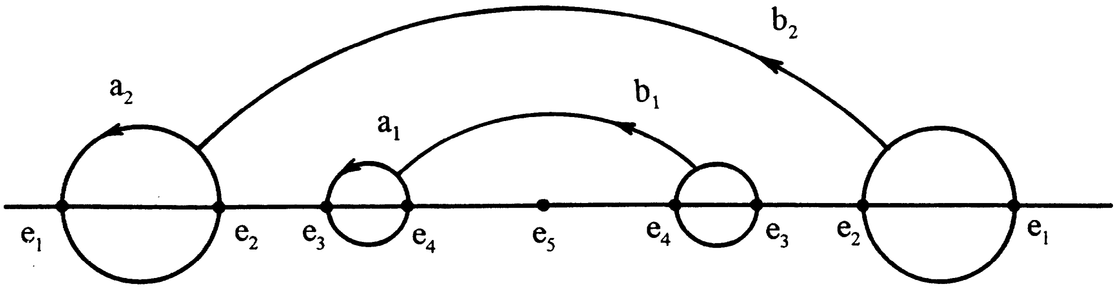

Fig. 5.14. The fundamental domain F^{I} of the Schottky group for an algebraic curve of genus two with the basis of cycles as shown in Fig. 5.12

groups G^{I} and G^{II}, corresponding to these two uniformizations of X, are different. Their fundamental domains F^{I} and F^{II} are shown in Fig. 5.14 and Fig. 5.15 respectively where the points e_i which are the images of the ramification points $E_i(E = \infty \to z = \infty)$ are also indicated. The involutions π and τ of the first uniformization are described above, the local parameter at ∞ equals $k = z$. For UII we also have $\pi z = -z$, but as explained earlier $\pi\tau z = \overline{z}$, $k = \mathrm{i}z$.

The reduction $B_n = -A_n$ simplifies the period matrix

$$
\begin{aligned}
B_{nm} &= \sum_{\sigma\in G_m\backslash G/G_n} \log\{B_m, A_m, \sigma B_n, \sigma A_n\} \\
&= \sum \log\left(\frac{B_m-\sigma B_n}{B_m-\sigma A_n}\right) + \sum \log\left(\frac{A_m-\sigma A_n}{A_m-\sigma B_n}\right) \\
&= \sum_{\sigma^*=\pi\sigma\pi} \log\left(\frac{A_m-\sigma^* A_n}{A_m-\sigma^*(-A_n)}\right) + \sum_{\sigma} \log\left(\frac{A_m-\sigma A_n}{A_m-\sigma(-A_n)}\right) \qquad (5.8.4)\\
&= \sum_{\sigma} \log\left(\frac{A_m-\sigma A_n}{A_m-\sigma(-A_n)}\right)^2 , \\
B_{nn} &= \log\mu_n + \sum_{\sigma\in G_n\backslash G/G_n, \sigma\neq I} \log\left(\frac{A_n-\sigma A_n}{A_n-\sigma(-A_n)}\right)^2 .
\end{aligned}
$$

Here we use an involution $\sigma \to \sigma^* = \pi\sigma\pi$ of the group G, which preserves the cosets. Convergence of the series in (5.8.4) is a corollary of convergence of the series $\sum(\sigma A_n - \sigma(-A_n))$. Due to (5.5.2,3), the other parameters are equal to

$$
\begin{aligned}
U_n &= \sum_{\sigma\in G/G_n} (\sigma A_n - \sigma(-A_n)) , \\
W_n &= \sum_{\sigma\in G/G_n} ((\sigma A_n)^3 - (\sigma(-A_n))^3) , \qquad (5.8.5)\\
\tilde{W}_n &= W_n - 3cU_n, \quad c = \sum_{\sigma\in G, \sigma\neq I} \gamma^{-2} ,
\end{aligned}
$$

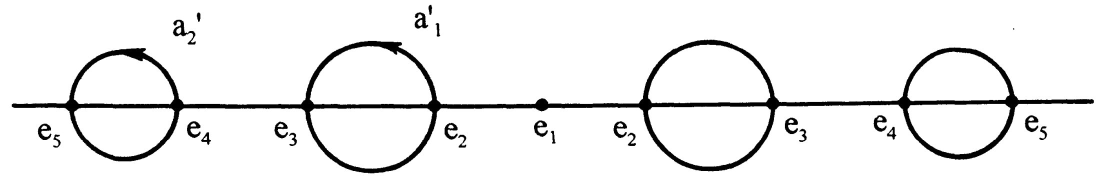

Fig. 5.15. The fundamental domain $F^{\rm II}$ of Schottky group for an algebraic curve of genus two with the basis of cycles depicted in Fig. 5.13

The description of the set of the uniformization parameters in this case is given by

$$
\begin{gathered}
0 < A_1 < \ldots < A_N < +\infty, \quad 0 < \sqrt{\mu_n} < 1, \quad n = 1, \ldots, N \quad , \\
\left(\frac{1-\sqrt{\mu}_n}{1+\sqrt{\mu_n}}\right)\left(\frac{1-\sqrt{\mu_{n+1}}}{1+\sqrt{\mu_{n+1}}}\right) > \frac{A_n}{A_{n+1}}, \quad n = 1, \ldots, N-1 \quad .
\end{gathered} \tag{5.8.6}
$$

The uniformization UII is equivalent to the choice of the local parameter $k = \mathrm{i}z$ at ∞, which in turn (5.8.3) leads to the transformation $x \to \mathrm{i}x$, $t \to -\mathrm{i}t$.

Theorem 5.6. All real nonsingular finite-gap solutions of the KdV equation are given by the following formulas:

$$
u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(\boldsymbol{U}x + \tilde{\boldsymbol{W}}t + \boldsymbol{D}) \quad , \tag{5.8.7}
$$

$$
u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(\mathrm{i}(\boldsymbol{U}x - \tilde{\boldsymbol{W}}t + \boldsymbol{D})) \quad . \tag{5.8.8}
$$

where $\boldsymbol{U}$, $\tilde{\boldsymbol{W}}$ are determined by (5.8.5), the parameters A_n, μ_n belong to the set (5.8.6) and $\boldsymbol{D} \in \mathbb{R}^N$ is an arbitrary vector. Formula (5.8.7) [as well as (5.8.8)] gives all the finite-gap solutions.

Now we return to formula (5.8.5), put $t = 0$ in it and derive expressions for the ends of spectrum bands of the integrable Schrödinger operator

$$
\left(-\frac{d^2}{dx^2} - u(x)\right)\varphi = \lambda\varphi
$$

with the potential (5.8.2) in terms of uniformization parameters. Formula (5.8.2) was obtained under the condition that the local parameter at ∞ equals $k = \mathrm{i}\sqrt{\lambda}$. Comparing it with $k = z$ (for $G^{\rm I}$) and $k = \mathrm{i}z$ (for $G^{\rm II}$), we have

$$
\begin{aligned}
\lambda^{\rm I}(z) &= -\sum_{\sigma\in G^{\rm I}}[(\sigma z)^2 - (\sigma 0)^2] + Q \quad , \\
\lambda^{\rm II}(z) &= \sum_{\sigma\in G^{\rm II}}[(\sigma z)^2 - (\sigma 0)^2] - Q \quad , \\
Q &= \sum_{\sigma\in G,\sigma\neq I}[\left(\frac{\alpha}{\gamma}\right)^2 - \left(\frac{\beta}{\delta}\right)^2]
\end{aligned} \tag{5.8.9}
$$

for G^{I} and G^{II} respectively. We have $E_{2N+1} = Q$ (for UI) and $E_I = -Q$ (for UII), (see Figs. 5.14, 15).

The other ends of the gaps are the images of the fixed points of the involution π on $\Omega/G : \sigma_n z = \pi z$. We have $2N$ points $z_n^{\pm}, n = 1, \dots, N$, each pair $z_n^{\pm}$ being a solution of the quadratic equation

$$(z - A_n)^2 = \mu_n(z + A_n)^2 \Rightarrow z_n^{\pm}, \quad z_n^- < z_n^+ \quad .$$

The gaps are given by (see Figs. 5.14,15)

$$\begin{aligned} &E^{\mathrm{I}}_{2N-2n+1} = \lambda^{\mathrm{I}}(z_n^+), \quad &&E^{\mathrm{I}}_{2N-2n+2} = \lambda^{\mathrm{I}}(z_n^-), \\ &E^{\mathrm{II}}_{2n} = \lambda^{\mathrm{II}}(z_n^-), &&E^{\mathrm{II}}_{2n+1} = \lambda^{\mathrm{II}}(z_n^+) \quad . \end{aligned}$$

The corresponding potentials for UI and UII are

$$\begin{aligned} u(x) &= 2\frac{\partial^2}{\partial x^2} \log \theta(\boldsymbol{U}x + \boldsymbol{D}) + 2c, \quad \text{for UI} \quad , \\ u(x) &= 2\frac{\partial^2}{\partial x^2} \log \theta(\mathrm{i}(\boldsymbol{U}x + \boldsymbol{D})) - 2c, \quad \text{for UII} \quad , \end{aligned}$$

with the constants determined by (5.8.5). At last, note that a shift $u \longrightarrow u - \alpha$, $\lambda \longrightarrow \lambda + \alpha$ leads to the general finite-gap spectrum.

5.9 Qualitative Analysis of Solutions of the KdV Equation

Since G is the Fuchsian group of the second kind, all the series of Sect. 5.8 converge. The smaller the circles C_n, C'_n, the better the convergence. The small circles, in turn, correspond to the small loops v_n, which implies a smallness of allowed bands of Fig. 5.12 and gaps of Fig. 5.13. In the paper [5.20], using the theta functional substitution technique, a detailed analysis of two-phase solutions in these limiting cases was carried out. The first terms of the series for B, $\boldsymbol{U}$, $\boldsymbol{W}$, with respect to a small parameter (size of zone), were obtained. They turned out to be sufficient for a physical interpretation of the solutions. The formulas of the present chapter seem to be even more suitable for such an investigation. In addition, they allow to define uniquely some physical characteristics of the multi-phase solution such as amplitudes, wave numbers and phase velocities of harmonics (see below). These characteristics can hardly be defined in the substitution approach since the modular transformation of the theta function changes them.

Let us consider (5.8.7, 8) in the limit $\mu \longrightarrow 0$. The solution (5.8.8), defined by UII in this limit, describes small amplitude waves. Indeed, this solution corresponds to a spectrum with small gaps which in turn implies the smallness of the C_n-circles of UII. Denoting the corresponding parameters of UII by A_n^{II}, μ_n^{II}, the first approximation gives

$$B_{nn}^{\mathrm{II}} \approx \log \mu_n^{\mathrm{II}}, \quad B_{nm}^{\mathrm{II}} \approx \left(\frac{A_m^{\mathrm{II}} - A_n^{\mathrm{II}}}{A_m^{\mathrm{II}} + A_n^{\mathrm{II}}}\right)^2 \quad ,$$

$$U_n^{\mathrm{II}} \approx 2A_n^{\mathrm{II}}, \quad \tilde{W}_n^{\mathrm{II}} \approx 2(A_n^{\mathrm{II}})^3 \quad .$$

The diagonal elements of the period matrix are much bigger than the off-diagonal matrix elements; therefore, the solution represents a sum of non-interacting small amplitude harmonics:

$$u(x,t) \approx -16 \sum^{N} (A_n^{\mathrm{II}})^2 \sqrt{\mu_n^{\mathrm{II}}} \cos\left(2A_n^{\mathrm{II}}(x - (A_n^{\mathrm{II}})^2 t) + D_n\right) \quad .$$

The phase speeds of harmonics are equal to $\tilde{W}_n^{\mathrm{II}}/U_n^{\mathrm{II}} \approx (A_n^{\mathrm{II}})^2$.

Similarly, the limit of small allowed bands is studied. It corresponds to the limit $\mu \to 0$ in (5.8.7). We denote the corresponding uniformization parameters of UI by A_n^{I}, μ_n^{I}. The first approximation

$$B_{nn}^{I} \approx \log \mu_n^{I}, \quad B_{nm}^{I} \approx \left(\frac{A_m^{I} - A_n^{I}}{A_m^{I} + A_n^{I}}\right)^2, \quad U_n^{I} \approx 2A_n^{I}, \quad \tilde{W}_n^{I} \approx 2(A_n^{I})^3$$

describes the interaction of N-soliton chains moving with the velocities $W_n^{\mathrm{I}}/U_n^{\mathrm{I}}$. A soliton chain represents a periodic infinite sequence of solitons. Solitons of different chains interact in pairs with the usual collisional phase shift.

One may associate a wave number to every soliton chain or harmonic. As a matter of fact, periods of the theta functions in (5.8.7, 8) are determined by the lattice

$$\Lambda = \{2\pi \mathrm{i} \boldsymbol{M} + \boldsymbol{N}B\}, \quad \boldsymbol{M}, \boldsymbol{N} \in \mathbb{Z}^N \quad ,$$

where B is real. For small amplitude waves the argument of the theta function in (5.8.8) is imaginary; hence, the wave numbers of the harmonics are determined by the imaginary part of the lattice Λ

$$K_n^{\mathrm{II}} = U_n^{\mathrm{II}} \quad . \tag{5.9.1}$$

The theta functional argument (5.8.7) of the soliton chain is real; therefore the wave numbers are given by

$$\boldsymbol{K}^{\mathrm{I}} = (K_1^{\mathrm{I}}, \ldots, K_N^{\mathrm{I}}) = 2\pi (B^{\mathrm{I}})^{-1} (\boldsymbol{U}^{\mathrm{I}})^T \quad . \tag{5.9.2}$$

The interpretation of an N-phase solution with a finite period matrix B is nontrivial, due to the possible modular transformations (2.5.12) changing the wave numbers and other characteristics of solitons. Usually this freedom is eliminated by fixing a certain "basic" period matrix obtained from a certain ratio of diagonal and off-diagonal matrix elements [5.21]. In this way the interpretation of degenerate cases is extended to finite B. We mention that this fixation of the "basic" period matrix is well-defined only for $N = 2$.

We extend the interpretation of formulas (5.8.7,8) presented above to all possible values of uniformization parameters. Let the expression (5.8.8) describe N interacting harmonics with the wave numbers $\boldsymbol{K}^{\mathrm{II}}$ and the phase speeds $\tilde{W}_n^{\mathrm{II}}/U_n^{\mathrm{II}}$, and let the expression (5.8.7) describe N interacting soliton chains with the wave numbers K^{I} and the speeds $\tilde{W}_n^{\mathrm{I}}/U_n^{\mathrm{I}}$. The distinctive property of $\boldsymbol{U}$ and B, defined by (5.8.4, 5), singling them out from all equivalent ones, is as follows:

$$0 < U_1 < \ldots < U_N \quad ,$$

$$-B_{n,1} < \ldots < -B_{n,n-1} < -B_{nn} > -B_{n,n+1} > \ldots > -B_{n,N} \quad .$$

Such an interpretation is imposed by the spectral problem. The difference to the above mentioned approach [5.20, 21] can be found only in the intermediate case of medium amplitudes. But the interaction picture in this case is so complicated that one of the interpretations suggested can hardly be preferred to the other.

The cycles of Fig. 5.12, 13 are connected by the symplectic transformation (see also (2.4.22, 2.5.12, 13))

$$a = d = 0 \ , \quad c^T b = -1 \ , \ c = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ \vdots & & & & & \vdots \\ -1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} . \tag{5.9.3}$$

The theta function is transformed according to

$$\begin{aligned} &\theta(\boldsymbol{z}'; B') = \chi \exp\left\{\frac{1}{2}\langle \boldsymbol{z}, B^{-1}\boldsymbol{z}\rangle\right\} \theta(\boldsymbol{z}; B) \quad , \\ &B' = 4\pi^2 (c^T)^{-1} B^{-1} c^{-1}, \quad \boldsymbol{z}' = 2\pi \mathrm{i} (c^{-1})^T B^{-1} \boldsymbol{z} \quad . \end{aligned} \tag{5.9.4}$$

Let $\boldsymbol{U}^{\mathrm{II}}$, $\tilde{\boldsymbol{W}}^{\mathrm{II}}$, $\boldsymbol{B}^{\mathrm{II}}$ be the parameters determining the N-phase solution (5.8.8). (We would like to mention that the corresponding basis is a', b' and $z' = \mathrm{i}(\boldsymbol{U}^{\mathrm{II}}x - \tilde{\boldsymbol{W}}^{\mathrm{II}}t) + \boldsymbol{D})$). The modular transformation (5.9.4) yields the representation

$$\begin{aligned} &u(x,t) = 2\frac{\partial^2}{\partial x^2} \log \theta\left(\boldsymbol{U}^{\mathrm{I}}x - \boldsymbol{W}'t + \boldsymbol{D}; B^{\mathrm{I}}\right) + \alpha \quad , \\ &B^{\mathrm{I}} = 4\pi^2 c^{-1} (B^{\mathrm{II}})^{-1} (c^T)^{-1}, \quad \alpha = 2\langle \boldsymbol{U}^{\mathrm{II}}; (B^{\mathrm{II}})^{-1} \boldsymbol{U}^{\mathrm{II}}\rangle \quad , \\ &\boldsymbol{U}^{\mathrm{I}} = \frac{1}{2\pi} B^I c^T \boldsymbol{U}^{\mathrm{II}}, \quad \boldsymbol{W}' = -\frac{1}{2\pi} B^I c^T \tilde{\boldsymbol{W}} \end{aligned} \tag{5.9.5}$$

for the same solution. This solution differs from (5.8.7) with A^{I}, μ^{I} by the transformation (5.8.3) and may be interpreted as an interaction of N-soliton chains moving with the speeds W_n'/U_n^{I} on the constant background α.

Let $\boldsymbol{U}$ be the solution characterized by the wave numbers $\boldsymbol{K}^{\mathrm{II}} = (K_1^{\mathrm{II}}, \ldots, K_N^{\mathrm{II}})$ in the interacting harmonics description. The passage (5.9.5) to the soliton interpretation combined with (5.9.1, 2) gives the following connection of wave numbers:

$$\boldsymbol{K}^{\mathrm{I}} = \boldsymbol{K}^{\mathrm{II}} c \,. \tag{5.9.6}$$

Consider a periodic small amplitude solution with the period 2π, describing the interaction of N first harmonics, i.e., none of the harmonics is omitted:

$$\boldsymbol{K}^{\mathrm{II}} = (1, 2, \ldots, N) \quad .$$

If we increase the amplitudes of all harmonics keeping the period fixed then at a certain time the soliton interpretation becomes more natural. Moreover, due to (5.9.6)

$$\boldsymbol{K}^{\mathrm{I}} = (1, \ldots, 1)$$

all chains have the same period 2π. This phenomenon is called the wave number paradox. For $N = 2$ it was discussed in [5.20].

The approach of present section was in [5.3, 4] applied to calculate and to draw plots of the finite-gap solutions of the KdV equations. It turns out that the suggested interpretation is in good agreement with the numerical experiments.

5.10 Solutions of the Sine-Gordon Equation

We considered above only the real Riemann surfaces of decomposing type. However, the finite-gap integration of a number of equations deals with the real Riemann surfaces of nondecomposing type. The sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{5.10.1}$$

is the best-known equation of this family. The Schottky uniformization allows us to describe effectively the real finite-gap solutions in this case also.

The real finite-gap solutions of the sine-Gordon equation are determined by the hyperelliptic curves

$$\mu^2 = \lambda \prod_{i=1}^{2N} (\lambda - E_i) \quad , \tag{5.10.2}$$

where $2k$ points $E_1 < \ldots < E_{2k} < 0$ lie on the real axis and the others

$$\overline{E_{2k+2n-1}} = E_{2k+2n}, \quad n = 1, \ldots, N-k$$

are mutually conjugated (see Sect. 4.5 for details). The curve X is of the nondecomposing type if $k \neq N$.

The canonical basis of cycles is fixed in Fig. 5.16. The cycle $\mathcal{L}$, going around the cut $[0, +\infty]$, is equal to $a_1 + \cdots + a_N$ and the anti-involution

$$\tau : \ (\lambda, \mu) \to (\overline{\lambda}, -\overline{\mu})$$

transforms the basis as indicated in (4.3.23):

$$\begin{aligned} \tau a_i &= -a_i, \quad i = 1, \dots, N \\ \tau b_j &= b_j, \quad j = 1, \dots, k \\ \tau b_j &= b_j - a_j, \quad j = k+1, \dots, g \quad . \end{aligned} \tag{5.10.3}$$

The cycle $\mathcal{L}$ fixes a branch of the square root $\sqrt{\lambda}$ on X. The normalized Abelian differentials $d\Omega_1$, $d\Omega_2$ are defined by their singularities

$$\begin{aligned} d\Omega_1 &\to d(\sqrt{\lambda}), \quad \lambda \to \infty \quad , \\ d\Omega_2 &\to d(\frac{1}{\sqrt{\lambda}}), \quad \lambda \to 0 \quad . \end{aligned} \tag{5.10.4}$$

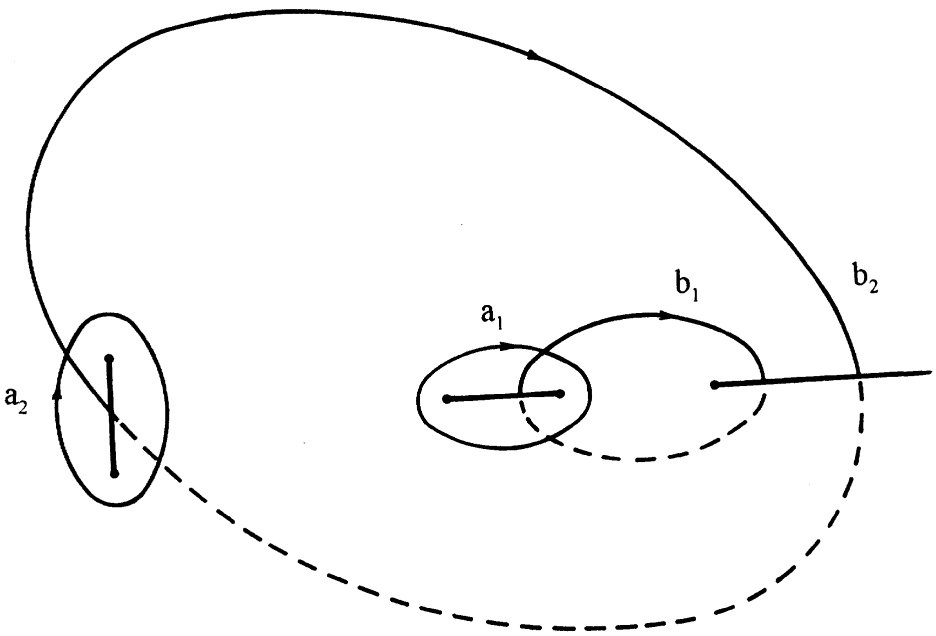

Fig. 5.16. The basis of cycles for an algebraic curve of genus two

All real finite-gap solutions of the sine-Gordon equation are determined by (see also (4.2.25, 27, 31)):

$$u(x,t) = 2\mathrm{i} \log \frac{\theta\left(\mathrm{i}\frac{(U-V)}{4}x + \mathrm{i}\frac{(U+V)}{4}t + \boldsymbol{D} + \pi \mathrm{i} \boldsymbol{\Delta}\right)}{\theta\left(\mathrm{i}\frac{(U-V)}{4}x + \mathrm{i}\frac{(U+V)}{4}t + \boldsymbol{D}\right)} \quad , \tag{5.10.5}$$

where $\boldsymbol{U}$, $\boldsymbol{V}$ are the vectors of b-periods of $d\Omega_1$, $d\Omega_2$ respectively, $\boldsymbol{\Delta} = (1, \dots, 1)$, $\boldsymbol{D} = \boldsymbol{D}_0 + \pi \mathrm{i}(\varepsilon_1, \dots, \varepsilon_k, 0, \dots, 0) + \pi \mathrm{i}/2(1, \dots, 1, 0, \dots, 0)$ (zeros on the last $N-k$ places), $\boldsymbol{D}_0$ is an arbitrary real vector, and $\varepsilon_1, \dots, \varepsilon_k$ are either 0 or 1 and enumerate the different components of solutions.

The curve (5.10.2), as well as the hyperelliptic M-curve of Sect. 5.8, is uniformized by the Schottky group with generators $\sigma_1, \dots, \sigma_N$, which is a subgroup of index 2 of the group with generators α_n and $\pi z = -z$: $\alpha_n^2 = I$,

$\sigma_n = \alpha_n \pi$. The reduction $B_n = -A_n$ is also valid. The only difference to Sect. 5.8 is that along with the positive μ with $0 < \mu_1, \ldots, < \mu_k < 1$ there are now $N - k$ negative ones $-1 < \mu_{k+1}, \ldots, < \mu_N < 0$. This means that the generators $\sigma_{k+1}, \ldots, \sigma_N$ map the upper half plane H onto the lower half-plane $\overline{H}$. Consequently, the group G is not a Fuchsian group, although the real axis remains an invariant circle of G. The fixed points on Ω/G of the hyperelliptic involution $\pi z = -z$, besides 0 and ∞, lie in pairs on every circle C_n. For $n \geq k$ they are the points of intersection of C_n with the real axis, and for $n > k$ they are complex conjugate. The anti-involution τ is, as above, $\tau z = \overline{z}$. The basis of cycles of this uniformization satisfies the condition (5.10.3). For $N = 2$, $k = 1$ it is shown in Fig. 5.17.

Although G is not a Fuchsian group of the second kind it is a group having an invariant circle. Hence, all the series of these sections converge.

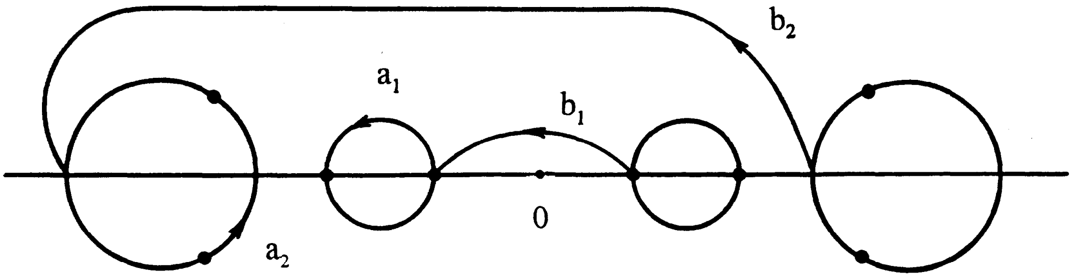

Fig. 5.17. The fundamental domain of the Schottky group uniformizing the curve in Fig. 5.16

The period matrix is given by the same expression (5.8.4). For the correct definition of U, V it is necessary to coordinatize the local parameters at the points $z = 0$ and $z = \infty$.

First of all, the function $\lambda(z)$ has a double pole at $z = \infty$, a double zero at $z = 0$ and satisfies the reduction $\lambda(\overline{z}) = \overline{\lambda}(z)$: therefore it is equal to

$$\lambda(z) = q \sum_{\sigma \in G} \left((\sigma z)^2 - (\sigma 0)^2\right) \quad , \tag{5.10.6}$$

where q is a real constant. The asymptotics of $\lambda(z)$ at the points $z = \infty$ and $z = 0$ is

$$\begin{aligned} &\lambda(z) \to qz^2, \quad z \to \infty \quad , \\ &\lambda(z) \to qcz^2, \quad z \to 0 \quad , \\ &c = \sum_{\sigma \in G} \delta^{-4}(1 - 2\beta\gamma) \quad . \end{aligned} \tag{5.10.7}$$

It is evident that the fixed oval of $\pi\tau$, which is situated over the cut $[0, \infty]$ in the λ-plane, is mapped onto the imaginary axis in the z-plane, therefore q in (5.10.6) is negative

$$q = -a^2 \quad .$$

Let us consider the Riemann surface X of the curve (5.10.2) as a two-sheeted cover of the λ-plane (two copies of the λ-planes glued along the cuts $[0,+\infty]$, $[E_1, E_2], \ldots,$ $[E_{2N-1}, E_{2N}]$). We fix the branch of the square root $\sqrt{\lambda}$ so that it changes its sign when we intersect the loops v_n situated over cuts $[E_{2n-1}, E_{2n}]$ and coinciding with the a-cycles. As was mentioned above, the loops v_n are mapped onto the circles C_n, C'_n. Therefore, on the fundamental domain F the function $\sqrt{\lambda}$ is single-valued. Finally, we have the following asymptotics:

$$\begin{aligned} &\sqrt{\lambda} \to \mathrm{i}az, \quad z \to \infty \quad , \\ &\sqrt{\lambda} \to \mathrm{i}az\sqrt{c}, \quad z \to 0 \quad , \end{aligned} \tag{5.10.8}$$

where $\sqrt{c}$ must be taken to be positive. Indeed, $\sqrt{\lambda}(\mathrm{i}y)$ is real-valued for $y \in \mathbb{R}$ and never vanishes except at $y = 0$.

The parameter a is not essential since a change of a is equivalent to a Lorentz transformation with respect to which the equation is invariant. Chosing $a = 1$ and comparing (5.10.4) and (5.10.8) we obtain the asymptotics

$$\begin{aligned} &d\Omega_1 \to \mathrm{i}\, dz, \quad z \to \infty \quad , \\ &d\Omega_2 \to -\frac{\mathrm{i}}{\sqrt{c}}\, d\frac{1}{z}, \quad z \to 0 \quad . \end{aligned}$$

In the usual way we calculate the b-periods of Abelian differentials of the second kind, using the reciprocity law (2.4.13) in terms of normalized holomorphic Abelian differentials

$$\begin{aligned} &U_n = \mathrm{i} \sum_{\sigma \in G/G_n} [\sigma(-A_n) - \sigma A_n] \quad , \\ &V_n = -\frac{\mathrm{i}}{\sqrt{c}} \sum_{\sigma \in G/G_n} \left[\frac{1}{\sigma(-A_n)} - \frac{1}{\sigma A_n}\right] \quad . \end{aligned} \tag{5.10.9}$$

Theorem 5.7. All real nonsingular finite-gap solutions of the sine-Gordon equation are given by the formulas (5.10.5, 9, 8.4, 10.7).

If X is considered as two copies of the λ-planes glued along the cuts $[0, \infty]$, $[E_{2n-1}, E_{2n}]$, then the "vertical" cuts $[E_{2i}, \overline{E}_{2i}]$ may be chosen in various ways. The order of the points of intersection of $[E_{2i}, \overline{E}_{2i}]$ with the real axis is not fixed. In a similar way the order of fixed points of the generators σ_n with positive and negative μ_n is not determined. An appropriate choice of generators changes this order, while preserving the condition $B_n = -A_n$.

We considered the uniformization of the curve (5.10.2) connected with the fixed half-basis of cycles $a_1, \ldots, a_N$. Sometimes, for example in constructing the action-angle variables [5.22], other half-bases may seem more convenient.

Remark. Due to the arguments given at the end of Sect. 4.3 the formula

$$u(x,t) = 2\mathrm{i}\log\frac{\theta\left(\frac{(U-V)}{4}x - \frac{(U+V)}{4}t + \boldsymbol{D} + \pi\mathrm{i}\boldsymbol{\Delta}\right)}{\theta\left(\frac{(U-V)}{4}x - \frac{(U+V)}{4}t + \boldsymbol{D}\right)}$$

is more convenient for the description of small-amplitude solutions of the sine-Gordon equation. In this representation the small-amplitude solutions are obtained in the limit of small generating circles of the Schottky group $\mu \to 0$ and can be investigated in exactly the same way as done in Sects. 5.7 and 9 for the KP and KdV equations.

5.11 Uniformization and Coverings

The finite-gap solutions of various nonlinear integrable equations are determined by special Riemann surfaces. These are hyperelliptic curves, which determine the solutions of the KdV, NS, sine-Gordon and many other equations, elliptic-hyperelliptic curves (two-sheeted coverings of elliptic curves), which generate the solutions of the XYZ Landau-Lifshitz equation, two-sheeted coverings, by which the finite-gap two-dimensional Schrödinger operators are constructed, and various curves corresponding to tops of all kinds (see Chap. 6). In addition, coverings (first of all of elliptic curves) generate interesting special finite-gap solutions of nonlinear integrable equations (see Chap.7). In all these cases of uniformization it is necessary to take into account specific characteristics of the curve. For this purpose we need a simple generalization of the Schottky groups.

Let $C_1, C_1', \ldots, C_N, C_N', S_1, \ldots, S_M$ be a set of $2N + M$ disjoint Jordan curves on $\mathbb{C}$, which comprise the boundary of a $2N + M$-connected domain $\widehat{F}$. The loxodromic transformations σ_n, $n = 1, \ldots, N$ transform the outside of a boundary curve C_n onto the inside of a boundary curve C_n', $\sigma_n C_n = C_n'$. The elliptic transformations α_m, $m = 1, \ldots, M$, $\alpha_m^{k_m} = 1$ transform the outside of S_m onto the inside of S_m. The transformations $\sigma_1, \ldots, \sigma_N$, $\alpha_1, \ldots, \alpha_M$ generate a generalized Schottky group $\widehat{G}$, $\widehat{F}$ is the fundamental domain of $\widehat{G}$. The factor $\Omega/\widehat{G}$, where Ω is the set of discontinuities of $\widehat{G}$, is a Riemann surface of genus N with $2M$ marked points z_m^1, z_m^2. Here $z_m^{1,2}$ are the fixed points of α_m. The local parameters at $z_m^{1,2}$ are equal to $(z - z_m^{1,2})^{1/k_m}$. If all the boundary curves $C_1, C_1', \ldots, C_N, C_N', S_1, \ldots, S_M$ are circles (the generalized Schottky group is called a classical one), the orders k_m of all elliptic generators necessarily equal 2.

Let G be subgroup of $\widehat{G}$ of finite index $d = [\widehat{G} : G]$. This means that the set of representatives $g_1, \ldots, g_d$ of left cosets $G/\widehat{G}$ is finite, i.e., an arbitrary element $\widehat{g}$ can be represented as follows: $\widehat{g} = g g_s$, $g_s \in \{g_1, \ldots, g_d\}$, $g \in G$. If $\widehat{G}$ is a group with fundamental domain $\widehat{F}$, then G is a Klein group with the fundamental domain

$$F = g_1\widehat{F} \cup \ldots \cup g_d\widehat{F} \quad ,$$

and the Riemann surface Ω/G is an N-sheeted covering of $\Omega/\widehat{G}$. A normal subgroup $G \subset \widehat{G}$ induces a normal covering $\Omega/G \to \Omega/\widehat{G}$, and the quotient group $\widehat{G}/G$ is an automorphism group of the covering.

As a matter of fact, in Sects. 5.8,10 we used the generalized Schottky groups when we described the uniformizations of hyperelliptic curves. Let $\widehat{G}$ be a group with elliptic generators $\pi, \alpha_1, \ldots, \alpha_N$, $\pi^2 = \alpha^2 = 1$, $\pi z = -z$ whose fundamental domain $\widehat{F}$ represents the right half-plane ($\mathrm{Re}\, z > 0$) with N circles $C_1, \ldots, C_N$ cut out. The transformation α_n maps the outside of C_n onto the inside of C_n. The subgroup G of index 2, which consists of all elements of $\widehat{G}$ of even graduation, is a usual Schottky group with generators $\sigma_n = \pi\alpha_n$. This group was used for integration of the KdV and sine-Gordon equations. In this case the set of representatives of left cosets $G/\widehat{G}$ is $\{I, \pi\}$, and $F = \widehat{F} \cup \pi\widehat{G}$ is the fundamental domain of G shown in Figs. 5.14,15,17.

Let us consider coverings of elliptic curves. The elliptic-hyperelliptic curves are described in complete analogy with the hyperelliptic case. The generalized Schottky group $\widehat{G}$ is generated by elliptic transformations π, $\alpha_1, \ldots, \alpha_M$, $\pi^2 = \alpha_m^2 = 1$ and by a loxodromic transformation $\widehat{\sigma}$. Fix π to be equal to $\pi z = -z$. The fundamental domain $\widehat{F}$ represents a half-plane with $M+2$ circles $C_1, C_1', S_1, \ldots, S_M$ cut out. The transformation α_m maps the outside of S_m onto the inside of S_m, and $\widehat{\sigma}$ maps the outside of C_1 onto the inside of C_1'. The subgroup G is generated by

$$\sigma_1 = \widehat{\sigma},\ \sigma_2 = \pi\widehat{\sigma}\pi,\ \sigma_3 = \pi\alpha_1, \ldots, \sigma_{M+2} = \pi\alpha_M \quad .$$

This group is a usual Schottky group with the fundamental domain $F = \widehat{F} \cup \pi\widehat{F}$ and the following restrictions to the fixed points A_i, B_i of the transformations σ_i;

$$A_1 = -A_2, \quad B_1 = -B_2, \quad A_i = -B_i, \quad i = 3, \ldots, M+2 \quad .$$

G is a normal subgroup of $\widehat{G}$, and Ω/G is a two-sheeted covering of an elliptic curve. The quotient group $\widehat{G}/G$ corresponds to an involution interchanging the sheets.

For $M = 0$ the Riemann surface Ω/G is of genus 2. The generators σ_1, σ_2 of the corresponding Schottky group satisfy the relation

$$\sigma_2 = \pi\sigma_1\pi, \quad A_2 = -A_1 \equiv -A, \quad B_2 = -B_1 \equiv -B \quad .$$

The transformation $z \to AB/z$ corresponds to the hyperelliptic involution of Ω/G. The holomorphic differentials ω_1, ω_2, normalized in the natural basis of cycles, are transformed into each other under the action of π : $\pi^*\omega_1 = \omega_2$. Hence the differential $w = \omega_1 + \omega_2$ is the normalized holomorphic differential of the elliptic curve $\Omega/\widehat{G}$ and $v = \omega_1 - \omega_2$ is the Prym differential of the covering $\Omega/G \to \Omega/\widehat{G}$. One can easily show that

$$\begin{aligned} w &= \sum_{\sigma\in\widehat{G}/G_1} \left[(z-\sigma B)^{-1} - (z-\sigma A)^{-1}\right] dz \quad , \\ v &= \sum_{\sigma\in\widehat{G}/G_1} (-1)^{\sigma_\pi} \left[(z-\sigma B)^{-1} - (z-\sigma A)^{-1}\right] dz \quad , \end{aligned} \tag{5.11.1}$$

where $\sigma \in \widehat{G}/G_1$ means that the summation is taken over all elements of $\widehat{G}$ of the form $\sigma = \ldots\pi$ (i.e., the word ends with the element π). The integer number σ_π equals the number of $\pi's$ in the representation of σ via $\widehat{\sigma}$ and π. Setting

$$\omega = \int_{b_1} w, \quad \omega' = \int_{b_1} v \quad ,$$

we have for the period matrix of Ω/G

$$B = \frac{1}{2}\begin{pmatrix} \omega+\omega' & \omega-\omega' \\ \omega-\omega' & \omega+\omega' \end{pmatrix} \quad .$$

Expressions (5.11.1) imply

$$\begin{aligned} \omega &= \log\mu_1 + \sum_{\sigma\in G_1\backslash\widehat{G}/G_1} \log\{\sigma B, \sigma A, B, A\} \quad , \\ \omega' &= \log\mu_1 + \sum_{\sigma\in G_1\backslash\widehat{G}/G_1} (-1)^{\sigma_\pi} \log\{\sigma B, \sigma A, B, A\} \quad , \end{aligned} \tag{5.11.2}$$

The case of a curve of genus 2, which is a two-sheeted covering of an elliptic curve, was considered above; N-sheeted coverings of genus 2 can be described in a similar way. Let us consider the case $N = 3$. The group $\widehat{G}$ is quite the same as for $N = 2$. The difference is in the choice of the subgroup G. For $N = 3$ it is generated by

$$\sigma_1 = \widehat{\sigma}^2, \quad \sigma_2 = \pi\widehat{\sigma}\pi, \quad \widehat{\sigma}\pi\widehat{\sigma}^{-1} \quad ,$$

and its fundamental domain $F = \widehat{F}\cup\pi\widehat{F}\cup\widehat{\sigma}\widehat{F}$ is shown in Fig. 5.18. The subgroup G is a generalized Schottky group with a fundamental domain consisting of five circles. The subgroup G is not normal.

The differential w (5.11.1) of $\Omega/\widehat{G}$ is holomorphic, and its periods are equal to

$$\left(\int_{a_1} w, \int_{a_2} w, \int_{b_1} w, \int_{b_2} w\right) = (2\pi\mathrm{i}, 2\pi\mathrm{i}, 2\omega, \omega) \quad .$$

As will be shown in Chap. 7, if one holomorphic differential of the curve of genus 2 is reduced to an elliptic one, then there exists another linearly independent holomorphic differential, which is also reduced to an elliptic differential. Let us calculate the periods of the last one.

Let $(2\pi\mathrm{i}, 2\pi\mathrm{i}, k\omega, \omega)$ and $(2\pi\mathrm{i}n_1 + m_1\tau, 2\pi\mathrm{i}n_2 + m_2\tau, 2\pi\mathrm{i}n_3 + m_3\tau, 2\pi\mathrm{i}n_4 + m_4\tau)$ be periods of these differentials, where $k, n_i, m_i \in \mathbb{Z}$. Calculating the

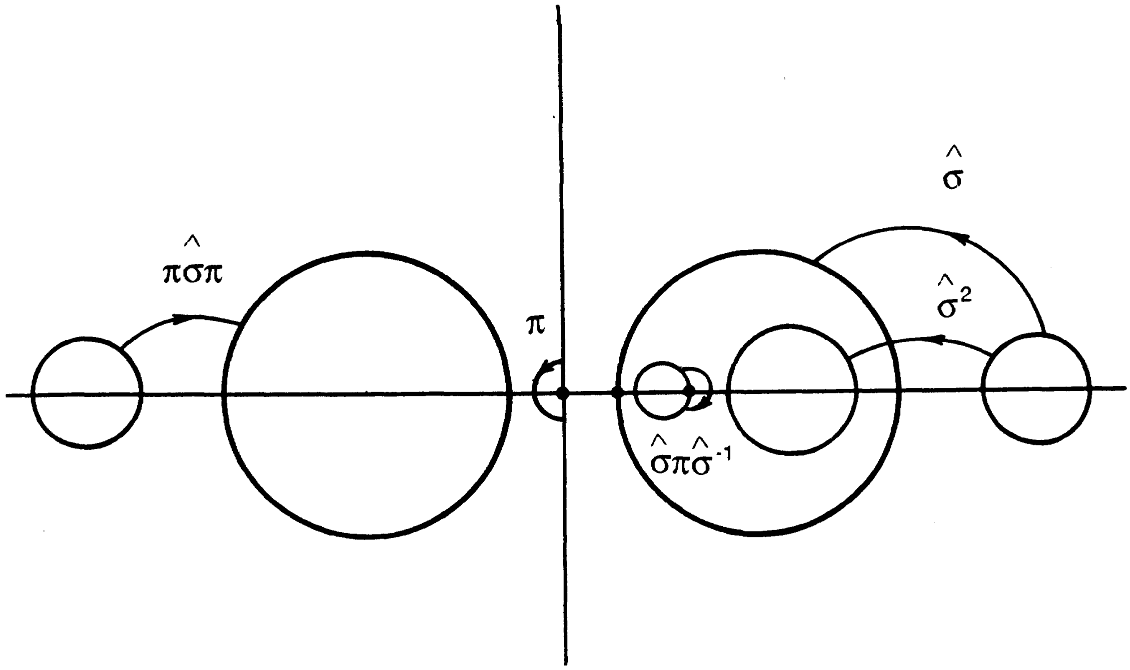

Fig. 5.18. The fundamental domain $F = \widehat{F} \cup \pi\widehat{F} \cup \widehat{\sigma}\widehat{F}$

period matrix and taking into account its symmetry and the incommensurability of $\omega, \tau, 2\pi\mathrm{i}, \omega\tau$, we have $n_2 = -kn_1$, $m_2 = -km_1$, $n_4 = -n_3$, $m_4 = -m_3$. Finally, in the new normalization, the periods of the second differential become equal to $(2\pi\mathrm{i}, -2\pi\mathrm{i}k, \omega', -\omega')$ and the period matrix of Ω/G is given by

$$B = \frac{1}{k+1}\begin{pmatrix} k^2\omega + \omega' & k\omega - \omega' \\ k\omega - \omega' & \omega + \omega' \end{pmatrix} .$$

In the case under consideration $k = 2$ and (5.11.1) yields the expression for w in terms of the basic holomorphic differentials $w = \omega_1 + \omega_2$. For the differential w and its period ω we have the same formulas (5.11.1, 2). As was shown, the second holomorphic differential, which reduces to an elliptic one, is equal to

$$\begin{aligned} v = \omega_1 - 2\omega_2 = &\sum_{\sigma \in G/G_1} \left[(z - \sigma B)^{-1} - (z - \sigma A)^{-1}\right] dz \\ &- 2 \sum_{\sigma \in G/G_2} \left[(z - \sigma\pi B)^{-1} - (z - \sigma\pi A)^{-1}\right] dz \quad , \end{aligned}$$

where the summation is taken over cosets generated by subgroups G_1, G_2 with generators σ_1, σ_2, respectively.

Note that the fundamental domain of $\widehat{G}$ is bounded by three circles; therefore all corresponding theta series converge due to the Schottky condition (Lemma 5.2).

Appendix 5.1 Description of the Schottky Space of M-Curves

As we have seen above (Sect. 5.1) the problem, which arises naturally in application of the Schottky uniformization to the finite-gap integration theory, is to describe explicitly the set $S = \{A_1, B_1, \mu_1, \dots, A_N, B_N, \mu_N\}$ of uniformization parameters. There are a lot of papers devoted to this problem [5.16, 19, 23-27]. Effective concrete results are obtained for the case of the Fuchsian group G. It was mentioned that this is the most important case generating real finite-gap solutions. For all uniformizations of the present chapter the set S can be described explicitly. Below we give a complete description of S for M-curves.

Let σ be a hyperelliptic transformation, characterized by the triple A, B, μ (5.1.1). A half-circle $\mathcal{L}(\sigma)$, lying in H and orthogonal to $\mathbb{R}$ with the ends at points A, B, is called an invariant line. The transformation σ maps $\mathcal{L}(\sigma)$ onto itself. The arrangement of invariant lines of generators and some of their products was investigated in the papers [5.18, 24, 25, 28]. Based on these results, a description of generator parameters $S = \{A_1, B_1, \mu_1, \dots, A_N, B_N, \mu_N\}$ was obtained in [5.19] (see also [5.16, 25], where slightly different parameterizations are used). It is evident that S depends on the choice of generators of G. In [5.19] the generator system of Fig. 5.4 for the Riemann surfaces of the signature $(0, n)$ was studied. We describe the set S for another system of generators corresponding to the cuts shown in Fig. 5.5 and the fundamental domain of Fig. 5.6. It allows us to present a complete description of S.

Lemma 5.8. Let $\sigma_1, \dots, \sigma_N$ be the generators of the Fuchsian Schottky group with the fundamental domain shown in Fig. 5.6. Then the invariant lines $\mathcal{L}(\sigma_n{+}1)$, $\mathcal{L}(\sigma_n)$, $\mathcal{L}(\sigma_n^{-1}\sigma_{n+1})$ are situated as shown in Fig. 5.6.

Proof. Considering the transformation $\sigma_1^{-1}\sigma_2$, one can easily show that

$$A_1 < \sigma_1^{-1}\sigma_2 A_2 < \sigma_1^{-1}\sigma_2 A_1 < \sigma_1^{-1}\sigma_2 Y < X < Y < A_2 \quad .$$

This means that the attractive fixed point of $\sigma_1^{-1}\sigma_2$ is inside the interval $[A_1, X]$ and the repelling fixed point is inside $[Y, A_2]$. In quite the same way other transformations $\sigma_n^{-1}\sigma_{n+1}$ are considered.

Lemma 5.9. The invariant lines $\mathcal{L}(\sigma_{n+1}), \mathcal{L}(\sigma_n), \mathcal{L}(\sigma_n^{-1}\sigma_{n+1})$ are situated as in Fig. 5.19, if and only if, the inequality

$$\{B_n, A_n, B_{n+1}, A_{n+1}\} > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}}\right)^2 \tag{5.A.1}$$

holds.

Proof. First of all let us make a linear transformation

$$z \to gz = \{z,\ A_n,\ B_{n+1},\ A_{n+1}\} \quad .$$

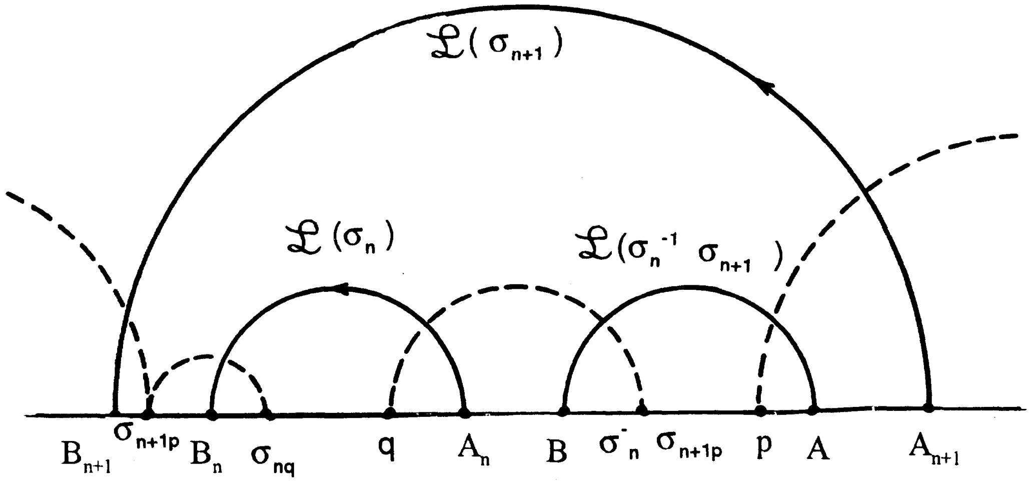

Fig. 5.19. The invariant lines $\mathcal{L}(\sigma_{n+1})$, $\mathcal{L}(\sigma_n)$, $\mathcal{L}(\sigma_n^{-1}\sigma_{n+1})$

Hence, $A_{n+1} \to \infty$, $B_{n+1} \to 0$, $A_n \to 1$, $B_n \to \lambda = \{B_n, A_n, B_{n+1}, A_{n+1}\}$. As elements of PSL(2, ℝ), the transformations $\tilde{\sigma} = g\sigma g^{-1}$ are

$$\tilde{\sigma}_{n+1} = \begin{pmatrix} \sqrt{\mu_{n+1}} & 0 \\ 0 & 1/\sqrt{\mu_{n+1}} \end{pmatrix} ,$$

$$\tilde{\sigma}_n = \frac{1}{1-\lambda}\begin{pmatrix} \sqrt{\mu_n} - \dfrac{\lambda}{\sqrt{\mu_n}} & \lambda\left(\dfrac{1}{\sqrt{\mu_n}} - \sqrt{\mu_n}\right) \\ -\dfrac{1}{\sqrt{\mu_n}} + \sqrt{\mu_n} & \dfrac{1}{\sqrt{\mu_n}} - \lambda\sqrt{\mu_n} \end{pmatrix} ,$$

$$\begin{aligned}\tilde{\sigma} &= \tilde{\sigma}_n^{-1}\tilde{\sigma}_{n+1} \\ &= \frac{1}{1-\lambda}\begin{pmatrix} \sqrt{\mu_{n+1}}\left(\dfrac{1}{\sqrt{\mu_n}} - \lambda\sqrt{\mu_n}\right) & \dfrac{-\lambda}{\sqrt{\mu_{n+1}}}\left(\dfrac{1}{\sqrt{\mu_n}} - \sqrt{\mu_n}\right) \\ \sqrt{\mu_{n+1}}\left(\dfrac{1}{\sqrt{\mu_n}} - \sqrt{\mu_n}\right) & \dfrac{1}{\sqrt{\mu_{n+1}}}\left(\sqrt{\mu_n} - \dfrac{\lambda}{\sqrt{\mu_n}}\right) \end{pmatrix} , \\ &0 < \sqrt{\mu_n},\ \sqrt{\mu_{n+1}},\quad 0 < \lambda < 1 \ .\end{aligned}$$

The fix points of $\tilde{\sigma}$ (denote them by $\tilde{A}, \tilde{B}$) are solutions of the quadratic equation

$$\begin{aligned} f(z) =& z^2\mu_{n+1}(1-\mu_n) \\ &+ z(\mu_n - \lambda - \mu_{n+1} + \lambda\mu_n\mu_{n+1}) + \lambda(1-\mu_n) = 0 \ . \end{aligned} \tag{5.A.2}$$

Let the invariant lines be located as illustrated in Fig. 5.20. Then both roots of (5.A.2) are larger than 1. Hence, $\partial f/\partial z|_{z=1} < 0$, which in turn yields

$$\lambda > \frac{\mu_n + \mu_{n+1} - 2\mu_n\mu_{n+1}}{1 - \mu_n\mu_{n+1}} \ . \tag{5.A.3}$$

The transformation $\tilde{\sigma}$ is hyperbolic in two cases: $\mathrm{tr}\tilde{\sigma} > 2$ and $\mathrm{tr}\tilde{\sigma} < -2$ or equivalently

$$\lambda < \left(\frac{\sqrt{\mu_n} - \sqrt{\mu_{n+1}}}{1 - \sqrt{\mu_n\mu_{n+1}}}\right)^2 \tag{5.A.4}$$

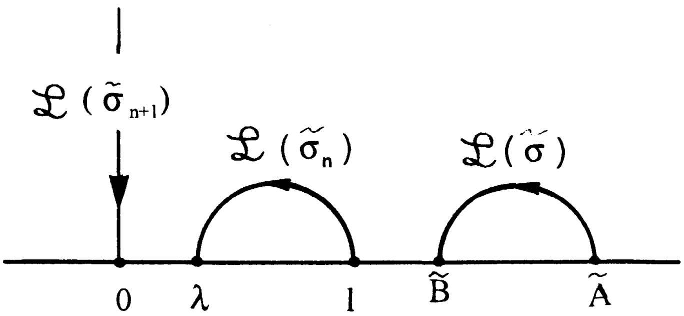

Fig. 5.20. The invariant lines of $\tilde{\sigma}_n, \tilde{\sigma}_{n+1}, \tilde{\sigma}_n^{-1}\tilde{\sigma}_{n+1}$

$$\lambda > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}}\right)^2 \tag{5.A.5}$$

However, (5.A.4) contradicts (5.A.3) which completes the proof of (5.A.5).

Conversely, if (5.A.5) is valid then $\tilde{\sigma}$ is the hyperbolic element with two fixed points which are the solutions of (5.A.5). Since (5.A.5) yields (5.A.3), both fixed points are bigger than 1. It remains to prove that $\tilde{A} > \tilde{B}$. Otherwise $\tilde{\sigma}^{-1}$ and $\tilde{\sigma}_n^{-1}$ map the interval $[1, \tilde{A}]$ into itself $\tilde{\sigma}^{-1}[1, \tilde{A}] \subset [1, \tilde{A}]$, $\tilde{\sigma}_n^{-1}[1, \tilde{A}] \subset [1, \tilde{A}]$. Hence, the product $\tilde{\sigma}_{n+1}^{-1} = \tilde{\sigma}^{-1}\tilde{\sigma}_n^{-1}$ possesses the same property. We see that there is a fixed point of $\tilde{\sigma}_{n+1}$ in the interval $[1, \tilde{A}]$. The resulting contradiction proves the true arrangement of fixed points of $\tilde{\sigma}$: $\tilde{A} > \tilde{B}$. The Lemma is proved, because it is formulated in terms of invariants of the transformation $\tilde{\sigma} = g\sigma g^{-1}$.

Lemma 5.10. Let invariant lines $\mathcal{L}(\sigma_{n+1})$, $\mathcal{L}(\sigma_n)$, $\mathcal{L}(\sigma_n^{-1}\sigma_{n+1})$, be situated as in Fig. 5.6, i.e., every triple as in Fig. 5.19. Then there exists a fundamental domain of G in the form shown in Fig. 5.6.

Proof. Let p be some point between A and B, then $B < \sigma_n^{-1}\sigma_{n+1}p < A$. It is evident that $\sigma_{n+1}p > B_{n+1}$ (Fig. 5.19). On the other hand, applying σ_n to $\sigma_n^{-1}\sigma_{n+1}p$, we have $\sigma_{n+1}p < B_n$. Hence the fundamental domain of the group with generators σ_n, σ_{n+1} is bounded by the half-circles shown in Fig. 5.19 by dotted lines.

In summary, the following theorem is proved.

Theorem 5.11. The following statements about the uniformization of Fig. 5.5, 6 are equivalent:

1) fundamental domain of G is as in Fig. 5.6,
2) invariant lines are arranged as in Fig. 5.6,
3) parameters of generators satisfy the inequalities

$$B_N < B_{N-1} < \ldots < B_1 < A_1 < \ldots < A_N \quad ,$$
$$0 < \sqrt{\mu_i} < 1, \quad i = 1, \ldots, N \quad ,$$
$$\{B_n, A_n, B_{n+1}, A_{n+1}\} > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}} \right)^2 \quad , n = 1, \ldots, N-1 \quad .$$

6. Theta Function Formulas for Classical Tops

The methods developed in the previous chapters are now applied to integrable systems of classical mechanics. We restrict ourselves to the tops investigated in the 19th century and do not discuss the numerous examples of integrable systems found recently with the help of the inverse scattering method. A good survey of these modern results can be found in [6.28, 29, 52]. This chapter is an improved version of the preprint [6.1] by *Bobenko*. The main results were reported in [6.2-5].

6.1 The Lax Equation and Analytic Properties of the Baker-Akhiezer Function

The analytic properties of the Baker-Akhiezer functions are deducible from the corresponding Lax representations

$$\frac{d}{dt}L(\lambda) + [L(\lambda), A(\lambda)] = 0 \quad . \tag{6.1.1}$$

Let

$$L = \Psi \hat{\mu} \Psi^{-1}$$

be the diagonal form of the matrix L, where $\hat{\mu}$ is the eigenvalue matrix. It satisfies the equation

$$\hat{\mu}_t = [\hat{\mu}, \Psi^{-1} A \Psi + \Psi^{-1} \Psi_t] \tag{6.1.2}$$

and, as a corollary, it does not depend on t since the RHS of (6.1.2) has a zero diagonal part.

The Baker-Akhiezer function is an eigenfunction of the operator L

$$L\psi = \mu\psi \quad . \tag{6.1.3}$$

Here L is an $(N \times N)$ matrix and ψ is an N-dimensional vector. The eigenvalues of L do not depend on t. Therefore, the characteristic polynomial

$$\det(L(\lambda) - \mu) = 0 \tag{6.1.4}$$

is also independent of t. It defines a spectral curve X. The suitably normalized ψ-function is an analytic function on the Riemann surface of the spectral curve (6.1.4).

Let us consider the simplest, but simultaneously the most general example where L and A are rational functions of λ, and there are no reductions. It is evident that X is the N-sheeted covering of the λ-plane. So we have N values $\mu^1, \ldots, \mu^N$ (every value is counted according to its multiplicity), corresponding to every λ. Respectively, we have N eigenvectors $\psi^i = \psi(\lambda, \mu^i)$ of the matrix L.

Let us consider the function

$$(\det \Psi)^2 = (\det(\psi^1, \ldots, \psi^N))^2 \quad . \tag{6.1.5}$$

It is a single-valued function of λ with the divisor of the poles of degree $2K$, where K is the degree of the divisor of the poles of ψ on X. The degree of the divisor of zeros of the function (6.1.5) is equal to the sum of all branch numbers $\sum \nu_i$. By equating these degrees and taking into account the Riemann-Hurwitz formula (2.2.1) we get

$$2K = \sum \nu_i = 2g - 2 + 2N \quad ,$$

where g is the genus of X. Finally we see that ψ has the divisor of the poles of the degree

$$K = g + N - 1 \quad .$$

Differentiating (6.1.3) by t, we see that the Baker-Akhiezer function satisfies the equation

$$(L - \mu)(\psi_t - A\psi) = 0 \quad ,$$

which in turn gives

$$\psi_t = A\psi + a(\lambda, t)\psi \quad .$$

Here $a(\lambda, t)$ is some scalar function, which can be eliminated by suitable renormalization of ψ. We see that ψ has essential singularities at the poles of A.

Finally we see that the Baker-Akhiezer function is a solution of the system

$$L\psi = \mu\psi \quad , \psi_t = A\psi \quad .$$

It is an analytic function on X, having the divisor of poles independent of t and the essential singularities at the poles of A.

Let us mention that the various possible reductions of $L - A$ pairs lead to the symmetry of the spectral curves and to the specific properties of ψ-functions. Below we consider $L - A$ pairs with reductions.

6.2 Integrable Systems

The Kirchhoff equations are important in classical mechanics and hydrodynamics:

$$\dot{p} = [p, \omega], \quad \dot{M} = [M, \omega] + [p, u], \quad \omega^i = \frac{\partial H}{\partial M_i}, \quad u^i = \frac{\partial H}{\partial p_i} \quad . \tag{6.2.1}$$

Here, [,] denotes the vector product in $\mathbb{R}^3$. Equations (6.2.1) are Hamilton's equations of motion [6.6]

$$\dot{f} = \{H, f\}, \quad \dot{f} = f_t \tag{6.2.2}$$

with respect to the following Poisson brackets:

$$\begin{gathered}\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0 \quad , \\ i, j, k = 1, 2, 3, \quad \epsilon_{123} = 1 \quad .\end{gathered} \tag{6.2.3}$$

Here ϵ_{ijk} is as usual a totally antisymmetric tensor. The Poisson brackets (6.2.3) are, in fact, the Lie-Poisson bracket for the Lie algebra $e(3)$ of the motion group $E(3)$ of Euclidean space. Notice that

$$f_1 = p^2 = \sum_i p_i^2, \quad f_2 = pM = \sum_i p_i M_i \tag{6.2.4}$$

are trivial integrals of motion (Casimir functions) for the Poisson bracket (6.2.3). Thus we have a four-dimensional phase space. For the corresponding system to be completely integrable, it is sufficient to possess one additional (besides the Hamiltonian) integral of motion K.

The rotation of a rigid body about a fixed point is described by (6.2.1). In this case the orthogonal frame is attached to the body and coincides with the axes of the inertia ellipsoid. The origin is chosen to be at the fixed point. The Hamiltonian is

$$H = \frac{1}{2}(I_1 M_1^2 + I_2 M_2^2 + I_3 M_3^2) + \Gamma_1 p_1 + \Gamma_2 p_2 + \Gamma_3 p_3 \quad . \tag{6.2.5}$$

Here M is the angular momentum of the body, p is the unit gravitational field vector, the constant vector (Γ_1, Γ_2, Γ_3) indicates the center of mass and I_1^{-1}, I_2^{-1}, I_3^{-1} are the main moments of inertia of the body.

The following cases are integrable:

(1.) *The Euler case:* $\Gamma = 0$, $K = M^2 = M_1^2 + M_2^2 + M_3^2$.

(2.) *The Lagrange case:* $I_1 = I_2$, $\Gamma_1 = \Gamma_2 = 0$, $K = M_3$.

(3.) *The Kowalewski case:* $I_1 = I_2 = I_3/2$, $\Gamma_3 = 0$.

(4.) *The Goryachev-Chaplygin case:* $I_1 = I_2 = I_3/4$, $\Gamma_3 = 0$ and the constant f_2 vanishes, i.e., $pM = 0$.

In the last case we have the integrable Hamiltonian system on only one integral level. The formulas for the additional integrals K for the Kowalewski and the Goryachev-Chaplygin cases are presented in the Sects. 6.3, 4.

We remark that in all integrable cases presented above K is a polynomial of M_i and p_j. *Ziglin* has shown [6.7] that there are no additional cases of integrability if K is a meromorphic function of M_i and p_j.

For quadratic Hamiltonians

$$H = \frac{1}{2}\sum(a_{ij}M_iM_j + 2b_{ij}M_ip_j + c_{ij}p_ip_j)$$

equations (6.2.1) coincide with the Kirchhoff equations of motion of a rigid body in an ideal incompressible liquid being at rest at infinity. The orthogonal frame is attached to the body and is chosen such that the inertia tensor is diagonal. In this case M and p are respectively the complete angular momentum and the complete impulse of the body-liquid system. The non-trivial integrable *Clebsch* [6.8] and [first] *Steklov* [6.9] cases are known and are the only cases with the additional quadratic integral K, the corresponding expressions for which are presented in Sects. 6.7, 8.

We also consider the Euler equations on the Lie algebra $SO(4)$, which have interesting applications in hydrodynamics. The Lie algebra $SO(4)$ is isomorphic to the direct sum of two copies of $SO(3)$. In the following we shall always use the isomorphism $SO(4) = SO(3) + SO(3)$. The Euler equations with the Hamiltonian

$$H = \frac{1}{2}\sum_{ij}(a_{ij}S_iS_j + 2b_{ij}S_jT_i + c_{ij}T_iT_j), \quad a_{ij} = a_{ji}, \quad c_{ij} = c_{ji} \tag{6.2.6}$$

and the Lie-Poisson bracket

$$\begin{aligned} &\{S_i, S_j\} = \epsilon_{ijk}S_k, \quad \{S_i, T_j\} = 0 \quad , \\ &\{T_i, T_j\} = \epsilon_{ijk}T_k, \quad i,j,k = 1,2,3 \end{aligned} \tag{6.2.7}$$

are given by

$$\dot{\boldsymbol{S}} = [\boldsymbol{S}, A\boldsymbol{S} + B^T\boldsymbol{T}], \quad \dot{\boldsymbol{T}} = [\boldsymbol{T}, B\boldsymbol{S} + C\boldsymbol{T}] \quad .$$

Here A, B, C denote the matrices of the coefficients of the Hamiltonian (6.2.6).

The two trivial integrals

$$g_1 = S^2 = \sum_i S_i^2, \quad g_2 = T^2 = \sum T_i^2 \tag{6.2.8}$$

show that as in the $e(3)$ case, we have four-dimensional orbits. The additional integral of motion K exists in the integrable *Manakov* [6.10] and [second] *Steklov* [6.11] cases.[1] These are the only cases with quadratic K. Recently another case of integrability with quadratic H and quartic K was found [6.16, 17].

[1] The Euler equations on $SO(4)$ of the special type (special A, B, C) describe the motion of a rigid body with an ellipsoidal cavity filled with liquid. A family of integrable cases of such systems depending on 3 parameters was found by *Steklov* [6.12]. When the problem of finding integrable Euler equations on $SO(4)$ was investigated later, these same integrable cases were found [6.13, 14, 15]. The number of arbitrary parameters increased to 6. We will refer to this case as the second Steklov case.

Another classical problem, integrable in terms of two-dimensional theta functions, is the Neumann system. The equations of motion are

$$[\boldsymbol{S}_{tt} + I\boldsymbol{S}, \boldsymbol{S}] = 0, \quad I = \mathrm{diag}(I_1, I_2, I_3), \quad S^2 = 1 \quad . \tag{6.2.9}$$

It describes the motion of a particle restricted to the unit sphere under the quadratic potential

$$U(S) = \frac{1}{2}\sum_i I_i S_i^2 \quad .$$

Below we construct theta functional formulas for all systems mentioned above except the Lagrange top (the Euler and the Lagrange tops are easily solved in elliptic functions and are investigated in detail).

6.3 Kowalewski Top

In her celebrated paper [6.18] published in 1889, *Kowalewski* found a new and highly nontrivial integrable case of motion of a heavy rigid body around a fixed point, completing the list of integrable tops. Two previous known integrable cases are Euler's top in which the stationary point coincides with the center of mass, and Lagrange's top which is axially symmetric. The third case discovered by Kowalewski is rather bizarre: the moments of inertia have a fixed ratio 2 : 2 : 1, and the center of mass lies in the equatorial plane of the top.

In this section we follow the paper [6.2], where calculations omitted here are presented.

6.3.1 Kowalewski's Paper

The starting point of Kowalewski's work was her observation that Euler's and Lagrange's tops are solved in terms of Jacobi functions. Therefore, her initial idea was to try to solve the equations of motion of a general heavy rigid body about a fixed point in terms of Abelian functions. However, Weierstrass pointed out that a general solution of this form does not exist in the general case and may be possible only for some particular geometries of the top [6.19]. Thus Kowalewski started her search for tops of this type.

She considered the equations of motion for the general top (6.2.5)

$$\begin{aligned} \dot{\boldsymbol{M}} &= [\boldsymbol{M}, I\boldsymbol{M}] + [\boldsymbol{p}, \boldsymbol{\Gamma}], \quad \dot{\boldsymbol{p}} = [\boldsymbol{p}, I\boldsymbol{M}] \quad , \\ I\boldsymbol{M} &= (I_1 M_1, I_2 M_2, I_3 M_3), \quad \boldsymbol{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3) \end{aligned} \tag{6.3.1}$$

and substituted the series

$$M_i = \frac{m_i}{t - t_0} + \ldots, \quad p_i = \frac{n_i}{(t - t_0)^2} + \ldots \tag{6.3.2}$$

in the neighborhood of the singularity point ($t_0 \in \mathbb{C}$) into the equations (6.3.1). The question was: for what kind of tops is the series (6.3.2) a general solution of equations (6.3.1) (i.e., which series have a sufficient number of independent constants).

In her paper Kowalewski obtained three remarkable results. First she proved (with some gaps which were filled in later, see comments in [6.20]) that the only tops with the property that the general solution is given by meromorphic functions of the complex time variable are Euler's and Lagrange's tops and a new top with the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - p_1 \quad . \tag{6.3.3}$$

She also found the additional integral

$$K = (M_1^2 - M_2^2 + 2p_1)^2 + 4(M_1 M_2 + p_2)^2 \tag{6.3.4}$$

for the top (6.3.3) which now carries her name. Finally, using a non-trivial change of variables, Kowalewski reduced the equations of motion to the Jacobi inversion problem for the hyperelliptic curve (Kowalewski curve) of genus 2 [2]

$$\mu^2 = ((\lambda - H)^2 - K/4)(\lambda((\lambda - H)^2 + (1 - K/4)) - (pM)^2) \quad . \tag{6.3.5}$$

Kowalewski's paper became very popular, especially the first part which attracted attention and was widely discussed and generalized (see the comments in [6.20]). At the same time the extremely technical part, devoted to explicit integration of the top, remained, for a long time, only a sequence of well-guessed substitutions and calculations. The relationship among the three problems considered by Kowalewski was also unclear. This was recently clarified in [6.21, 22, 23].

6.3.2 The Lax Pair for the Kowalewski Top

Let us consider the Kowalewski gyrostat (KG). This is a system with the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2 + 2\gamma M_3) - p_1 \quad . \tag{6.3.6}$$

Theorem 6.1. The Kowalewski gyrostat, as defined by the Hamiltonian (6.3.6), is completely integrable and admits a Lax representation $dL/dt+[L,A]=0$ given by

$$L=\frac{\mathrm{i}}{\lambda}\begin{bmatrix} 0 & p_- & 0 & -p_3 \\ -p_+ & 0 & p_3 & 0 \\ 0 & -p_3 & 0 & -p_+ \\ p_3 & 0 & p_- & 0 \end{bmatrix} + \mathrm{i}\begin{bmatrix} -\gamma & 0 & M_- & 0 \\ 0 & \gamma & 0 & -M_+ \\ M_+ & 0 & -2M_3-\gamma & -2\lambda \\ 0 & -M_- & 2\lambda & 2M_3+\gamma \end{bmatrix},$$

[2] Here and below, with no loss of generality, we shall assume in the sequel that $p^2 = 1$.

$$A = \frac{\mathrm{i}}{2}\begin{bmatrix} 2M_3+\gamma & 0 & M_- & 0 \\ 0 & -2M_3-\gamma & 0 & -M_+ \\ M_+ & 0 & -2M_3-\gamma & -2\lambda \\ 0 & -M_- & 2\lambda & 2M_3+\gamma \end{bmatrix} \quad , \tag{6.3.7}$$

$$p_\pm = p_1 \pm \mathrm{i}p_2, \quad M_\pm = M_1 \pm \mathrm{i}M_2 \quad .$$

These matrices obey the symmetry relations

$$L(-\lambda) = \eta L(\lambda)\eta, \quad \eta = \mathrm{diag}(1,-1,1,-1) = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad , \tag{6.3.8}$$

$$L(\lambda)^T = -\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} L(\lambda) \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad . \tag{6.3.9}$$

We recall the definition of the Pauli matrices σ_i:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

The invariants of the matrix $L(\lambda)$ are integrals of motion in involution H, $f_1 = p^2 = 1$, $f_2^2 = (pM)^2$ and

$$K = (M_1^2 - M_2^2 + 2p_1)^2 + 4(M_1M_2 + p_2)^2 \\ - 4\gamma((M_3+\gamma)(M_1^2 + M_2^2) + 2M_1p_3) \quad .$$

This integral is an extension of the integral (6.3.4) found by Kowalewski and was discussed in [6.24, 25]. The Lax pair (6.3.7) as well as broader generalizations of the Kowalewski top and the corresponding Lax pairs were obtained by *Reyman* and *Semenov-Tian-Shansky* [6.53].

6.3.3 The Spectral Curve

Let us now turn to the original Kowalewski top where $\gamma = 0$. We shall consider complex equations of motion. The Lax equations are linearizable on the Jacobian of the spectral curve $\hat{X}$ defined by the equation

$$\det(L(\lambda) - \mu) = 0 \quad . \tag{6.3.10}$$

The characteristic equation (6.3.10) for the Lax matrix of the KT takes the form

$$\mu^4 - 2d_1(\lambda^2)\mu^2 + d_2(\lambda^2) = 0 \quad ,$$

$$d_1(z) = z^{-1} - 2H + 2z \quad ,$$

$$d_2(z) = z^{-2} + 4[(Mp)^2 - H)]z^{-1} + K \quad .$$

The symmetries (6.3.8, 9) give rise to two commuting involutions τ_1, τ_2 on $\hat{X}$

$$\tau_1(\lambda,\mu) = (-\lambda,\mu), \quad \tau_2(\lambda,\mu) = (\lambda,-\mu) \quad , \tag{6.3.11}$$

which in turn induce the coverings $\hat{X} \to X$ and $X \to E$ given by the relations $z = \lambda^2$ and $y = \mu^2$. So the curves X and E are defined by the equations

$$\mu^4 - 2d_1(z)\mu^2 + d_2(z) = 0 \tag{6.3.12}$$

and

$$y^2 - 2d_1(z)y + d_2(z) = 0 \quad .$$

The covering $\hat{X} \to X$ is unramified and thus is determined by a cycle $\mathcal{L}$ (mod 2) on X : a loop γ on X lifts to a closed loop on $\hat{X}$ if and only if $\langle\gamma, \mathcal{L}\rangle = 0$ (mod 2), where $\langle\gamma, \mathcal{L}\rangle$ is the intersection number. To put it another way, the function $\lambda = \sqrt{z}$ acquires a factor $(-1)^{\langle\gamma,\mathcal{L}\rangle}$ upon a circuit of γ.

For later use, we must have a closer look at the cover $X \to E$. The elliptic curve E is a two-sheeted covering of the z-plane. There are two points $\infty_\pm$ at "infinity" where z has simple poles, and one point 0 where z has a double zero. The function y has a simple pole at ∞_+, a simple zero at ∞_-, a double pole at 0, and hence two other simple zeros at some points P_1, P_2. The branch points of the function $\mu = \sqrt{y}$ on E are therefore ∞_+, ∞_-, P_1, P_2. Thus X is obtained by glueing together two copies of E along suitable cuts $[\infty_+, \infty_-]$ and $[P_1, P_2]$.

We choose the cut $[\infty_+, \infty_-]$ such that the function $\lambda = \sqrt{z}$ becomes unramified on $E\backslash[\infty_+, \infty_-]$ (notice that $\infty_\pm$ are the only branch points of $\sqrt{z}$).

The curve $\hat{X}$ may be thought of as the Riemann surface of the function $\lambda = \sqrt{z}$ on X. One can always choose a canonical basis in $\mathrm{H}_1(X, \mathbb{Z})$ so that

$$\pi a_1 = -a_3, \quad \pi b_1 = -b_3, \quad \pi a_2 = -a_2, \quad \pi b_2 = -b_2 \quad ,$$

where $\pi\colon (\mu, z) \to (-\mu, z)$, and also $a_2 = \mathcal{L}$ (mod 2) (Fig. 6.1).

We may now identify $\hat{X}$ with two copies of X glued together along $\mathcal{L}\colon \hat{X} = X^{(1)} \cup_{\mathcal{L}} X^{(2)}$. It is natural to choose the contour $\mathcal{L}$ such that $\pi\, \mathcal{L} = -\mathcal{L}$ (the minus sign denotes reversed orientation). The involution τ_1 acts on $\hat{X}$ by permuting the sheets $X^{(i)}$.

The final thing we need is the behaviour of μ near the points of $\hat{X}$ where $\lambda = \infty$ or $\lambda = 0$. These are the points $\infty_\pm^{(i)}$ and $0_\pm^{(i)}$ on the sheets $C^{(i)}$. If we arrange the points $\infty_\pm^{(i)}$ onto a 4-tuple $(\infty_-^{(1)}, \infty_-^{(2)}, \infty_+^{(1)}, \infty_+^{(2)})$, the 4 branches of μ near $\lambda = \infty$ can be combined into a row-vector

$$\mu(\lambda) \sim (0, 0, 2\lambda, -2\lambda) + o(1) \quad . \tag{6.3.13}$$

In a similar way, with respect to the ordering $(0_-^{(1)}, 0_+^{(2)}, 0_+^{(1)}, 0_-^{(2)})$ (this particular ordering is convenient for the calculations in Sect. 6.3.7), we have

$$\mu(\lambda) \sim -\epsilon\lambda^{-1}(1, -1, 1, -1) \tag{6.3.14}$$

near $\lambda = 0$, with $\epsilon = \pm 1$ depending on the location of $\mathcal{L}$. It is always possible to choose $\mathcal{L}$ such that $\epsilon = 1$.

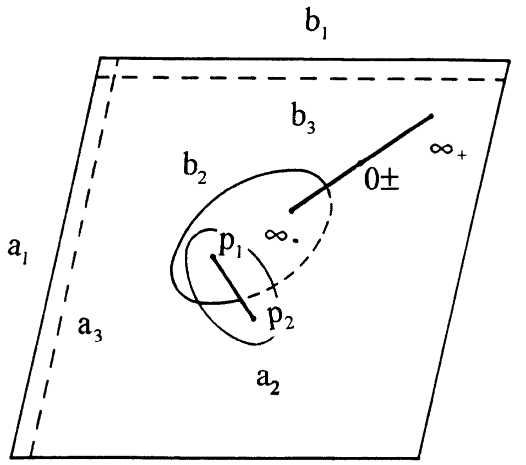

Fig. 6.1. A plane model of the elliptic curve E. The cycles a_i, b_i are depicted relative to the representation of X as a two-sheeted cover of E: continuous lines show parts of the cycles on the upper sheet while dotted lines show their parts on the lower sheet

6.3.4 Analyticity Properties of the Baker-Akhiezer Function

The Baker-Akhiezer function, defined as a solution of the linear system

$$L(\lambda(P))\psi(P) = \mu(P)\psi(P), \quad \frac{\partial}{\partial t}\psi(P) = A(\lambda(P))\psi(P) \quad , \tag{6.3.15}$$

has certain analyticity properties as a vector-valued function on $\hat{X}$. We can also require ψ to be symmetric with respect to the first of the involutions (6.3.11)

$$\psi(\tau_1 P) = \eta\psi(P) \quad . \tag{6.3.16}$$

This enables us to regard ψ as a double-valued function on the curve $X = \hat{X}/\tau_1$, which makes all calculations much simpler.

Let us now state the analyticity properties of ψ.

1. ψ is meromorphic on $\hat{X}$ except at $\lambda = \infty$ and $\psi\ \exp(-t\mu/2)$ is meromorphic on $\hat{X}$ except at $\lambda = 0$.
2. The divisor of poles of ψ, denoted by $\hat{\mathcal{D}}$, has degree 8 and is time independent.
3. ψ satisfies the symmetry condition (6.3.16).

The divisor $\hat{\mathcal{D}}$ is not, however, completely determined by these conditions. If f is a meromorphic function on X and $(f) \leq \hat{\mathcal{D}}$ on $\hat{X}$, then ψ can be replaced by $f\psi$. Using this freedom, we can fix two points of $\hat{\mathcal{D}}$ to be $\infty_+^{(1)}$ and $\infty_+^{(2)}$. Then $\hat{\mathcal{D}}$ is the pull-back to $\hat{X}$ of a divisor $\mathcal{D} \cup \infty_+$ on X, and $\deg \mathcal{D} = 3$.

The behaviour of ψ near $\lambda = \infty$ can be reformulated in a more convenient matrix form. Let $\Psi(\lambda)$ be the 4×4 matrix whose jth column is the value of ψ on the jth sheet of $\hat{X} \to \{\lambda\}$ near $\lambda = \infty$ (the ordering of sheets corresponds to the ordering of points over $\lambda = \infty$, described in Sect. 6.3.3). We can then write $\Psi(\lambda)$ as

$$\Psi(\lambda, t) = (\Phi + S\lambda^{-1} + \ldots)\mathrm{diag}(1, 1, \lambda e^{\lambda t}, -\mathrm{i}\lambda e^{-\lambda t}) \tag{6.3.17}$$

(the factor $-\mathrm{i}$ in the last entry of (3.17) is taken for notational convenience). Denoting

$$L(\lambda) = L_{-1}\lambda^{-1} + L_0 + L_1\lambda, \quad A(\lambda) = A_0 + A_1\lambda \quad ,$$

we have from (6.3.15,17,13)

$$L_1 = 2\Phi \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix} \Phi^{-1}, \quad L_0 = [S\Phi^{-1}, L_1] \quad , \tag{6.3.18}$$

$$A_1 = \Phi \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix} \Phi^{-1}, \quad A_0 = \Phi_t\Phi^{-1} - [S\Phi^{-1}, A_1] \quad . \tag{6.3.19}$$

The symmetry condition (6.3.16) takes the form (notice that τ_1 permutes the sheets):

$$\Psi(-\lambda) = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \Psi(\lambda) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad ,$$

which gives the symmetry relations for Φ and S:

$$\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \Phi \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \Phi, \quad \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} S \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = -S \quad .$$

Combined with (6.3.18) this implies that $\Phi(t)$ has the form

$$\Phi(t) = c\ \mathrm{diag}(q_1(t), q_2(t), 1, 1) \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & -\mathrm{i} \\ & & -\mathrm{i} & 1 \end{pmatrix} \quad .$$

We can set $c = 1$. The relation (6.3.19) yields differential equations for $q_i(t)$:

$$\frac{dq_1}{dt} = \mathrm{i}M_3q_1, \quad \frac{dq_2}{dt} = -\mathrm{i}M_3q_2 \quad ,$$

so that

$$q_1(t) = \alpha\ \exp(\mathrm{i}\int^t M_3dt), \quad q_2(t) = \beta\ \exp(-\mathrm{i}\int^t M_3dt) \quad . \tag{6.3.20}$$

In a similar way, arranging the 4 eigenvectors $\psi(0_\pm^{(i)})$ into a 4×4 matrix $\Psi(0)$ according to the ordering of the points $0_\pm^{(i)}$ described in Sect. 6.3.3, we have

$$\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \Psi(0) \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \Psi(0) \tag{6.3.21}$$

and using (6.3.14,15), we find

$$L_{-1} = -\epsilon\, \Psi(0) \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \Psi^{-1}(0) \quad . \tag{6.3.22}$$

The strategy of our further computations will be as follows. Using the symmetry property (6.3.16) of ψ , we reformulate the problem entirely in terms of the curve X: the functions ψ_1, ψ_3, $\lambda\psi_2$, $\lambda\psi_4$ are single-valued functions on X. The properties of ψ stated above allow us to write explicit formulas for ψ_3, ψ_4, which in turn serve to compute the coefficients S_{ij} of S for $i, j = 3, 4$. From (6.3.18) we have the relation

$$M_3 = -\mathrm{i}S_{33} - S_{43} \quad , \tag{6.3.23}$$

which by (6.3.20) yields expressions for q_1, q_2. After that we can write down the remaining components ψ_1, ψ_2. To determine the constant factors that occur in these formulas, we must use the second symmetry relation (6.3.9). Combining it with (6.3.18), we come down to the Prym condition for the divisor $\mathcal{D}$ and determine the Baker-Akhiezer function completely. Finally to derive the evolution of $\boldsymbol{p}(t)$ we use (6.3.22).

6.3.5 Explicit Formulas for the Baker-Akhiezer Function

We now begin to implement the program outlined above. First of all we have to introduce certain Abelian differentials.

Let $d\Omega$ be a normalized Abelian differential of the second kind on X with a pole at ∞_+ such that

$$\Omega(P) = \int^P d\Omega = \lambda + O(\lambda^{-1}) \quad \text{as} \quad P \to \infty_+$$

(recall that there is a well-defined branch of λ on $X \backslash \mathcal{L}$; its sign is specified by requiring that $\mu \sim 2\lambda$ at ∞_+). Let us denote by

$$\boldsymbol{V} = (V_1, V_2, V_3), \quad V_j = \int_{b_j} d\Omega$$

the b-period vector of $d\Omega$.

Let $d\Omega_3$ be a normalized Abelian differential of the third kind which has simple poles at ∞_+ and ∞_- with residues 1 and -1, respectively. We choose a path ℓ from ∞_+ to ∞_- and normalize $d\Omega_3$ by the condition $\int_{a_j} d\Omega_3 = 0$, where the cycles a_j are supposed not to intersect ℓ. It is easily checked that $d\Omega_3$ is the pullback to X of a differential on E given by

$$d\Omega_3 = \frac{(1 + qz^{-1})dz}{y - d_1(z)}$$

with some constant q, so that $\pi^* d\Omega_3 = d\Omega_3$ $(\pi(z, \mu) = (z, -\mu))$. We put

$$\Omega_3(P) = \int^P d\Omega_3$$

and fix the constant of integration by the condition

$$e^{\Omega_3(P)} = \lambda + O(1) \quad \text{as} \quad P \to \infty_+ \quad .$$

We need the values of the multi-valued functions $\Omega(P)$, $e^{\Omega_3(P)}$, $\int^P \omega$ at the points $\infty_\pm$ and $0_\pm$. For that purpose we specify the choice of the path ℓ joining ∞_+ and ∞_- coinciding with the cut $[\infty_+, \infty_-]$ (see Fig. 6.1), i.e.,
(a) its projection to E passes through 0 and is symmetric with respect to $0 \in E$,
(b) the cycle $\ell - \pi\ell$ is homologous to a_2,
(c) ℓ does not intersect the ramification contour $\mathcal{L}$.

Since the periods of $d\Omega$, $d\Omega_3$, ω_1 and ω_3 over a_2 are all zero, the multi-valued analytic functions $\Omega(P)$, $e^{\Omega_3(P)}$, $\int^P \omega_{1,3}$ have single-valued branches in a neighborhood of the contour $\ell \cup \pi\ell$. Also we set $\int^{\infty_+} \omega_j = 0$. Standard calculations [6.2] show that

$$\int_{\infty_+}^{\infty_-} \omega = \int_\ell \omega = (r, \pi\mathrm{i}, -r), \qquad \int_{b_2} d\Omega_3 = \pi\mathrm{i} \quad ,$$

$$\Omega(\infty_-) = 0, \quad e^{\Omega_3(P)} = -\frac{a^2}{\lambda} + O(1) \quad \text{as} \quad P \to \infty_- \quad ,$$

where r and a are defined as follows:

$$r = \int_{b_1} d\Omega_3 = -\int_{b_3} d\Omega_3, \quad a = e^{\Omega_3(0_+)} = -e^{\Omega_3(0_-)} \quad .$$

In particular we see that $e^{\Omega_3(P)}$ changes sign when analytically continued along b_2.

Let $\boldsymbol{D}$ be a vector in $\mathbb{C}^3$ such that the divisor of $\theta(\int_{\infty_+}^P \omega + \boldsymbol{D})$ on X coincides with the divisor $\mathcal{D}$ introduced above.

Theorem 6.2. The Bakher-Akhiezer function $\psi(P,t)$ is given by

$$\psi_1 = q_1 \frac{\theta(\int^P \omega + \boldsymbol{V}t + \boldsymbol{D})\theta[\epsilon](\boldsymbol{D}+\boldsymbol{R})}{\theta(\int^P \omega + \boldsymbol{D})\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})} e^{\Omega(P)t},$$

$$\psi_2 = q_2 \frac{\theta[\epsilon](\int^P \omega + \boldsymbol{V}t + \boldsymbol{D})\theta[\epsilon](\boldsymbol{D}+\boldsymbol{R})}{\theta(\int^P \omega + \boldsymbol{D})\theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})} e^{\Omega(P)t},$$

$$\psi_3 = \frac{\theta[\epsilon](\int^P \omega + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})\theta(\boldsymbol{D})}{\theta(\int^P \omega + \boldsymbol{D})\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})} e^{\Omega(P)t + \Omega_3(P)},$$

$$\psi_4 = -\mathrm{i} \frac{\theta(\int^P \omega + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})\theta(\boldsymbol{D})}{\theta(\int^P \omega + \boldsymbol{D})\theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})} e^{\Omega(P)t + \Omega_3(P)}, \tag{6.3.24}$$

where

$$R=(r,0,r), \quad \epsilon=\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix} \quad .$$

The proof is a straightforward corollary of the analytical properties of $\theta(\int^P \omega)$, $\Omega(P)$, $\Omega_3(P)$ displayed above and of the Baker-Akhiezer function described in Sect. 6.3.4. Recall that now we do not take into account the second symmetry relation (6.3.9).

The expressions (6.3.24) for ψ_3 and ψ_4 enable us to calculate $M_3(t)$.

Lemma 6.3.

$$M_3=-\mathrm{i}\frac{\partial}{\partial t}\log\frac{\theta[\epsilon](Vt+D+R)}{\theta(Vt+D+R)} \quad .$$

Proof. From (6.3.23) we get

$$M_3=\lim_{P\to\infty_+}\lambda(-\mathrm{i}\psi_3(P)-\psi_4(P))=$$

$$-\mathrm{i}\left.\left(\frac{\frac{\partial}{\partial k}\theta[\epsilon](\int^P \omega+Vt+D+R)}{\theta[\epsilon](\int^P\omega+Vt+D+R)}-\frac{\frac{\partial}{\partial k}\theta(\int^P \omega+Vt+D+R)}{\theta(\int^P\omega+Vt+D+R)}\right)\right|_{P=\infty_+}$$

where $k=\lambda^{-1}$ is a local parameter at $P=\infty_+$. The derivative $\partial/\partial k$ may be replaced by $\partial/\partial t$ due to the usual reciprocity law for $d\Omega$ and ω.

The integrand in (6.3.20) turns out to be an exact derivative and thus we get

$$q_1(t)=\alpha\frac{\theta[\epsilon](Vt+D+R)}{\theta(Vt+D+R)}, \quad q_2(t)=\beta\frac{\theta(Vt+D+R)}{\theta[\epsilon](Vt+D+R)} \quad , \tag{6.3.25}$$

where the constants of integration α, β are still to be determined.

6.3.6 The Prym Condition

It is now time to take into account the second symmetry condition (6.3.9), which is done best in the resulting formulas for the solutions. Substituting (6.3.24,25) into (6.3.18), we have the coefficients of L_0:

$$(L_0)_{13}=-2S_{13}=-2\alpha\frac{\theta(Vt+D)\theta[\epsilon](D+R)}{\theta(Vt+D+R)\theta(D)} \quad ,$$

$$(L_0)_{42}=-2\mathrm{i}S_{31}q_2^{-1}=\frac{2\mathrm{i}}{\beta}a^2\frac{\theta(Vt+D+2R)\theta(D)}{\theta(Vt+D+R)\theta[\epsilon](D+R)} \quad ,$$

$$(L_0)_{24}=-2\mathrm{i}S_{23}=-2\mathrm{i}\beta\frac{\theta[\epsilon](Vt+D)\theta[\epsilon](D+R)}{\theta[\epsilon](Vt+D+R)\theta(D)} \quad ,$$

$$(L_0)_{31} = 2\mathrm{i}S_{41}q_1^{-1} = -\frac{2a^2}{\alpha}\frac{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{D}+2\boldsymbol{R})\theta(\boldsymbol{D})}{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{D}+\boldsymbol{R})\theta[\epsilon](\boldsymbol{D}+\boldsymbol{R})} \quad .$$

The relation (6.3.9) implies $(L_0)_{13} = -(L_0)_{42}$, $(L_0)_{24} = -(L_0)_{31}$, which gives

$$\alpha\beta = \mathrm{i}a^2\frac{\theta(\boldsymbol{D})^2\theta(\boldsymbol{V}t+\boldsymbol{D}+2\boldsymbol{R})}{\theta(\boldsymbol{D}+\boldsymbol{R})^2\theta(\boldsymbol{V}t+\boldsymbol{D})} = \mathrm{i}a^2\frac{\theta(\boldsymbol{D})^2\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{D}+2\boldsymbol{R})}{\theta(\boldsymbol{D}+\boldsymbol{R})^2\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{D})} \quad .$$

For this equality to hold identically, the theta functions depending on t must cancel out:

$$\theta(\boldsymbol{V}t+\boldsymbol{D}+2\boldsymbol{R}) = c\,\theta(\boldsymbol{V}t+\boldsymbol{D})$$

with some constant c. Since $\pi^*\boldsymbol{V} = -\boldsymbol{V}$, $\pi^*\boldsymbol{R} = \boldsymbol{R}$ and, moreover,

$$\theta(-\boldsymbol{u}) = \theta(\boldsymbol{u}) = \theta(\pi^*\boldsymbol{u}) \quad ,$$

this implies

$$\boldsymbol{D} = \boldsymbol{P} - \boldsymbol{R}, \quad \pi^*\boldsymbol{P} = -\boldsymbol{P} \quad .$$

These relations give

$$\alpha = \Delta a\frac{\theta(\boldsymbol{P}-\boldsymbol{R})}{\theta[\epsilon](\boldsymbol{P})}, \quad \beta = \frac{\mathrm{i}a}{\Delta}\frac{\theta(\boldsymbol{P}-\boldsymbol{R})}{\theta[\epsilon](\boldsymbol{P})} \quad ,$$

$$M_+ = \frac{2\mathrm{i}a}{\Delta}\frac{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P}-\boldsymbol{R})}{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})}, \quad M_- = 2\mathrm{i}\Delta a\frac{\theta(\boldsymbol{V}t+\boldsymbol{P}-\boldsymbol{R})}{\theta(\boldsymbol{V}t+\boldsymbol{P})} \quad ,$$

where the constant Δ is still to be determined.

6.3.7 The Poisson Vector

We still have to calculate the Poisson vector $\boldsymbol{p}(t)$. We will use (6.3.22) and bear in mind that $p_1^2+p_2^2+p_3^2 = 1$. In view of (6.3.21), we write $\Psi(0)$ as

$$\Psi(0) = \mathcal{L}\begin{pmatrix} \mathcal{A} & \sigma_3\mathcal{A}\sigma_1 \\ \mathcal{B} & \sigma_3\mathcal{B}\sigma_1 \end{pmatrix}\mathcal{R} \quad ,$$

where

$$\mathcal{L} = \mathrm{diag}\left(\frac{\Delta}{\theta(\boldsymbol{V}t+\boldsymbol{P})}, \frac{\mathrm{i}}{\Delta\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})}, -\frac{1}{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})}, \frac{\mathrm{i}}{\theta(\boldsymbol{V}t+\boldsymbol{P})}\right),$$

$$\begin{aligned}\mathcal{R} =& a\,\theta(\boldsymbol{P}-\boldsymbol{R}) \\ &\times \mathrm{diag}\left(\frac{e^{\Omega(0_-)t}}{\theta(\int^{0_-}\omega+\boldsymbol{P}-\boldsymbol{R})}, \frac{e^{\Omega(0_+)t}}{\theta(\int^{0_+}\omega+\boldsymbol{P}-\boldsymbol{R})},\right. \\ &\qquad\left.\frac{e^{\Omega(0_+)t}}{\theta(\int^{0_+}\omega+\boldsymbol{P}-\boldsymbol{R})}, \frac{e^{\Omega(0_-)t}}{\theta(\int^{0_-}\omega+\boldsymbol{P}-\boldsymbol{R})}\right),\end{aligned}$$

$$\mathcal{A} = \begin{pmatrix} \theta(\int^{0-} \omega + \boldsymbol{V}t + \boldsymbol{P} - \boldsymbol{R}) & \theta(\int^{0+} \omega + \boldsymbol{V}t + \boldsymbol{P} - \boldsymbol{R}) \\ \theta[\epsilon](\int^{0-} \omega + \boldsymbol{V}t + \boldsymbol{P} - \boldsymbol{R}) & -\theta[\epsilon](\int^{0+} \omega + \boldsymbol{V}t + \boldsymbol{P} - \boldsymbol{R}) \end{pmatrix} ,$$

$$\mathcal{B} = \begin{pmatrix} \theta[\epsilon](\int^{0-} \omega + \boldsymbol{V}t + \boldsymbol{P}) & -\theta[\epsilon](\int^{0+} \omega + \boldsymbol{V}t + \boldsymbol{P}) \\ \theta(\int^{0-} \omega + \boldsymbol{V}t + \boldsymbol{P}) & \theta(\int^{0+} \omega + \boldsymbol{V}t + \boldsymbol{P}) \end{pmatrix} . \tag{6.3.26}$$

To verify these formulas we recall that $\psi_j(0_\pm^{(1)}) = (-1)^j \psi_j(0_\pm^{(2)})$ and $e^{\Omega_3(0_\pm)} = \pm a$. Also, it can easily be shown that

$$\int^{0\pm} \omega = \frac{1}{2}\boldsymbol{R} \pm \boldsymbol{C}, \quad \pi^* \boldsymbol{C} = -\boldsymbol{C} \quad . \tag{6.3.27}$$

This and the relation $\theta(\boldsymbol{u}) = \theta(-\pi^* \boldsymbol{u})$ imply

$$\mathcal{B} = \sigma_1 \mathcal{A} \quad . \tag{6.3.28}$$

Therefore (6.3.22) can be written as

$$L_{-1} = -\mathcal{L} W \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} W^{-1} \mathcal{L}^{-1}$$

with

$$W = \begin{pmatrix} \mathcal{A} & \sigma_3 \mathcal{A} \sigma_1 \\ \sigma_1 \mathcal{A} & \sigma_3 \sigma_1 \mathcal{A} \sigma_1 \end{pmatrix}, \quad W^{-1} = \frac{1}{2} \begin{pmatrix} \mathcal{A}^{-1} & \mathcal{A}^{-1}\sigma_1 \\ \sigma_1 \mathcal{A}^{-1} \sigma_3 & -\sigma_1 \mathcal{A}^{-1} \sigma_1 \sigma_3 \end{pmatrix}$$

(we have assumed that $\epsilon = 1$ in (6.3.22), [see (6.3.14)]). After simple calculations we find

$$L_{-1} = -\mathcal{L} \begin{pmatrix} S_1\sigma_1 + S_2\sigma_2 & S_3\sigma_3\sigma_1 \\ S_3\sigma_1\sigma_3 & S_1\sigma_1 - S_2\sigma_2 \end{pmatrix} \mathcal{L}^{-1} \quad ,$$

where the S_j are defined by

$$\sum S_j \sigma_j = \mathcal{A}\ \sigma_3 \mathcal{A}^{-1} \quad .$$

By equating the matrix coefficients $(L_{-1})_{32} = (L_{-1})_{14}$, we finally get $\Delta^2 = 1$ and

$$M_+ = \frac{\theta(\boldsymbol{V}t + \boldsymbol{P})}{\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{P})}(S_1 + \mathrm{i}S_2), \quad M_- = \frac{\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{P})}{\theta(\boldsymbol{V}t + \boldsymbol{P})}(S_1 - \mathrm{i}S_2) \quad ,$$

$$M_3 = -\Delta\ S_3 \quad .$$

Now we can sum up our calculations.

Theorem 6.4. The general solution of the equations of motion for the Kowalewski top is given by

$$M_+ = 2\mathrm{i}a \frac{\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{P} - \boldsymbol{R})}{\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{P})}, \quad M_- = 2\mathrm{i}a \frac{\theta(\boldsymbol{V}t + \boldsymbol{P} - \boldsymbol{R})}{\theta(\boldsymbol{V}t + \boldsymbol{P})} \quad ,$$

$$M_3 = -\mathrm{i}\frac{\partial}{\partial t}\log\frac{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})}{\theta(\boldsymbol{V}t+\boldsymbol{P})} \quad , \tag{6.3.29}$$

$$p_+ = 2\frac{\theta(\boldsymbol{V}t+\boldsymbol{P})}{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})}\frac{AB}{AD+BC}, \quad p_- = 2\frac{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})}{\theta(\boldsymbol{V}t+\boldsymbol{P})}\frac{CD}{AD+BC} \quad ,$$

$$p_3 = \frac{BC-AD}{AD+BC} \quad ,$$

where

$$\epsilon = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad a = e^{\Omega_3(0_+)} \quad ,$$

$$A = \theta\left(\int_{\infty_+}^{0_-}\omega + \boldsymbol{V}t + \boldsymbol{P}\right), \quad B = \theta\left(\int_{\infty_+}^{0_+}\omega + \boldsymbol{V}t + \boldsymbol{P}\right) \quad ,$$

$$C = \theta[\epsilon]\left(\int_{\infty_+}^{0_-}\omega + \boldsymbol{V}t + \boldsymbol{P}\right), \quad D = \theta[\epsilon]\left(\int_{\infty_+}^{0_+}\omega + \boldsymbol{V}t + \boldsymbol{P}\right) \quad .$$

6.3.8 The Geometry of Liouville Tori

The remaining indeterminacy in (6.3.29) for the dynamical variables (the change of sign $\Delta \to -\Delta$ or the permutation $0_+ \longleftrightarrow 0_-$) reflects the freedom in reconstructing the Lax matrix (6.3.7) from the algebraic data. It is easily verified that this freedom amounts to conjugation $L \to ULU^{-1}$ by a matrix U of the form

$$U = \mathrm{diag}(1, 1, -1, -1) \quad .$$

This is equivalent to a renormalization of the Baker-Akhiezer function $\psi \to U\psi$ and induces a symmetry of the Kowalewski top:

$$\boldsymbol{M} \to B\boldsymbol{M}, \quad \boldsymbol{p} \to -B\boldsymbol{p}, \quad B = \mathrm{diag}(-1, -1, 1) \quad . \tag{6.3.30}$$

Clearly, (6.3.30) leaves the Hamiltonian invariant but changes the sign of $f_2 = (\boldsymbol{pM})$. Recall that only the square $(\boldsymbol{pM})^2$ is a spectral invariant.

We may summarize the situation as follows.

Theorem 6.5. If $(\boldsymbol{pM}) \neq 0$, the common level surface of the spectral invariants H, K, $(\boldsymbol{pM})^2$ consists of two components (Liouville tori) each of which is an affine part of an Abelian variety isomorphic to $\mathrm{Prym}_\pi X$. These components are permuted by the transformation (6.3.30).

If $(\boldsymbol{pM}) = 0$, the curve E degenerates into a rational curve which is a two-sheeted cover of the z-plane. The curve X is given by the equation

$$(\mu^2 - d_1(z))^2 = 4(z^2 - 2Hz + 1 + H^2 - \frac{K}{4})$$

and has genus 2. In the variables $u = \sqrt{2}\mu z x$, $x = 1/2(\mu^2 - 1/z)$ it takes the usual hyperelliptic form

$$u^2 = x(x^2 + 2Hx + \frac{K}{4})(x^2 + 2Hx + \frac{K}{4} - 1) \quad .$$

Notice that it is different from the Kowalewski curve (6.3.5) with $(\boldsymbol{pM}) = 0$. The fact that there are various hyperelliptic curves associated with the Kowalewski top which are different from the classical Kowalewski curve was pointed out in [6.22]. The motion of the top linearizes on the Jacobians of these curves which are isogeneous to one another.

For $(\boldsymbol{pM}) = 0$ the mapping of the Liouville torus to J(X) becomes an unramified two-sheeted covering. The corresponding theta functional formulas are presented in [6.2].

6.3.9 Reduction of Two-Dimensional Theta Functions

Since the Kowalewski flow on J(X) is parallel to the Prym variety of the cover $X \to E$, it is desirable to express the dynamics entirely in terms of the theta functions related to this Prym variety. The Prymian has polarization (2.1) and its period matrix is [6.51]

$$\Pi = \begin{pmatrix} 2\int_{b_1}(\omega_1 + \omega_3) & \int_{b_2}(\omega_1 + \omega_3) \\ 2\int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{pmatrix} \quad .$$

Let $1/2B_0$ be the period of E

$$B_0 = \int_{b_1}(\omega_1 - \omega_3) \quad .$$

We write the Prym vectors $\boldsymbol{V}$ and $\boldsymbol{P}$ entering (6.3.29) and C defined by (6.3.27) as

$$\boldsymbol{V} = (\frac{v_1}{2},\ v_2,\ \frac{v_1}{2}), \quad \boldsymbol{P} = (\frac{p_1}{2},\ p_2,\ \frac{p_1}{2}), \quad C = (\frac{c_1}{2},\ c_2,\ \frac{c_1}{2})$$

where we have used the notation

$$\boldsymbol{w} = (v_1 t + p_1,\ v_2 t + p_2), \quad \boldsymbol{c} = (c_1,\ c_2) \quad .$$

Then we have the following expressions for the theta functions occurring in (6.3.29)

$$\theta(\boldsymbol{V}t+\boldsymbol{P})=\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(0;B_0)+\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(0;B_0)\quad,$$

$$\theta(\boldsymbol{V}t+\boldsymbol{P}-\boldsymbol{R})=\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(2r;B_0)$$
$$+\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(2r;B_0)\quad,$$

$$\theta(\boldsymbol{V}t+\boldsymbol{P}+\int_{\infty_+}^{0_\pm}\omega)=\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(\boldsymbol{w}\pm\boldsymbol{c};\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(r;B_0)$$
$$+\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(\boldsymbol{w}\pm\boldsymbol{c};\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(r;B_0)\quad,$$

$$\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})=\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(0;B_0)$$
$$+\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(0;B_0)\quad,$$

$$\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P}-\boldsymbol{R})=\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(2r;B_0)$$
$$+\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(\boldsymbol{w};\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(2r;B_0)\quad,$$

$$\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P}+\int_{\infty_+}^{0_\pm}\omega)=\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(\boldsymbol{w}\pm\boldsymbol{c};\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(r;B_0)$$
$$+\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(\boldsymbol{w}\pm\boldsymbol{c};\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(r;B_0)\quad.$$

Remark. Adding to W a period of the form

$$\begin{pmatrix}2&0\\0&1\end{pmatrix};\Pi\Big)\begin{pmatrix}M\\N\end{pmatrix},\qquad M,N\in\mathbb{Z}^2$$

does not change the solutions (6.3.29). This shows that the mapping of the Liouville torus to $\mathrm{Prym}_\pi X$ is one-to-one, as was already mentioned in Theorem 6.5.

6.4 The Goryachev-Chaplygin Top

Now the system under consideration is a special case of the motion of a heavy rigid body with a fixed point, discovered by *Goryachev* and *Chaplygin* [6.26] in 1900. It represents a symmetric top with the principal moments of inertia satisfying $I_1^{-1} : I_2^{-1} : I_3^{-1} = 1 : 1 : 1/4$ and the centre of mass located in the equatorial plane. The Hamiltonian of the Goryachev-Chaplygin top (GCT) is given by

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1 \quad , \tag{6.4.1}$$

where M is the angular momentum and p is the field strength vector in the moving frame (Sect. 6.2).

The system (6.4.1) admits an extra integral of motion provided that the Casimir function f_2 (6.2.1) for the Poisson brackets (6.2.3) vanishes

$$M_1 p_1 + M_2 p_2 + M_3 p_3 = 0 \quad . \tag{6.4.2}$$

A more general system described by

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2 + 4\gamma M_3) - 2p_1$$

is called the Goryachev-Chaplygin gyrostat (GCG). It is also integrable if $(\boldsymbol{M}\boldsymbol{p}) = 0$ [6.27]. We mention also two papers where GCT is studied in a different way. In [6.30] the R-matrix technique is used to solve both the classical and the quantum problems. In [6.31] the geometry of the complexified Liouville tori for GCT is thoroughly studied using the general technique developed in [6.21]. In particular, a close connection is established in [6.31] between GCT and the periodic Toda lattice with three particles.

Here we follow the paper [6.3]. Let us note that (compared with [6.3]) similar but slightly more complicated formulas for these solutions were obtained in [6.45].

6.4.1 The Lax Pair for the Goryachev-Chaplygin Top

There is an interesting connection between GCT and the Kowalewski top (KT) on the Lax representation level.

An important observation of [6.3] is that removing the first column and the first row of the Lax matrix (6.3.7) we get the Lax matrix for the GCG

$$L = \mathrm{i}\begin{pmatrix} \frac{2}{3}\gamma & \frac{p_3}{\lambda} & -M_+ \\ -\frac{p_3}{\lambda} & -2M_3 - \frac{4}{3}\gamma & -2\lambda - \frac{p_+}{\lambda} \\ -M_- & \frac{p_-}{\lambda} + 2\lambda & 2M_3 + \frac{2}{3}\gamma \end{pmatrix} \quad . \tag{6.4.3}$$

Put

$$A = \mathrm{i}\begin{pmatrix} -3M_3 - \frac{2}{3}\gamma & 0 & -M_+ \\ 0 & -2M_3 - \frac{2}{3}\gamma & -2\lambda \\ -M_- & 2\lambda & 2M_3 + \frac{4}{3}\gamma \end{pmatrix} \quad . \tag{6.4.4}$$

Then the Lax equation is equivalent to the Hamiltonian equation with the Hamiltonian (6.4.1), provided that the constraint (6.4.2) is satisfied.

For future use we introduce the notation

$$L = L_{-1}\lambda^{-1} + L_0 + L_1\lambda$$

for the coefficients of the Lax matrix (6.4.3).

In the following we shall consider only the GCT case with $\gamma = 0$. Formulas for the general case may be easily obtained in quite the same way.

6.4.2 The Spectral Curve

Let $\hat{X}$ denote the spectral curve given by the equation $\det(L(\lambda) - \mu I) = 0$. The symmetry relation

$$L(-\lambda) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} L(\lambda) \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{6.4.5}$$

gives rise to an involution on $\hat{X}$

$$\tau : (\lambda, \mu) \to (-\lambda, \mu) \quad .$$

It is natural to consider the quotient curve $X = \hat{X}/\tau$ given by

$$\mu^3 + \mu(2H - 4z - \frac{1}{z}) - 2iG = 0 \quad z = \lambda^2 \quad ,$$

where $H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1$ is the Hamiltonian and $G = M_3(M_1^2 + M_2^2) + 2M_1 p_3$ is the Goryachev-Chaplygin integral. It is equivalent to the Chaplygin curve [6.26]

$$y^2 = (\mu^3 + 2H\mu - 2iG)^2 - 16\mu^2, \quad y = 8z\mu - \mu^3 - 2H\mu + 2iG \quad .$$

Note that we always assume $(\boldsymbol{M}\boldsymbol{p}) = 0$, $\boldsymbol{p}^2 = 1$.

The spectral curve $\hat{X}$ is a three-sheeted covering of the λ-plane Λ and is also a double cover of $X = \hat{X}/\tau$

$$\begin{array}{ccccc} (\lambda, \mu) & \hat{X} & \xrightarrow{2:1} & X = \hat{X}/\tau & (\mu, z) \\ & 3:1 \downarrow & & \downarrow 3:1 & \\ (\lambda) & \Lambda & \xrightarrow{2:1} & \mathbb{C} & (z) \end{array}$$

We denote the points of $\hat{X}$ with $\lambda = 0$ and $\lambda = \infty$ by

$\lambda = 0$	0^{I}	$\mu = 0$	$\lambda = \infty$	∞^{I}	$\mu = 0$
	0^{II}	$\mu \sim -\lambda^{-1}$		∞^{II}	$\mu \sim -2\lambda$
	0^{III}	$\mu \sim \lambda^{-1}$		∞^{III}	$\mu \sim 2\lambda.$

X is a three-sheeted covering of the z-plane. We denote the points with $z = 0$ and $z = \infty$ by 0_1, 0_2, and ∞_1, ∞_2, in such a way that 0_2, ∞_2 are the branch points of the covering $X \to \mathbb{C} \ni z$. This covering is unramified at 0_1, ∞_1 and $\mu(0_1) = \mu(\infty_1) = 0$.

The function $\lambda = \sqrt{z}$ is double-valued on X and changes sign when analytically continued along a closed path which intersects a certain contour $\mathcal{L}$. Here $\mathcal{L}$ is a contour connecting the points 0_1 and ∞_1 and determined by the covering $\hat{X} \to X$. Glueing two copies of X along $\mathcal{L}$, we obtain $\hat{X}$. The condition

$$\mu \sim -2\lambda \quad \text{at} \quad \infty_2, \quad \mu \sim -\lambda^{-1} \quad \text{at} \quad 0_2$$

uniquely fixes $\mathcal{L}$ and the branch of λ.

6.4.3 Analytic Properties of the Baker-Akhiezer Function

Our main goal is to construct explicitly the Baker-Akhiezer function $\psi(P) = (\psi_1, \psi_2, \psi_3)^T$ which is analytic on $\hat{X}$ and satisfies

$$L\psi = \mu\psi, \quad \psi_t = A\psi \quad . \tag{6.4.6}$$

We may assume that ψ satisfies the symmetry relation (6.4.5)

$$\psi(\tau P) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \psi(P), \quad P \in \hat{X} \quad . \tag{6.4.7}$$

Hence, the component ψ_2 may be regarded as a single-valued function on X, while ψ_1 and ψ_3 are double-valued on X and change sign when analytically continued along a closed path intersecting $\mathcal{L}$. We may assume that the ψ_i are defined on $X \backslash \mathcal{L}$ and satisfy the symmetry relation

$$\psi_i^+(P) = (-1)^i \psi_i^-(P) \tag{6.4.8}$$

for P belonging to the cut $\mathcal{L}$. In other words, ψ_1, ψ_3 acquire a factor $(-1)^{\langle\gamma, \mathcal{L}\rangle}$ upon a circuit of γ. Here $\langle\gamma, \mathcal{L}\rangle$ is the intersection number.

Let us define a matrix-valued function

$$\Psi(\lambda) = (\psi(P^{\mathrm{I}}), \quad \psi(P^{\mathrm{II}}), \quad \psi(P^{\mathrm{III}})) \quad ,$$

where P^{I}, P^{II}, P^{III} are the inverse images of λ with respect to the mapping $\hat{X} \to \Lambda$. We mark them so that

$$P^{\mathrm{I,II,III}} \to \infty^{\mathrm{I,II,III}} \quad \text{at } \lambda \to \infty \quad ,$$
$$P^{\mathrm{I,II,III}} \to 0^{\mathrm{I,II,III}} \quad \text{at } \lambda \to 0$$

holds. The function $\Psi(\lambda)$ is defined on the domain $U = U_0 \cup U_\infty$ which is a union of two simply-connected domains with the points $\lambda = 0$ and $\lambda = \infty$, respectively. These domains also do not contain the branch points of the covering $\hat{X} \to \Lambda$ and are invariant with respect to the involution $\lambda \to -\lambda$.

The reduction (6.4.7) can be rewritten in terms of Ψ:

$$\Psi(-\lambda) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \Psi(\lambda) \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \quad . \tag{6.4.9}$$

To see this, note $\tau 0^{\mathrm{II}} = 0^{\mathrm{III}}$, $\tau\infty^{\mathrm{II}} = \infty^{\mathrm{III}}$, $\tau 0^{\mathrm{I}} = 0^{\mathrm{I}}$, $\tau\infty^{\mathrm{I}} = \infty^{\mathrm{I}}$.

According to (6.4.6,8) it is natural to determine the asymptotics of $\Psi(\lambda)$ at $\lambda \to \infty$ and $\lambda \to 0$ as

$$\Psi \underset{\lambda\to\infty}{=} (\Phi + S\lambda^{-1} + \ldots) \begin{pmatrix} 1 & & \\ & e^{-2\lambda t} & \\ & & e^{2\lambda t} \end{pmatrix} \begin{pmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad ,$$

$$\Psi \underset{\lambda\to 0}{=} T \begin{pmatrix} \lambda^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} .$$

Then the reduction (6.4.8) gives

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} S \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = S \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} ,$$

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} T \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = T \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} .$$

The coefficients of $L(\lambda) = L_{-1}\lambda^{-1} + L_0 + L_1\lambda$ are related to the matrices Φ, S, T in these expansions by

$$\begin{aligned} L_{-1} &= -T \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} T^{-1} , \\ L_0 &= -2 \left[S\Phi^{-1}, \begin{pmatrix} 0 & & \\ & 1 & \mathrm{i} \\ & -\mathrm{i} & 0 \end{pmatrix} \right] , \\ L_1 &= 2\Phi \begin{pmatrix} 0 & & \\ & -1 & \\ & & 1 \end{pmatrix} \Phi^{-1} , \end{aligned} \tag{6.4.10}$$

which gives

$$\Phi = \begin{pmatrix} q & & \\ & 1 & 1 \\ & -\mathrm{i} & \mathrm{i} \end{pmatrix} , \quad \frac{q_t}{q} = -3\mathrm{i}M_3 .$$

In the usual way all these analytical properties can be reformulated for the vector function ψ on X.

With a suitable normalization, the Baker-Akhiezer function has the following properties which characterize it completely:

(1.) ψ is analytic on $X\backslash\mathcal{L}$, satisfies the symmetry relations (6.4.8) on $\mathcal{L}$ and is meromorphic on $X\backslash\infty_2$.

(2.) In the neighborhood of the points 0_1, ∞_1, ∞_2, ψ has the following asymptotic behaviour:

$$\psi \sim \begin{pmatrix} O(\lambda^{-1}) \\ O(1) \\ O(\lambda^{-1}) \end{pmatrix} \quad \text{for} \quad P \to 0_1,$$

$$\psi \sim \begin{pmatrix} q\lambda + O(\lambda^{-1}) \\ O(1) \\ O(\lambda^{-1}) \end{pmatrix} \quad \text{for} \quad P \to \infty_1,$$

$$\psi \sim \left(\begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix} + O(\lambda^{-1}) \right) \exp(-2\lambda t) \quad \text{for} \quad P \to \infty_2.$$

(3.) The divisor of poles of ψ $\mathcal{D} = P_1 + P_2$ has degree 2 and does not depend on t.

(4.) The normalization constant q above satisfies the differential equation $q_t/q = -3\mathrm{i}M_3$; hence,
$q = \alpha \ \exp(-3\mathrm{i} \int^t M_3 dt)$.

The Baker-Akhiezer function ψ with these properties satisfies (6.4.6), where L and A are almost the Lax matrices for GCT, the only exception being that the condition

$$(L_{-1})_{12} = -(L_{-1})_{21} \tag{6.4.11}$$

is not automatically fulfilled. (This condition will be imposed in the last stage of the computation. As we shall see, it amounts to a suitable choice of the integration constant α.)

It is useful to present the expressions (6.4.10) in more detail

$$\begin{aligned} &M_3 = -S_{32} - \mathrm{i}S_{22}, \quad M_+ = 2S_{12}, \quad M_- = 2S_{21}/q \quad , \\ &(L_{-1})_{12} = -\frac{T_{12}}{T_{22}}, \quad (L_{-1})_{21} = \frac{T_{22}T_{31}}{T_{11}T_{32} - T_{31}T_{12}} \quad , \\ &p_+ = \mathrm{i}\frac{T_{22}T_{11}}{T_{31}T_{12} - T_{11}T_{32}}, \quad p_- = \mathrm{i}\frac{T_{32}}{T_{22}} \quad . \end{aligned} \tag{6.4.12}$$

6.4.4 Construction of the Baker-Akhiezer Function

To write down explicit formulas for the function ψ_1, ψ_2 and ψ_3, we must define a number of standard objects on X. Let $\Omega_3(P)$, $\Delta(P)$ and $\Omega(P)$ be the normalized Abelian integrals of the third and the second kind, respectively, which are uniquely specified by their behaviour in the neighborhoods of the points ∞_1, ∞_2, 0_2

$$\begin{array}{cccc} P & e^{\Omega_3(P)} & e^{\Delta(P)} & \Omega(P) \\ \infty_1 & a & \lambda^2 + O(1) & f \\ \infty_2 & \lambda + b + O(\lambda^{-1}) & d\lambda^{-1} + O(\lambda^{-2}) & -2\lambda + O(\lambda^{-1}) \\ 0_2 & c\lambda + O(\lambda^{-2}) & e & O(1) \ . \end{array} \tag{6.4.13}$$

Let us denote

$$\boldsymbol{R} = \int_b d\Omega_3, \quad \boldsymbol{\Delta} = \int_b d\Delta, \quad \boldsymbol{V} = \int_b d\Omega \quad . \tag{6.4.14}$$

There are some useful relations between the different constants in (6.4.13) and (6.4.14). Comparing the singularities, we get

$$\lambda^2/\mu = e^{3\Omega_3(P)}e^{2\Delta(P)} \quad , \tag{6.4.15}$$

which implies $3\boldsymbol{R}+2\boldsymbol{\Delta} \equiv 0$ modulo the periods. Let us choose the paths $[\infty_2, 0_2]$, $[\infty_1, \infty_2]$ such that an exact equality holds; i.e.,

$$3\boldsymbol{R} + 2\boldsymbol{\Delta} = 0 \quad . \tag{6.4.16}$$

Also, using (6.4.15) we get

$$e^{\Delta(P)} = \frac{d}{\lambda}(1 - \frac{3b}{\lambda} + \ldots), \quad P \to \infty_2 \quad ,$$

where b is the constant term in the expansion of e^{Ω_3} at ∞_2 (6.4.13). Using the general properties of Abelian integrals we obtain

$$f = 3b, \quad d = ae \quad . \tag{6.4.17}$$

Choose $\boldsymbol{D} \in \mathrm{J}(X)$ such that the divisor of zeros of $\theta(\int^P \omega + \boldsymbol{D})$ on X is precisely $\mathcal{D}$, the divisor introduced in the definition of ψ above. We are now in a position to write down the explicit formulas for ψ.

Theorem 6.6. The function ψ is given by the following formulas:

$$\begin{aligned}
\psi_1 =& \frac{q}{a\lambda} \frac{\theta(\int^P \omega + \boldsymbol{V}t + \boldsymbol{D} - \frac{1}{2}\boldsymbol{R})\theta(\boldsymbol{D} + \frac{3}{2}\boldsymbol{R})}{\theta(\int^P \omega + \boldsymbol{D})\theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})} \\
&\times \exp(\Omega(P)t + \Omega_3(P) + \Delta(P) - ft) \quad , \\
\psi_2 =& \frac{\theta(\int^P \omega + \boldsymbol{V}t + \boldsymbol{D})\theta(\boldsymbol{D})}{\theta(\int^P \omega + \boldsymbol{D})\theta(\boldsymbol{V}t + \boldsymbol{D})} \exp(\Omega(P)t) \quad , \\
\psi_3 =& -\frac{\mathrm{i}}{\lambda} \frac{\theta(\int^P \omega + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})\theta(\boldsymbol{D})}{\theta(\int^P \omega + \boldsymbol{D})\theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})} \exp(\Omega(P)t + \Omega_3(P)) \quad .
\end{aligned} \tag{6.4.18}$$

The expression for the function ψ_1 can also be written in a different form,

$$\begin{aligned}
\psi_1(P) &= q(\varphi(P) - \mathrm{i}\varphi(\infty_2)\psi_3(P)) \quad , \\
\varphi(P) &= b\lambda \frac{\theta(\int^P \omega + \boldsymbol{V}t + \boldsymbol{D} - \boldsymbol{R})\theta(\boldsymbol{D} + 3/2\boldsymbol{R})}{\theta(\int^P \omega + \boldsymbol{D})\theta(\boldsymbol{V}t + \boldsymbol{D} + 1/2\boldsymbol{R})} \\
&\qquad \times \exp(\Omega(P)t - \Omega_3(P) - ft) \quad .
\end{aligned} \tag{6.4.19}$$

6.4.5 Formulas for Dynamical Variables

Substituting asymptotics of ψ_2, ψ_3 at ∞_2 into (6.4.22), we obtain

$$\mathrm{i}M_3 = -\frac{1}{2}\frac{\partial}{\partial t} \log \frac{\theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})}{\theta(\boldsymbol{V}t + \boldsymbol{D})} - b \quad ,$$

$$q = \alpha \, \exp(3bt) \left(\frac{\theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{R})}{\theta(\boldsymbol{V}t + \boldsymbol{D})} \right)^{3/2} ,$$

where we have used the form $\int^P \omega = \frac{1}{2\lambda} V + \ldots$ of the Abel map near ∞_2.

Now we must satisfy the last condition of (6.4.11). To obtain the expressions for $(L_{-1})_{12}$ and $(L_{-1})_{21}$, we use (6.4.18,19), respectively. We get

$$\begin{aligned}
(L_{-1})_{12} &= -\frac{\psi_1(0_2)}{\psi_2(0_2)} \\
&= -\frac{ce}{a} q e^{-ft} \frac{\theta(Vt + D + 1/2R)\theta(Vt + D)\theta(D + 3/2R)}{\theta^2(Vt + D + R)\theta(D)} \quad , \\
(L_{-1})_{21} &= -\frac{\psi_2(0_2)}{\varphi(0_2) q} \\
&= -\frac{c}{aq} e^{ft} \frac{\theta(Vt + D + R)\theta(Vt + D + 1/2R)\theta(D)}{\theta^2(Vt + D)\theta(D + 3/2R)} \quad .
\end{aligned}$$

This implies that

$$\alpha^2 = -\frac{1}{e} \frac{\theta^2(D)}{\theta^2(D + 3/2R)} \quad .$$

To compute $p_\pm$ we also use both (6.4.18,19) for ψ,

$$p_- = \mathrm{i} \frac{\psi_3(0_2)}{\psi_2(0_2)}, \quad p_+ = \varphi(\infty_2) \frac{\psi_2(0_2)}{\varphi(0_2)} \quad .$$

Finally, taking into account (6.4.16, 17), we obtain the following theorem

Theorem 6.7. The general solution of the GCT is given by the following formulas:

$$\begin{aligned}
M_+ &= 2\mathrm{i}\sqrt{e} \frac{\theta(Vt + D - 1/2R)}{\theta(Vt + D)} \left(\frac{\theta(Vt + D + R)}{\theta(Vt + D)} \right)^{1/2} , \\
M_- &= -2\mathrm{i}\sqrt{e} \frac{\theta(Vt + D + 3/2R)}{\theta(Vt + D + R)} \left(\frac{\theta(Vt + D)}{\theta(Vt + D + R)} \right)^{1/2} , \\
M_3 &= \frac{\mathrm{i}}{2} \frac{\partial}{\partial t} \log \frac{\theta(Vt + D + R)}{\theta(Vt + D)} + b\mathrm{i} \quad , \\
p_+ &= c \frac{\theta(Vt + D - R)\theta(Vt + D + R)}{\theta^2(Vt + D)} \quad , \\
p_- &= c \frac{\theta(Vt + D + 2R)\theta(Vt + D)}{\theta^2(Vt + D + R)} \quad , \\
p_3 &= -\sqrt{e} \frac{c}{a} \frac{\theta(Vt + D + 1/2R)}{[\theta(Vt + D)\theta(Vt + D + R)]^{1/2}} \quad .
\end{aligned} \tag{6.4.20}$$

The square roots in (6.4.20) are quite unusual. Their presence is predicted by Painlevé analysis of the equations of motion, which shows that the leading powers of singularities in t are half integers [6.31]. The sign change of the square root in (6.4.20) leads to the transformation $M_1 \to -M_1$, $M_2 \to -M_2$, $p_3 \to -p_3$ preserving the equations of motion.

The paths $[\infty_2, \infty_1]$, $[\infty_2, 0_2]$ are already fixed (6.4.16); the constants a, c and e are defined by the integrals upon these very paths.

6.5 Integration of the Lax Representations with the Spectral Parameter on an Elliptic Curve. XYZ Landau-Lifshitz Equation

All other tops considered below possess Lax representations with spectral parameter varying on an elliptic curve. In this section we describe the integration process in this case, using the papers [6.4, 32]. The Lax representations of all examples considered below are of matrix dimension 2×2. The general theory for an arbitrary matrix case is constructed in [6.33].

We use the uniformization of the spectral parameter suggested in [6.34]

$$w_1(u) = \frac{1}{\operatorname{sn}(u,k)}, \quad w_2(u) = \frac{\operatorname{dn}(u,k)}{\operatorname{sn}(u,k)}, \quad w_3(u) = \frac{\operatorname{cn}(u,k)}{\operatorname{sn}(u,k)} \quad ,$$

$$w_\alpha^2 - w_\beta^2 = J_\beta - J_\alpha, \quad (J_1, J_2, J_3) = (0, k^2, 1) \quad .$$

Here sn, cn, dn are the Jacobi elliptic functions of the module k. The variable u varies on the torus $\hat{E}$ which is a parallelogram with the lattice 4K, $4\mathrm{i}K'$ (here K is the complete elliptic integral of the module k). Let us denote by E the "quarter" of $\hat{E}$, the torus with the lattice $2K$, $2\mathrm{i}K'$.

The general form of the Lax representation with the spectral parameter on an elliptic curve in the case of 2×2 matrix dimension is as follows:

$$\begin{aligned} L(u) &= \sum_{\alpha=1}^{3} \sum_{s=1}^{N} \sum_{k=1}^{N_s} L_\alpha^{s,k}(t) f_\alpha^k(u - u_s)\sigma_\alpha \quad , \\ A(u) &= \sum_{\alpha=1}^{3} \sum_{s=1}^{N} \sum_{k=1}^{N_s} A_\alpha^{s,k}(t) f_\alpha^k(u - u_s)\sigma_\alpha \quad , \end{aligned} \tag{6.5.1}$$

$$f_\alpha^k(u) = \begin{cases} \dfrac{w_1 w_2 w_3}{w_\alpha} \left[\dfrac{w_1^2 + w_2^2 + w_3^2}{3}\right]^n (u) & \text{at } k = 2n+2 \quad , \\ w_\alpha \left[\dfrac{w_1^2 + w_2^2 + w_3^2}{3}\right]^n (u) & \text{at } k = 2n+1 \quad . \end{cases}$$

The functions $f_\alpha^k(u)$ have a pole of the kth order at the point $u = 0$ and satisfy the important reduction

$$\begin{aligned} f_\alpha^k(u + 2K)\sigma_\alpha &= \sigma_3 f_\alpha^k(u)\sigma_\alpha\sigma_3 \quad , \\ f_\alpha^k(u + 2\mathrm{i}K')\sigma_\alpha &= \sigma_1 f_\alpha^k(u)\sigma_\alpha\sigma_1 \quad . \end{aligned} \tag{6.5.2}$$

The functions $f^k_\alpha(u-u_s)$ generalize the function $1/(\lambda-\lambda_s)$ to the elliptic case.

The matrices $L(u)$, $A(u)$(6.5.1) obey the symmetry relations (6.5.2). This implies that the spectral curve X

$$\mu^2 = \det L(u) \tag{6.5.3}$$

is a two-sheeted covering of the torus E. Let us choose the canonical basis of cycles in a natural way as illustrated in Fig. 6.2. The projections of the cycles a_1, b_1 on E form the canonical basis of cycles of E corresponding to the shifts on $2iK'$ and 2K respectively.

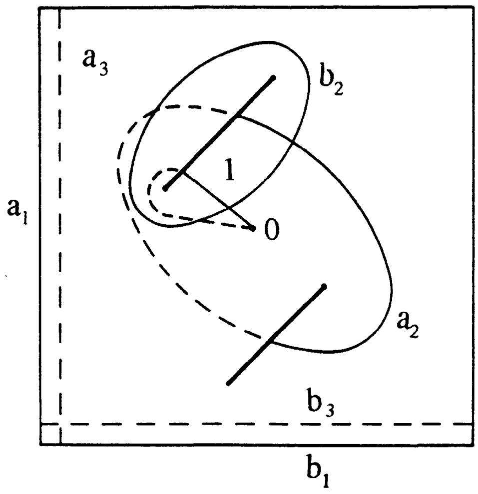

Fig. 6.2. A canonical basis of cycles

Let us also define the necessary Prym differentials [6.51, 32]. Denote by

$$\nu_1 = \frac{1}{2}(\omega_1 + \omega_{n+1}), \quad \nu_i = \omega_i, \quad i = 2, \dots, n$$

the odd differentials with respect to the involution $\pi : (\mu, u) \to (-\mu, u)$. These differentials differ from the canonical Prym differentials by the normalization. Their period matrix

$$\mathit{\Pi}_{ij} = \int_{b_j} d\nu_i, \quad i,j = 1, \dots, n$$

is in a simple way related to the canonical period matrix $\prod$ of the Prym variety $\mathrm{Prym}_\pi X$

$$\mathit{\Pi} = 2\begin{pmatrix} 1/2 & \\ & I \end{pmatrix} \prod \begin{pmatrix} 1/2 & \\ & I \end{pmatrix} . \tag{6.5.4}$$

It is this matrix which defines the corresponding theta function we use below.

Let us denote u_i^+, $u_i^- = \pi u_i^+$ the points of X with projections $X \to E$ equal to u_i.

The normalized Prym integrals of the second kind $\Omega(P)$

$$\pi^* d\Omega = -d\Omega, \quad \int_{a_j} d\Omega = 0, \quad j = 1, \dots, n$$

are determined by the asymptotics at the poles

$$\Omega(P) = \pm \sum_k^{N_s} a_{s,k}(u - u_s)^{-k} + O(1), \quad P \to u_s^\pm \quad .$$

Here $a_{s,k}$ are constants. We denote by

$$V_j = \int_{b_j} d\Omega, \quad j = 1, \dots, n$$

the b-period vector.

Theorem 6.8. The Baker-Akhiezer function corresponding to the Lax pair (6.5.1) is given by

$$\psi_1 = \frac{\theta(\int_{P_0}^P \boldsymbol{\nu} + \boldsymbol{V}t + \boldsymbol{D}; \Pi)}{\theta(\int_{P_0}^P \boldsymbol{\nu} + \boldsymbol{D}; \Pi)} \exp\left(t \int_{P_0}^P d\Omega\right) \quad ,$$

$$\psi_2 = \frac{\theta(\int_{P_0}^P \boldsymbol{\nu} + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{\Delta}; \Pi)}{\theta(\int_{P_0}^P \boldsymbol{\nu} + \boldsymbol{D}; \Pi)} \exp\left(t \int_{P_0}^P d\Omega\right) \quad , \tag{6.5.5}$$

$$\boldsymbol{\nu} = (\nu_1, \dots, \nu_n), \quad \boldsymbol{\Delta} = \pi \mathrm{i}(1, 0, \dots, 0) = \int_{a_1} \boldsymbol{\nu}, \quad \boldsymbol{D} \in \mathbb{C}^n \quad .$$

The complete proof of this theorem is given in [6.4]. Here we only remark that the Baker-Akhiezer function (6.5.5) satisfies the reduction corresponding to (6.5.2). Indeed, for the analytical continuation along the cycles a_1 and b_1 we have

$$M_{b_1} \psi(P) = \sigma_3 \psi(P), \quad M_{a_1} \psi(P) = \sigma_1 \psi(P) m(P) \quad ,$$

where $m(P)$ is the following function:

$$m(P) = \frac{\theta\left(\int_{P_0}^P \boldsymbol{\nu} + \boldsymbol{D}; \Pi\right)}{\theta\left(\int_{P_0}^P \boldsymbol{\nu} + \boldsymbol{D} + \boldsymbol{\Delta}; \Pi\right)} \quad .$$

In the neighborhood of the point u_s the matrix function of $u \in E$

$$\Psi(u) = (\psi(u^+), \psi(u^-))$$

is well-defined. Here $u^{\pm}$ are the two preimages of the point u with respect to the projection $X \to E$. They are uniquely determined by the conditions $u^{\pm} \to u_s^{\pm}$ when $u \to u_s$. Let us also consider the diagonal matrix

$$\hat{\mu} = \begin{pmatrix} \mu(u^+) & \\ & \mu(u^-) \end{pmatrix} .$$

As usual, the following equalities are valid

$$L\Psi = \Psi\hat{\mu}, \quad \Psi_t = A\Psi \quad . \tag{6.5.6}$$

Substitution of the asymptotics of Ψ and $\hat{\mu}$ near u_s

$$\Psi(u) = (\Phi_s + O(u-u_s)) \exp\left(\sum_k^{N_s} a_{s,k}(u-u_s)^{-k} t\sigma_3\right) \quad ,$$

$$\hat{\mu} = l\sigma_3(u-u_s)^{-N_s} + \dots, \quad u \to u_s \quad ,$$

in (6.5.6) determines $L_\alpha^{s,k}$ and $A_\alpha^{s,k}$. In particular, for the first coefficients we have

$$L_1^{s,N_s} = l\frac{C_sD_s - A_sB_s}{A_sD_s - B_sC_s}, \quad L_2^{s,N_s} = -il\frac{C_sD_s + A_sB_s}{A_sD_s - B_sC_s} \quad ,$$

$$L_3^{s,N_s} = l\frac{A_sD_s + B_sC_s}{A_sD_s - B_sC_s}, \quad \Phi_s = \begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix} \quad , \tag{6.5.7}$$

$$l^2 = \sum_\alpha (L_\alpha^{s,N_s})^2 \quad .$$

$$\Phi_s\sigma_3\Phi_s^{-1} = \frac{1}{a_{s,N_s}} \sum_\alpha A_\alpha^{s,N_s}\sigma_\alpha = \frac{1}{l}\sum_\alpha L_\alpha^{s,N_s}\sigma_\alpha \quad .$$

Calculating Φ_s and reducing A_s, B_s, C_s, D_s by common multipliers we see that L_α^{s,N_s} are given by the expressions (6.5.7), where

$$A_s = \theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{\epsilon}_s; \Pi), \quad B_s = \theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{\epsilon}_s + \boldsymbol{r}_s; \Pi) \quad ,$$

$$C_s = \theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{\epsilon}_s + \boldsymbol{\Delta}; \Pi), \quad D_s = \theta(\boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{\epsilon}_s + \boldsymbol{r}_s + \boldsymbol{\Delta}; \Pi) \quad ,$$

$$\boldsymbol{\epsilon}_s = \int_{P_0}^{u_s^+} \boldsymbol{\nu}, \quad \boldsymbol{r}_s = \int_{u_s^+}^{u_s^-} \boldsymbol{\nu} \quad .$$

Here the projection of the integration path in $\boldsymbol{r}_s$ onto E should be homologically equivalent to zero. The elliptic integral $u = \int du$ calculated along the path of $\boldsymbol{\epsilon}_s$ should be equal to

$$\int_{P_0}^{u_s^+} du = u_s - P_0 \pmod{4K, 4iK'}$$

modulo period lattice of the "big" torus E.

The solutions presented above may be considered as the finite-gap solutions of integrable nonlinear equations with the Lax representations with elliptic spectral parameter. To construct such an equation, we should introduce a new variable x with respect to which the Baker-Akhiezer function satisfies the similar equation $\psi_x = B\psi$. Here $B(u)$ is the matrix elliptic function of the same structure as $A(u)$, i.e., the reductions (6.5.2) are valid for $B(u)$ also. The compatibility condition

$$B_t - A_x + [B, A] = 0$$

gives the nonlinear integrable equation.

The additional condition that ψ is the eigenfunction of some matrix $L(u)$ means that ψ is an analytic function on X which is two-sheeted covering of E. All finite-gap solutions are obtained by the choice of all possible $L(u)$ or, equivalently, of all possible two-sheeted coverings of E.

The most important example of an equation of this kind is the completely anisotropic XYZ Landau-Lifshitz equation

$$\boldsymbol{S}_t = [\boldsymbol{S}, \boldsymbol{S}_{xx}] + [\boldsymbol{S}, I\boldsymbol{S}], \quad S_1^2 + S_2^2 + S_3^2 = 1 \quad .$$

Here the square brackets denote the vector product and $I\boldsymbol{S}$ – the vector with the coordinates $(I_1S_1,\ I_2S_2,\ I_3S_3)$. This equation describes nonlinear waves in ferromagnetics. The zero curvature representation for it was found in 1979 by *Sklyanin* and *Borovik* [6.34]:

$$\begin{aligned}
B(u) &= -\mathrm{i}\varrho \sum_\alpha S_\alpha w_\alpha(u)\sigma_\alpha \quad , \\
A(u) &= 2\mathrm{i}\varrho^2 \sum_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u) S_\alpha \sigma_\alpha - \sum_\alpha w_\alpha(u)[S, S_x]_\alpha \sigma_\alpha \quad , \qquad (6.5.8)\\
\varrho &= \frac{1}{2}\sqrt{I_3 - I_1}, \quad k = \sqrt{\frac{I_2 - I_1}{I_3 - I_1}}, \quad I_1 < I_2 < I_3 \quad .
\end{aligned}$$

The corresponding Ψ-function has the following singularity at $u = 0$

$$\Psi(u) = (\Phi + O(u)) \exp\left(-\mathrm{i}\varrho x \sigma_3 \frac{1}{u} + 2\mathrm{i}\varrho^2 t \sigma_3 \frac{1}{u^2}\right) \quad .$$

Finally we obtain the following:

Theorem 6.9. The finite-gap solutions of the Landau-Lifshitz equation are given by [6.4]:

$$\begin{aligned}
&S_1 = \frac{CD - AB}{AD - BC}, \quad S_2 = -\mathrm{i}\frac{CD + AB}{AD - BC}, \quad S_3 = \frac{AD + BC}{AD - BC} \quad , \\
&A = \theta(\boldsymbol{U}x + \boldsymbol{V}t + \boldsymbol{D}; \Pi), \quad B = \theta(\boldsymbol{U}x + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{r}; \Pi) \quad , \\
&C = \theta(\boldsymbol{U}x + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{\Delta}; \Pi), \quad D = \theta(\boldsymbol{U}x + \boldsymbol{V}t + \boldsymbol{D} + \boldsymbol{r} + \boldsymbol{\Delta}; \Pi) \quad .
\end{aligned}$$

Here all the parameters are determined by an arbitrary Riemann surface X which is a two-sheeted cover of E. The vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ are the b-period vectors of the normalized Prym differentials of the second kind

$$\begin{aligned} U_i &= \int_{b_i} d\Omega_1, \quad \Omega_1 \to \mp(\mathrm{i}\varrho u^{-1} + O(1)) \quad , \\ V_i &= \int_{b_i} d\Omega_2, \quad \Omega_2 \to \pm(2\mathrm{i}\varrho^2 u^{-2} + O(1)) \quad . \end{aligned} \qquad u \to 0^{\pm}$$

The integral $\boldsymbol{r}$ is equal to

$$\boldsymbol{r} = \int_{0+}^{0^-} \boldsymbol{\nu}$$

and the path of integration should be fixed in such a way that for an elliptic integral the equality $\int_{0+}^{0^-} du \equiv 0 \pmod{4K,\ 4\mathrm{i}K'}$ holds.

6.6 Curves of Lower Genera. The Euler Case and the Neumann System

For the curves of lower genera ($n = 1, 2$) the formulas of the previous section can be simplified. For this purpose we use the addition formula (2.6.8) for theta functions

$$\theta(\boldsymbol{z}_1; \Pi)\theta(\boldsymbol{z}_2; \Pi) = \sum_{\boldsymbol{\delta}} \theta \begin{bmatrix} \boldsymbol{\delta} \\ 0 \end{bmatrix} (\boldsymbol{z}_1 + \boldsymbol{z}_2; 2\Pi)\theta \begin{bmatrix} \boldsymbol{\delta} \\ 0 \end{bmatrix} (\boldsymbol{z}_1 - \boldsymbol{z}_2; 2\Pi) \quad ,$$

where the sum is taken over all n-dimensional vectors $\boldsymbol{\delta}$ with coordinates 0, 1.

For $n = 1$ we have

$$\begin{aligned} L_1^{s,N_s} &= l\frac{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z)\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (r_s)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (r_s)} \quad , \\ L_2^{s,N_s} &= -\mathrm{i}l\frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z)\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (r_s)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (r_s)} \quad , \\ L_3^{s,N_s} &= l\frac{\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z)\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (r_s)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (r_s)} \quad , \end{aligned} \tag{6.6.1}$$

$$z = 2Vt + 2D + 2\epsilon_s + r_s \quad .$$

Here and below in this section we use the notation $\theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(x) = \theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(x; 2\Pi)$.
For $n = 2$ the formulas are more complicated

$$L_1^{s,N_s} = -l\frac{\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(r_s) + \theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(r_s) + \theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(r_s)},$$

$$L_2^{s,N_s} = -\mathrm{i}l\frac{\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(r_s) + \theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z)\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(r_s) + \theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(r_s)}, \tag{6.6.2}$$

$$L_3^{s,N_s} = l\frac{\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(r_s) + \theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(r_s) + \theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(r_s)},$$

$$\boldsymbol{z} = 2\boldsymbol{V}t + 2\boldsymbol{D} + 2\epsilon_s + \boldsymbol{r} \quad .$$

Let us now consider the Euler top and the Neumann system, which have the simplest elliptic $L - A$ pairs. The $L - A$ pair for the equations of motion of the Euler top

$$\boldsymbol{M}_t = [\boldsymbol{M}, I\boldsymbol{M}], \quad I = \mathrm{diag}(I_1, I_2, I_3), \quad \varrho = \frac{1}{2}\sqrt{I_3 - I_1}, \quad (\boldsymbol{M}, \boldsymbol{M}) = 1$$

is as follows:

$$L = -\mathrm{i}\varrho\sum_\alpha M_\alpha w_\alpha(u)\sigma_\alpha \quad ,$$

$$A = 2\mathrm{i}\varrho^2\sum_\alpha\frac{w_1w_2w_3}{w_\alpha}(u)M_\alpha\sigma_\alpha, \quad k = \sqrt{\frac{I_2 - I_1}{I_3 - I_1}} \quad .$$

The spectral curve is given by the equation (6.5.3) and corresponds to the case $n = 1$. The solutions are given by the expressions (6.6.1), where $l = 1$ and $\Omega \to \pm 2\mathrm{i}\varrho^2 u^{-2}$, $u \to 0^\pm$.

The solutions of the Landau-Lifshitz equation independent of t are the solutions of the Neumann system [6.37]

$$\boldsymbol{S}_{tt} + I\boldsymbol{S} = \lambda(t)\boldsymbol{S} \quad , \boldsymbol{S}^2 = 1 \quad . \tag{6.6.3}$$

This system was solved by *C. Neumann* [6.38] using the method of separation of variables. The equation (6.6.3) was considered in connection with the finite-gap potentials [6.39, 40, 41]. In particular, the generalization of the system (6.6.3) to a higher dimension case was solved. The $L - A$ pair for the system (6.6.3) is equal to

$$L = 2\mathrm{i}\varrho^2 \sum_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u) S_\alpha \sigma_\alpha - \mathrm{i}\varrho \sum_{\alpha\beta\gamma} w_\alpha(u) S_\beta S_{t\gamma} \epsilon_{\alpha\beta\gamma} \sigma_\alpha \quad ,$$

$$A = -\mathrm{i}\varrho \sum_\alpha S_\alpha w_\alpha(u) \sigma_\alpha \quad .$$

The function $\det L(u)$ is even and has a pole of fourth order at the point $u = 0$. The spectral curve corresponds to the case $n = 2$, and possesses an involution $\tau u = -u$. We denote by $u = p_1, p_2, q_1, q_2$ the branch points of the covering $X \to E$. The Prym differentials are odd with respect to the involution

$$\tau^* \boldsymbol{\nu} = -\boldsymbol{\nu} \quad .$$

For the vector $\boldsymbol{r}$ we have

$$\boldsymbol{r} = \int_l \boldsymbol{\nu} = \int_{\tau l} \boldsymbol{\nu} = -\boldsymbol{r} + \int_{a_2} \boldsymbol{\nu} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 0 \\ \pi \mathrm{i} \end{pmatrix} \quad ,$$

since $\tau l = l - a_2$ (Fig. 6.2). The one-half of theta constants $\theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} (r; 2\Pi)$ in (6.2) becomes equal to zero. Finally we obtain the following formulas:

$$S_1 = -\frac{\theta \begin{bmatrix} 1\ 0 \\ 0\ 1 \end{bmatrix} (z; 2\Pi) \theta \begin{bmatrix} 1\ 0 \\ 0\ 1 \end{bmatrix}}{\theta \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix} (z; 2\Pi) \theta \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix}} \quad ,$$

$$S_2 = \mathrm{i} \frac{\theta \begin{bmatrix} 0\ 0 \\ 0\ 1 \end{bmatrix} (z; 2\Pi) \theta \begin{bmatrix} 0\ 0 \\ 0\ 1 \end{bmatrix}}{\theta \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix} (z; 2\Pi) \theta \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix}} \quad ,$$

$$S_3 = \frac{\theta \begin{bmatrix} 0\ 0 \\ 1\ 1 \end{bmatrix} (z; 2\Pi) \theta \begin{bmatrix} 0\ 0 \\ 1\ 1 \end{bmatrix}}{\theta \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix} (z; 2\Pi) \theta \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix}} \quad ,$$

where $\theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} (0; 2\Pi), \quad z = 2Vt + D,$

$$V_n = \int_{b_n} d\Omega, \quad \Omega \to \mp \mathrm{i}\varrho u^{-1}, \quad u \to 0^\pm \quad .$$

The Prym differentials $2\nu_1$, $2\nu_2$ in this case are the holomorphic differentials of the Riemann surface $X/\pi\tau$ of genus 2. The involution π is a hyperelliptic involution of $X/\pi\tau$ with 0, K, $\mathrm{i}K'$, $K + \mathrm{i}K'$, $p = p_1 \equiv -p_2$, $q = q_1 \equiv -q_2$ being the fixed points of π. It is easy to see that $2\nu_1, 2\nu_2$ are normalized, so the matrix 2Π is exactly the period matrix of $X/\pi\tau$.

6.7 Manakov and Clebsch Cases

It was mentioned in [6.37] that the one-phase solutions (depending on the combination $x + vt$) of the Landau-Lifshitz equation and of the asymmetric chiral O(3)-field equation are the solutions of the Clebsch and Manakov cases of integrability respectively. In this way the Lax representations for these tops came from known zero curvature representation for the Landau-Lifshitz equation (6.5.8) and asymmetric ciral O(3)-field equation [6.35].

The Manakov Case. The Lax representation is as follows:

$$
\begin{aligned}
&L_M(u) = \sum_\alpha \{S_\alpha w_\alpha(u-\kappa) + T_\alpha w_\alpha(u+\kappa)\}\,\sigma_\alpha/2\mathrm{i} \quad ,\\
&A_M(u) = c_1 A_M^{(1)} + c_2 A_M^{(2)} \quad ,\\
&A_M^{(1)} = \sum_\alpha S_\alpha w_\alpha(u-\kappa)\sigma_\alpha/2\mathrm{i} \quad ,\\
&A_M^{(2)} = -\sum_\alpha \left\{ S_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u-\kappa) + T_\alpha w_\alpha(2\kappa) w_\alpha(u-\kappa)\right\}\sigma_\alpha/2\mathrm{i} \quad .
\end{aligned}
\tag{6.7.1}
$$

The Lax equation (6.1.1) with the matrices (6.7.1) describes the Hamiltonian system with the Poisson bracket (6.2.7) and the Hamiltonian

$$
\begin{aligned}
&H = c_1 H_1 + c_2 H_2, \quad H_1 = \sum w_\alpha S_\alpha T_\alpha, \quad w_\alpha \equiv w_\alpha(2\kappa) \quad ,\\
&H_2 = \frac{1}{2}\sum \left(-w_\alpha^2 (S_\alpha^2 + T_\alpha^2) + 2\frac{w_1 w_2 w_3}{w_\alpha} S_\alpha T_\alpha \right) \quad .
\end{aligned}
\tag{6.7.2}
$$

We see that the spectral curve corresponds to $n = 2$. The general solution of the Manakov case is given by (6.6.2):

$$
\begin{aligned}
&S_\alpha = \epsilon_\alpha S \frac{\theta[\boldsymbol{i}_\alpha](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[\boldsymbol{i}_\alpha](\boldsymbol{r}+\boldsymbol{\delta}) + \theta[\boldsymbol{j}_\alpha](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[\boldsymbol{j}_\alpha](\boldsymbol{r}+\boldsymbol{\delta})}{\theta[m](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[m](\boldsymbol{r}+\boldsymbol{\delta}) + \theta[n](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[n](\boldsymbol{r}+\boldsymbol{\delta})} ,\\
&T_\alpha = \epsilon_\alpha T \frac{\theta[\boldsymbol{i}_\alpha](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[\boldsymbol{i}_\alpha](\boldsymbol{r}-\boldsymbol{\delta}) + \theta[\boldsymbol{j}_\alpha](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[\boldsymbol{j}_\alpha](\boldsymbol{r}-\boldsymbol{\delta})}{\theta[m](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[m](\boldsymbol{r}-\boldsymbol{\delta}) + \theta[n](\boldsymbol{z}_0 + 2\boldsymbol{V}t)\theta[n](\boldsymbol{r}-\boldsymbol{\delta})}\\
&\epsilon_1 = -1, \quad \epsilon_2 = -\mathrm{i}, \quad \epsilon_3 = 1 \quad ,\\
&[i_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [i_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad [i_3] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad ,\\
&[j_1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad [j_2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad [j_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ,\\
&[m] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad [n] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad ,
\end{aligned}
$$

$$\boldsymbol{r} = \int_{-\kappa+}^{\kappa^-} \boldsymbol{\nu}, \quad \boldsymbol{\delta} = \int_{\kappa+}^{-\kappa^+} \boldsymbol{\nu}, \quad \boldsymbol{z}_0 \in \mathbb{C}^2, \quad V_i = \int_{b_i} d\Omega \quad .$$

Here S and T are constants (6.2.8) and the normalized Abelian integral Ω is determined by the asymptotics

$$\int d\Omega \to \pm\frac{1}{2\mathrm{i}} S\left(-\frac{c_2}{(u-x)^2} + \frac{c_1}{u-x} + \dots\right), \quad u \to \kappa^{\pm} \quad .$$

The Clebsch Case. The Lax representation is as follows:

$$L_c(u) = \sum \left\{ p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} + M_\alpha w_\alpha \right\} \frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$A_c(u) = d_1 A_c^{(1)} + d_2 A_c^{(2)} \quad ,$$

$$A_c^{(1)} = \sum p_\alpha w_\alpha \sigma_\alpha / 2\mathrm{i} \quad , \tag{6.7.3}$$

$$A_c^{(2)} = \sum \left\{ p_\alpha w_\alpha (w^2 + J_\alpha - 2J) + M_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \right\} \frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$w^2 = (w_1^2 + w_2^2 + w_3^2)/3, \quad J = (J_1 + J_2 + J_3)/3, \quad w_\alpha = w_\alpha(u) \quad .$$

It is a Hamiltonian system with the Poisson bracket (6.2.3) and the Hamiltonian

$$\begin{aligned} H &= d_1 H_1 + d_2 H_2, \quad H_1 = \frac{1}{2} \sum (J_\alpha p_\alpha^2 + M_\alpha^2) \quad , \\ H_2 &= \frac{1}{2} \sum \left(\frac{J_1 J_2 J_3}{J_\alpha} p_2^2 - J_\alpha M_\alpha^2 \right) \quad . \end{aligned} \tag{6.7.4}$$

The solution formulas are obtained by the isomorphism [6.5, 36] of this case and the Manakov case:

$$p_\alpha = w_\alpha (S_\alpha - T_\alpha), \quad M_\alpha = \frac{w_1 w_2 w_3}{w_\alpha} (S_\alpha + T_\alpha) \quad ,$$

$$c_1 = \frac{w_1^2 w_2^2 + w_1^2 w_3^2 + w_2^2 w_3^2}{2 w_1 w_2 w_3} (d_1 - w_1^2 d_2) + 2 w_1 w_2 w_3 d_2 \quad ,$$

$$c_2 = d_1 - w_1^2 d_2, \quad w_\alpha = w_\alpha(\kappa) \quad .$$

Direct integration of the $L - A$ pair (6.7.3) by the technique of Sect. 6.5 yields the formulas (6.6.2) for p_α, where $l = p$ (6.2.4), $\boldsymbol{r} = \int_{0+}^{0^-} \boldsymbol{\nu}$ and the vector $\boldsymbol{V}$ is determined by the normalized integral with the singularity

$$\int d\Omega \to \pm\frac{1}{2\mathrm{i}} \left(\frac{d_2}{u^3} p + \frac{d_2}{u^2} \frac{pM}{p^2} - \frac{d_1}{u} p + \dots \right), \quad u \to 0^{\pm} \quad .$$

These formulas were obtained by *Kötter* [6.42] (see also [6.9]) for $d_2 = 0$. The expressions for M_α obtained in this way are more complicated and we do not present them here.

Remark 6.10. Adding to $\boldsymbol{V}t$ a period vector of the form

$$\left(\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\middle| \Pi \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}\boldsymbol{N}\\ \boldsymbol{M}\end{pmatrix}, \quad \boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^2 \tag{6.7.5}$$

does not change the solutions. Let us normalize the period matrix (6.7.5) changing a basis in $\mathbb{C}^2$. We obtain that the period lattice is given by the normalized matrix

$$\left(\begin{matrix}2 & 0\\ 0 & 1\end{matrix}\middle| 2\prod\right) , \tag{6.7.6}$$

where $\prod$ and Π are connected by (6.5.4). The matrix (6.7.6) is exactly the period matrix of the Prym variety $\mathrm{Prym}_\pi(X)$. It shows that the mapping of the Liouville torus to $\mathrm{Prym}_\pi(X)$ is one-to-one. This fact was established in [6.34, 32].

Remark 6.11. All integrable cases considered in the present and next sections depend on 6 arbitrary parameters. We obtain additional parameters adding the Casimir functions f_1, f_2, g_1, g_2 to Hamiltonians (6.7.2,4). Furthermore, the bracket (6.2.3) is invariant with respect to the transformation $p_\alpha \to ap_\alpha$ (for all α together), which changes a Hamiltonian. We also remark that for any integral K of the Clebsch top (as well as of the first Steklov case of integrability [see below])

$$\sum_\alpha \partial K/\partial J_\alpha$$

is an integral of motion. Therefore, the transformation $J_\alpha \to J_\alpha + \Delta$ (for all α together) preserves integrability. Combined with the transformation $p_\alpha \to ap_\alpha$ mentioned above, it guarantees integrability of the Clebsch and first Steklov cases with arbitrary J_α.

6.8 The Steklov Cases

The integrable Steklov case of motion of rigid body in liquid was solved by *Kötter* [6.44]. In his paper he used implicitly the Lax representation with an elliptic spectral parameter [6.28]. Various modifications of the Lax pairs for the Steklov cases were suggested in [6.5, 45, 46].

The Second Steklov Case. It possesses the following Lax representation:

$$L_{\mathrm{II}}(u) = \sum \left\{ S_\alpha w_\alpha(u) + \frac{1}{2} T_\alpha (w_\alpha(u-\kappa) + w_\alpha(u+\kappa)) \right\} \frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$A_{\mathrm{II}}(u) = c_1 A_{\mathrm{II}}^{(1)} + c_2 A_{\mathrm{II}}^{(2)} \quad , \tag{6.8.1}$$

$$A_{\mathrm{II}}^{(1)} = 2 \sum S_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u) \frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$A_{\mathrm{II}}^{(2)} = \sum T_\alpha \left\{ w_\alpha(u-\kappa) - w_\alpha(u+\kappa) \right\} \frac{\sigma_\alpha}{2\mathrm{i}} \quad .$$

The Lax equation (6.1.1) with the matrices (6.8.1) describes the Hamiltonian system with the Poisson bracket (6.2.7) and the Hamiltonian

$$H = c_1 H_1 + c_2 H_2 \quad ,$$

$$H_1 = \sum \left(w_2^2(\kappa) S_\alpha^2 - 2 \frac{w_1 w_2 w_3}{w_\alpha}(\kappa) S_\alpha T_\alpha \right) \quad ,$$

$$H_2 = \sum \left(- \frac{w_1 w_2 w_3}{w_\alpha^2}(\kappa) T_\alpha^2 + 2 w_\alpha(\kappa) S_\alpha T_\alpha \right) \quad .$$

The $L - A$ pair satisfies the reduction

$$L(-u) = -L(u), \quad A(-u) = A(u) \quad . \tag{6.8.2}$$

Note that the factor E/i (where i is the involution $iu = -u$) is a rational curve. The matrix L, multiplied by $w_1 w_2 w_3(u)$

$$L \to w_1 w_2 w_3 L$$

becomes a function on E/i. So the Steklov cases (see also the first Steklov case below) possess Lax representations with a rational spectral parameter.

The spectral curve X (6.5.3) corresponds to $n = 3$ of Sect. 6.5. It also has an involution

$$\tau : (\mu, u) \to (-\mu, -u) \quad ,$$

which is a corollary of (6.8.2). This involution has two fixed points 0^+ and 0^- with $u = 0$ (Fig. 6.3). The factor X/τ is a curve of genus 2. The involution π changing the sheets of the cover $X \to E$ is the hyperelliptic involution of X/τ. Its fixed points are p_1, q_1, p_2, K, $\mathrm{i}K'$, $K + \mathrm{i}K'$.

We shall specify the parameters determining the Baker-Akhiezer function to satisfy the reduction

$$\psi(\tau P) = \psi(P) \quad . \tag{6.8.3}$$

Then the reduction (6.8.2) is automatically fulfilled.

One can always choose a canonical basis of cycles such that (Fig. 6.3)

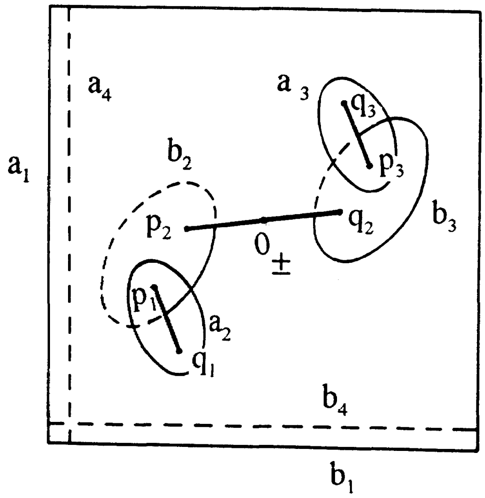

Fig. 6.3. The spectral curve X

$$\tau a_1 = a_4, \quad \tau b_1 = b_4, \quad \tau a_2 = a_3, \quad \tau b_2 = b_3 \quad .$$

The Prym differentials $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^T$ (Sect. 6.5) satisfy the equality

$$\tau^* \boldsymbol{\nu} = T\boldsymbol{\nu} \quad , T = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \quad .$$

The asymptotics at the singularity points of μ and of the normalized Abelian integral of the second kind Ω, determining the velocity vector $\boldsymbol{V}$, are as follows:

$$\begin{array}{lll} \mu \to \pm \dfrac{S}{2\mathrm{i}u}, & \Omega \to \pm c_1 \dfrac{S}{\mathrm{i}u^2} & \text{at } P \to 0^\pm \\ \mu \to \pm \dfrac{T}{4\mathrm{i}(u-\kappa)}, & \Omega \to \pm c_2 \dfrac{T}{2\mathrm{i}(u-\kappa)} & \text{at } P \to \kappa^\pm \\ \mu \to \mp \dfrac{T}{4\mathrm{i}(u+\kappa)}, & \Omega \to \pm c_2 \dfrac{T}{2\mathrm{i}(u+\kappa)} & \text{at } P \to -\kappa^\pm . \end{array}$$

Hence, the equality

$$\tau^* d\Omega = d\Omega$$

holds. For the b-periods we have

$$\begin{array}{l} T\Pi T = \Pi \\ \boldsymbol{V} = T\boldsymbol{V} \end{array} \Rightarrow \Pi = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \gamma & \delta \\ \beta & \delta & \gamma \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_2 \end{pmatrix} \quad . \tag{6.8.4}$$

Let us fix the fixed point of τ as the starting point in all integrals $P_0 = 0^+$ or 0^-. The symmetry of the period matrix gives

$$\theta(Tx; \Pi) = \theta(x; \Pi) \quad .$$

This in turn yields

$$\theta(\int_{P_0}^{\tau P} \boldsymbol{\nu} + \boldsymbol{V}t + \boldsymbol{D}; \Pi) = \theta(T(\int_{P_0}^{\tau P} \boldsymbol{\nu} + \boldsymbol{V}t + \boldsymbol{D}); \Pi) = \theta(\int_{P_0}^{P} \boldsymbol{\nu} + \boldsymbol{V}t + \boldsymbol{D}; \Pi)$$

if the vector $\boldsymbol{D}$ is also symmetric

$$\boldsymbol{D} = T\boldsymbol{D} \quad . \tag{6.8.5}$$

Thus we obtain that (6.8.3) is equivalent to (6.8.5).

The even part of $\mathrm{Prym}_\pi(X)$ with respect to τ is a two-dimensional Abelian torus. We see that the flow $\boldsymbol{V}t$ is restricted to this torus. Let us give the solutions in terms of two-dimensional theta functions. For this purpose we use the theta function reduction technique, see Chap. 7. Let us make a substitution of the theta function's summation variable

$$\boldsymbol{m} = N(\boldsymbol{n} + \boldsymbol{\delta}), \quad N = \begin{pmatrix} 1 & & \\ & 1 & -1 \\ & 1 & 1 \end{pmatrix} \quad .$$

Here $\boldsymbol{n} \in \mathbb{Z}^3$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3) = \{(0,0,0), (0, 1/2, 1/2)\}$. The matrix $N^T \Pi N$ consists of two blocks

$$N^T \Pi N = \begin{pmatrix} \Pi_+ & 0 \\ 0 & \Pi_- \end{pmatrix}, \quad \Pi_+ = \begin{pmatrix} \alpha & 2\beta \\ 2\beta & 2(\gamma + \delta) \end{pmatrix}, \quad \Pi_- = 2(\gamma - \delta).$$

Hence, the following equality holds

$$\begin{aligned} \langle \Pi m, m \rangle + 2\langle x, m \rangle = & \langle \Pi_+(n_1 + \delta_1, n_2 + \delta_2), (n_1 + \delta_1, n_2 + \delta_2) \rangle \\ & + 2\langle (x_1, x_2 + x_3), (n_1 + \delta_1, n_2 + \delta_2) \rangle \\ & + \langle \Pi_-(n_3 + \delta_3), n_3 + \delta_3 \rangle + 2\langle x_3 - x_2, n_3 + \delta_3 \rangle \quad . \end{aligned}$$

which yields the representation of the 3-dimensional theta function in terms of 2-dimensional and 1-dimensional theta functions:

$$\begin{aligned} \theta(x; \Pi) = \; & \theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ((x_1, x_2 + x_3); \Pi_+) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (x_3 - x_2; \Pi_-) \\ & + \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ((x_1, x_2 + x_3); \Pi_+) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (x_3 - x_2; \Pi_-) \quad . \end{aligned} \tag{6.8.6}$$

The structures (6.8.4,5) of the vectors $\boldsymbol{V}$ and $\boldsymbol{D}$ prove that the 1-dimensional theta functions in (6.8.6) are constants.

The First Steklov Case. The corresponding $L - A$ pair and the Hamiltonian are given by

$$L_{\mathrm{I}}(u) = \sum \left\{ p_\alpha w_\alpha (w^2 + \frac{J_\alpha - J}{2}) + \frac{1}{2} M_\alpha w_\alpha \right\} \frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$A_{\mathrm{I}}(u) = d_1 A_{\mathrm{I}}^{(1)} + d_2 A_{\mathrm{I}}^{(2)}, \quad A_{\mathrm{I}}^{(1)} = -2\sum p_\alpha \frac{w_1 w_2 w_3}{w_\alpha}\frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$A_{\mathrm{I}}^{(2)} = \sum \left\{ 2p_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(w^2 + J - \frac{J_\alpha}{2}) + M_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \right\} \frac{\sigma_\alpha}{2\mathrm{i}} \quad ,$$

$$w_\alpha = w_\alpha(u), \quad H = d_1 H_1 + d_2 H_2 \quad ,$$

$$H_1 = \frac{1}{2}\sum((J_\alpha^2 + 2\frac{J_1 J_2 J_3}{J_\alpha})p_\alpha + 2J_\alpha p_\alpha M_\alpha - M_\alpha^2) \quad ,$$

$$H_2 = \frac{1}{2}\sum(J_\alpha(J_1^2 + J_2^2 + J_3^2 - J_\alpha^2)p_\alpha^2 + 2\frac{J_1 J_2 J_3}{J_\alpha}p_\alpha M_\alpha + J_\alpha M_\alpha^2) \quad . \tag{6.8.7}$$

The solution formulas can easily be obtained using the isomorphism [6.5] of this case with the second Steklov case:

$$p_\alpha = S_\alpha, \quad M_\alpha = S_\alpha(w_\alpha^2(\kappa) - 3w^2(\kappa)) - 2T_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(\kappa) \quad ,$$

$$c_1 = d_1 - w^2(\kappa)d_2, \quad c_2 = -w_1 w_2 w_3(\kappa)d_2 \quad .$$

Direct integration of the $L - A$ pair (6.8.7) yields the same formula for p_α.

Remark 6.12. Adding to $\boldsymbol{v}t = (v_1, v_2)^T t$ a period vector of the form

$$\left(\begin{array}{cc|cc} 1 & 0 & 2\alpha & 2\beta \\ 0 & 1 & 2\beta & \gamma+\delta \end{array}\right)\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}, \quad \boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^2 \tag{6.8.8}$$

does not change the solutions. Since $2\nu_1$, $\nu_2 + \nu_3$ are the normalized holomorphic differentials of the Riemann surface X/τ, the matrix (6.8.8) is the period matrix of X/τ. It shows that the mapping of the Liouville torus to $J(X/\tau)$ is one-to-one. This fact was established in another way in [6.21].

6.9 Complete Description of Motion in the Rest Frame

Up to now we have described the motion of tops in the moving frame attached to the body. But for a complete description of the rotation it is necessary to describe it in the rest frame.

It is convenient to use the isomorphism of an algebra of vectors in $\mathbb{R}^3$ with a vector multiplicaton and an algebra of traceless 2×2 matrices with a commutation operation

$$\begin{aligned} &\boldsymbol{X} = (X_1, X_2, X_3) \longleftrightarrow X = \sum_\alpha X_\alpha \frac{\sigma_\alpha}{2\mathrm{i}} \\ &[\boldsymbol{X}, \boldsymbol{Y}] \longleftrightarrow [X, Y] \quad . \end{aligned} \tag{6.9.1}$$

Everywhere below X means the matrix (6.9.1). The coordinates X and X' of a vector in the moving and the rest frames, respectively, are connected by the transformation

$$X = GX'G^{-1} \tag{6.9.2}$$

with some 2×2 matrix G . Our aim is to determine this connection matrix.

The equations of motion of a heavy rigid body about a fixed point in the moving frame attached to the body are as follows:

$$M_t = [M, (-G_t G^{-1})] + [p, L], \quad p_t = [p, (-G_t G^{-1})] \quad . \tag{6.9.3}$$

Here $L = \sum L_\alpha \sigma_\alpha / 2\mathrm{i}$ and L_α are the constant coordinates of the center of mass in the moving frame with an origin at the fixed point. Comparing (6.2.1) with (6.9.3) we get

$$G_t G^{-1} = -\sum_\alpha \frac{\partial H}{\partial M_\alpha} \frac{\sigma_\alpha}{2\mathrm{i}} \quad . \tag{6.9.4}$$

We also fix the third axis of the rest frame, assuming it to be the gravity vector. Combined with (6.9.2) it gives

$$G\sigma_3 G^{-1} = \sum p_\alpha \sigma_\alpha \quad . \tag{6.9.5}$$

An arbitrary solution of (6.9.4) satisfying (6.9.5) may differ by a constant diagonal gauge factor

$$G \to GC, \quad [C, \sigma_3] = 0 \quad . \tag{6.9.6}$$

The remaining freedom (6.9.6) corresponds to the so far unspecified two axes of the rest frame.

The motion of the rigid body in liquid is described in a similar way. In this case the vectors

$$\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3), \quad \Omega_\alpha = \partial H / \partial M_\alpha \quad ,$$

$$\boldsymbol{v} = (v_1, v_2, v_3), \quad v_\alpha = \partial H / \partial p_\alpha$$

are the angular and translation velocities of the rigid body in the moving frame attached to the body [6.9]. As above, for the heavy tops the rotation of the fixed frame to the moving frame is determined by the matrix G satisfying (6.9.4). Let us choose the third axis of the rest frame coinciding with the (constant) momentum $\boldsymbol{p}$. Then for G we have

$$G\sigma_3 G^{-1} = \frac{1}{p} \sum p_\alpha \sigma_\alpha \quad . \tag{6.9.7}$$

The remaining freedom in G is the same as for the heavy tops (6.9.6).

The velocity of translation movement in the rest frame $\boldsymbol{v}' = (v_1', v_2', v_3')$ is equal to

$$\sum_\alpha v'_\alpha \sigma_\alpha = G^{-1} \sum_\alpha v_\alpha \sigma_\alpha G = G^{-1} \sum_\alpha \frac{\partial H}{\partial p_\alpha} \sigma_\alpha G \quad .$$

To find $G(t)$ we still have to solve the linear differential equation (6.9.4). It turns out, however, that the Baker-Akhiezer functions contain more information than the Euler-Poisson equations themselves and allows us to find $G(t)$ without solving (6.9.4).

The Kowalewski Top. Let us consider the equation $\Psi_t = A\Psi$ at $\lambda = 0$. Observe that $A(\lambda = 0)$ decomposes into two 2×2 blocks, which essentially coincide with the angular velocity.

In particular, the matrix

$$\varphi = \begin{pmatrix} \psi_1(0_-) & \psi_1(0_+) \\ \psi_3(0_-) & \psi_3(0_+) \end{pmatrix}$$

satisfies

$$\varphi_t = -\frac{1}{2\mathrm{i}} \sum \frac{\partial H}{\partial M_\alpha} \sigma_\alpha \varphi = -\frac{1}{2\mathrm{i}} (M_1\sigma_1 + M_2\sigma_2 + 2M_3\sigma_3)\varphi \quad .$$

From (6.3.26) we find

$$\varphi = a\theta(\boldsymbol{P} - \boldsymbol{R}) \begin{pmatrix} \dfrac{1}{\theta(\boldsymbol{V}t + \boldsymbol{P})} & 0 \\ 0 & -\dfrac{1}{\theta[\epsilon](\boldsymbol{V}t + \boldsymbol{P})} \end{pmatrix}$$

$$\times \mathcal{A} \begin{pmatrix} \dfrac{e^{bt}}{\theta(\int_{\infty_+}^{0_-} \omega + \boldsymbol{P} - \boldsymbol{R})} & 0 \\ 0 & \dfrac{e^{-bt}}{\theta(\int_{\infty_+}^{0_+} \omega + \boldsymbol{P} - \boldsymbol{R})} \end{pmatrix} ,$$

where $\pm b = \int_{\infty_+}^{0_\mp} d\Omega$ and $\mathcal{A}$ is given by (6.3.26) [(6.3.28)]

$$\mathcal{A} = \begin{pmatrix} \theta\left(\displaystyle\int_{\infty_+}^{0_-} \omega + \boldsymbol{V}t + \boldsymbol{P}\right) & \theta\left(\displaystyle\int_{\infty_+}^{0_+} \omega + \boldsymbol{V}t + \boldsymbol{P}\right) \\ \theta[\epsilon]\left(\displaystyle\int_{\infty_+}^{0_-} \omega + \boldsymbol{V}t + \boldsymbol{P}\right) & -\theta[\epsilon]\left(\displaystyle\int_{\infty_+}^{0_+} \omega + \boldsymbol{V}t + \boldsymbol{P}\right) \end{pmatrix} .$$

It is easily checked that the time evolution of the Poisson vector $\boldsymbol{p}$ is given by

$$\boldsymbol{p}(t) = \frac{1}{2\mathrm{i}} \varphi \sigma_3 \varphi^{-1} \quad .$$

So reducing φ by the constant right factor, we have

$$G = \begin{pmatrix} \frac{1}{\theta(\boldsymbol{V}t+\boldsymbol{P})} & \\ & -\frac{1}{\theta[\epsilon](\boldsymbol{V}t+\boldsymbol{P})} \end{pmatrix} \mathcal{A} \begin{pmatrix} e^{bt+b_0} & \\ & e^{-bt-b_0} \end{pmatrix} \quad ,$$

where $b_0 = \text{const.}$

By inverting, we also obtain the evolution of the top in the rest frame. For example, the motion of the symmetry axis of the top in the rest frame is given by

$$\sum L'_\alpha \sigma_\alpha = \begin{pmatrix} e^{-bt-b_0} & \\ & e^{bt+b_0} \end{pmatrix} \mathcal{A}^{-1} \sigma_3 \mathcal{A} \begin{pmatrix} e^{bt+b_0} & \\ & e^{-bt-b_0} \end{pmatrix} \quad ,$$

where L'_α are the coordinates of the unit vector directed along the axis of the top.

For the Clebsch and Steklov cases we restrict ourselves to the case $d_{\mathrm{II}} = 0$, i.e., $H = H_I$ for both systems.

The Clebsch Case. Substituting the asymptotics

$$\Psi(u) = (\Phi + Qu + O(u^2)) \exp\left(\frac{\sigma_3}{2\mathrm{i}}\frac{p}{u}t\right)$$

into the equations

$$L\Psi = \Psi\hat{\mu}, \quad \Psi_t = A_{\mathrm{I}}\Psi$$

we obtain

$$p\Phi\sigma_3\Phi^{-1} = \sum p_\alpha\sigma_\alpha \quad ,$$

$$\sum M_\alpha\sigma_\alpha = \left[Q\Phi^{-1}, \sum p_\alpha\sigma_\alpha\right] + \frac{pM}{p^2}\sum p_\alpha\sigma_\alpha \quad , \tag{6.9.8}$$

$$2\mathrm{i}\ \Phi_t\Phi^{-1} = \left[\sum p_\alpha\sigma_\alpha, Q\Phi^{-1}\right] \quad .$$

From these formulas we get

$$\Phi_t\Phi^{-1} = -\frac{1}{2\mathrm{i}}\left(\sum M_\alpha\sigma_\alpha - \frac{pM}{p^2}\sum p_\alpha\sigma_\alpha\right) \quad . \tag{6.9.9}$$

The equalities (6.9.8, 9) show that $G(t)$ satisfying (6.9.4, 7) can be easily obtained using Φ with the help of multiplication by the right factor

$$G = \Phi\ \exp\left(-\frac{\sigma_3}{2\mathrm{i}}\frac{pM}{p}t\right)$$

(we consider the case $H = H_{\mathrm{I}}$). Finally, we have

$$
\begin{aligned}
G &= \begin{pmatrix} \theta(\boldsymbol{V}t+\boldsymbol{D};\Pi) & \theta(\boldsymbol{V}t+\boldsymbol{D}+\boldsymbol{r};\Pi) \\ \theta(\boldsymbol{V}t+\boldsymbol{D}+\boldsymbol{\Delta};\Pi) & \theta(\boldsymbol{V}t+\boldsymbol{D}+\boldsymbol{r}+\boldsymbol{\Delta};\Pi) \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{bt+b_0} & \\ & e^{-bt-b_0} \end{pmatrix} \quad ,
\end{aligned}
\tag{6.9.10}
$$

$$
b = -\frac{1}{2\mathrm{i}}\frac{pM}{p} + \frac{1}{2}\int_{0-}^{0+} d\Omega \quad .
$$

Remark 6.13. We ignored this fact, but in reality the Ψ-function constructed here satisfies the equation $\psi_t = A\psi + a(\lambda, t)\psi$ (Sect. 6.1), since it was determined up to a scalar factor depending on t. Nevertheless the connection (6.9.2) of the bases with (6.9.8) is valid since (6.9.2) is invariant with respect to this multiplication.

The Steklov Case. The analogous expressions for the Steklov case are as follows:

$$
\Psi(u) = (\Phi + Su + Qu^2 + O(u^3)) \exp\left(\frac{\sigma_3}{2\mathrm{i}}\left(-\frac{2p}{u^2}t\right)\right) \quad ,
$$

$$
\hat{\mu} = \left(\frac{p}{u^3} + \frac{pM}{2p}\frac{1}{u} + O(u)\right)\frac{\sigma_3}{2\mathrm{i}} \quad ,
$$

$$
p\Phi\sigma_3\Phi^{-1} = \sum p_\alpha\sigma_\alpha \quad ,
$$

$$
\left[\sum p_\alpha\sigma_\alpha, Q\Phi^{-1}\right] = \frac{pM}{2p^2}\sum\left(p_\alpha\sigma_\alpha - \frac{1}{2}M_\alpha\sigma_\alpha\right) \quad ,
$$

$$
\begin{aligned}
2\mathrm{i}\Phi_t\Phi^{-1} &= -2\left[\sum p_\alpha\sigma_\alpha, Q\Phi^{-1}\right] + \sum p_\alpha\sigma_\alpha(J - J_\alpha) \\
&= \sum (M_\alpha - J_\alpha p_\alpha)\sigma_\alpha + \sum p_\alpha\sigma_\alpha\left(J - \frac{pM}{p^2}\right) \quad ,
\end{aligned}
$$

$$
G = \Phi \, \exp\left(-\frac{\sigma_3}{2\mathrm{i}}\left(Jp - \frac{pM}{p}\right)t\right) \quad .
$$

The final result in this case is given by the same formula (6.9.10) as for the Clebsch case. The difference is that in the Steklov case the theta functions in (6.9.10) are 3-dimensional (Sect. 6.8) and the constant b is determined by the slightly different expression

$$
b = -\frac{1}{2\mathrm{i}}\left(Jp - \frac{pM}{p}\right) + \frac{1}{2}\int_{0_-}^{0+} d\Omega \quad .
$$

7. Algebro-Geometric Principles of Superposition of Finite-Gap Solutions of Integrable Nonlinear Equations

In this chapter we are concerned with the following question: what kind of specialization of the parameters enables one to express the finite-gap solution of an algebro-geometric integrable equation with partial derivatives, associated with a non-singular algebraic curve X_g of genus g,[1] through Abelian functions of lower genera, in particular, through elliptic functions? By answering this question, we could reduce finite-gap solutions to computationally simpler functions and single out practically important special classes of solutions which are periodic in one of the variables or satisfy a given boundary condition. A solution to this problem was proposed in 1982 [7.1a], using the theory of reduction of Abelian integrals and Riemann theta functions to lower genera that was initiated by Weierstrass (see, for example, [7.2 a, b]). It was understood that the solution of an algebro-geometric integrable equation may be expressed in terms of Abelian functions of lower genera provided the moduli of the algebraic curve belong to some subset which is dense in moduli space. Specifically, for genus $g = 2$, such a reduction is possible in a countable number of cases in which the curve X_2 covers N-sheetedly the torus X_1, where $N = 2, 3, \ldots$; furthermore, the Riemann matrix Π of the curve X_2 is reduced to the form $\Pi = \begin{pmatrix} 2\pi\mathrm{i} & 0 & B_{11} & 2\pi\mathrm{i}/N \\ 0 & 2\pi\mathrm{i} & 2\pi\mathrm{i}/N & B_{22} \end{pmatrix}$. Although the problem of reducing finite-gap solutions to lower genera is basically solved by referring to Weierstrass classical theory of reduction, the derivation of the resulting formulas is a technically complicated procedure that requires application of addition theorems of the N-th order theta functions (Sect. 2.5).

As mentioned in [7.3 a, b], these technical difficulties could be surmounted in a special case of curves X_g that have a group of non-trivial automorphism $G = \{g_i\}$. In this case the action of each element $g_i \in G$ on the basis of cycles in $H_1(X_g, \mathbb{Z})$, $g : H_1(X_g, \mathbb{Z}) \to H_1(X_g, \mathbb{Z})$ is associated with the transformation $\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{Tr}(2g, \mathbb{Z})$ that leaves the B-matrix invariant. So, in view of (1.5.13), the B-matrix satisfies the relations

$$B(2\pi\mathrm{i}d_j + c_jB) = 2\pi\mathrm{i}\,(2\pi\mathrm{i}b_j + a_jB) \quad \forall g_j \in G \quad ,$$

[1] Throughout this chapter the lower index of X is used to denote the genus of X.

which are used to define the special structure of the B-matrix. This approach, developed in [7.3], is simple and straightforward, so that it is quite effective in the cases where it is applicable. For instance, it is possible in a number of cases to derive numerical values for the B-matrix, starting from symmetry considerations only.

The reader can find a discussion of different aspects of the reduction problem for finite-gap solutions as presented in this chapter in [7.4].

Here we give the Weierstrass reduction theory for the case of genus $g = 2$ and describe the reduction of theta functions to lower genera for curves with automorphisms. As applications we consider the isospectral deformation of elliptic finite-gap potentials which are the solutions in terms of elliptic functions for a number of algebro-geometrically integrable nonlinear equations and classical dynamic systems. We use the formalism of the theory of theta functions (*Krazer* [7.2a], *Krazer* and *Wirtinger* [7.2b], *Igusa* [7.5], *Fay* [7.6], *Griffith* and *Harris* [7.7], *Farkas* and *Kra* [7.8], *Mumford* [7.9]) and give brief proofs of the propositions directly related to the theory of reduction. The original results presented here were previously reported in [7.1, 3].

7.1 The Simplest Reduction Case

We illustrate the present approach by considering an example of the reduction of a hyperelliptic integral to the elliptic integrals described by Jacobi and Krazer's monograph [7.2 a]. By making in the elliptic integral

$$J = \int \frac{d\xi}{\sqrt{\xi(1-\xi)(1-c^2\xi)}}$$

a rational substitution of order $N = 2$,

$$\xi = \frac{(1-\alpha)(1-\beta)\lambda}{(\lambda-\alpha)(\lambda-\beta)} \quad , \tag{7.1.1}$$

we obtain

$$J = -\sqrt{(1-\alpha)(1-\beta)} \int \frac{(\lambda^2-\alpha\beta)d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-\alpha)(\lambda-\beta)(\lambda-\alpha\beta)\varphi(\lambda)}}, \tag{7.1.2}$$

where $\varphi(\lambda) = (\lambda-\alpha)(\lambda-\beta) - c^2(1-\alpha)(1-\beta)\lambda$. If the constant c in the last equality is chosen so that the function $\varphi(\lambda)$ is a complete square, i.e., we set

$$c^2 = c_\pm^2 = -\frac{(\sqrt{\alpha}\pm\sqrt{\beta})^2}{(1-\alpha)(1-\beta)} \quad , \tag{7.1.3}$$

(7.1.2) takes the form

$$\begin{aligned}&\int \frac{d\xi}{\sqrt{\xi(1-\xi)(1-c_{\pm}^2\xi)}}\\&= -\sqrt{(1-\alpha)(1-\beta)}\int \frac{(\lambda \pm \sqrt{\alpha\beta})d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-\alpha)(\lambda-\beta)(\lambda-\alpha\beta)}} \quad .\end{aligned} \tag{7.1.4}$$

Equation (7.1.4) examplifies the reduction of independent hyperelliptic integrals to elliptic ones made by a rational change of variables (7.1.1). In other words, (7.1.1) and (7.1.4) imply that there exist covers $\pi^{\pm}$,

$$X_1^{(+)} \xleftarrow{\pi^+} X_2 \xrightarrow{\pi^-} X_1^{(-)} \quad ,$$

of a hyperelliptic curve of genus 2, $X_2 = (\mu, \lambda)$,

$$\mu^2 = \lambda(\lambda-1)(\lambda-\alpha)(\lambda-\beta)(\lambda-\alpha\beta) \quad , \tag{7.1.5}$$

over elliptic curves (tori)

$$\eta_{\pm}^2 = \xi(1-\xi)(1-c_{\pm}^2\xi) \tag{7.1.6}$$

with moduli $c_{\pm}$, defined by (7.1.3).

The important property of the curve (7.1.5) is the presence of a non-trivial automorphism T of second order with a fixed point,

$$T : (\mu, \lambda) \to \left(\frac{\mu(\alpha\beta)^{3/2}}{\lambda^3}, \ \frac{\alpha\beta}{\lambda}\right) \quad . \tag{7.1.7}$$

The automorphism T makes it possible to determine the factor X_2/T, which can easily be shown by going from (7.1.5) to a conformally equivalent curve

$$\widetilde{\mu}^2 = (\widetilde{\lambda}^2 - e_1^2)(\widetilde{\lambda}^2 - e_2^2)(\widetilde{\lambda}^2 - e_3^2) \quad , \tag{7.1.8}$$

using the transformation

$$\lambda = \frac{e_1 + e_2}{e_2 - e_1}\,\frac{\widetilde{\lambda} - e_1}{\widetilde{\lambda} + e_1} \quad . \tag{7.1.9}$$

It is seen that the curve X_2/T is defined by the formula

$$\eta^2 = (\xi - e_1^2)(\xi - e_2^2)(\xi - e_3^2) \quad .$$

When the curve X_2 is defined in this way, (7.1.4) has the form

$$\begin{aligned}&\int \frac{\widetilde{\lambda}d\widetilde{\lambda}}{\widetilde{\mu}} = \frac{1}{2}\int \frac{d\xi}{\sqrt{(\xi-e_1^2)(\xi-e_2^2)(\xi-e_3^2)}} \quad ,\\&\int \frac{d\widetilde{\lambda}}{\widetilde{\mu}} = \frac{1}{2}\int \frac{d\xi}{\sqrt{\xi(\xi-e_1^2)(\xi-e_2^2)(\xi-e_3^2)}} \quad .\end{aligned} \tag{7.1.10}$$

The elementary example described above leads us to the formulation of the general reduction problem of Abelian integrals and Riemann theta functions to lower genera that was given by Weierstrass (see *Krazer* [7.2 a]). The first results along these lines were produced by Königsberger and Kowalewski [7.2 a]; the problem was also treated by Appel, Gursa, Burkhardt, Bolza, Krazer, Picard, Pringsheim, Poincaré, Humbert, Hermite and others [7.2 a], and was recently discussed within the present-day context by *Igusa* [7.10] and *Martens* [7.11]. These papers have considerably influenced our treatment as presented in this chapter.

The algebraic curves for which Abelian integrals reduce to lower genera sometimes have non-trivial automorphisms. In particular, for the case of genus $g = 2$ the restrictions imposed by the Riemann-Hurwitz formula (2.2.1) enable us to conclude that the automorphism (7.1.8) is only possible for the given type of automorphisms with fixed points. With $g > 2$, there is a much larger number of curves with additional symmetries which were discussed and classified by *Klein* [7.12], *Kuribayashi* and *Komiya* [7.13], *Accola* [7.14], and *Horiuchi* et al. [7.15].

7.2 Poincaré Theorem

In this section we formulate the Poincaré theorem about the "complete reducibility" of the Riemann matrix to block form under the action of the group $\mathrm{Tr}(2g, \mathbb{Q})$ and discuss its relation to coverings over a torus.

The point $B \in \mathcal{U}_g$ is said to be *reducible relatively to the action of the group* $\mathrm{Tr}(2g, \mathbb{Q})$, if there is an element $\sigma \in \mathrm{Tr}(2g, \mathbb{Q})$ such that

$$\sigma \cdot B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix} ,$$

where $B' \in \mathcal{U}_{g'}$, $B'' \in \mathcal{U}_{g''}$. Following [7.10, 11] we formulate the Poincaré theorem about "complete reducibility".

Theorem 7.1. (Poincaré [7.16]) Let Π_1 be a Riemann matrix (2.5.3) such that for some complex $(g_0 \times 2g_0)$-matrix Π_0 and some $(2g_1 \times 2g_0)$-integral matrix M, where $1 \le g_0 < g_1$, the upper $(g_0 \times 2g_1)$ submatrix of the matrix Π_1 can be written as $\Pi_0 M^T$. Then there exists an element $\sigma \in \mathrm{Tr}(2g_1, \mathbb{Q})$ and a point $B_{g_0} \in \mathcal{U}_{g_0}$ such that the matrix $(2\pi \mathrm{i} I_{g_0}, B_{g_0})$ is necessarily equivalent to the Riemann matrix Π_0 and

$$\sigma \cdot B_{g_1} = \begin{pmatrix} B_{g_0} & 0 \\ 0 & B' \end{pmatrix} , \tag{7.2.1}$$

where $\Pi_1 = \varrho(2\pi \mathrm{i} I_{g_1}, B_{g_1})$ and ϱ is a complex invertible $(g_1 \times g_1)$-matrix.

At the points reducible relatively to the group $\mathrm{Tr}(2g, \mathbb{Z})$ the theta functions $\theta[\varepsilon](z; B)$ are decomposed into a product of theta functions of dimensions g_0

and g' with period matrices B_{g_0} and B', and at the points reducible relatively to the group $\mathrm{Tr}(2g, \mathbb{Q})$, the theta functions are decomposed into a sum of the products of theta functions of dimensions g_0 and g' with matrices NB_{g_0}, NB' for some positive integer N. In what follows we will be concerned with this case only.

In the special case $g_0 = 1$, the Poincaré theorem is reduced to the Weierstrass theorem:

Theorem 7.2. (Weierstrass, see *Kowalewski* [7.17]) Suppose that $g_0 = 1$ and the condition of the Poincaré theorem are valid. Then and only then exists an element $\sigma \in \mathrm{Tr}(2g_1, \mathbb{Z})$ and a point $B_{g_1} \in \mathcal{U}_{g_1}$ such that

$$\sigma \cdot B_{g_1} = \begin{pmatrix} B_{11} & 2\pi i k/N & 0 & \cdots & 0 \\ 2\pi i k/N & & & & \\ 0 & & & & \\ \vdots & & B' & & \\ 0 & & & & \end{pmatrix} , \tag{7.2.2}$$

where k is the positive integer $1 \le k < N$ and

$$N = Pf(M^T Q M) \tag{7.2.3}$$

and Pf is a Pfaffian.[2]

Thus, we have

Corollary 7.3. Let the B-matrix be of the form (7.2.2). Then the reduction formula holds

$$\begin{aligned} \theta^N(z; B) = \sum_{p_N} c_{p_N} \theta \begin{bmatrix} 2p_{N1} \\ 2kp_{N2} \end{bmatrix} (Nz_1; NB_{11}) \\ \times \theta \begin{bmatrix} 2p_{N2} & 2p_{N3} & \cdots & 2p_{Ng} \\ 2kp_{N1} & 0 & \cdots & 0 \end{bmatrix} (N\hat{z}; N\hat{B}) \quad , \end{aligned} \tag{7.2.4}$$

in which $\hat{z} = (z_2, \ldots, z_g)$, the summation is over the whole set of representatives $\mathbb{Z}^g/N$, $p_N = (p_{N1}, p_{N2}, \ldots, p_{Ng})$, and the constants c_{p_N} are equal to

[2] We recall that the Pfaffian $Pf(A)$ of the determinant $|A|$ of a skew-symmetric matrix A of degree $n = 2m$ is said to be a form of degree m with respect to the elements of the determinant such that $|A| = (Pf(A))^2$.

$$
\begin{aligned}
c_{p_N} &= \exp\left(\frac{\pi \mathrm{i}}{2} k p_{N1} p_{N2}\right) \\
&\times \sum_{\substack{p_M \\ M=1,\ldots,N-1}} \theta\begin{bmatrix} 2p_N - 2p_{N-1} \\ 0 \end{bmatrix}(0; N(N-1)B) \\
&\times \theta\begin{bmatrix} 2p_{N-1} - 2p_{N-2} \\ 0 \end{bmatrix}(0; (N-1)(N-2)B) \times \cdots \\
&\times \theta\begin{bmatrix} 2p_3 - 2p_2 \\ 0 \end{bmatrix}(0; 6B)\theta\begin{bmatrix} 2p_2 \\ 0 \end{bmatrix}(0; 2B) \quad ,
\end{aligned} \tag{7.2.5}
$$

where $\boldsymbol{p}_M = (p_{M1}, \ldots, p_{Mg})$, $M = N, \ldots, 2$.

The proof follows from the Koizumi formula (see (2.4.4) and [7.18]) and the following equality:

$$
\theta[\varepsilon]\left(\boldsymbol{z}; \begin{pmatrix} B_{11} & 2\pi \mathrm{i} m & 0 & \cdots & 0 \\ 2\pi \mathrm{i} m & & & & \\ 0 & & & & \\ \vdots & & \hat{B} & & \\ 0 & & & & \end{pmatrix}\right) = (-1)^{-\varepsilon_1' \varepsilon_2' m/2} \tag{7.2.6}
$$

$$
\times \theta\begin{bmatrix} \varepsilon_1' & \varepsilon_2' & \varepsilon_3' & \cdots & \varepsilon_g' \\ \varepsilon_1'' + \varepsilon_2' m & \varepsilon_2'' + \varepsilon_1' m & \varepsilon_3'' & \cdots & \varepsilon_g'' \end{bmatrix}\left(\boldsymbol{z}; \begin{pmatrix} B_{11} & 0 \cdots 0 \\ 0 & \\ \vdots & \hat{B} \\ 0 & \end{pmatrix}\right).
$$

Thus, (7.2.6) implies that when the condition of the Weierstrass theorem is valid, the g-dimensional theta function $\theta[\varepsilon](\boldsymbol{z}; B)$ is expressed through $g-1$-dimensional theta functions and Jacobian theta functions.

We show that the condition of the Poincaré theorem holds, if the Riemann matrix Π_{g_1} is generated by an algebraic curve which is an N-sheeted covering over an algebraic curve of genus $g_0 < g_1$.

Let $\pi : X_{g_1} \to X_{g_0}$ be an arbitrary N-sheeted branched cover of the compact Riemann surfaces X_{g_1} and X_{g_0} of genera $g_1 > g_0 \geq 1$. We write down the canonical bases of homologies

$$
(\gamma^{(i)}, \ldots, \gamma^{(i)}_{2g_i}) = (a^{(i)}_1, \ldots, a^{(i)}_{g_i}; b^{(i)}_1, \ldots, b^{(i)}_{g_i}) \in H_1(X_{g_i}, \mathbb{Z}), \quad i = 0, 1
$$

with intersection matrices

$$
Q_i = (\gamma^{(i)}_k \circ \gamma^{(j)}_l) = \begin{pmatrix} 0 & I_{g_i} \\ -I_{g_i} & 0 \end{pmatrix}, \quad i = 0, 1 \quad , \tag{7.2.7}
$$

assuming that the bases $\gamma^{(0)}$ and $\gamma^{(1)}$ do not intersect the branch points of the cover and their images. We define the holomorphic differentials $\omega^{(i)} = (\omega^{(i)}_1, \ldots, \omega^{(i)}_{g_i})$ dual to $\gamma^{(i)}$, $i = 0, 1$, i.e., the holomorphic differentials normalized so that the Riemann matrix Π_{g_j} has the form $(2\pi \mathrm{i} I_{g_j}; B_{g_j})$, $B_{g_j} \in \mathcal{U}_{g_j}$, $j = 0, 1$.

The cover π induces the mapping of homologies, $M : H_1(X_{g_1}, \mathbb{Z}) \to H_1(X_{g_0}, \mathbb{Z})$, which is described by the $(2g_1 \times 2g_0)$ integral matrix (M_{ij}) by the formula

$$M\gamma_i^{(1)} = \sum_{j=1}^{2g_0} M_{ij}\gamma_j^{(0)} \quad , \tag{7.2.8}$$

where $\gamma_i^{(1)}$ is the basis cycle of $H_1(X_{g_1}, \mathbb{Z})$. Under the action of π^{-1} every basis holomorphic differential $\omega_i^{(0)} \in H_1(X_{g_0}, \mathbb{Z})$ transforms into a linear combination of basis differentials $\omega_j^{(1)} \in H_1(X_{g_1}, \mathbb{Z})$. The $(g_0 \times g_1)$ complex matrix φ can thus be defined by

$$\varphi \int\limits_{\gamma_i^{(1)}} \omega^{(1)} = \int\limits_{\sum_{j=1}^{2g_0} M_{ij}\gamma_j^{(0)}} \omega^{(0)}, \quad i = 1, \ldots, g_1 \quad . \tag{7.2.9}$$

Equation (7.2.9), written in matrix form

$$\varphi \Pi_{g_1} = \Pi_{g_0} M^T \tag{7.2.10}$$

represents the condition of Theorem 7.1.

We can also show that for the matrix M the following equation holds (see for example [7.14]):

$$NQ^{(0)} = M^T Q^{(1)} M \quad , \tag{7.2.11}$$

where $Q^{(i)}$, $i = 0, 1$, are intersection matrices (7.2.7) and N is the number of sheets of the cover π. When $g_0 = 1$, (7.2.11) has the form (7.2.3). We denote $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, where M_i are $(g_1 \times g_0)$-matrices. Then (7.2.11) leads to the equations

$$\begin{aligned} &M_2^T M_1 - M_1^T M_2 = 0 \quad , \\ &M_4^T M_1 - M_3^T M_2 = I_{g_0} N \quad , \\ &M_4^T M_3 - M_3^T M_4 = 0 \quad , \end{aligned} \tag{7.2.12}$$

which, for $g_0 = 1$, result in a single equation

$$\sum_{j=1}^{g} M_{j,1} M_{g_1+j,2} - M_{g_1+1,1} M_{j,2} = N \quad . \tag{7.2.13}$$

Our further discussion is based on the Weierstrass Theorem 7.2 for covers:

Theorem 7.4. Suppose that an algebraic curve X_g covers N-sheetedly a torus X_1. Then there is an element $\sigma \in \mathrm{Tr}(2g, \mathbb{Z})$ and a point $B \in \mathcal{U}_g$ such that the representation (7.2.2) in which k is the positive integer, $1 \leq k < N$ and B is a $((g-1) \times (g-1))$-matrix of $\mathcal{U}_{g-1}$ holds.

7.3 Coverings of Genus 2 Over a Torus. Humbert Varieties

In this section we prove the Weierstrass theorem of Sect. 7.2 for genus $g = 2$ and describe the conditions of reduction in the moduli space of algebraic curves of genus 2.

Theorem 7.5. (Weierstrass theorem for genus 2) In order for the algebraic curve X_2 to cover N-sheetedly the torus X_1, it is necessary and sufficient that there are an element $\sigma \in \mathrm{Tr}(4, \mathbb{Z})$ and a point $B \in \mathcal{U}_2$ such that

$$\sigma \cdot B = \begin{pmatrix} B_{11} & 2\pi \mathrm{i}/N \\ 2\pi \mathrm{i}/N & B_{22} \end{pmatrix} \quad . \tag{7.3.1}$$

Proof. (*Krazer* [7.2 a]). Suppose that there is a cover $\pi : X_2 \to X_1$. Then, on X_2 there is a holomorphic differential $\boldsymbol{\omega}$, reducible to an elliptic differential with periods v and v'. Decomposing the differential into holomorphic differentials ω_1 and ω_2, $\boldsymbol{\omega} = c_1\omega_1 + c_2\omega_2$, normalized in this basis $(\boldsymbol{a}, \boldsymbol{b}) \in H_1(X_2, \mathbb{Z})$, and calculating its $\boldsymbol{a}$ and $\boldsymbol{b}$ periods, we have the equations

$$\begin{aligned} c_1 &= M_{11}v + M_{12}v' \quad , \\ c_2 &= M_{21}v + M_{22}v' \quad , \\ c_1B_{11} + c_2B_{12} &= M_{31}v + M_{32}v' \quad , \\ c_1B_{12} + c_2B_{22} &= M_{41}v + M_{42}v' \quad . \end{aligned} \tag{7.3.2}$$

The condition for which the equations (7.3.2) are compatible for the unknowns c_1, c_2, v_1, v_2 is

$$\det \begin{pmatrix} 2\pi \mathrm{i} I_g & B \\ & M^T \end{pmatrix} = 0 \quad , \tag{7.3.3}$$

where M is a (4×2)-integral matrix

$$M = \begin{pmatrix} M_{11} M_{12} \\ M_{21} M_{22} \\ M_{31} M_{32} \\ M_{41} M_{42} \end{pmatrix} \quad , \tag{7.3.4}$$

or, in expanded form,

$$\frac{1}{2\pi i}(\alpha B_{11} + \beta B_{12} + \gamma B_{22}) + \frac{\delta}{(2\pi i)^2}(B_{12}^2 - B_{11}B_{22}) + \varepsilon = 0 \quad , \tag{7.3.5}$$

where

$$\begin{aligned} &\alpha = M_{12}M_{41} - M_{11}M_{42}, \quad \gamma = M_{21}M_{32} - M_{31}M_{22} \quad , \\ &\delta = M_{12}M_{21} - M_{11}M_{12}, \quad \varepsilon = M_{31}M_{42} - M_{41}M_{32} \quad , \\ &\beta = M_{11}M_{32} - M_{12}M_{31} - (M_{21}M_{42} - M_{22}M_{41}) \quad . \end{aligned} \tag{7.3.6}$$

The condition (7.2.13) must hold for the numbers M_{ij}, i.e.,

$$M_{11}M_{32} - M_{12}M_{31} + (M_{21}M_{42} - M_{22}M_{41}) = N \quad . \tag{7.3.7}$$

Substituting from the first four equalities (7.3.6) the quantities

$$\begin{aligned} M_{21} &= \frac{M_{11}\gamma + M_{31}\delta}{M_{11}M_{32} - M_{12}M_{31}}, \quad M_{22} = \frac{M_{12}\gamma + M_{32}\delta}{M_{11}M_{32} - M_{12}M_{31}} \quad , \\ M_{41} &= \frac{M_{11}\varepsilon - M_{31}\alpha}{M_{11}M_{32} - M_{12}M_{31}}, \quad M_{42} = \frac{M_{12}\varepsilon - M_{32}\alpha}{M_{11}M_{32} - M_{12}M_{31}} \quad , \end{aligned} \tag{7.3.8}$$

we obtain the equation

$$M_{21}M_{42} - M_{22}M_{41} = -\frac{\varepsilon\delta + \alpha\gamma}{M_{11}M_{32} - M_{12}M_{31}} \quad . \tag{7.3.9}$$

With this equation taken into account, the fifth of the equations (7.3.6) transforms into a quadratic equation with the roots

$$\begin{aligned} M_{11}M_{32} - M_{12}M_{31} &= \frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4(\varepsilon\delta + \alpha\gamma)}) \quad , \\ M_{21}M_{42} - M_{11}M_{22} &= \frac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4(\varepsilon\delta + \alpha\gamma)}) \quad . \end{aligned} \tag{7.3.10}$$

From this equation and (7.3.7) we have

$$N^2 = \beta^2 - 4(\varepsilon\delta + \alpha\gamma) \quad . \tag{7.3.11}$$

To complete the proof, we have to find an element $\sigma \in \mathrm{Tr}(4, \mathbb{Z})$ such that (7.3.1) is valid. For this purpose we find a matrix T such that

$$M^T T = \begin{pmatrix} N & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad . \tag{7.3.12}$$

We consider the matrices A_1, B_1, D_{12}, C_{12}, $A_2 = D_{12}A_1D_{12}$, $B_2 = D_{12}B_1D_{12}$, where

$$\begin{aligned} A_1 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \\ C_{12} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad . \end{aligned} \tag{7.3.13}$$

For the sake of completeness, we also give the matrices

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad . \tag{7.3.14}$$

The action of the matrix A_i, $i = 1,2$ on M^T amounts to adding the $(2+i)$-th column of the matrix M^T to the i-th column. As a result of the action of the matrix B_i, $i = 1,2$, the i-th column of the matrix M^T is added to the $(2+i)$-th column, and the i-th column is replaced by the $(2+i)$-th column multiplied by -1. Under the action of the matrix D_{12}, the first and the second columns of the matrix M^T are exchanged, as are the third and the fourth columns. Finally, under the action of C_{12} the first column of the matrix M^T is added to the second, and the fourth column is subtracted from the third. The six matrices introduced are enough to reduce the arbitrary matrix M^T to the form (7.3.12). The matrix $T^T = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ maps a basis into a basis in $H_1(X_2, \mathbb{Z})$. Therefore, the appropriate matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Tr}(4, \mathbb{Z})$ transforms the B-matrix to the form (7.3.1). The proposition proven can be inverted (*Krazer* [7.2 a]).

Corollary 7.6. If a holomorphic differential on the curve X_2 is reduced to an elliptic differential, then the other independent differential is also reduced to it.

We can thus conclude that the curve X_2 is an N-sheeted covering over two tori X_1 and X_1'; the resultant relations are represented in the following commutative diagram

$$\begin{array}{ccccc} X_1' & \xleftarrow{\pi'} & X_2 & \xrightarrow{\pi} & X_1 \\ \downarrow{\scriptstyle A'} & & \downarrow{\scriptstyle A_2} & & \downarrow{\scriptstyle A} \\ J(X_1') & \xleftarrow{\psi'} & J(X_2) & \xrightarrow{\psi} & J(X_2) \end{array}$$

in which the mappings π, π' introduced above, A', A_2, A are Abel maps, and ψ, ψ' are the mappings of the Jacobians.

We note that while proving the Weierstrass theorem for genus 2, we introduced the varieties

$$\begin{aligned} \mathbf{H}_\Delta = \{B | \frac{1}{2\pi \mathrm{i}}(\alpha B_{11} + \beta B_{12} + \gamma B_{22}) + \frac{\delta}{(2\pi \mathrm{i})^2}(B_{12}^2 \\ - B_{11}B_{22}) + \varepsilon = 0, \quad \Delta = \beta^2 - 4\alpha\gamma - 4\varepsilon\delta \in N\} \quad . \end{aligned} \tag{7.3.15}$$

Proposition 7.7. (*Krazer* [7.2 a]) The quantity Δ is invariant under the action of the group $\mathrm{Tr}(4, \mathbb{Z})$.

The variety $\mathbb{H}_\Delta$ is called an *Humbert variety with the invariant* Δ [7.19]. In the present case of reduction, the invariant Δ is a complete square $\Delta = N^2$, $N \in \mathbb{N}$. The other cases are discussed by *van der Geer* [7.19]. In the simplest case $\Delta = 1$ $\mathbb{H}_1$ is a variety of singularities, i.e., a set of points B such that the matrix is equivalent to a diagonal one and thus reducible with respect to $\mathrm{Tr}(4, \mathbb{Z})$ (Sect. 7.1). In what follows we discuss other examples of Humbert varieties.

7.4 Expression of Moduli for a Genus 2 Curve in Terms of Theta Constants

In this section we specify the description of hyperelliptic curves, given in Sect. 1.8, to cover the case when $g = 2$. This is necessary for the construction of applications of the Weierstrass theorem.

As is known, a non-singular curve X_2 of genus 2 is always hyperelliptic [7.8]. Following *Bolza* [7.20], we write down the curve X_2 as

$$\widetilde{\mu}^2 = \frac{4\prod\limits_{j=1}^{6}{}'(\theta_1[\delta_j] + \theta_2[\delta_j]\widetilde{\lambda})}{\pi^4 \prod\limits_{k=1}^{10}{}'' \theta[\varepsilon_k]} \quad , \tag{7.4.1}$$

where the products $\prod'$ and $\prod''$ are extended to all odd and all even characteristics. The Lemma given below is derived from works by *Rosenhain* [7.21] and *Bolza* [7.20].

Lemma 7.8. Let the curve X_2 have a basis fixed in $H_1(X_2, \mathbb{Z})$ by six characteristics $[A_j] = \int_{Q_1}^{Q_j} \omega$, where $Q_1, \dots, Q_6$ are the branching points of X_2, and $\omega = (\omega_1, \omega_2)$ are normalized holomorphic differentials. Then, the curve $\widetilde{X}_2 = (\widetilde{\mu}, \widetilde{\lambda})$, realized as (7.4.1), is conformally equivalent to the curve

$$\mu^2 = \lambda(\lambda+1)(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) \quad , \tag{7.4.2}$$

where $\lambda(Q_6) = \infty$, $\lambda(Q_5) = 0$, $\lambda(Q_4) = -1$, $\lambda(Q_{4-i}) = \lambda_i$, $i = 1, 2, 3$,

$$\lambda_i = -\frac{\theta^2[\varepsilon_j]\theta^2[\varepsilon_k]}{\theta^2[\delta_j]\delta^2[\varepsilon_k]}, \quad i \neq j \neq k = 1,2,3 \quad . \tag{7.4.3}$$

$$\begin{aligned} &[\varepsilon_1] = [A_4 A_5 A_6], \quad [\delta_1] = [A_4] \quad , \\ &[\varepsilon_2] = [A_2], \quad [\delta_2] = [A_2 A_5 A_6] \quad , \\ &[\varepsilon_3] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\delta_3] = [A_5 A_6] \quad , \end{aligned} \tag{7.4.4}$$

the differentials $d\widetilde{\lambda}/\widetilde{\mu}$ and $\widetilde{\lambda}d\widetilde{\lambda}/\widetilde{\mu}$ are normalized on X_2, and the normalization constants c_{ij}, $i,j=1,2$ of the holomorphic differentials $\omega_k=(c_{k1}\lambda+c_{k2})d\lambda/\mu$, $k=1,2$ normalized on (7.4.2), are equal to

$$\begin{aligned} c_{11} &= -(1/2\pi^2\kappa_2)\theta_2[\varrho_1], \quad c_{12}=(\kappa_1/2\pi^2\kappa_2^2)\theta_2[\varrho_2] \quad , \\ c_{21} &= (1/2\pi^2\kappa_2)\theta_1[\varrho_1], \quad c_{22}=-(\kappa_1/2\pi^2\kappa_2^2)\theta_1[\varrho_2] \quad , \end{aligned} \tag{7.4.5}$$

where

$$\begin{aligned} &[\varrho_1]=[A_3A_5A_6], \quad [\varrho_2]=[A_3] \quad , \\ &\kappa_1=\theta[\varepsilon_1]\theta[\varepsilon_2]\theta[\varepsilon_3], \quad \kappa_2=\theta[\delta_1]\theta[\delta_2]\theta[\delta_3] \quad , \end{aligned} \tag{7.4.6}$$

$$\kappa_1/\kappa_2=(\lambda_1\lambda_2\lambda_3)^{1/4}\equiv\kappa \quad , \tag{7.4.7}$$

and the sum of the characteristics in (7.4.4,6) is written multiplicatively.

Proof. We set

$$\lambda=\kappa\frac{\theta_1[\varrho_1]+\theta_2[\varrho_1]\widetilde{\lambda}}{\theta_1[\varrho_2]+\theta_2[\varrho_2]\widetilde{\lambda}} \quad , \tag{7.4.8}$$

where the quantity $\kappa=\kappa_1/\kappa_2$ and the characteristics $[\varrho_i]$ are determind by (7.4.6,7). By substituting (7.4.8) into (7.4.1), we obtain

$$\begin{aligned} \widetilde{\mu}^2 &= \kappa(\lambda\theta_2[\varrho_2]-\kappa\theta_2[\varrho_1])^{-6}D^2([\varrho_1],[\varrho_2]) \\ &\quad \times\lambda\prod_{j=3}^{6}(\lambda D([\varrho_j],[\varrho_2])-\kappa D([\varrho_j],[\varrho_1])) \quad , \end{aligned}$$

where $[\varrho_1],\ldots,[\varrho_6]$ are the six independent odd characteristics and

$$D([\varrho_i],[\varrho_j])=\theta_1[\varrho_i]\theta_2[\varrho_j]-\theta_1[\varrho_j]\theta_2[\varrho_i] \quad .$$

Using the Rosenhain formulas (2.6.4) (an extended version of these formulas is given in App. 7.1), we transform the last equation into the form (7.4.2), where λ_i, $i=1,2,3$ are determined by (7.4.3), and

$$\mu=-\frac{\pi^2}{2}\frac{\prod'''\theta^2[\varepsilon]}{\kappa_2}\frac{\widetilde{\mu}}{(\theta_1[\varrho_2]+\theta_2[\varrho_2]\widetilde{\lambda})^3}$$

with the product $\prod'''$ being expanded to four even characteristics unequal to the characteristics (7.4.4). Now, in line with Sect. 2.8, we write the following two representations in a fixed basis of homologies for the coordinate λ that has a second-order zero at the point Q_5 and a second-order pole at the point Q_6:

$$
\begin{aligned}
\lambda &= \frac{\theta^2[A_4A_5A_6]\theta^2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\int_{Q_1}^{Q}\omega; B)}{\theta^2[A_4]\theta^2[A_5A_6](\int_{Q_1}^{Q}\omega; B)} \\
&= \frac{\theta^2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\theta^2[A_4A_5A_6](\int_{Q_1}^{Q}\omega; B)}{\theta^2[A_5A_6]\theta^2[A_4](\int_{Q_1}^{Q}\omega; B)} \quad .
\end{aligned}
\tag{7.4.9}
$$

By substituting $Q = Q_3, Q_5$ into the first of equations (7.4.9), we obtain two formulas from equations (7.4.3). To avoid indeterminate forms in calculating λ_2, we substitute $Q = Q_4$ into the second of formulas (7.4.9) (Sect. 2.8) and obtain the characteristics indicated in (7.4.4).

To prove the propositions on holomorphic differentials, we note that (2.8.17) yield for the curve (7.4.1):

$$
\begin{aligned}
&\widetilde{c}_{12} = \widetilde{c}_{21} = \frac{\pi^2}{2}\left(\prod_{k=1}^{10}{}''\,\theta[\varepsilon_k]\right)^{1/2}\left(\prod_{l=1}^{6}{}'\,\theta_2[\delta_l]\right)^{-1/2} \quad , \\
&\widetilde{c}_{11} = \widetilde{c}_{22} = 0 \quad ,
\end{aligned}
\tag{7.4.10}
$$

i.e., the differentials $d\widetilde{\lambda}/\widetilde{\mu}$ and $\widetilde{\lambda}d\widetilde{\lambda}/\widetilde{\mu}$ are indeed normalized. We can use this result and the transformation (7.4.8), to find the normalization constants c_{ij} for the curve (7.4.2) in the form (7.4.5).

Remark 7.9. Lemma 7.8 is also proved for any other choice of the zero and pole of the function λ (the choice we made is due to the application of the sine-Gordon equation (Sect. 7.9).

Remark 7.10. The curve (7.4.1) has a symmetry under the transformation

$$
\left(\lambda; \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22}\end{pmatrix}\right) \to \left(\frac{1}{\lambda}; \begin{pmatrix} B_{22} & B_{12} \\ B_{12} & B_{11}\end{pmatrix}\right) \quad , \tag{7.4.11}
$$

which converts the canonical differentials $d\widetilde{\lambda}/\widetilde{\mu}$ and $\widetilde{\lambda}d\widetilde{\lambda}/\widetilde{\mu}$ into each other.

Remark 7.11. Equations (7.4.5-7) can be obtained by fixing the curve in the form (7.4.2) and determining some basis in $H_1(X_2, \mathbb{Z})$ on it. Then, for the divisor $\mathcal{D} = P_1 + P_2 - 2Q_1$, we write down (2.8.11)

$$
\begin{aligned}
&\sqrt{(\lambda^{(1)} - \lambda(Q_k))(\lambda^{(2)} - \lambda(Q_k))} \\
&\quad = h_k \frac{\theta[A_3A_5A_k](A(\mathcal{D}); B)}{\theta[A_3A_5A_6](A(\mathcal{D}), B)}, \quad k = 1, \dots, 5 \quad ,
\end{aligned}
\tag{7.4.12}
$$

where h_k are constants. By making $\lambda^{(1)}$ tend to $\lambda(Q_k)$ and $\lambda^{(2)}$ to $\lambda(Q_k)$, $k = 1,\ldots,5$ in (7.4.12), and removing indeterminate forms on the right-hand sides of the equations, we have the overdetermined set of equations

$$\begin{aligned}
c_{11}\theta_1[A_3A_5A_6] + c_{12}\theta_2[A_3A_5A_6] &= 0 \quad ,\\
c_{12}\theta_1[A_1A_3A_5] + c_{22}\theta_2[A_1A_3A_5] &= 0 \quad ,\\
(c_{11}+c_{12})\theta_1[A_2A_3A_5] + (c_{21}+c_{22})\theta_2[A_2A_3A_5] &= 0 \quad ,\\
(c_{11}\lambda_1+c_{12})\theta_1[A_3^2A_5] + (c_{21}\lambda_1+c_{22})\theta_2[A_3^2A_5] &= 0 \quad ,\\
(c_{11}\lambda_2+c_{12})\theta_1[A_3A_4A_5] + (c_{21}\lambda_2+c_{22})\theta_2[A_3A_4A_5] &= 0 \quad ,\\
(c_{11}\lambda_3+c_{12})\theta_1[A_3A_5^2] + (c_{21}\lambda_3+c_{22})\theta_2[A_3A_5^2] &= 0 \quad .
\end{aligned} \tag{7.4.13}$$

Relations (7.4.5) are obtained by solving these equations by the aid of (7.4.3) for the projections of the branch points λ_i, Rosenhain formulas (2.6.4) and Thomae formulas (2.8.15).

To conclude the description of curve (7.4.1), we give (2.8.13) as specialized for this case. For this purpose we write

$$Z[\varepsilon](\lambda) = \theta_1[\varepsilon] + \theta_2[\varepsilon]\lambda, \quad \mathcal{D} = P_1 + P_2 - Q_1 - Q_2 \quad ,$$

where $p_i = (\mu^{(i)}, \lambda^{(i)})$, $i = 1,2$ are two points in the general position and $Q_j = (0, E_j)$, $j = 1,2$ are two branch points. Then

$$\begin{aligned}
\frac{\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))} &= \frac{Z\begin{bmatrix}1&1\\0&1\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&1\\0&1\end{bmatrix}(\lambda^{(2)})}{Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(2)})} \quad ,\\
\frac{\theta^2\begin{bmatrix}1&0\\1&1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))} &= -\frac{Z\begin{bmatrix}0&1\\0&1\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}0&1\\0&1\end{bmatrix}(\lambda^{(2)})}{Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(2)})} \quad ,\\
\frac{\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))} &= -\frac{Z\begin{bmatrix}0&1\\1&1\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}0&1\\1&1\end{bmatrix}(\lambda^{(2)})}{Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(2)})} \quad ,\\
\frac{\theta^2\begin{bmatrix}0&1\\0&1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))} &= \frac{Z\begin{bmatrix}1&0\\1&1\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&0\\1&1\end{bmatrix}(\lambda^{(2)})}{Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(2)})} \quad ,\\
\frac{\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))} &= \frac{Z\begin{bmatrix}1&0\\1&0\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&0\\1&0\end{bmatrix}(\lambda^{(2)})}{Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(1)})Z\begin{bmatrix}1&1\\1&0\end{bmatrix}(\lambda^{(2)})} \quad .
\end{aligned} \tag{7.4.14}$$

To conclude this section, we discuss the condition under which the curve is non-degenerate.

Lemma 7.12. (see, e.g., *Dubrovin* [7.22]). Let B be a matrix of the periods of curve X_g of genus g, and $\hat{\theta}_{ij}\begin{bmatrix}\varepsilon'\\0\end{bmatrix}$, $\hat{\theta}\begin{bmatrix}\varepsilon'\\0\end{bmatrix}$ theta constants $\hat{\theta}_{ij}\begin{bmatrix}\varepsilon'\\0\end{bmatrix} = (\partial^2/\partial z_i \partial z_j)\theta\begin{bmatrix}\varepsilon'\\0\end{bmatrix}(z;2B)|_{z=0}$, $\hat{\theta}\begin{bmatrix}\varepsilon'\\0\end{bmatrix} = \theta\begin{bmatrix}\varepsilon'\\0\end{bmatrix}(0;2B)$. Then

$$\mathrm{rank}(\hat{\theta}_{ij}\begin{bmatrix}\varepsilon'\\0\end{bmatrix}, \hat{\theta}\begin{bmatrix}\varepsilon'\\0\end{bmatrix}) = \frac{g(g+1)}{2} + 1 \quad , \tag{7.4.15}$$

where $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_g)$ are all possible vectors with coordinates 0 or 1.

In the present case when $g = 2$, the matrix $(\hat{\theta}_{ij}\begin{bmatrix}\varepsilon'\\0\end{bmatrix}, \hat{\theta}\begin{bmatrix}\varepsilon'\\0\end{bmatrix})$ is a (4×4)-matrix, and the condition (7.4.15) was shown by *Sasaki* [7.23] to reduce by application of the Rosenhain formulas (2.6.4) to the condition for the modular form of weight 5 with respect to $\mathrm{Tr}(4, \mathbb{Z})$ to be non-vanishing

$$g_1(B) = \prod_{|\varepsilon|=1} \theta[\varepsilon] \neq 0 \quad . \tag{7.4.16}$$

The vanishing of each of 10 even theta constants determines the component of a set of "boundaries", $\partial\mathbb{F}_2$ of the variety $\mathbb{F}_2 = \mathcal{U}_2/\mathrm{Tr}(4, \mathbb{Z})$. Each of these components is associated with the coalescence of the branch points of the curve X_2. To describe $\partial\mathbb{F}_2$ explicitly, we consider the singular submanifold S_2,

$$S_2 = \mathcal{U}_1 \times \mathcal{U}_1 = \left\{ B \,\middle|\, \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \right\}$$

in $\mathcal{U}_2$. It is obvious that for even characteristics $[\varepsilon]$,

$$\theta\begin{bmatrix}\varepsilon'_1 & \varepsilon'_2\\ \varepsilon''_1 & \varepsilon''_2\end{bmatrix}(0;B) = \begin{cases} \theta\begin{bmatrix}\varepsilon'_1\\ \varepsilon''_1\end{bmatrix}(0;B_{11})\theta\begin{bmatrix}\varepsilon'_2\\ \varepsilon''_2\end{bmatrix}(0;B_{22}), & \text{if} \quad [\varepsilon] \neq \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \\ 0, & \text{if} \quad [\varepsilon] = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}. \end{cases}$$

Equations (7.4.3) for branch point projections imply that S_2 is associated with the coalescence of three branch points. By acting on S_2 with the group $\mathrm{Tr}(4, \mathbb{Z})$ such that the characteristic $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ goes into the other nine even characteristics, we get nine more components of the set $\partial\mathbb{F}_2$. The union of these ten components is the Humbert variety $\mathbb{H}_1$.

Taking into account the formula

$$\lim_{B_{22}\to-\infty} \theta[\varepsilon] = \begin{cases} \theta\begin{bmatrix}\varepsilon_1' \\ \varepsilon_1''\end{bmatrix}, & \text{if} \quad \varepsilon_2' = 0 \quad , \\ 0, & \text{if} \quad \varepsilon_2' = 1 \quad , \end{cases} \tag{7.4.17}$$

we can see that the even theta constant $\theta[\varepsilon]$ vanishes on the set

$$\mathbb{D}_2 = \{B | B_{11} \to -\infty \quad \text{or} \quad B_{22} \to -\infty\} \quad , \tag{7.4.18}$$

which corresponds to the coalescence of two branch points. It follows from (7.4.17), with the action of $\mathrm{Tr}(4, \mathbb{Z})$ taken into account, that every component of $\mathbb{D}_2$ [7.18].

By expanding $g_1(B)$ in a series in the neighbourhood of $\mathbb{H}_1$ and $\mathbb{D}_2$, we can see that the divisor of zeros of $g_1(B)$ is exactly $4\mathbb{D}_2 + 2\mathbb{H}_1$.

7.5 Reduction Scheme for Genus 2

In this section we describe the operators needed to reduce the theta functions and Abelian integrals on an N-sheeted covering of $X_2 = (\mu, \lambda)$ over a torus to Jacobian theta functions and elliptic integrals on a torus X_1.

7.5.1. Choice of a Basis. Our discussion is based on the formula(2.6.4). It is most useful, if the basis of homologies in $H_1(X_2, \mathbb{Z})$ is fixed so that the B-matrix has the form (7.3.1). We show the procedure of going over to this basis from any fixed one.

We fix the basis of homologies as indicated in Fig. 7.1. According to (2.8.2), the half-periods $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_6$ are then equal to

$$\begin{gathered} \boldsymbol{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad , \\ \boldsymbol{A}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{A}_5 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_6 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad . \end{gathered} \tag{7.5.1}$$

Consider the Humbert surface $\mathbb{H}_\Delta$ with $\Delta = N^2 > 1$. In the fixed homology basis we can propose a possible form of the matrix M as

$$M = \begin{pmatrix} N-1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \quad , \tag{7.5.2}$$

i.e.,

$$\pi(a_1, a_2; b_1, b_2) = ((N-1)\tilde{a}, -\tilde{a}; \tilde{b}, -\tilde{b}) \; , \quad (\tilde{a}, \tilde{b}) \in H_1(X_1, \mathbb{Z})$$

and the corresponding component of $\mathbb{H}_\Delta$ is found with the help of (7.3.6) as

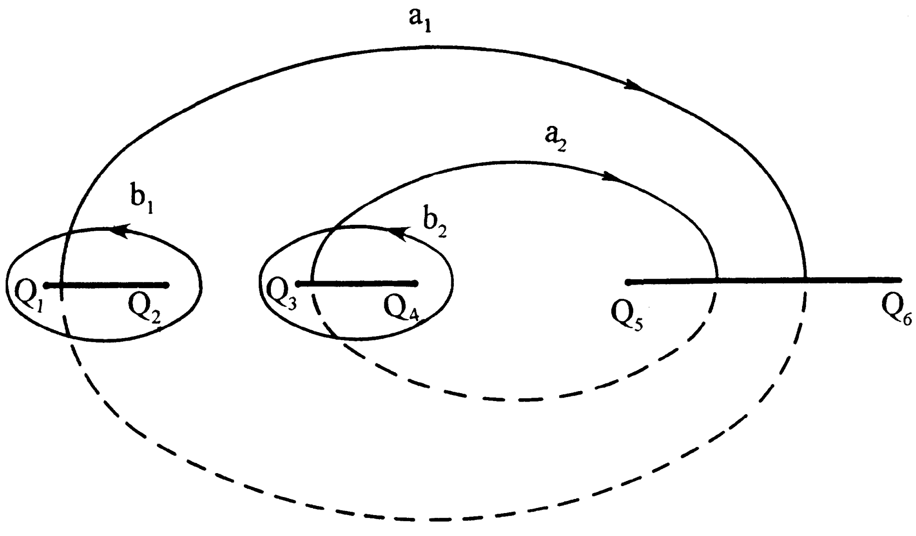

Fig. 7.1. Homology basis in $H_1(X_2, \mathbb{Z})$, where X_2 is a Riemann surface of genus two

$$(N-1)B_{11} + (N-2)B_{12} - B_{22} = 0 \quad . \tag{7.5.3}$$

To come to the homology basis for which B-matrix has the form (7.3.1) we have to find the transformation T such that (7.3.12) is valid. This transformation can be constructed with the help of the matrices (7.3.13,14). We find that in the considered case $T^T = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given as

$$T^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix} . \tag{7.5.4}$$

Under the action of $\sigma \in \mathrm{Tr}(4, \mathbb{Z})$ given by (7.5.4) the matrix B and basis characteristics (7.5.1) are transformed within the law of transformation for the theta function (2.5.12-14) to the following form

$$\begin{aligned} B^\# &= \begin{pmatrix} B^\#_{11} & 2\pi \mathrm{i}/N \\ 2\pi \mathrm{i}/N & B^\#_{22} \end{pmatrix} \quad , \\ B^\#_{11} &= \frac{1}{N}((N-1)B_{11} - B_{22}), \quad B^\#_{22} = \frac{1}{N(B_{11} + B_{22})} \quad , \end{aligned} \tag{7.5.5}$$

$$\begin{aligned} A^\#_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\#_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\#_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad , \\ A^\#_4 &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A^\#_5 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A^\#_6 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad . \end{aligned} \tag{7.5.6}$$

We write (7.4.3-6) in this basis

$$\lambda_1 = \frac{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}}{\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}, \quad \lambda_2 = \frac{\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}}{\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}},$$
$$\lambda_3 = \frac{\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}}{\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}, \tag{7.5.7}$$

$$c_{11} = \frac{\kappa\theta_2\begin{bmatrix}1&0\\1&1\end{bmatrix}}{2\pi^2\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}},$$
$$c_{12} = \frac{-\kappa\theta_2\begin{bmatrix}1&0\\1&0\end{bmatrix}}{2\pi^2\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}},$$
$$c_{21} = \frac{-\kappa\theta_1\begin{bmatrix}1&0\\1&1\end{bmatrix}}{2\pi^2\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}}, \tag{7.5.8}$$
$$c_{22} = \frac{\kappa\theta_1\begin{bmatrix}1&0\\1&0\end{bmatrix}}{2\pi^2\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}},$$
$$\kappa = \frac{\kappa_1}{\kappa_2} = \frac{\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}}{\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}}.$$

7.5.2. Reduction Condition. The conditions (7.3.15) that distinguish in $\mathcal{U}_2$ the Humbert variety $\mathbb{H}_{N^2}$, $N = 2, 3, \ldots$ are reduction conditions. In this subsection we show the procedure of reformulating these conditions in equivalent form – as conditions on theta constants and as conditions on branch point projections. To write the reduction condition as conditions on theta constants, we fix a homology basis in which the B-matrix has the form (7.3.1). The obvious equality

$$\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\left(0;\begin{pmatrix}NB_{11}&0\\0&NB_{22}\end{pmatrix}\right) = 0$$

and the formula (7.2.6) that has the form

$$\begin{aligned}&\theta\begin{bmatrix}\varepsilon_1' & \varepsilon_2' \\ \varepsilon_1'' & \varepsilon_2''\end{bmatrix}\left(z;\begin{pmatrix}B_{11} & 2\pi\mathrm{i} \\ 2\pi\mathrm{i} & B_{22}\end{pmatrix}\right)\\ &\quad=(-1)^{-\varepsilon_1'\varepsilon_2'/2}\theta\begin{bmatrix}\varepsilon_1' & \varepsilon_2' \\ \varepsilon_1''+\varepsilon_2' & \varepsilon_2''+\varepsilon_1'\end{bmatrix}\left(z;\begin{pmatrix}B_{11} & 0 \\ 0 & B_{22}\end{pmatrix}\right)\end{aligned} \tag{7.5.9}$$

for genus $g=2$ yield the condition

$$\theta\begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix}\left(0;\begin{pmatrix}NB_{11} & 2\pi\mathrm{i} \\ 2\pi\mathrm{i} & NB_{22}\end{pmatrix}\right)=0 \quad . \tag{7.5.10}$$

Making in (7.5.10) the Nth order transformation with the use of (2.6.4), we have the required relation between theta constants. This relation determines only one of the components of the Humbert variety $\mathbb{H}_{N^2}$. To get the rest of the components, it is necessary to act by the group $\mathrm{Tr}(4,\mathbb{Z})$ on the theta constants involved in the reduction condition. To derive the reduction condition in terms of branch point projections, we must use the Thomae formulas (2.8.15) that relate even theta constants to branch point projections.

7.5.3. Description of a Covering. To describe a covering, we must make up a list of the reduced theta constants contained in (7.5.5-8) and then write down these formulas in terms of Jacobian theta constants with moduli NB_{11} and NB_{22}.

7.5.4. Reduction of Holomorphic Differentials to Elliptic Ones. The above discussion shows that normalized holomorphic differentials reduce exactly to elliptic differentials. To describe the reduction of holomorphic differentials, it is sufficient to express the normalization constants c_{ij}, $i,j=1,2$ through Jacobian theta constants with moduli NB_{11} and NB_{22} with the help of the formulas derived in Sect. 7.4.

7.5.5. Description of Covers $\pi: X_2 \to X_1$, $\pi': X_2 \to X_1'$. In Sect. 7.1, we have given the algebraic derivation of a covering that belongs to Jacobi (see *Krazer* [7.2]). It allows a generalization (quite complicated, though) to the case of N-sheeted coverings, $N>2$. Attempting to produce a complete theory of reduction that uses theta functions, we give the derivation of covers π, π' based on Picard's idea used by *Bolza* [7.20] to describe 4-sheeted coverings. First of all we note that it is convenient to represent the covering as (7.4.1), because in that case the reduction made for one of the independent holomorphic differentials can be used to derive a reduction for another independent differential by employing the transformation (7.4.11).

We consider an N-sheeted covering of genus $g=2$ over a torus, realized as (7.4.1) and the divisor $\mathcal{D}=P_1+P_2-Q_1-Q_2$, where $P_i=(\widetilde{\mu}^{(i)},\widetilde{\lambda}^{(i)})$, $i=1,2$ are two points of the general position and $Q_j=(0,E_j)$, $j=1,2$ are two branch points. Let $\boldsymbol{A}(\mathcal{D})=\boldsymbol{\zeta}=(\zeta_1,\zeta_2)\in J(X_2)$, i.e.,

$$\zeta_1=\int_{Q_1}^{P_1}\frac{d\widetilde{\lambda}}{\widetilde{\mu}}+\int_{Q_2}^{P_2}\frac{d\widetilde{\lambda}}{\widetilde{\mu}},\quad \zeta_2=\int_{Q_1}^{P_1}\frac{\widetilde{\lambda}d\widetilde{\lambda}}{\widetilde{\mu}}+\int_{Q_2}^{P_2}\frac{\widetilde{\lambda}d\widetilde{\lambda}}{\widetilde{\mu}} \quad . \tag{7.5.11}$$

We consider the elliptic function ξ over the torus $X_1 = (\eta, \xi)$,

$$\eta^2 = \xi(1-\xi)(1-k^2\xi), \quad k = \frac{\theta^2\begin{bmatrix}1\\0\end{bmatrix}(0; NB_{11})}{\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0; NB_{11})},$$
$$\xi = \frac{\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0; NB_{11})\,\theta^2\begin{bmatrix}1\\1\end{bmatrix}(N\zeta_1; NB_{11})}{\theta^2\begin{bmatrix}1\\0\end{bmatrix}(0; NB_{11})\,\theta^2\begin{bmatrix}0\\1\end{bmatrix}(N\zeta_1; NB_{11})}. \tag{7.5.12}$$

The equality

$$2N\pi\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0; NB_{11})\,\zeta_1 = \int_0^{\xi}\frac{d\xi}{\sqrt{\xi(1-\xi)(1-k^2\xi)}},$$

and (7.5.11) yield an equation

$$\int_{Q_1}^{P_1}\frac{d\widetilde{\lambda}}{\widetilde{\mu}} + \int_{Q_2}^{P_2}\frac{d\widetilde{\lambda}}{\widetilde{\mu}} = \frac{1}{2N\pi\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0; NB_{11})}\int_0^{\xi}\frac{d\xi}{\sqrt{\xi(1-\xi)(1-k^2\xi)}}, \tag{7.5.13}$$

which defines ξ as an algebraic function of $\widetilde{\lambda}^{(1)}$ and $\widetilde{\lambda}^{(2)}$. We set

$$\widetilde{\lambda}^{(2)} = E_2 = -\frac{\theta_1\begin{bmatrix}1&1\\1&0\end{bmatrix}}{\theta_2\begin{bmatrix}1&1\\1&0\end{bmatrix}}. \tag{7.5.14}$$

Equation (7.5.13) then takes the form

$$\int_{Q_1}^{P_1}\frac{d\widetilde{\lambda}}{\widetilde{\mu}} = \frac{1}{2N\pi\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0; NB_{11})}\int_0^{\xi}\frac{d\xi}{\sqrt{\xi(1-\xi)(1-k^2\xi)}}, \tag{7.5.15}$$

where $Q_1 = (0, E_1)$ and

$$E_1 = -\frac{\theta_1\begin{bmatrix}0&1\\0&1\end{bmatrix}}{\theta_2\begin{bmatrix}0&1\\0&1\end{bmatrix}}.$$

To define ξ as a function of $\widetilde{\lambda}$, we make use of the identity

$$-\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(N\zeta_1;NB_{11})}{\theta\begin{bmatrix}0\\0\end{bmatrix}(N\zeta_1;NB_{11})}=\frac{\theta\begin{bmatrix}1&0\\1&1\end{bmatrix}\left(N\zeta;\begin{pmatrix}NB_{11}&2\pi\mathrm{i}\\2\pi\mathrm{i}&NB_{22}\end{pmatrix}\right)}{\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\left(N\zeta;\begin{pmatrix}NB_{11}&2\pi\mathrm{i}\\2\pi\mathrm{i}&NB_{22}\end{pmatrix}\right)}. \tag{7.5.16}$$

Using the Nth order transformations with the use of (2.6.6) on the right-hand side, we obtain a function that depends on relations between the theta function and the B-matrix such as (7.3.1). By imposing the conditions (7.5.14) and using (7.4.14), we can write down these relations as an Nth-order relational function of $\widetilde{\lambda}=\widetilde{\lambda}^{(1)}$. For reference, we list here (7.4.14) under the condition (7.5.14)

$$\theta\begin{bmatrix}0&1\\0&1\end{bmatrix}(A(\mathcal{D}))=0 \quad ,$$

$$\frac{\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))}=\frac{\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}}{\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}}\frac{\theta_1\begin{bmatrix}1&0\\1&0\end{bmatrix}+\theta_2\begin{bmatrix}1&0\\1&0\end{bmatrix}\widetilde{\lambda}}{\theta_1\begin{bmatrix}1&1\\0&1\end{bmatrix}+\theta_2\begin{bmatrix}1&1\\1&0\end{bmatrix}\widetilde{\lambda}},$$

$$\frac{\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))}=-\frac{\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}}\frac{\theta_1\begin{bmatrix}1&1\\0&1\end{bmatrix}+\theta_2\begin{bmatrix}1&1\\0&1\end{bmatrix}\widetilde{\lambda}}{\theta_1\begin{bmatrix}1&0\\1&1\end{bmatrix}+\theta_2\begin{bmatrix}1&0\\1&1\end{bmatrix}\widetilde{\lambda}},$$

$$\begin{aligned}\frac{\theta^2\begin{bmatrix}0&1\\1&1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}(A(\mathcal{D}))}&=-\frac{\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}}{\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}}\\&\times\frac{\left(\theta_1\begin{bmatrix}0&1\\1&1\end{bmatrix}+\theta_2\begin{bmatrix}0&1\\1&1\end{bmatrix}\widetilde{\lambda}\right)\left(\theta_1\begin{bmatrix}0&1\\0&1\end{bmatrix}+\theta_2\begin{bmatrix}0&1\\0&1\end{bmatrix}\widetilde{\lambda}\right)}{\left(\theta_1\begin{bmatrix}1&0\\1&1\end{bmatrix}+\theta_2\begin{bmatrix}1&0\\1&1\end{bmatrix}\widetilde{\lambda}\right)\left(\theta_1\begin{bmatrix}1&1\\1&0\end{bmatrix}+\theta_2\begin{bmatrix}1&1\\1&0\end{bmatrix}\widetilde{\lambda}\right)}\quad ,\end{aligned} \tag{7.5.17}$$

$$\begin{aligned}\frac{\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}(A(\mathcal{D}))}&=-\frac{\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}}\\&\times\frac{\left(\theta_1\begin{bmatrix}1&1\\1&0\end{bmatrix}+\theta_2\begin{bmatrix}1&1\\1&0\end{bmatrix}\widetilde{\lambda}\right)\left(\theta_1\begin{bmatrix}1&1\\0&1\end{bmatrix}+\theta_2\begin{bmatrix}1&1\\0&1\end{bmatrix}\widetilde{\lambda}\right)}{\left(\theta_1\begin{bmatrix}1&0\\1&1\end{bmatrix}+\theta_2\begin{bmatrix}1&0\\1&1\end{bmatrix}\widetilde{\lambda}\right)\left(\theta_1\begin{bmatrix}1&0\\1&0\end{bmatrix}+\theta_2\begin{bmatrix}1&0\\1&0\end{bmatrix}\widetilde{\lambda}\right)}\quad ,\end{aligned}$$

$$\frac{\theta^2\begin{bmatrix}0\,1\\1\,1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(A(\mathcal{D}))}=-\frac{\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}}{\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}}$$

$$\times\frac{\left(\theta_1\begin{bmatrix}0\,1\\0\,1\end{bmatrix}+\theta_2\begin{bmatrix}0\,1\\0\,1\end{bmatrix}\widetilde{\lambda}\right)\left(\theta_1\begin{bmatrix}0\,1\\1\,1\end{bmatrix}+\theta_2\begin{bmatrix}0\,1\\1\,1\end{bmatrix}\widetilde{\lambda}\right)}{\left(\theta_1\begin{bmatrix}1\,0\\1\,1\end{bmatrix}+\theta_2\begin{bmatrix}1\,0\\1\,1\end{bmatrix}\widetilde{\lambda}\right)\left(\theta_1\begin{bmatrix}1\,0\\1\,0\end{bmatrix}+\theta_2\begin{bmatrix}1\,0\\1\,0\end{bmatrix}\widetilde{\lambda}\right)} \quad .$$

7.5.6. Reduction of Abelian Differentials of the Second and Third Types. To reduce Abelian differentials of the second and third types to elliptic differentials with singularities, it is necessary to use (2.7.4,7) and the reductions of theta functions and holomorphic differentials derived above.

7.6 Example: 2- and 4-Sheeted Covers of Genus 2 over a Torus

7.6.1. 2-Sheeted Covers. We derive the reduction condition. For this purpose we use (2.6.8) to transform the condition (7.5.10)

$$\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\left(0;\begin{pmatrix}2B_{11} & 2\pi\mathrm{i}\\ 2\pi\mathrm{i} & 2B_{22}\end{pmatrix}\right)=0 \tag{7.6.1}$$

to the form

$$\theta^2\begin{bmatrix}1\,0\\0\,1\end{bmatrix}=\theta^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix} \quad . \tag{7.6.2}$$

Condition (7.6.2) determines one of the components of the Humbert variety $\mathbb{H}_4$. The variety $\mathbb{H}_4$ was described in modular forms by *Hirzebruch* and *van der Geer* [7.24], who proved that it involves 15 components. The vanishing of 15 differences $x_i - x_j$, $i,j=1,\ldots,6$ determines these components. More specifically,

$$\begin{aligned}&\mathbb{H}_4=\{B|g_4(B)=0\} \quad ,\\ &g_4(B)=g_1(B)\prod_{1\le i<j\le 6}(x_i-x_j)=\chi_{35} \quad ,\end{aligned} \tag{7.6.3}$$

where χ_{35} is the modular form of weight 35 with respect to $\mathrm{Tr}(4,\mathbb{Z})$, derived by *Igusa* [7.25].

We write the reduction conditions in terms of branch point projections. Using the rules described in Sect. 2.8, we define the basis (7.5.5) and assign the characteristics $\begin{bmatrix}1\,0\\0\,1\end{bmatrix}$ and $\begin{bmatrix}1\,0\\0\,0\end{bmatrix}$ that appear in the reduction condition (7.6.2) to the partitions of numbers $1,\ldots,6$ into two subsets $\{1,4,5\}\cup\{2,3,6\}$ and

$\{1,4,6\} \cup \{2,3,5\}$, respectively. Using the Thomae formulas (2.8.15), we rewrite (7.6.2) as "double quotients"

$$\frac{(E_5 - E_4)(E_5 - E_1)}{(E_5 - E_3)(E_5 - E_2)} = \frac{(E_6 - E_4)(E_6 - E_1)}{(E_6 - E_3)(E_6 - E_2)} \quad . \tag{7.6.4}$$

Condition (7.6.4) is just the reduction condition for the case $N = 2$. Going over to a curve with branches at infinity, realized as (7.4.2), we write down the condition (7.6.4) for this case

$$\lambda_2 = \lambda_1 \lambda_3 \quad . \tag{7.6.5}$$

Condition (7.6.5) also follows from (7.5.7,6.2).

We find the explicit form of a 2-sheeted covering. For this purpose we use (2.6.8) to make up a list of theta constants (Appendix 7.3), denoting by $\vartheta_j = \vartheta_j(0; B_{11}/\pi \mathrm{i})$, $\widetilde{\vartheta}_j = \vartheta_j(0; B_{22}/\pi \mathrm{i})$, $j = 2,3,4$, the Jacobian theta constants. By substituting reduced theta constants into (7.4.1), we have

$$\eta^2 \sqrt{k'^2 - \widetilde{k}^2} = 4\pi^2 \widetilde{\vartheta}_3^2 (1 + \zeta^2)(k + \widetilde{k}'\zeta^2)(\widetilde{k} + k'\zeta^2) \quad , \tag{7.6.6}$$

where $\zeta = \widetilde{\lambda}\vartheta_3^2/\widetilde{\vartheta}_3^2$ and

$$k = \frac{\vartheta_2^2}{\vartheta_3^2}, \quad k' = \frac{\vartheta_4^2}{\vartheta_3^2}, \quad \widetilde{k} = \frac{\widetilde{\vartheta}_2^2}{\widetilde{\vartheta}_3^2}, \quad \widetilde{k}' = \frac{\widetilde{\vartheta}_4^2}{\widetilde{\vartheta}_3^2} \quad .$$

From (7.5.15) we obtain the following formulas of reduction of holomorphic differentials

$$\begin{aligned} \frac{\sqrt{k'^2 - \widetilde{k}^2}\, d\zeta}{\sqrt{(1 + \zeta^2)(k + \widetilde{k}'\zeta^2)(\widetilde{k} + k'\zeta^2)}} &= \frac{1}{2} \frac{d\xi}{\sqrt{\xi(1 - \xi)(1 - k^2\xi)}} \quad , \\ \frac{\sqrt{k'^2 - \widetilde{k}^2}\, \zeta d\zeta}{\sqrt{(1 + \zeta^2)(k + \widetilde{k}'\zeta^2)(\widetilde{k} + k'\zeta^2)}} &= \frac{1}{2} \frac{d\xi'}{\sqrt{\xi'(1 - \xi')(1 - \widetilde{k}^2\xi')}} \quad . \end{aligned} \tag{7.6.7}$$

Explicit formulas for the covers π, π' are obvious for the curve (7.6.6).

7.6.2. 4-Sheeted Coverings [2.20]. We first derive a reduction condition for this case. Using the equation

$$\theta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(0; \begin{pmatrix} 4B_{11} & 2\pi \mathrm{i} \\ 2\pi \mathrm{i} & 4B_{22} \end{pmatrix} \right) = 0 \tag{7.6.8}$$

and twice applying (2.6.9) we have the required reduction condition

$$\theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \theta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \theta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \theta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0 \quad . \tag{7.6.9}$$

To write the condition (7.6.9) in terms of branch point projections, we make use of the Thomae formulas (2.8.15). In the fixed homology basis (7.4.7), we assign the characteristics $\begin{bmatrix} 0\,0 \\ 0\,0 \end{bmatrix}$, $\begin{bmatrix} 0\,0 \\ 1\,1 \end{bmatrix}$, $\begin{bmatrix} 0\,0 \\ 1\,0 \end{bmatrix}$, $\begin{bmatrix} 0\,0 \\ 0\,1 \end{bmatrix}$ to the partitions of numbers $1,\dots,6$ into two groups $\{1,3,5\} \cup \{2,4,6\}$; $\{1,3,6\} \cup \{3,4,5\}$; $\{1,2,5\} \cup \{3,4,6\}$, and $\{2,4,5\} \cup \{1,3,6\}$, respectively. We then use (7.6.8) and (2.8.15) to derive the reduction condition

$$\begin{aligned} &\sqrt[4]{(E_3 - E_1)(E_5 - E_1)(E_4 - E_2)(E_6 - E_2)(E_6 - E_4)(E_5 - E_3)} \\ &+ \sqrt[4]{(E_2 - E_1)(E_6 - E_2)(E_6 - E_1)(E_5 - E_3)(E_5 - E_4)(E_4 - E_3)} \\ =& \sqrt[4]{(E_2 - E_1)(E_5 - E_2)(E_5 - E_1)(E_4 - E_3)(E_6 - E_3)(E_6 - E_4)} \\ &+ \sqrt[4]{(E_5 - E_2)(E_5 - E_4)(E_4 - E_2)(E_3 - E_1)(E_6 - E_3)(E_6 - E_1)} \end{aligned} \tag{7.6.10}$$

if the curve X_2 has no branch points at infinity, and

$$\begin{aligned} &\sqrt[4]{(E_3 - E_1)(E_4 - E_2)}\{\sqrt[4]{(E_5 - E_1)(E_5 - E_3)} \\ &- \sqrt[4]{(E_5 - E_2)(E_5 - E_4)}\} \\ =& \sqrt[4]{(E_2 - E_1)(E_4 - E_3)}\{\sqrt[4]{(E_5 - E_1)(E_5 - E_2)} \\ &- \sqrt[4]{(E_5 - E_3)(E_5 - E_4)}\} \end{aligned} \tag{7.6.11}$$

if its branch point Q_6 is located at infinity.

We now describe a 4-sheeted covering. For this purpose we denote $\alpha = -\mathrm{i}\vartheta_3^2$, $\beta = \vartheta_2^2$, $\gamma = \vartheta_4^2$, $\alpha' = \mathrm{i}\widetilde{\vartheta}_3^2$, $\beta' = \widetilde{\vartheta}_4^2$, $\gamma' = \widetilde{\vartheta}_2^2$ where $\vartheta_j = \vartheta_j(0; 2B_{11}/\pi\mathrm{i})$, $\vartheta_j = \vartheta_j(0; 2B_{22}/\pi\mathrm{i})$, $j = 2, 3, 4$, and introduce three functions

$$\begin{aligned} F_1 &= (\beta^2 B - \gamma^2 C) - 2\mathrm{i}A\alpha\alpha'\widetilde{\lambda} - (\beta'^2 B - \gamma'^2 C)\widetilde{\lambda}^2 \quad , \\ F_2 &= (\gamma^2 C - \alpha^2 A) - 2\mathrm{i}B\beta\beta'\widetilde{\lambda} - (\gamma'^2 C - \alpha'^2 A)\widetilde{\lambda}^2 \quad , \\ F_3 &= (\alpha^2 A - \beta^2 B) - 2\mathrm{i}C\gamma\gamma'\widetilde{\lambda} - (\alpha'^2 A - \beta'^2 B)\widetilde{\lambda}^2 \quad , \end{aligned} \tag{7.6.12}$$

where $A = -\alpha\alpha' + \beta\beta' + \gamma\gamma'$, $B = \alpha\alpha' - \beta\beta' + \gamma\gamma'$, $C = \alpha\alpha' + \beta\beta' - \gamma\gamma'$. For the curve (7.4.1) we then have a representation

$$\widetilde{\mu}^2 = \frac{4\pi^2 F_1 F_2 F_3}{ABC(BC + CA + AB)} \quad , \tag{7.6.13}$$

and the normalized holomorphic differentials $d\widetilde{\lambda}/\widetilde{\mu}$, $\widetilde{\lambda}d\widetilde{\lambda}/\widetilde{\mu}$ are reduced to elliptic differentials by fourth-order rational substitutions. Let us find the substitutions. We denote $\widehat{\hat{\theta}}[\varepsilon](z) = \theta[\varepsilon](z; 4B)$, $\hat{\theta}[\varepsilon](z) = \theta[\varepsilon](z; 2B)$. The following formulas are valid:

$$\begin{aligned} 2\widehat{\hat{\theta}}\begin{bmatrix} 1\,1 \\ 1\,1 \end{bmatrix}\widehat{\hat{\theta}}\begin{bmatrix} 1\,0 \\ 1\,1 \end{bmatrix}(4\zeta) = &+\hat{\theta}\begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix} 0\,1 \\ 1\,1 \end{bmatrix}(2\zeta) \\ &- \hat{\theta}\begin{bmatrix} 0\,1 \\ 0\,1 \end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix} 0\,1 \\ 1\,0 \end{bmatrix}(2\zeta) \quad , \end{aligned} \tag{7.6.14}$$

$$2\hat{\hat{\theta}}\begin{bmatrix}0&1\\1&0\end{bmatrix}\hat{\theta}\begin{bmatrix}0&0\\1&0\end{bmatrix}(4\zeta) = +\hat{\theta}\begin{bmatrix}0&1\\0&1\end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix}0&1\\1&1\end{bmatrix}(2\zeta) + \hat{\theta}\begin{bmatrix}0&1\\1&0\end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix}0&1\\0&0\end{bmatrix}(2\zeta) \ ,$$

$$4\hat{\theta}\begin{bmatrix}0&0\\0&0\end{bmatrix}\hat{\theta}\begin{bmatrix}0&1\\0&0\end{bmatrix}(2\zeta) = -\theta^2\begin{bmatrix}0&1\\0&1\end{bmatrix}(\zeta) + \theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}(\zeta) - \theta^2\begin{bmatrix}0&1\\1&1\end{bmatrix}(\zeta) + \theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}(\zeta) \ ,$$

$$4\hat{\theta}\begin{bmatrix}0&0\\1&1\end{bmatrix}\hat{\theta}\begin{bmatrix}0&1\\0&0\end{bmatrix}(2\zeta) = +2\theta\begin{bmatrix}0&1\\0&1\end{bmatrix}(\zeta)\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}(\zeta) + 2\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}(\zeta)\theta\begin{bmatrix}0&1\\1&1\end{bmatrix}(\zeta) \ , \qquad (7.6.15)$$

$$4\hat{\theta}\begin{bmatrix}0&0\\0&1\end{bmatrix}\hat{\theta}\begin{bmatrix}0&1\\0&1\end{bmatrix}(2\zeta) = +2\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}(\zeta)\theta\begin{bmatrix}0&1\\0&1\end{bmatrix}(\zeta) + 2\theta\begin{bmatrix}0&1\\1&1\end{bmatrix}(\zeta)\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}(\zeta) \ ,$$

$$4\hat{\theta}\begin{bmatrix}0&0\\1&0\end{bmatrix}\hat{\theta}\begin{bmatrix}0&1\\1&0\end{bmatrix}(2\zeta) = +2\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}(\zeta)\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}(\zeta) - 2\theta\begin{bmatrix}0&1\\0&1\end{bmatrix}(\zeta)\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}(\zeta) \ .$$

Using (7.5.6, 10) for $N = 4$, we write the coordinate ξ over the torus

$$\xi = \frac{1}{k^2}\frac{\theta^2\begin{bmatrix}1&0\\1&1\end{bmatrix}\left(4\zeta;\begin{pmatrix}4B_{11} & 2\pi i\\ 2\pi i & 4B_{22}\end{pmatrix}\right)}{\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}\left(4\zeta;\begin{pmatrix}4B_{11} & 2\pi i\\ 2\pi i & 4B_{22}\end{pmatrix}\right)} \ . \qquad (7.6.16)$$

We now use (7.6.14) and the equations $\hat{\hat{\theta}}\begin{bmatrix}1&1\\1&1\end{bmatrix} = -i\vartheta_2\tilde{\vartheta}_2$, $\hat{\hat{\theta}}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \vartheta_3\tilde{\vartheta}_2$, to obtain the following equation from (7.6.16):

$$\xi = -\frac{1}{k^2}\left\{\frac{\hat{\theta}\begin{bmatrix}0&1\\0&0\end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix}0&1\\1&1\end{bmatrix}(2\zeta) - \hat{\theta}\begin{bmatrix}0&1\\0&1\end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix}0&1\\1&0\end{bmatrix}(2\zeta)}{\hat{\theta}\begin{bmatrix}0&1\\0&1\end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix}0&1\\1&1\end{bmatrix}(2\zeta) + \hat{\theta}\begin{bmatrix}0&1\\1&0\end{bmatrix}(2\zeta)\hat{\theta}\begin{bmatrix}0&1\\0&0\end{bmatrix}(2\zeta)}\right\}^2 . \qquad (7.6.17)$$

Imposing the condition (7.6.8), we use (7.6.15) in (7.6.17). Thus we have

$$\xi = \frac{1}{k^2}\frac{\theta^2\begin{bmatrix}0&1\\1&1\end{bmatrix}(\zeta)}{\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}(\zeta)}$$

$$\times\left\{\frac{\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\Theta(\zeta)-2\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}(\zeta)}{\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\Theta(\zeta)+2\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&1\end{bmatrix}(\zeta)}\right\}^2 \tag{7.6.18}$$

where

$$\Theta(\zeta)=\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}-\theta^2\begin{bmatrix}0&1\\1&1\end{bmatrix}+\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix} \quad .$$

From (7.5.17) we find that

$$\frac{\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}(A(\mathcal{D}))}=\frac{CF_1}{AF_3},\qquad \frac{\theta^2\begin{bmatrix}0&1\\1&1\end{bmatrix}(A(\mathcal{D}))}{\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}(A(\mathcal{D}))}=\frac{CF_2}{BF_3}$$

such that (7.6.18) transforms to

$$\xi=\frac{1}{k^2}\frac{A}{B}\frac{F_2}{F_1} \tag{7.6.19}$$

$$\times\left\{\frac{\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(BCF_1+ACF_2+ABF_3)-2\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}BCF_1}{\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(BCF_1+ACF_2+ABF_3)-2\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}ACF_2}\right\}^2$$

We further note that

$$\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}=\beta'^2\gamma^2-\beta^2\gamma'^2=\gamma'^2\alpha^2-\gamma^2\alpha'^2$$
$$=\alpha'^2\beta^2-\alpha^2\beta'^2 \quad ,$$

$$BCF_1+CAF_2+ABF_3=2(\beta'^2\gamma-\beta^2\gamma'^2)F_4 \quad ,$$

where

$$F_4=(\alpha^3\alpha'+\beta^3\beta'+\gamma^3\gamma')+\mathrm{i}(\gamma^2\beta'^2-\gamma'^2\beta^2)\widetilde{\lambda}$$
$$+(\alpha\alpha'^3+\beta\beta'^3+\gamma\gamma'^3)\widetilde{\lambda}^2 \quad .$$

Equation (7.6.19), therefore, takes the form

$$\xi=-\frac{1}{k^2}\frac{F_2}{F_1}\left\{\frac{F_4-F_1}{F_4+F_2}\right\}^2 \quad . \tag{7.6.20}$$

Now, to complete the derivation of the formula, we have to find a common factor for the quadratic trinomials $F_4 - F_1$ and $F_4 + F_2$. Indeed, the roots of the equation $F_4 - F_1 = 0$ are $\mathrm{i}(\alpha^2 A - \beta^2 B)/C\gamma\gamma'$ and $\gamma B\beta\beta'/(\gamma'^2 C - \alpha'^2 A)$, and the roots of the equation $F_4 + F_2 = 0$ are $\mathrm{i}A\alpha\alpha'/(\beta'^2 B - \gamma'^2 C)$ and $\mathrm{i}(\alpha^2 A - \beta^2 B)/C\gamma\gamma'$. So, we have proved the validity of the reduction formula

$$
\begin{aligned}
\frac{d\widetilde{\lambda}}{\widetilde{\mu}} &= \sqrt{ABC(BC + AC + AB)}\frac{d\lambda}{\sqrt{F_1 F_2 F_3}} \\
&= \frac{\mathrm{i}}{4}\frac{d\xi}{\sqrt{\xi(1-\xi)(\alpha^2 + \beta^2\xi)}} \quad ,
\end{aligned}
\tag{7.6.21}
$$

$$
\begin{aligned}
\xi =& K\frac{[(\gamma'^2 C - \alpha'^2 A)\widetilde{\lambda}^2 + 2\mathrm{i}\beta\beta' B\widetilde{\lambda} - (\gamma^2 C - \alpha^2 A)]}{[(\beta'^2 B - \gamma'^2 C)\widetilde{\lambda}^2 + 2\mathrm{i}\alpha\alpha' A\widetilde{\lambda} - (\beta^2 B - \gamma^2 C)]} \\
&\times \left(\frac{(\gamma'^2 C - \alpha'^2 A)\widetilde{\lambda} - \mathrm{i}\beta\beta' B}{(\beta'^2 B - \gamma'^2 C)\widetilde{\lambda} - \mathrm{i}\alpha\alpha' A}\right)^2 \quad ,
\end{aligned}
\tag{7.6.22}
$$

$$
K = \frac{\alpha^4\alpha'^2 A^2(\gamma^2 C - \alpha^2 A - 3\beta^2 B)^2}{\beta^4\beta'^2 B^2(\gamma^2 C - \beta^2 B - 3\alpha^2 A)^2} \quad .
$$

To derive a reduction formula for the second independent holomorphic differential and a formula for the coordinate ξ' of the cover $\pi' : X_2 \to X_1'$, similar to (7.6.22), we must replace $(\alpha, \beta, \gamma) \to (\alpha', \beta', \gamma')$, $\widetilde{\lambda} \to -1/\widetilde{\lambda}$ in (7.6.21, 22).

7.7 Elliptic Finite-Gap Potentials

It is established (see for example Sect. 3.6) that the spectrum of the Schrödinger operator $\mathcal{H} = d^2/dx^2 - u(x)$, with a periodic Lamé potential

$$
u(x) = g(g+1)\wp(x), \quad g \in N \quad ,
\tag{7.7.1}
$$

where $\wp$ is the elliptic Weierstrass function, consists of g gaps of an absolutely continuous spectrum. The Lamé potentials were studied by *Hermite* [7.26], *Halphen* [7.27], *Ince* [7.28] and others. *Treibich* and *Verdier* [7.29] have recently formulated a geometric approach to describe elliptic finite-gap potentials. Thus they found a new series of such potentials different from the potentials (7.7.1).

On the other hand, *Its* and *Matveev* [7.30] have derived a general formula that expresses the arbitrary g-gap potential through a meromorphic function on the Jacobian $J(X_g)$ of the hyperelliptic curve X_g of genus g. Here again the question arises (similar to that stated in the introduction to this Chapter): under what conditions on the moduli of the hyperelliptic curve X_g does the general finite-gap potential [7.30], which is a quasi-periodic function of x reduce to the periodic elliptic function x?

In this section we give an answer to this question within the format of the Weierstrass reduction theory developed above. This theory enables us to use the general theta function formula of Sect. 3 to derive not only the Lamé potential, but also a new series of Treibich-Verdier potentials.

We consider a nonsingular hyperelliptic curve X_g of genus g

$$\mu^2 = \prod_{k=1}^{2g+1}(\lambda - E_k) \quad , \tag{7.7.2}$$

where $E_k = \pi(Q_k)$, $k = 1,\dots,2g+1$, $k \neq l$, k, $l = 1,\dots,2g+1$, $\pi(Q_{2g+2}) = \infty$, and define the basis $(a,b) \in H_1(X_g,\mathbb{Z})$. According to [7.30], the g-gap potential $u(x,t)$ of a one-dimensional Schrödinger equation, which is associated with the curve (7.7.2), is defined by the formula

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\theta(\boldsymbol{U}x + \boldsymbol{V}t + \boldsymbol{A}(D) - \boldsymbol{K}; B) + C \tag{7.7.3}$$

in which t is the isospectral deformation parameter or the time in the Korteweg - de Vries equation

$$u_t = 6uu_x - u_{xxx} \quad , \tag{7.7.4}$$

$\boldsymbol{U}, \boldsymbol{V} \in \mathbb{C}^g$ are the "winding vectors" defined by

$$U_j = -2\mathrm{i}c_{1j}, \quad V_j = 8\mathrm{i}(c_{2j} + \frac{1}{2}c_{1j}\sum_{k=1}^{2g+1} E_k), \quad j = 1,\dots,g \quad , \tag{7.7.5}$$

where the c_{ij}, $i,j = 1,\dots,g$ are the normalization constants of holomorphic differentials $\boldsymbol{\omega} = (\omega_1,\dots,\omega_g)$, $\mathcal{D} = P_1 + \dots + P_g$ is a non-special divisor, $\boldsymbol{K}$ is the vector of the Riemann constants amd the constant C is equal to

$$C = \sum_{i=1}^{g}\oint_{a_i}\lambda\omega_i \quad . \tag{7.7.6}$$

Definition. *The g-gap potential $u(x)$, associated with the curve X_g, will be called N-elliptic, if the curve X_g is an N-sheeted cover of a torus.*

Theorem 7.13. The g-gap N-elliptic potential associated with the curve X_g equipped with a homology basis in which the matrix B has the form (7.2.2), is defined by the formula

$$u(x,t) = -\frac{2}{N}\frac{\partial^2}{\partial x^2}\log\sum_{p_N} C_{p_N}\theta\begin{bmatrix}2p_{N1}\\ 2kp_{N2}\end{bmatrix}(NU_1x + NV_1t + NW_1; BN) \tag{7.7.7}$$

$$\times\,\theta\begin{bmatrix}2p_{N2} & 2p_{N3} & \dots 2p_{Ng}\\ 2kp_{N1} & 0 & \dots 0\end{bmatrix}(N\hat{\boldsymbol{U}}x + N\hat{\boldsymbol{V}}t + N\hat{\boldsymbol{W}}; N\hat{B}) + C \quad ,$$

in which the summation is over the whole set of representatives $(1/N)\mathbb{Z}^g$, $\boldsymbol{U} = (U_1,\hat{\boldsymbol{U}})$, $\boldsymbol{V} = (V_1,\hat{\boldsymbol{V}})$, $\boldsymbol{W} = (W_1,\hat{\boldsymbol{W}}) = \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K}$, is a non-special divisor on X_g; $\hat{B}$ is a matrix of degree $g-1$, formed from B by crossing out the first line, equal to $(B_{11}, 2\pi i k/N, 0, \dots, 0)$ and the first column; the constants $C p_N$ are defined by (7.2.5).

The proof follows immediately from (7.7.3, 2.4, 5).

From (7.7.7) it follows that the N-elliptic potential is a quasi-periodic function of x. The following theorem enables one to describe elliptic periodic potentials.

Theorem 7.14. For the g-gap potential to be an elliptic periodic potential on the torus X_1, the following conditions are necessary and sufficient:

(i) the hyperelliptic curve X_g associated with the potential $u(x)$ is non-singular and is an N-sheeted cover of the torus X_1;

(ii) the following equations

$$U_2 = U_3 = \cdots = U_g = 0. \tag{7.7.8}$$

are solvable for hyperelliptic points in the interior of the set $\mathcal{U}_g$ in a homology basis in $H_1(X_g, \mathbb{Z})$ such that the B-martix has the form (7.2.2).

Proof. A sufficiency condition. We choose a basis in $H_1(X_g, \mathbb{Z})$ such that the B-matrix has the form (7.2.2). Under the conditions (7.7.8) the g-gap N-elliptic potential $u(x,t)$ is written as

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \times \tag{7.7.9}$$

$$\log\theta\left(U_1 x + V_1 t + W_1, \boldsymbol{V} t + \hat{\boldsymbol{W}}; \begin{pmatrix} B_{11} & 2\pi i k/N & 0\dots 0 \\ 2\pi i k/N & & \\ 0 & & \\ \vdots & \hat{B} & \\ 0 & & \end{pmatrix} \right)$$

where $\boldsymbol{U} = (U_1, \hat{\boldsymbol{U}})$, $\boldsymbol{V} = (V_1, \hat{\boldsymbol{V}})$, $\boldsymbol{W} = (W_1, \hat{\boldsymbol{W}})$, and $\hat{B} \in \mathcal{U}_{g-1}$. We denote $U_1 = \pi i \omega$ and define the half-period ω' such that $\omega'/\omega = N B_{11}/2\pi i$. The transformational properties of the theta function (2.5.10) imply that $u(x+2\omega, t) = u(x,t)$, $u(x+2\omega', t) = u(x,t)$, i.e., $u(x,t)$ is a doubly periodic function of x.

A necessary condition. We suppose that the g-gap potential given by (7.7.3) is elliptic in x with periods 2ω, $2\omega'$, $\mathrm{Im}\frac{\omega'}{\omega} > 0$. Then it follows from (7.7.3) and transformational properties of theta functions (2.5.10) that the following set of $2g$ equations has to be valid

$$\begin{aligned} 2\omega U_j &= N_j + \sum_{i=1}^{g} M_i B_{ij}, j = 1, \dots, g, \\ 2\omega' U_k &= N'_j + \sum_{i=1}^{g} M'_i B_{ij}, k = 1, \dots, g, \\ &N_i, N'_k, M_i, M'_i \in \mathbb{Z} \end{aligned} \tag{7.7.10}$$

The compatibility condition for (7.7.10) has the form

$$\operatorname{rank}\begin{pmatrix} f_1 & 0\dots & 0 & 1 & 0\\ 0 & f_2\dots & 0 & 1 & 0\\ \vdots & \ddots & \vdots & \vdots & \\ 0 & \dots & f_g & 1 & 0\\ f_1' & 0\dots & 0 & 0 & 1\\ 0 & f_2'\dots & 0 & 0 & 1\\ \vdots & \ddots & \vdots & \vdots & \\ 0 & \dots & f_g' & 0 & 1 \end{pmatrix} \le g+2, \tag{7.7.11}$$

where $f_i = N_i + \sum_{k=1}^{g} M_k B_{ki}, f_i' = N_i' + \sum_{k=1}^{g} M_k' B_{ki}, i = 1,\dots,g$, is equivalent to the compatibility condition for (7.2.10),

$$\operatorname{rank}\begin{pmatrix} 1 & 0 & \dots & 0 & M_{11} & M_{12}\\ 0 & 1 & \dots & 0 & M_{21} & M_{22}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & M_{g-1,1} & M_{g-1,2}\\ B_{11} & B_{12} & \dots & B_{1g} & M_{g1} & M_{g2}\\ B_{21} & B_{22} & \dots & B_{2g} & M_{g+1,1} & M_{g+1,2}\\ \vdots & \vdots & \ddots & \vdots & \dots & \dots\\ B_{g1} & B_{g2} & \dots & B_{gg} & M_{2g,1} & M_{2g,2} \end{pmatrix} \le g+2 \quad , \tag{7.7.12}$$

with $g_1 = g$, $g_0 = 1$ and the matrix M of the form

$$M = \begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\\ \vdots & \vdots\\ M_{g-1,1} & M_{g-1,2}\\ M_{g1} & M_{g2}\\ M_{g+1,1} & M_{g+1,2}\\ \vdots & \vdots\\ M_{2g,1} & M_{2g,2} \end{pmatrix} = \begin{pmatrix} M_1 & -M_1'\\ M_2 & -M_2'\\ \vdots & \vdots\\ M_g & -M_g'\\ -N_1 & N_1'\\ \vdots & \vdots\\ -N_g & N_g' \end{pmatrix}, \tag{7.7.13}$$

Then according to Theorem 7.4 the curve X_g covers N-sheetedly a torus X_1, where the number of sheets N equals to

$$N = \sum_{k=1}^{g} \left(M_k N_k' - M_k' N_k\right). \tag{7.7.14}$$

Therefore the condition i) of the theorem is valid.

Following Theorem 7.2 let us transform the B-matrix into the form (7.7.2).The matrix (7.7.13) is transformed to the form

$$M = \begin{pmatrix} N & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \tag{7.7.15}$$

Returning to (7.7.10) we find that condition ii) is valid in the new basis. The theorem is proved.

Corollary 7.15. The N-elliptic periodic potential (7.7.9) has N poles $x_1(t), \dots, x_N(t)$ on the torus $X_1 = \mathbb{C}/\omega\mathbb{Z} \oplus \omega'\mathbb{Z}$, where $\omega = \pi\mathrm{i}/U_1$, $\omega' = NB_{11}/2U_1$ and is thus representable as

$$u(x,t) = 2\sum_{i=1}^{N} \wp(x - x_j(t)) \quad , \tag{7.7.16}$$

where $\wp$ is an elliptic Weierstrass function on the torus X_1.

Proof. We consider the function

$$f(x,t) = \tag{7.7.17}$$

$$\frac{\theta\left(U_1x + V_1t + W_1, \boldsymbol{V}t + \boldsymbol{W}; \begin{pmatrix} B_{11} & 2\pi\mathrm{i}k/N & 0\dots0 \\ 2\pi\mathrm{i}k/N & & \\ 0 & & \\ \vdots & \hat{B} & \\ 0 & & \end{pmatrix}\right)}{\vartheta_3\left(\frac{U_1x + V_1t + W_1}{2\pi\mathrm{i}}; \frac{B_{11}}{2\pi\mathrm{i}}\right)} .$$

The function (7.7.17) is nonconstant and meromorphic on the torus $X_1 = \mathbb{C}/\omega\mathbb{Z}\oplus \omega'\mathbb{Z}$, where $\omega = \pi\mathrm{i}/U_1$, $\omega' = NB_{11}/2U_1$. This is easily seen from (2.5.10); a shift by 2ω leaves (7.7.17) unchanged, while a shift by $2\omega'$ results in the factor $\exp(-(1/2)B_{11} - (U_1x + V_1t + W_1))$ appearing in the numerator and the denominator. The zeros of the function (7.7.17) on the torus X_1 are known, because they are zeros of the Jacobian theta function ϑ_3 : $\frac{(U_1x+V_1t+W_1)}{2\pi\mathrm{i}} = \frac{1}{2} + \frac{(2k+1)}{2}\frac{B_{11}}{2\pi\mathrm{i}}$, $k = 0, \dots, N-1$. Therefore, the theta function has in its numerator exactly N zeros on X_1, and consequently the potential $u(x,t)$ has N poles $x_1(t), \dots, x_N(t)$. Using the formula (7.7.9) and the expansion of the Weierstrass $\wp$-function near the pole, $\wp(x) = 1/x^2 + o(x^2)$, we derive (7.7.10).

Corollary 7.16. The N functions $x_j(t)$, $j = 1, \dots, N$ are roots of the equation

$$\sum_{m,n}^{N-1} A_{mn}(t)\theta\begin{bmatrix}2m/N\\2kn/N\end{bmatrix}(NU_1x+NV_1t+NW_1;NB_{11})=0 \quad , \tag{7.7.18}$$

with the functions $A_{mn}(t)$ equal to

$$\begin{aligned}A_{mn}(t) = &\sum_{p_{N3},\dots,p_{Ng}} C_{\{m/N,n/N,p_{N3},\dots,p_{Ng}\}}\\ &\times\theta\begin{bmatrix}2kn/N & p_{N3},\dots,0\\ 2m/N & 0,\dots,0\end{bmatrix}(N\hat{V}t+N\hat{W};N\hat{B}) \quad ,\end{aligned} \tag{7.7.19}$$

where the constants $C_{\{.\}}$ are defined by equations (7.2.5), and the vectors $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}=\boldsymbol{A}(\mathcal{D})+\boldsymbol{K}$ and the matrix $\hat{B}$ are defined by stating Theorem 7.13.

Proof. Equality $\theta=0$, where θ is the theta function in the numerator of (7.7.17), is valid strictly for $x_1(t),\dots,x_N(t)$. Equation $\theta=0$, is changed to the equality $\theta=0$ by applying the Koizumi addition theorem (2.6.6).

We now consider a 2-gap N-elliptic potential $u(x,t)$, associated with a non-degenerate curve X_2 equipped with a homology basis in $H_1(X_2,\mathbb{Z})$ such that $B_{12}=2\pi\mathrm{i}/N$. According to Theorem 7.14 and formulas (7.7.5,5.7), in order for the potential $u(x,t)$ to be an elliptic function of x, it is sufficient that the modular equation

$$\theta_1\begin{bmatrix}1&0\\1&1\end{bmatrix}\left(0;\begin{pmatrix}B_{11} & 2\pi\mathrm{i}/N\\ 2\pi\mathrm{i}/N & B_{22}\end{pmatrix}\right)=0 \tag{7.7.20}$$

to be solvable in the interior of U_2. We denote by $\vartheta_i=\vartheta_i\left(0;\frac{NB_{11}}{2\pi\mathrm{i}}\right)$ and by $\widetilde{\vartheta}_i=\vartheta_i\left(0;\frac{NB_{22}}{2\pi\mathrm{i}}\right)$, $i=2,3,4$ the Jacobian theta constants.

Example. $g=2$, $N=2$. For this case (7.7.20) can be reduced by (2.6.8) (App. 7.3) to the form

$$\vartheta_4^2\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\left(0;\begin{pmatrix}B_{11} & \pi\mathrm{i}\\ \pi\mathrm{i} & B_{22}\end{pmatrix}\right)=0 \quad .$$

This equation may be satisfied only at the boundary of moduli space and consequently, periodic 2-gap 2-elliptic potentials such as (7.7.9) do not exist.

Example. $g=2$, $N=4$. In this case the condition (7.7.20) can be reduced by (2.4.8) (Appendix 7.3) to the equation

$$\vartheta_4^2(\vartheta_3^2\widetilde{\vartheta}_3^2+\vartheta_2^2\widetilde{\vartheta}_4^2-\vartheta_4^2\widetilde{\vartheta}_2^2)=2\vartheta_3^4\widetilde{\vartheta}_2^2 \quad ,$$

which implies that

$$\widetilde{k}=k'(1-4k^2), \quad \text{or equivalently,} \quad \widetilde{k}'=k(1-4k'^2) \quad , \tag{7.7.21}$$

where k and $\widetilde{k}$ are Jacobian moduli, $k = \mathrm{i}\vartheta_2^2/\vartheta_2^2$, $\widetilde{k} = \widetilde{\vartheta}_3^2/\widetilde{\vartheta}_2^2$. Thus, for $k \neq 0, 1 \pm 1/2$ the condition (7.7.8) is valid. According to Corollary (7.15), the appropriate 2-gap potential must have four poles. Let $\mathcal{D} = P_1 + P_2 - 2Q_6$ and $P_1 = (0, E_1)$, $P_2 = (0, E_6)$. Then, $[\boldsymbol{A}(\mathcal{D}) + \boldsymbol{K}] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in the basis (7.5.5) and the potential is defined by

$$u(x,0) = -2\frac{\partial^2}{\partial x^2}\log\theta\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\left(\frac{\pi \mathrm{i} x}{\omega}, 0; \begin{pmatrix} B_{11} & \pi\mathrm{i}/2 \\ \pi\mathrm{i}/2 & B_{22} \end{pmatrix}\right) + C \quad . \tag{7.7.22}$$

The function $\theta\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}(\frac{\pi \mathrm{i} x}{\omega}, 0; B)$ has a third-order zero at $x = 0$ (this is verified by expansion in powers of x, with the condition (7.7.20), written for $N = 4$, taken into account) and a single zero at $x = \omega'$

$$\begin{aligned} \theta\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}(\pi\mathrm{i}\omega'/\omega, 0; B) &= \theta\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}(2B_{11}, 0; B) \\ &= -\theta\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}(2B_{11}, \pi\mathrm{i}; B) = -\theta\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}(2B_{11}, 2B_{12}; B) \\ &= -\exp(-2B_{11})\theta\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}(0; B) = 0 \quad . \end{aligned}$$

Therefore, for the potential (7.7.9) the representation (7.7.16) takes the form $u(x,0) = 6\wp(x) + 2\wp(x - \omega')$, and consequently, $u(x,0)$ coincides with one of the *Treibich-Verdier potentials* [7.29]

$$u(x,0) = 6\wp(x) + 2[\wp(x - \omega_i) - e_i], \quad i = 1,2,3 \quad . \tag{7.7.23}$$

We find the curve X_2, corresponding to the potentials (7.7.23). For this purpose we fix the homology basis (7.5.5) and calculate the branch point projections λ_1, λ_2, λ_3 with the use of formulas (7.5.6) under the conditions of (7.7.21) and $B_{12} = \pi\mathrm{i}/2$. Using the table of theta constants for a 4-sheeted covering (Appendix 7.3), we conclude that the curve $X_2 = (\mu, \lambda)$ is realized as

$$\mu^2 = \lambda(\lambda - 1)(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \quad , \tag{7.7.24}$$

$$\begin{aligned} &\lambda_1 = \left(\frac{k' + \sqrt{1 - 4k^2}}{k - \sqrt{1 - 4k^2}}\right)^2, \quad \lambda_2 = \left(\frac{k + \sqrt{1 - 4k'^2}}{k - \sqrt{1 - 4k^2}}\right)^2 \quad , \\ &\lambda_3 = \left(\frac{k - \sqrt{1 - 4k'^2}}{k - \sqrt{1 - 4k^2}}\right)^2 \quad , \end{aligned} \tag{7.7.25}$$

where $k^2 = (e_2 - e_3)/(e_1 - e_3)$, $k'^2 = (e_1 - e_2)/(e_2 - e_3)$, $\wp(\omega_j) = e_j$, $j = 1,2,3$. The curve (7.7.24) may also be realized as

$$\mu^2 = (\lambda - E_1)(\lambda - E_2)(\lambda - E_3)(\lambda - E_4)(\lambda - E_5) \quad , \tag{7.7.26}$$

where

$$\begin{aligned} E_1 &= -6e_i \quad , \\ E_{2,3} &= -2e_j - e_k \pm 2\sqrt{(2e_k + e_j)(7e_j + 5e_k)} \quad , \\ E_{4,5} &= -2e_k - e_j \pm 2\sqrt{(2e_j + e_k)(7e_k + 5e_j)} \quad , \end{aligned} \tag{7.7.27}$$

where $i \neq j \neq k = 1, 2, 3$, and each of the curves (7.7.26, 27) corresponds to ith potential (7.7.23).

Remark 7.17. We have given the derivation of the curve (7.7.18, 19), based entirely on the theory of theta functions. We now outline another approach. Every 2-gap potential satisfies the *Novikov* equation [7.31],

$$\frac{\delta}{\delta u}(I_2 + c_1 I_1 + c_0 I_0 + c_{-1} I_{-1}) = 0 \quad , \tag{7.7.28}$$

where $I_{-1} = \overline{u}$, $I_0 = \overline{u^2}$, $I_1 = 2\overline{u^3} + \overline{u'^2}$, $I_2 = 5\overline{u^{(4)}} + 10\overline{uu'} + \overline{u''^2}$ are the integrals of motion of the KdV equation, $u_t = 6uu_x - u_{xxx}$, and the bar stands for the averaging operation

$$\overline{f} = \lim_{L \to \infty} \int_{-L}^{L} f(x) dx \quad .$$

We normalize a two-gap potential by expansion near $x = 0$,

$$u(x, 0) = \frac{6}{x^2} + ax^2 + bx^4 + cx^6 + dx^8 + \cdots \quad . \tag{7.7.29}$$

To such a potential corresponds the curve

$$\begin{aligned} \mu^2 = {} & \lambda^5 - (35/2)a\lambda^3 - (63/2)b\lambda^2 + [(567/8)a^2 \\ & + (297/4)c]\lambda + 1377/4)ab - (1287/2)d \quad . \end{aligned} \tag{7.7.30}$$

By making appropriate calculations, we can see that the potentials (7.7.23) satisfy (7.7.28), and the curves (7.7.26,27) are the same as (7.7.30).

7.8 Elliptic Solutions of the KdV Equation and Particle Dynamics

We consider a periodic N-elliptic g-gap potential $u(x) = u(x, 0)$ and its isospectral deformation, i.e., the deformation conserving the spectrum of the Schrödinger operator $\mathcal{H} = d^2/dx^2 - u(x)$,

$$u(x,0) \to u(x,t) = 2\sum_{j=1}^{N} \wp(x - x_j(t)) + c \quad , \tag{7.8.1}$$

where $x_j(t)$, $j = 1, \dots, N$ are functions of time such that $u(x,t)$ satisfies the KdV equation (7.7.4). The description of an isospectral deformation of the potential $u(x)$ is known (*Airault, McKean* and *Moser* [7.32 a], as well as [7.32 b, c]), to reduce to integrating a completely integrable system of Calogero-Moser particles $x_1, \dots, x_N$ with a pair interaction potential $\wp(x_i - x_j)$ $(i \neq j)$, $i, j = 1, \dots, N$ and a Hamiltonian such as

$$h = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i \neq j} \wp(x_i - x_j) \quad . \tag{7.8.2}$$

More specifically, the particles $x_1, \dots, x_N$ are on the locus

$$\mathcal{L}_N = \left\{ (x_1, \dots, x_N) \,\middle|\, \sum_{i \neq j}^{N} \wp'(x_i - x_j) = 0, \quad j = 1, \dots, N \right\} \tag{7.8.3}$$

and evolve according to the law

$$\dot{x}_i = -12\sum_{j \neq i} \wp(x_i - x_j), \quad i = 1, \dots, N. \tag{7.8.4}$$

The geometry of the locus $\mathcal{L}_N$ has not been discussed as yet. It has been established that the locus $\mathcal{L}_N$ is non-empty if N is a "triangular" number, i.e., $N = g(g+1)/2$ [7.32 a]. *Treibich* and *Verdier* [7.29] have recently proved the existence of components of $\mathcal{L}_N$ for non-triangular numbers N, too.

The problem of integrating the system (7.8.2), stated in the most general form, was solved by *Krichever* [7.33], who described the elliptic solutions to the Kadomtsev-Petviashvili equation, associated with the algebraic curve of the general position. Here we make use of the results of [7.33], restricting the treatment (and the restriction is non-trivial) to the present case of a hyperelliptic curve. For this purpose we assign for the Baker-Akhieser function Ψ (Chap. 4) an ansatz such that

$$\Psi(x, x_1(t), \dots, x_N(t); \alpha) = e^{kx} \sum_{j=1}^{N} b_j \Phi(x - x_j; \alpha) \quad , \tag{7.8.5}$$

where b_j, $j = 1, \dots, N$ are the functions to be defined, (k, α) are the coordinates of a hyperelliptic curve and the function $\Phi(x; \alpha)$ is the solution to the equation

$$\left(\frac{d^2}{dx^2} - 2\wp(x) \right) \Phi = \lambda \Phi$$

and is defined by the formulas

$$\Phi = \Phi(x;\alpha) = \frac{\sigma(\alpha - x)}{\sigma(\alpha)\sigma(x)} \exp[\zeta(\alpha)x] \quad ,$$

in which σ and ζ are Weierstrass functions.

We substitute the ansatz (7.8.5) into the equation

$$\mathrm{L}\Psi = \lambda\Psi, \quad L = \frac{\partial^2}{\partial x^2} + 2\sum_{j=1}^{N} \wp(x - x_j) \quad , \tag{7.8.6}$$

$$A\Psi = \frac{\partial \Psi}{\partial t} \quad , \tag{7.8.7}$$

$$A = \frac{\partial^3}{\partial x^3} - 3\sum_{i=1}^{N} \wp(x - x_j) - \frac{3}{2}\sum_{j=1}^{N} \wp'(x - x_j) \quad ,$$

whose compatibility conditions result in the Lax representation $\partial L/\partial t = [L, A]$ for the KdV equation (7.7.4) with solutions of the form (7.8.1). Using the expansions

$$\Phi(x;\alpha) = \frac{1}{x} - \frac{1}{2}\wp(\alpha)x + \frac{1}{6}\wp'(\alpha)x^2 + \dots \quad ,$$
$$\wp(x) = \frac{1}{x^2} + \frac{1}{20}g_2 x^2 + \dots \quad ,$$

we equate the principal parts of the poles of equations (7.8.6,7) in the neighbourhood of the points x_j, $j = 1, \dots, N$. As a result, we have a set of homogeneous linear equations with respect to $b_1, \dots, b_N$, whose compatibility conditions are (7.8.3,4) and also

$$\det L = 0 \quad , \tag{7.8.8}$$

where $L = (L_{ij})$,

$$L_{ij} = (1 - \delta_{ij})\Phi_{ij} + k\delta_{ij}, \quad i, j = 1, \dots, N \quad ,$$

$$\det M = 0 \quad , \tag{7.8.9}$$

where $M = (M_{ij})$,

$$M_{ij} = \delta_{ij}(k^2 - 2\sum_{l \neq j} \wp_{jl} + \wp(\alpha) - \lambda) - 2(1 - \delta_{ij})\Phi'_{ij}, \quad i, j = 1, \dots, N \quad ,$$

and $\Phi_{ij} = \Phi(x_i - x_j; \alpha)$, $i \neq j$, $\wp_{kl} = \wp(x_k - x_l)$, $k \neq l$.

We now discuss the equalities (7.8.8,9) in more detail. Equality (7.8.8) defines the algebraic curve X_g,

$$X_g : R(k, \alpha) = 0 \quad , \tag{7.8.10}$$

$$R(k,\alpha) = k^N + \sum_{j=1}^{N} k^{N-j} r_j(\alpha, x_1, \dots, x_N) \quad ,$$

where r_j, $j = 1, \dots, N$ are the elliptic functions in α defined on X_1. For the first two functions r_j,

$$r_1 = -\sum_{\substack{i \neq j \\ i,j=1,\dots,N}} \Phi_{ij}\Phi_{ji}, \quad r_2 = \sum_{\substack{i \neq j \neq k \\ i,j,k=1,\dots,N}} \Phi_{ij}\Phi_{jk}\Phi_{ki} \quad ,$$

we give the explicit expressions

$$r_1 = \binom{N}{2}\wp(\alpha) - \sum_{1 \le i \le N} \wp_{ij}, \quad r_2 = -\binom{N}{3}\wp'(\alpha) \quad ,$$

derived by applying the addition theorem for Weierstrass $\wp$-functions.

The curve (7.8.8) will be called a *Krichever curve*. Its role in constructing elliptic solutions is as follows: by describing it explicitly, one can construct a corresponding theta function and determine the coordinates of particles $x_1, \dots, x_N$ through the zeros of the theta function ([7.31] and Sect. 7.7). We, therefore, describe the properties of the Krichever curve.

Proposition 7.18. The Krichever curve has the following properties:

(i) The Krichever curve is an N-sheeted cover of the torus X_1

(ii) In the neighborhood of the point $\alpha = 0$, the Krichever curve can be represented as

$$(k - \frac{N-1}{\alpha} + f_n(\alpha)) \prod_{j=1}^{N-1} (k + \frac{1}{\alpha} + f_j(\alpha)) = 0 \quad , \tag{7.8.11}$$

where $f_1, \dots, f_N$ are functions of α regular at $\alpha = 0$, i.e., the meromorphic function k defined on a curve has a simple pole with the principal part $(N-1)/\alpha$ on one of the sheets (this sheet will below be referred to as the "upper" sheet) and simple poles with the principal parts $-1/\alpha$ on the of $N-1$ sheets.

(iii) The genus g of the Krichever curve X_g is equal to $p+1$, where $2p$ is the number of the zeros of the function $\partial R/\partial k$ that do not lie over the point $\alpha = 0$ and differ from the zeros of the function $\partial R/\partial \alpha$.

(iv) The Krichever curve admits an involution $(k, \alpha) \to (-k, -\alpha)$.

(v) For $x_1 \dots, x_N$ on the locus $\mathcal{L}_N$, the Krichever curve is hyperelliptic.

Proof. The proof of (i) follows from a representation of the curve as in (7.8.8); the function $\Phi(x; \alpha)$ is doubly periodic in α and the coefficients of the polynomial (7.8.10) for powers k have no essential singularities. To prove (ii), we note that for $\alpha = 0$ the curve is represented as

$$\det \begin{vmatrix} k & 1/\alpha + o(\alpha) & \dots & 1/\alpha + o(\alpha) \\ 1/\alpha + o(\alpha) & k & \dots & 1/\alpha + o(\alpha) \\ \vdots & \vdots & & \vdots \\ 1/\alpha + o(\alpha) & 1/\alpha + o(\alpha) & \dots & k \end{vmatrix} = 0 \quad .$$

Thus, we have the representation (7.8.11).

(iii) follows from the Riemann-Hurwitz formula (2.2.1) and the fact that the branch points of the covering are exactly the zeros of the function $\partial R/\partial k$ that do not lie over $\alpha = 0$ [7.34].

The existence of the involution $(k, \alpha) \to (-k, -\alpha)$ on the curve (7.8.8) [7.34] follows from the representation of the curve as a determinant and the equality

$$-\widetilde{\Phi}(x, -\alpha) = \widetilde{\Phi}(-x, \alpha), \quad \widetilde{\Phi}(x, \alpha) = \sigma(\alpha - x)/\sigma(\alpha)\sigma(x) \quad .$$

Finally, to prove (v) [7.1 i], the curve should be assigned a meromorphic function of the second order. If and only if the points $x_1, \dots, x_N$ are on the locus (7.8.3), the equality (7.8.9) holds and, when in the neighbourhood of the point $\alpha = 0$, is written as

$$\det \begin{vmatrix} k^2 + 1/\alpha^2 - \lambda & 2/\alpha^2 & \dots & 2/\alpha^2 \\ 2/\alpha^2 & k^2 + 1/\alpha^2 - \lambda & \dots & 2/\alpha^2 \\ \vdots & \vdots & & \vdots \\ 2/\alpha^2 & 2/\alpha^2 & \dots & k^2 + 1/\alpha^2 - \lambda \end{vmatrix} = 0.$$

From this and item (ii) it follows that the function λ has no poles on the lower sheets and has a second-order pole on an upper sheet with the principal part N^2/α^2.

It is to be noted that, as shown by *Smirnov* [7.34], the properties of the Krichever curve are sufficient for it to be restored without applying the elimination procedure.

We have discussed above the potential (7.8.1) in which the particles $x_1, \dots, x_N$ are at the general position on the locus. Let us now consider an elliptic potential $u(x)$ at the special points of a locus.

$$\begin{aligned} u(x) &= \sum_{i=1}^{M} g_i(g_i + 1)\wp(x - x_i) + c \quad , \\ \sum_{i=1}^{M} g_i(g_i + 1) &= 2N \quad . \end{aligned} \tag{7.8.12}$$

Starting from work by *Hermite* [7.26] and *Halphen* [7.27], we choose an ansatz (corresponding to this potential) such that

$$\Psi(x, x_1, \dots, x_k; \alpha) = \sum_{j=1}^{M} e^{kx} \left\{ a_0 \Phi(x - x_j) + \sum_{l=1}^{g_j - 1} a_l \frac{\partial^l}{\partial x^l} \Phi(x - x_j) \right\} . \tag{7.8.13}$$

Making operators similar to those described above, it is possible to construct the Krichever curve in the cases when the potential (7.8.12) and the ansatz (7.8.13) are compatible. To illustrate this point, we consider

Example. [7.1 i]. Let $u(x)$ be a Treibich-Verdier potential (7.7.23). We substitute the ansatz (7.8.13) with $M = 1$, $x_j = \omega_j$, $g_i = 2$ into the Schrödinger equation with the potential (7.7.23). As a result, we come to a set of linear equations

$$\begin{aligned}
&a_0 = ka_1,\\
&(3k^2 - 3\wp(\alpha) + \lambda)a_1 + 6\Phi(\omega_j;\alpha)a_2 = 0,\\
&(-k^3 - 3\wp(\alpha)k - k\lambda + 2\wp'(\alpha))a_1 + 6\Phi'(\omega_j;\alpha)a_2 = 0,\\
&\left(-\frac{3}{2}\wp(\alpha)^2 + \frac{3}{2}k^2\wp(\alpha) - 6e_j^2 + \frac{1}{2}g_2 + \frac{1}{2}\lambda\wp(\alpha)\right)a_1\\
&\quad + (2\Phi''(\omega_j;\alpha) - 2\Phi'(\omega_j;\alpha)k - \Phi(\omega_j;\alpha)k^2 - \Phi(\omega_j;\alpha)\lambda)a_2 = 0.
\end{aligned}\tag{7.8.14}$$

The compatibility conditions for the set (7.8.14) are

$$\begin{aligned}
&(\wp'(\alpha) - 2(\wp(\alpha) - e_j)k)\lambda = 2(\wp(\alpha) - e_j)k^3 - 3k^2\wp'(\alpha)\\
&\quad + 6\wp(\alpha)(\wp(\alpha) - e_j)k - \wp'(\alpha)(\wp(\alpha) - 4e_j)\quad ,\\
&3(\wp(\alpha) - e_j)k^4 - 3\wp'(\alpha)k^3 - 12e_j(\wp(\alpha) - e_j)k^2\\
&\quad + 4(\wp(\alpha) - e_j)k^2\lambda + 3\wp(\alpha)\wp'(\alpha)k - \wp'(\alpha)k\lambda\\
&\quad - 2(\wp(\alpha) - e_j)(\wp(\alpha) + 2e_j)\lambda + (\wp(\alpha) - e_j)\lambda^2\\
&\quad - 3(\wp(\alpha))^3 + 15^2e_j - \wp(\alpha)(48e_j^2 - 3g_2) + 36e_j^3 - 3g_2e_j = 0\quad .
\end{aligned}\tag{7.8.15}$$

By excluding the variable $\wp$ from (7.8.15), we have the Krichever curves

$$\begin{aligned}
&k^4 - 3(2\wp(\alpha) - e_j)k^2 + 4\wp'(\alpha)k - 3\wp^2(\alpha)\\
&\quad + 3e_j\wp(\alpha) - 3e_ie_l = 0,\qquad j \neq i \neq l = 1,2,3\quad .
\end{aligned}\tag{7.8.16}$$

Each of these curves is birationally equivalent to the corresponding curve (7.7.26, 27); this birational equivalence is given by

$$\begin{aligned}
&(\mu,\lambda) = \left(\frac{6}{k^2}(k^3 - 3\wp(\alpha)k - \wp'(\alpha))(2k^4 - 6\wp(\alpha)k^2 - 2\wp'(\alpha)k + 9e_i),\right.\\
&\qquad \left.\frac{2(\wp(\alpha) - e_i)k^3 + 6\wp(\alpha)(\wp(\alpha) - e_i)k - \wp'(\alpha)(3k^2 + \wp(\alpha) - 4e_i)}{\wp'(\alpha) - 2\wp(\alpha)k + 2e_jk}\right)
\end{aligned}$$

$$\begin{aligned}
&(k,\wp(\alpha)) = \left(\frac{3\mu}{4(\lambda - 15/2e_i)(\lambda + 6e_j)},\right.\\
&\qquad \left. e_j - \frac{(\lambda - E_2))\lambda - E_3)(\lambda - 3e_i + 9e_j)^2}{16(\lambda - E_1)(\lambda - (15/2)e_i)^2}\right)
\end{aligned}\tag{7.8.17}$$

each of which can be obtained by excluding appropriate variables from (7.8.15).

To conclude this section we give another approach [7.1 e] describing the functions $x_1, \ldots, x_N$ that employs covers and enables one to produce the most complete results for genus $g = 2$. For genus $g = 2$ we consider the Jacobian inversion problem whose solution amounts to integrating the KdV equation

$$\begin{aligned} \int_{\infty}^{\lambda^{(1)}} \omega_1 + \int_{\infty}^{\lambda^{(2)}} \omega_1 &= U_1 x + V_1 t + W_1 \quad , \\ \int_{\infty}^{\lambda^{(1)}} \omega_2 + \int_{\infty}^{\lambda^{(2)}} \omega_2 &= U_2 x + V_2 t + W_2 \quad . \end{aligned} \tag{7.8.18}$$

Let $u(x,t)$ be a 2-gap N-elliptic potential that satisfies the conditions of Theorem 7.14. Then each of the Abelian integrals in equations (7.8.18) is reduced to an elliptic integral of some rational degree N by the substitutions $\mathcal{P}_1(\lambda)$, $\mathcal{P}_2(\lambda)$, and, with an appropriate choice of a homology basis $U_2 = 0$.

Next we express $\lambda^{(1)}$, $\lambda^{(2)}$ for a solution such as (7.8.1) to the KdV equation through Weierstrass $\wp$-functions. For this purpose we use the "trace formulas" (see Sect. 3.6 and [7.35]),

$$\begin{aligned} u(x,t) &= -2(\lambda^{(1)} + \lambda^{(2)}) + \sum_{j=1}^{2g+1} E_j \quad , \\ \frac{1}{8}\left(3u^2(x,t) - u_{xx}(x,t)\right) &= \lambda^{(1)}\lambda^{(2)} \\ &\quad - \frac{1}{2}\sum_{i,j} E_i E_j + \frac{3}{8}\left(\sum_{j=1}^{2g+1} E_j\right)^2 \quad . \end{aligned} \tag{7.8.19}$$

By setting $\sum_{j=1}^{j=2g+1} E_j = 0$, without loss of generality, and making x tend to x_j, $j = 1, \ldots, N$ in the solution (7.8.12), we find

$$\begin{aligned} \lambda^{(2)}(x_j + \varepsilon, t) &= \frac{1}{\varepsilon^2} + o(1) \quad , \\ \lambda^{(1)}(x_j + \varepsilon, t) &= -3\left(\sum_{k \neq j} \wp_{jk} + c\right) \quad . \end{aligned} \tag{7.8.20}$$

Therefore, the equations (7.8.18) in which $x = x_j$ and the hyperelliptic integrals are reduced to elliptic integrals, transform to

$$\mathcal{P}_1(\lambda^{(1)}(x_j)) = \wp(\alpha x_j + \beta t + \gamma) \quad , \tag{7.8.21}$$

$$\mathcal{P}_2(\lambda^{(1)}(x_j)) = \tilde{\wp}(\kappa t + \varrho) \quad , \tag{7.8.22}$$

where $\wp$ and $\tilde{\wp}$ are the Weierstrass functions defined on tori over which X_2 is an N-sheeted covering, and α, β, γ, κ, ϱ are the constants that appear under reduction. Eliminating the $\lambda^{(1)}$ from (7.8.21,22), we have an algebraic equation with respect to $\wp$ with coefficients depending on $\tilde{\wp}$. Here we describe the integration of a system of particles $x_1(t),\ldots,x_N(t)$; its specific feature is that (7.8.22) involves no variable x_j under the sign of the $\tilde{\wp}$-function because of the condition $U_2 = 0$.

Proposition 7.19. Finding the functions $x_1,\ldots,x_N$ for a 2-gap N-elliptic potential that satisfies the conditions of Theorem 7.14 reduces to solving an algebraic equation of the Nth degree (7.8.22) for $\lambda^{(1)}$ and to the subsequent quadratures

$$\dot{x}_j = 4\lambda^{(1)}(t), \quad j = 1,\ldots,M \quad . \tag{7.8.23}$$

Proof. The KdV equation implies that $\dot{x}_j = -12\sum_{k\neq j}\wp_{jk} + 4c$. Using formulas (7.8.20), we obtain (7.8.23).

Example. To illustrate our point, we construct an isospectral deformation for the 2-gap Treibich-Verdier potential (7.7.23). We first find the coordinates $\wp = \mathcal{P}_1(\lambda)$, $\tilde{\wp} = \mathcal{P}_2(\lambda)$ of coverings π, π', $(\wp',\wp) \xleftarrow{\pi} X_2 \xrightarrow{\pi'} (\tilde{\wp}',\tilde{\wp})$. For this purpose we make use of (7.6.21) for a 4-sheeted covering over a torus. By imposing the condition (7.7.21) on the moduli of such a covering, we have from (7.6.21, 22)

$$\left(\lambda + \frac{3}{2}e_j\right)\frac{d\lambda}{\mu} = \mathrm{i}\frac{d\wp}{2\wp'} \quad , \tag{7.8.24}$$

$$\begin{aligned} \wp - e_l &= -\frac{(\lambda - E_2)(\lambda - E_3)(\lambda - a_{jl})^2}{16(\lambda - E_1)(\lambda - b_j)^2} \quad , \\ \frac{d\lambda}{\mu} = \frac{\mathrm{i}d\tilde{\wp}}{3\tilde{\wp}'}, \quad \tilde{\wp} - \tilde{e}_l &= -\frac{(\lambda - E_2)(\lambda - E_3)(\lambda - c_{jl})^2}{16(\lambda - E_1)} \quad , \end{aligned} \tag{7.8.25}$$

where $a_{jl} = 3(e_j - 3e_l)$, $b_j = (15/2)e_j$, $c_{jl} = 5(e_j + e_l)$, $j \neq l$,

$$\begin{aligned} 3\tilde{e}_1 &= 2(e_1 - e_3)^3 - (e_1 - e_2)(5e_2 - 2e_3)^2 \quad , \\ 3\tilde{e}_2 &= -(e_1 - e_3)^3 + 2(e_1 - e_2)(5e_2 - 2e_3)^2 \quad , \\ 3\tilde{e}_3 &= -(e_1 - e_3)^3 - (e_1 - e_2)(5e_2 - 2e_3)^2 \quad . \end{aligned}$$

The reduction formulas (7.8.24) are given in the present context to make our discussion complete. Using (7.8.25), we derive, within the scheme presented, an equation of the fourth degree in $\lambda^{(1)} = \lambda$,

$$(\lambda - E_2)(\lambda - E_3)(\lambda - c_{jl})^2 + 4(\lambda - E_1)(\tilde{\wp}(8\mathrm{i}t) - \tilde{e}_l) = 0 \quad . \tag{7.8.26}$$

We note that since there is a remarkable relation between solutions such as (7.7.10) to the KdV equation and the dynamics of a Calogero-Moser system on a line [7.32], formula (7.8.26) gives a complete description of the trajectories for the third flow of a system (7.8.2), restricted to stationary points of the second flow for a Treibich-Verdier potential (7.7.23).

7.9 Elliptic Solutions of the Sine-Gordon Equation

In this section we describe the solutions in elliptic functions for the sine-Gordon equation [7.35], written in the laboratory system of coordinates

$$u_{tt} - u_{xx} + \sin u = 0 \quad . \tag{7.9.1}$$

The appropriate general finite-gap solutions were constructed in Sect. 4.2 (4.2.25-27). Here it is convenient to write these formulas as

$$u(x,t) = 2\mathrm{i} \log \frac{\theta(\boldsymbol{U}^{+}x + \boldsymbol{U}^{-}t + \boldsymbol{U}_0 + \boldsymbol{K} + \int_0^{Q_0} \omega; B)}{\theta(\boldsymbol{U}^{+}x + \boldsymbol{U}^{-}t + \boldsymbol{U}_0 + \boldsymbol{K} + \int_\infty^{Q_0} \omega; B)} \quad , \tag{7.9.2}$$

where $\boldsymbol{K}$ is the vector of Riemann constants, Q_0 is an arbitrary point of curve X_g, and the relation between the vectors $\boldsymbol{U}^{\pm}$ and $\boldsymbol{U}_0$ and the parameters of finite-gap solutions, used in Sect. 4.2, is given by the equation

$$\begin{aligned} U_j^{\pm} &= \mathrm{i}V_j \mp \frac{\mathrm{i}}{16} W_j = 2\mathrm{i}(c_{j1} \pm \frac{1}{16\sqrt{p_0}} c_{jg}), \quad j = 1, \dots, g \quad , \\ p_0 &= E_1 E_2 \cdots E_{2g} \quad , \\ \boldsymbol{U}_0 &= \boldsymbol{D} - \boldsymbol{K} + \int_{Q_0}^{0} \omega + \pi \mathrm{i} \boldsymbol{\Delta} \quad . \end{aligned} \tag{7.9.3}$$

In this section we shall consider only real finite-gap solutions to the (7.9.1). Therefore the curve X_g will always be real, implying (Sect. 4.2) that in (7.9.3) the following inequality holds:

$$\sqrt{p_0} > 0 \quad . \tag{7.9.4}$$

Just as in the case of elliptic finite-gap potentials (Sect. 7.7), we have for the solutions of (7.9.1) the following

Theorem 7.20. For the g-gap solution of equation (7.9.1) to be doubly periodic in the variable x, it is necessary and sufficient that following conditions are valid:
(i) curve (7.9.3) is non-singular and is an N-sheeted cover of the torus X_1;
(ii) the equations

$$U_2^{+} = U_2^{+} = \cdots U_g^{+} = 0 \tag{7.9.5}$$

are solvable for hyperelliptic points of the set $\mathcal{U}_g$, in a homology basis in $H_1(X_g, \mathbb{Z})$ such that the B-matrix has the form (7.2.2).

We note that since the expression for the "winding vectors" (7.9.3) involves a free parameter $\sqrt{p_0}$ that satisfies the condition (7.9.4), one of the equations (7.9.5), for example, $U_2^+ = 0$, can be satisfied by setting

$$p_0 = c_{2g}^2/16^2 c_{21}^2 \quad . \tag{7.9.6}$$

Therefore, for the case of genus $g = 2$ there exists an enumerable number of 2-gap solutions doubly periodic in x that correspond to $N = 2, 3, \ldots$. We turn to the case of genus $g = 2$.

We restrict ourselves to discussing only real solutions of genus $g = 2$ that belong to one of the three types (see, for example, [7.1 b, 34] and also Chap. 4):

$$\begin{aligned} &\text{i Type}: && E_4 < E_3 < E_2 < E_1 < 0 \quad ; \\ &\text{II Type}: && E_4 < E_3 < 0, \quad E_1 = \overline{E}_2 \neq E_2 \quad ; \\ &\text{III Type}: && E_1 < \overline{E}_2 \neq E_2, \quad E_3 = \overline{E}_4 \neq E_4 \quad . \end{aligned}$$

Since we consider below the solutions in terms of elliptic functions, it is convenient to fix a homology basis such as (7.5.5). Then, for the solution (7.9.2), we have the formula

$$u(x,t) = 2\mathrm{i}\log\frac{\theta[\varrho_0](\boldsymbol{U}^+x + \boldsymbol{U}^-t + \boldsymbol{U}_0; B)}{\theta[\varrho_\infty](\boldsymbol{U}^+x + \boldsymbol{U}^-t + \widetilde{\boldsymbol{U}}; B)} \quad , \tag{7.9.7}$$

where $[\varrho_0] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $[\varrho_\infty] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\widetilde{\boldsymbol{U}} = \boldsymbol{U}_0^{(0)} + \boldsymbol{K} + \int_0^{Q_0} \omega + 1/2\varrho_0 + 1/2\boldsymbol{B}_1$, $\boldsymbol{B}_1 = (B_{11}, B_{12})$. Using the second-order transformations (2.6.8) we write (7.9.7) as

$$\begin{aligned} &\operatorname{arctg}\left(\frac{u-u_0}{4}\right) = \mathrm{i}\frac{\hat\theta\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}(\zeta)\,\hat\theta\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(\zeta)}{\hat\theta\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}(\zeta)\,\hat\theta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}(\zeta)} \quad , \\ &\zeta = \boldsymbol{U}^+x + \boldsymbol{U}^-t + \widetilde{\boldsymbol{U}}, \quad u_0 = 2\mathrm{i}\ \log\frac{\theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \quad , \\ &\hat\theta[\varepsilon](\zeta) = \theta[\varepsilon](\zeta; 2B) \quad . \end{aligned} \tag{7.9.8}$$

Equations (7.9.3, 5.7) give expressions for the components of the "winding vectors" in terms of theta constants

$$\begin{aligned} U_j^\pm = &\frac{\mathrm{i}(-1)^{i-1}\sqrt[8]{p_0}}{\pi^2\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\theta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \\ &\times\left(\theta_j\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \mp \frac{1}{16\sqrt[4]{p_0}}\theta_j\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right), \quad j = 1, 2 \quad . \end{aligned} \tag{7.9.9}$$

We consider a 2-sheeted covering of genus $g=2$, i.e., we give the B-matrix as $B=\begin{pmatrix} B_{11} & \pi\mathrm{i} \\ \pi\mathrm{i} & B_{22}\end{pmatrix}$. By calculating the components of the "winding vectors" (7.9.12) in terms of theta constants (Appendix 7.3), we find that (7.9.6) is solvable for $p_0 = 1/16^4$, and the following equations are valid:

$$\begin{aligned} &U_1^+ = \mathrm{i}\vartheta_4^2/\pi(\vartheta_3^2\widetilde{\vartheta}_2^2 - \widetilde{\vartheta}_3^2\vartheta_4^2) = \alpha, \quad U_2^+ = 0 \quad , \\ &U_1^- = 0, \quad U_2^- = \mathrm{i}\widetilde{\vartheta}_2^2/\pi(\vartheta_3^2\widetilde{\vartheta}_2^2 - \widetilde{\vartheta}_3^2\vartheta_4^2) = \beta \quad , \end{aligned} \tag{7.9.10}$$

in which $\vartheta_j = \vartheta_j(0; B_{11}/\pi\mathrm{i})$, $\widetilde{\vartheta}_j = \vartheta_j(0, B_{22}/\pi\mathrm{i})$, $j=2,3,4$, are Jacobi theta constants. These equations and the transformational properties of the theta function (2.5.10) imply that the solutions $u(x,t)$ of the sine-Gordon equation defined by (7.9.4,5) are doubly periodic functions not only in x, but also in t with independent periods $\omega = 2\pi\mathrm{i}/\alpha$, $\omega' = 2B_{11}/\alpha$ and $\widetilde{\omega} = 2\pi\mathrm{i}/\beta$, $\widetilde{\omega}' = 2B_{22}/\beta$. Using (7.9.8), the solution (7.9.5) can be represented as

$$\operatorname{arctg}[u(x,t)/4] = F(\alpha x)G(\beta t) \quad , \tag{7.9.11}$$

where F and G are elliptic functions. This solution to the sine-Gordon equation finds numerous applications in physics.

Consider a 4-sheeted covering of genus $g=2$, i.e., we give a matrix such as $B=\begin{pmatrix} B_{11} & \pi\mathrm{i}/2 \\ \pi\mathrm{i}/2 & B_{22}\end{pmatrix}$. Using (7.9.8) and (2.4.8), we obtain a quasi-periodic solution in terms of elliptic functions

$$\operatorname{arctg}[(u-u_0)/4] = \tag{7.9.12}$$

$$\frac{\sum\limits_{\varepsilon_1',\varepsilon_2'=0,1} (-1)^{\varepsilon_2'/2} c_\varepsilon \vartheta\begin{bmatrix}\varepsilon_1'+1\\ \varepsilon_2'+1/2\end{bmatrix}\left(\frac{\zeta_1}{\pi\mathrm{i}};\frac{2B_{11}}{\pi\mathrm{i}}\right)\vartheta\begin{bmatrix}\varepsilon_2'+1/2\\ \varepsilon_1'+1\end{bmatrix}\left(\frac{\zeta_2}{\pi\mathrm{i}};\frac{2B_{22}}{\pi\mathrm{i}}\right)}{\sum\limits_{\varepsilon_1',\varepsilon_2'=0,1} c_\varepsilon \vartheta\begin{bmatrix}\varepsilon_1'\\ \varepsilon_2'+1/2\end{bmatrix}\left(\frac{\zeta_1}{\pi\mathrm{i}};\frac{2B_{11}}{\pi\mathrm{i}}\right)\vartheta\begin{bmatrix}\varepsilon_2'+1/2\\ \varepsilon_1'\end{bmatrix}\left(\frac{\zeta_2}{\pi\mathrm{i}};\frac{2B_{22}}{\pi\mathrm{i}}\right)},$$

where

$$c_\varepsilon = \exp \mathrm{i}\pi(\varepsilon_2' + 1/4\varepsilon_1' + 1/2\varepsilon_1'\varepsilon_2')\vartheta\begin{bmatrix}\varepsilon_1'\\ 1/2+\varepsilon_2'\end{bmatrix}\widetilde{\vartheta}\begin{bmatrix}\varepsilon_2'+1/2\\ \varepsilon_1'\end{bmatrix} \quad ,$$

and the components of the "winding vectors" are calculated in terms of reduced theta constants, using (7.9.9) and the tables of theta constants in Appendix 7.3. This solution is doubly periodic in x (but quasi-periodic in t) under the condition (7.9.6).

Other types of the solutions of (7.9.1) in terms of elliptic functions may be found in [7.3 a, b].

7.10 On One Periodic Solutions of the Kowalewski Problem

The general solution of the problem of motion of Kovalewski's gyroscope [7.37] was described in Chap. 5. Here we are concerned with some periodic motions of this gyroscope. The abundant literature [7.38 a, b] devoted to studies of different special motions of Kowalewski's gyroscope gives only cases of periodic motions that arise when the Kowalewski curve degenerates (Sect. 5.2). The case reported in the present section corresponds to a nondegenerate Kowalewski curve.

The Kowalewski equations in integral form are Jacobi inversion problem

$$\begin{aligned} &\int_{P_0}^{P_1} \frac{d\lambda}{\mu} + \int_{P_0}^{P_2} \frac{d\lambda}{\mu} = -8\mathrm{i}c \quad , \\ &\int_{P_0}^{P_1} \frac{\lambda d\lambda}{\mu} + \int_{P_0}^{P_1} \frac{\lambda d\lambda}{\mu} = -\sqrt{2}\mathrm{i}(t - t_0) \quad , \end{aligned} \tag{7.10.1}$$

where $P_i = (\mu_i, s_i) \in X_2$, $i = 1, 2$, and s_1, s_2 are *Kowalewski variables*, formulated for the *Kowalewski curve* $X_2 = (\mu, \lambda)$, $\mu^2 = \prod_{k=1}^{k=5}(\lambda - E_k)$, which is defined by

$$\mu^2 = \left[(\lambda - H)^2 - \frac{k}{4}\right] \lambda \left[(\lambda - H)^2 + \left(1 - \frac{k}{4}\right)\right] - (\boldsymbol{pM})^2 \quad , \tag{7.10.2}$$

where H, $\boldsymbol{p}$, $\boldsymbol{M}$ and k are the integrals of motion. By solving the inversion problem (7.10.1), we can express $s_1 + s_2$ and $s_1 s_2$ through theta functions on the Jacobian $J(X_2)$.

Proposition 7.21. For the motion of Kowalewski's gyroscope to be periodic and described by elliptic time functions, it is necessary and sufficient that the Kowalewski curve (7.10.2) is

(i) an N-sheeted cover of the torus X_1,
(ii) one of the components of the vector $\boldsymbol{U} = -\sqrt{2}\mathrm{i}(c_{11}, c_{21})$, where c_{ij} are the normalization constants of holomorphic differentials ω, is zero in u_2, in a homology basis for which $B_{12} = 2\pi\mathrm{i}/N$, and
(iii) the parameters H, k and pM are related by

$$H + k \geq (\boldsymbol{pM})^2 \quad . \tag{7.10.3}$$

Proof. Equations (7.10.1) imply that $s_1 + s_2$ and $s_1 s_2$ are expressed in terms of the theta functions of curve (7.10.2). Repeating the proof for Theorem 7.14, we conclude that the functions $s_1 + s_2$ and $s_1 s_2$ are elliptic time functions. According to (7.5.1), condition (iii) ensures the reality of motion.

As an example, we consider a curve of genus $g = 2$,

$$\mu^2 = (\lambda^2 - 3g_2)(\lambda + 3e_1)(\lambda + 3e_2)(\lambda + 3e_3) \quad , \tag{7.10.4}$$

where $e_i = \wp(\omega_i)$, $i = 1,2,3$ are parameters in the theory of elliptic Weierstrass functions. The branch point projections of the curve (7.10.4) $\pm\sqrt{3g_2}$, $-3e_i$, $i = 1,2,3$ coincide with the boundaries of the gaps of the 2-gap Lamé potential $u(x,0) = 6\wp(x)$. This is verified by employing the Novikov equation (7.7.28). The curve (7.10.4) 3-sheetedly covers the tori $X_1 = (\wp', \wp)$ and $\widetilde{X}_1 = (\widetilde{\wp}', \widetilde{\wp})$, $X_1 \xleftarrow{\pi} X_2 \xrightarrow{\widetilde{\pi}} \widetilde{X}_1$,

$$\begin{aligned} &\wp'^2 = 4\wp^3 - g_2\wp - g_3, \quad \widetilde{\wp}'^2 = 4\widetilde{\wp}^3 - \widetilde{g}_2\widetilde{\wp} - \widetilde{g}_3 \quad , \\ &\widetilde{g}_2 = 4(3g_2^2 + 27g_3^2)/g_2^2, \quad \widetilde{g}_3 = 72(g_3g_2^3 - 3g_3^3)/g_2^3 \quad . \end{aligned} \tag{7.10.5}$$

Specifically, the following formulas [7.1 d] are valid:

$$\begin{aligned} &(\wp', \wp) = \left(\frac{2\mu}{27}\frac{\lambda^3 - 9g_2\lambda - 54g_3}{(\lambda^2 - 3g_2)}, \frac{\lambda^3 + 27g_3}{9(\lambda^2 - 3g_2)}\right) \quad , \\ &(\widetilde{\wp}', \widetilde{\wp}) = \left(\sqrt{\frac{2}{27g_2^3}}\mu(4\lambda^2 - 3g_2), \frac{1}{3g_2}(4\lambda^3 - 9g_2\lambda^2 + 9g_3)\right) \quad . \end{aligned} \tag{7.10.6}$$

This description of covers π and $\widetilde{\pi}$ follows from the reduction of holomorphic differentials

$$\begin{aligned} &\frac{d\lambda}{\sqrt{(\lambda^2 - a)(8\lambda^3 - 6a\lambda - b)}} = \frac{1}{3}\frac{d\xi}{\sqrt{(2a\xi - b)(\xi^2 - a)}} \quad , \\ &\xi = (4\lambda^3 - 3a\lambda)/a \quad , \end{aligned} \tag{7.10.7}$$

$$\begin{aligned} &\frac{\lambda d\lambda}{\sqrt{(\lambda^2 - a)(8\lambda^3 - 6a\lambda - b)}} = \frac{1}{2\sqrt{3}}\frac{d\eta}{\sqrt{\eta^3 - 3a\eta + b}} \quad , \\ &\eta = (\lambda^3 - b)/3(\lambda^2 - a) \quad , \end{aligned} \tag{7.10.8}$$

to elliptic differentials by using third-order transformations [7.26]. In (7.10.7, 8),we set $a = 3g_2$, $b = -54g_3$, $\xi = \widetilde{\wp}$, $\eta = 6\wp$ and using the notation introduced above, it follows that

$$\frac{d\lambda}{\mu} = \frac{4}{3\sqrt{3g_2}}\frac{d\widetilde{\wp}}{\widetilde{\wp}'}, \quad \frac{\lambda d\lambda}{\mu} = -\frac{2}{3}\frac{d\wp}{\wp'} \quad . \tag{7.10.9}$$

We shall need the following Lemma on the isospectral deformation of a 2-gap Lamé potential:

Lemma 7.22. Let $u(x,t) = 2\sum_{i=1}^{3}\wp(x - x_i(t))$ be a solution to the KdV equation, $u_t = 6uu_x - u_{xxx}$, where the functions $x_j(t)$ satisfy the condition $x_1(t) + x_2(t) + x_3(t) = 0$. Then the quantities $\wp_{ij} = \wp(x_i - x_j)$, $i \neq j$, $i,j = 1,2,3$ are roots of the cubic equation

$$4\wp^3 - g_2\wp - \frac{1}{3}g_3 + \frac{1}{2}g_2\widetilde{\wp}(6\sqrt{3g_2}it) = 0 \quad , \tag{7.10.10}$$

where $\widetilde{\wp}$ is a Weierstrass $\wp$-function with invariants (7.10.5).

Proof. We consider for the curve (7.10.4) the Jacobi inversion problem whose solution amounts to integrating the KdV equation

$$\begin{aligned} &\int_{\infty}^{\lambda^{(1)}} \frac{d\lambda}{\mu} + \int_{\infty}^{\lambda^{(2)}} \frac{d\lambda}{\mu} = -8\mathrm{i}(t - t_0) \quad , \\ &\int_{\infty}^{\lambda^{(1)}} \frac{\lambda d\lambda}{\mu} + \int_{\infty}^{\lambda^{(2)}} \frac{\lambda d\lambda}{\mu} = 2\mathrm{i}x \end{aligned} \tag{7.10.11}$$

where $\lambda^{(1)} + \lambda^{(2)} = u(x,t)/2$. Using the "trace formulas"(7.8.19), we have expansions $\lambda^{(1)} = 1/\varepsilon^2 + o(1)$, $\lambda^{(2)} = 3(\wp_{jk} + \wp_{ji}) + o(\varepsilon)$, $i \neq j \neq k$ for $x = x_j + \varepsilon$, $\varepsilon \to 0$. Now we use these expansions to derive from (7.10.11), for $x = x_j$

$$\begin{aligned} &\int_{\infty}^{3(\wp_{jk}+\wp_{ji})} \frac{d\lambda}{\mu} = -8\mathrm{i}(t - t_0) \quad , \\ &\int_{\infty}^{3(\wp_{jk}+\wp_{ji})} \frac{\lambda d\lambda}{\mu} = 2\mathrm{i}x_j, \quad j = 1,2,3 \quad . \end{aligned} \tag{7.10.12}$$

The reduction formula (7.10.7) implies that the quantities $\wp_{ij}$, $i \neq j$, $i,j = 1,2,3$, are roots of (7.10.10) and satisfy the equations

$$\begin{aligned} &\wp_{ij} + \wp_{jk} + \wp_{ki} = 0 \quad , \\ &\wp_{ij}\wp_{jk} + \wp_{ij}\wp_{ki} + \wp_{jk}\wp_{ki} = -g_2/4 \quad . \end{aligned} \tag{7.10.13}$$

We show that the second of the equations (7.10.12) is satisfied identically. Indeed, the reduction formula (7.10.8) implies

$$\wp(3x_j) = \frac{-\wp_{ki}^3 + g_3}{3\wp_{ki}^2 - g_2} \quad . \tag{7.10.14}$$

Since $x_1 + x_2 + x_3 = 0$, $\wp_{ij}'^2 = \wp_{jk}'^2 = \wp_{ki}'^2$ (7.10.14) is transformed by the addition theorem for Weierstrass $\wp$-functions to equation

$$\wp_{ki} + \frac{\wp_{ki}'^2}{(2\wp_{ij} + \wp_{ki})^2} = \frac{\wp_{ki}^3 - g_3}{g_2 - 3\wp_{ki}^2} \quad ,$$

which is satisfied identically, because from (7.10.13) we find that $2\wp_{ij} + \wp_{ki} = \pm\sqrt{g_2 - 3\wp_{ki}}$.

We note that the description of the Jacobian $J(X_2)$, equivalent to (7.10.10), was produced in [7.34], proceeding from different considerations.

We turn back to the Kowalewski problem. As is known [7.38 a], the motion of Kowalewski's gyroscope is real, if it belongs to one of the five types

Type $1 - \infty \le s_2 < E_3 < E_2 < E_1 < E_4 \le s_1 \le E_5$,

Type $2 - \infty \le s_2 < E_3 < \min(E_2, E_4) < \max(E_2, E_4) < E_1 \le s_1 \le E_5$,

Type $3 - \infty \le s_2 \le E_4 < E_3 < E_2 < E_1 \le s_1 \le 5$,

$$\left.\begin{array}{ll}\text{Type} & 4 - \infty \leq s_2 \leq E_1 < E_4 \leq s_1 \leq E_5, \\ \text{Type} & 5 - \infty \leq s_2 \leq E_4 < E_1 \leq s_1 \leq E_4,\end{array}\right\} \overline{E}_2 = E_3 \neq E_2 \quad ,$$

where $E_1, \ldots, E_5$ are the branching points of (7.10.2). In what follows we consider the motion belonging to Type I.

Theorem 7.23. Suppose that $k > 0$, the inequality (7.10.3) is valid and

$$\begin{aligned} 22H &= 5(7\sqrt{k} + 4\sqrt{k+33}) \quad , \\ (pM) &= \left(\frac{3}{5}H - \frac{\sqrt{k}}{2}\right)\left(\frac{4}{25}H^2 + \frac{2}{5}\sqrt{k}H + 1\right) \quad , \end{aligned} \tag{7.10.15}$$

and the initial conditions are chosen as $s_1(0) = E_5$, $s_2(0) = E_3$. Then the motion of Kowalewski's gyroscope is real, periodic and described by elliptic functions as

$$\begin{aligned} s_{1,2}(t) &= \frac{4}{5}H \\ &+ \frac{1}{2}\left\{\sum_{j=1}^{3} \wp_j \pm \sqrt{9g_2 + \sum_{j=1}^{3} \wp_j^2 - 10 \sum_{1 \leq < j \leq 3} \wp_i \wp_j}\right\} \quad , \end{aligned} \tag{7.10.16}$$

where $\wp_j = \wp(\mathrm{i}t/\sqrt{2} + \omega_j | \omega, \omega')$, $\omega_1 = \omega$, $\omega_2 = \omega + \omega'$, $\omega_3 = \omega'$, $\wp(\omega_i) = e_i$, $i = 1, 2, 3$ and

$$e_{1,2} = \frac{1}{2}\left(\frac{H}{15} - \frac{\sqrt{k}}{6}\right) \pm \frac{1}{3}\sqrt{H\frac{k}{2}}, \quad e_3 = \frac{\sqrt{k}}{6} - \frac{H}{15} \quad . \tag{7.10.17}$$

Proof. We make in (7.10.1) a substitution $s_i = \widetilde{s}_i + \delta$, $i = 1, 2$ with the constant δ and the parameters H, k, pM chosen so that in the new variables the roots $\widetilde{E}_i$ of the polynomial μ^2 coincide with the boundaries of the gaps of a 2-gap lamé potential (7.10.2); namely, we satisfy the conditions

$$\widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_1 = 0, \quad \widetilde{E}_5 = -\widetilde{E}_3 \quad , \tag{7.10.18}$$

$$\widetilde{E}_1^2 + \widetilde{E}_2^2 + \widetilde{E}_4^2 = \frac{3}{2}\widetilde{E}_5^2 \quad , \tag{7.10.19}$$

The definition of the Kowalewski curve (7.10.2) yields the following equations

$$E_5 = H + \sqrt{k}/2, \quad E_4 = H - \sqrt{k}/2 \quad , \tag{7.10.20}$$

$$E_1 + E_2 + E_3 = 2H \quad , \tag{7.10.21}$$

$$E_1E_2 + E_1E_3 + E_2E_3 = 3H^2 - \frac{k}{4} + 1 \quad , \tag{7.10.22}$$

$$E_1E_2E_3 = (pM)^2 \quad . \tag{7.10.23}$$

Thus we find that $\delta = 4H/5$ and $E_3 = 3H/5 - \sqrt{k}/2$. By substituting E_3 and δ into the condition (7.10.19) and using it in calculations (7.10.20,23), we have the condition

$$\frac{11}{50}H^2 - \frac{7}{30}\sqrt{k}H + \frac{1}{12}k - \frac{2}{3} = 0 \quad ,$$

and solving it, we obtain an expression for (7.10.15). The compatibility condition (7.10.21-23) and the above expression for E_3 is found as the second of equalities (7.10.15) by substituting E_3 into the cubic equation $\lambda((\lambda - H)^2 + (1 - k/4)) = (pM)^2$. We can see that the conditions imposed on the parameters are met at least near $k = 0$.

We fix the initial conditions $s_i(0)$, $i = 1, 2$ that correspond to Type I of real solutions and set $\widetilde{s}_1(0) = \sqrt{3g_2}$, $\widetilde{s}_2(0) = -\sqrt{3g_2}$. The integration paths in formulas (7.10.1) can be chosen such that the constant c is given by

$$c = -\mathrm{i}\frac{\widetilde{\omega} + \widetilde{\omega}'}{6\sqrt{3g_2}} \quad . \tag{7.10.24}$$

Next we set

$$2(\widetilde{s}_1 + \widetilde{s}_2) = -2\sum_{j=1}^{3} \wp\left(\frac{it}{\sqrt{2}} - \alpha_j\right) \quad , \tag{7.10.25}$$

where the isospectral deformation of a 2-gap Lamé potential is on the right-hand side of the equation. To calculate the constants $\alpha_j = \alpha_j(c)$, $j = 1, 2, 3$, we make use of the relation (7.10.10), which, for the value (7.10.24) of the constant (it is an analog of the time variable in the KdV equation), reduces to

$$4\wp_{ij}^3 - g_2\wp_{ij} - \frac{1}{3}g_3 + \frac{1}{9}g_2\widetilde{e}_2 = 0 \quad . \tag{7.10.26}$$

Since $\widetilde{e}_{1,3} = \pm\sqrt{3g_2} + 3g_3/g_2$, $e_2 = -6g_3/g_2$, the equality (7.10.26) takes the form $\wp_{ij}'^2 = 0$; it thus follows that the constants α_j can be taken to be equal to ω_j, $j = 1, 2, 3$. The relation between the quantities $\wp(\omega_i) = e_i$ and the integrals of motion is derived from the reduction formulas. Equation (7.10.16) follows from (7.10.25) and the "trace formulas" (7.8.19).

7.11 Normal Coverings

In the preceding sections we have, basically, given an answer to the question: when are multi-dimensional theta functions of algebraic curves and Abelian integrals reduced to lower genera? While it is possible to describe all algebraic curves of genus 2 that cover elliptic curves, such an effective procedure is not available for Riemann surfaces of genera $g > 2$. We therefore consider an important (from the practical viewpoint) special class of algebraic curves whose theta functions are reducible. These are the curves X_g, which have a non-trivial group G of birational automorphisms and, consequently, are normal coverings over X_g/G, $\quad \pi^* : X_g \to X_g/G$.

For algebraic curves of genus $g > 2$, the order N of group G is finite, and $N \leq 84(g-1)$ by the *Hurwitz theorem* [7.8]. The upper bound in this inequality is attained for curves of genus $g \leq 7$ only if $g = 3$ (*Klein curve* [7.12], for which $N = 168$ and *Macbeth curve* [7.39], for which $N = 504$). For genera $g \leq 3$ there is a complete classification of Riemann surfaces with non-trivial groups of automorphisms that was obtained not so long ago for genus $g = 3$ by *Kuribayashi* and *Komiya* [7.13]. The examples of hyperelliptic curves with non-trivial symmetries that represent finite subgroups of the rotation group are given by *Horiuchi* [7.15].

We give a simple argument showing that in the case when the curve X_g has an automorphism T, there are additional restrictions on the period matrix B, resulting in a reduction of appropriate Riemann theta functions.

Suppose that on X_g with a fixed homology basis an automorphism T, $T : H_1(X_g, \mathbb{Z}) \to H_1(X_g, \mathbb{Z})$ acts according to the formulas

$$\begin{aligned} Ta_i &= \sum_{j=1}^{g} \{d_{ij}a_j + c_{ij}b_j\}, \quad i = 1, \dots, g \quad , \\ Tb_i &= \sum_{j=1}^{g} \{b_{ij}a_j + a_{ij}b_j\}, \quad i = 1, \dots, g \quad , \end{aligned} \tag{7.11.1}$$

in which a, b, c, d are such $(g \times g)$ integral matrices that the composite matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Tr}(2g, \mathbb{Z})$. Let $\omega = (\omega_1, \dots, \omega_g) \in \mathcal{H}^1(X_g, \mathbb{Z})$ be a basis of holomorphic differentials dual to a canonical homology basis. The representation T^* of the automorphism T in $\mathcal{H}^1(X_g, \mathbb{Z})$ acts according to the rule $T^*\omega(P) = \omega(TP)$, $P \in X_g$. Since the period matrix B remains unchanged during this process, (2.4.21) leads us to the equation

$$B(2\pi \mathrm{i} d + cB) = 2\pi \mathrm{i}(2\pi \mathrm{i} b + aB) \quad . \tag{7.11.2}$$

Relations (7.11.2) between matrix elements of B give a restriction on the period matrix of an algebraic curve that has non-trivial automorphism.

If on the algebraic curve X_g there acts a group of automorphisms G with generators T_k, $k = 1, \dots, s$, whose representation in $H_1(X_g, \mathbb{Z})$ is given by the generating matrices $a^{(k)}$, $b^{(k)}$, $c^{(k)}$, $d^{(k)}$, $\sigma^{(k)} = \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{pmatrix} \in \mathrm{Tr}(2g, \mathbb{Z})$ the B-matrix of such a Riemann surface must satisfy a set of equations

$$B(2\pi \mathrm{i} d^{(k)} + c^{(k)}B) = 2\pi \mathrm{i}(2\pi \mathrm{i} b^{(k)} + a^{(k)}B), \quad k = 1, \dots, s \quad . \tag{7.11.3}$$

It is possible to get a numerical value of a B-matrix for some curves, starting only from similar symmetry considerations, i.e., from (7.11.3). Furthermore, there are two ways of deriving the relations (7.11.1) between the "old" and the "new" basis. In the first case the Riemann surface is represented as a covering over a complex plane, and the second as a fundamental polygon of a *Fuchs group*, and the necessary relations (7.11.1) are provided by an analysis of the action of automorphism on the canonical basis of this polygon. The first approach is very convenient and effective for hyperelliptic curves; exactly this method was used in obtaining the numerical values of the B-matrix in [7.40] and [7.3 a-c]. An automorphic approach was, probably, first used by *Poincaré* [7.41] in order to calculate the period matrices for Klein curve. The technical errors committed in [7.41] were rectified in [7.40]. Another solution to the problem of calculating the B-matrix of a Klein curve was found in [7.3 a], using Baker's calculations [7.42]. Reference [7.3 a] also gives the reduction of appropriate theta functions to one-dimensional ones.

As a recent physical example of application of the present approach to the reduction, we refer to [7.43]. These works give examples of the reduction of theta functions of non-hyperelliptic curves such as

$$x^N y^N + x^N + y^N + 1/k^2 = 0, \quad N \geq 3 \quad , \tag{7.11.4}$$

related to the recently discovered integrable chiral model of Potts. The resulting four-dimensional theta function for $N = 3$ can be represented as a sum of 12 terms each of which is the product of four one-dimensional theta functions.

7.12 The Appel Theorem

There are several methods of representing in terms of theta functions of lower dimensions (Sect. 7.2) the theta functions constructed by the matrix B satisfying the condition of the Poincaré Theorem 7.1. The procedure of proving these expansions is the same in all cases: using a special form of the period matrix B, it is possible to replace the summation over the lattice, $\mathbb{Z}^g$, in formula (2.5.7) by summation over some sublattices. Next we use the *Appel Theorem* [7.44] (see also [7.3 a, b]) that enables us to express the g-dimensional theta function in terms of (g–1)-dimensional theta functions and Jacobi theta functions under certain conditions imposed on the period matrix (these conditions are a special case of the conditions of the Weierstrass and Poincaré Theorems in Sect. 7.2). To state the theorem, we give some definitions.

We suppose that the last column of the period matrix B satisfies the relations

$$\begin{aligned} m_j B_{jg} &= q_j, \quad j = 1, \ldots, \nu \quad , \\ m_j B_{jg} &= m_g B_{gg} + q_j, \quad j = \nu + 1, \ldots, g-1 \quad , \end{aligned} \tag{7.12.1}$$

where $m_k, q_j \in \mathbb{Z}$. We note that it is always possible to take $m_g > 0$, $m_j > 0$, $l = 1, \ldots, \nu$. If $m_k < 0$ for $k > \nu$, we make the following transformation: we invert the signs for z_k (argument of the theta function) and for n_k [summation index in (2.5.7)]. Under the transformation the theta function (2.4.30) remains unchanged, and m_k in (7.12.1) changes its sign and becomes positive. So, we can always think that in (7.12.1) $m_k \in \mathbb{N}$, $k = 1, \dots, g$, i.e., the change of signs described above is already affected. Furthermore, we agree to choose the smallest possible numbers m_k for which the (7.12.1) are still valid; in particular, it $B_{kg} = 0$, we take $m_k = 1$.

Theorem 7.24. (*Appel* [7.44]). Let the last column of the matrix B satisfy the conditions (7.12.1). Then

$$\begin{aligned} \theta^{(g)}(z; B) = & \sum_{r \in \mathbb{Z}^g(m)} \exp\left(\frac{1}{2}\langle Br, r\rangle + \langle r, z\rangle\right) \\ & \times \theta^{(g-1)}(y; A)\vartheta_3\left(\frac{y_g}{2\pi i}\middle|\frac{B_{gg}m_g^2}{2\pi i}\right) \quad , \end{aligned} \tag{7.12.2}$$

where the summation over r stands for a finite sum over $r = (r_1, \dots, r_g)$, $0 \le r_k \le m_k - 1$, $k = 1, \dots, g$, the indices (g) and $(g-1)$ indicate the dimension of an appropriate theta function, and the rest of the parameters are given by

$$\begin{aligned} & y = (y_1, \dots, y_{g-1}) = (\tilde{y}_1, \dots, \tilde{y}_\nu, \tilde{y}_{\nu+1} - \tilde{y}_g, \dots, \tilde{y}_{g-1} - \tilde{y}_g) \quad , \\ & \tilde{y}_k = m_k z_k + \frac{1}{2} m_k \frac{\partial}{\partial r_k}\langle Br, r\rangle, \quad k = 1, \dots, g \end{aligned} \tag{7.12.3}$$

$$A_{ii} = \begin{cases} m_i^2 B_{ii}, & \text{if} \quad i \le \nu \quad , \\ m_i^2 B_{ii} - m_g^2 B_{gg}, & \text{if} \quad i > \nu \quad , \end{cases}$$

$$A_{ij} = \begin{cases} m_i m_j B_{ij}, & \text{if} \quad i \quad \text{or} \quad j \le \nu \quad , \\ m_i m_j B_{ij} - m_g^2 B_{gg}, & \text{if} \quad i, j > \nu \quad . \end{cases}$$

In what follows we make use of a special case of this Theorem in which $q_i = 0$, $j = 1, \dots, g$.

Theorem 7.25. [7.3 b]. Let the last column of the matrix B satisfy the condition (7.12.1), and $q_j = 0$, $j = 1, \dots, g$. Then

$$\begin{aligned} \theta^{(g)}(z; B) = & \sum_{z \in \mathbb{Z}^g(m)} \theta^{(g-1)}\begin{bmatrix}\alpha \\ 0\end{bmatrix}(y; A) \\ & \times \theta^{(1)}\begin{bmatrix}\delta \\ 0\end{bmatrix}(m_g z_g; m_g^2 B_{gg}) \quad , \end{aligned} \tag{7.12.4}$$

where $\alpha = (\alpha_1, \dots, \alpha_{g-1})$, $\alpha_j = r_j/m_j$, $j = 1, \dots, g-1$, $\delta = \sum_{j=\nu+1}^{g} r_j/m_j$, the matrix A is given, as before, by the expression (7.12.3), and

$$y_i = \begin{cases} m_j z_j, & \text{if} \quad j = 1, \dots, \nu \quad , \\ m_j z_j - m_g z_g, & \text{if} \quad j = \nu + 1, \dots, g - 1 \quad . \end{cases} \tag{7.12.5}$$

We note that the condition of the Appel Theorem depends on the choice of a canonical homology basis, i.e., the B-matrix that satisfies the condition (7.12.1) in one canonical basis no longer satisfies it in another basis. We will not discuss here the problem of constructing the general transformation $\sigma \in \mathrm{Tr}(2g, \mathbb{Z})$, which can be used to derive a homology basis needed for the Appel Theorem to be applied. We only note that this basis can be easily found for a number of cases [7.3 a, b].

7.13 Reduction of the Theta Functions of Normal Coverings

For a number of definite classes of normal coverings, it is possible to indicate some homology basis and to examine in it the reduction of appropriate theta functions. In particular, *Fay* [7.6] considered two reducible cases: (1) the involution $T : Xg \to Xg$ has n pairs of fixed points; (2) cyclic automorphism $T : Xg \to Xg$ of the pth order ($T^p = 1$) has no fixed points.

Here we discuss the first case. Let Xg_1 be a Riemann surface of genus g_1, equipped with the involution $T : Xg_1 \to Xg_1$ with fixed points $Q_1, \dots, Q_{2n}$. We denote by Xg_0 the quotient $Xg_0 = Xg_1/\langle T \rangle$, and by g_0 the genus of Riemann surface Xg_0. We define a two-sheeted branch cover $\pi : Xg_1 \to Xg_0$ over Xg_0. According to the Riemann-Hurwitz formula (2.2.1), the genus g_1 is equal to $g_1 = 2g_0 + n - 1$. The canonical homology basis in $H_1(Xg, \mathbb{Z})$, $(a_1, \dots, a_{g_0}, a_{g_0+1}, \dots, a_{g_0-1}, a'_1, \dots, a'_{g_0}; b_1, \dots, b_{g_0}, b_{g_0+1}, \dots, b_{g_0+n-1}, b'_1, \dots, b'_{g_0})$ can be chosen such that $(a_1, \dots, a_{g_0}; b_1 \dots, b_{g_0})$ is a canonical basis in $H_1(Xg_0, \mathbb{Z})$ and

$$\begin{aligned} &a'_\nu + Ta_\nu = b'_\nu + Tb_\nu = 0, \quad 1 \le \nu \le g_0 \quad , \\ &a_i + Ta_i = b_i + Tb_i = 0, \quad g_0 + 1 \le i \le g_0 + n - 1 \quad . \end{aligned}$$

When $g = 3$, $g_0 = 1$, $n = 2$, see the homology basis in Fig. 7.2. When $1 \le \nu \le g_0$ and $g_0 + 1 \le i \le g_0 + n - 1$, the following equations are valid for dual normalized holomorphic differentials $\omega_1, \dots, \omega_{g_0}, \omega_{g_0+1}, \dots, \omega'_1, \dots, \omega'_{g_0}$:

$$\begin{aligned} &\omega_\nu(P) = -\omega'_\nu(TP) \quad , \\ &\omega_i(P) = -\omega_i(TP), \quad P \in Xg_1 \quad . \end{aligned} \tag{7.13.1}$$

The holomorphic differentials on Xg_0, dual to the canonical basis, $(a_1, \dots, a_{g_0}; b_1 \dots, b_{g_0}) \in H_1(Xg_0, \mathbb{Z})$ are equal to $v_\nu = \omega_\nu - \omega'_\nu$, $\nu = 1, \dots, g_0$, and the expressions

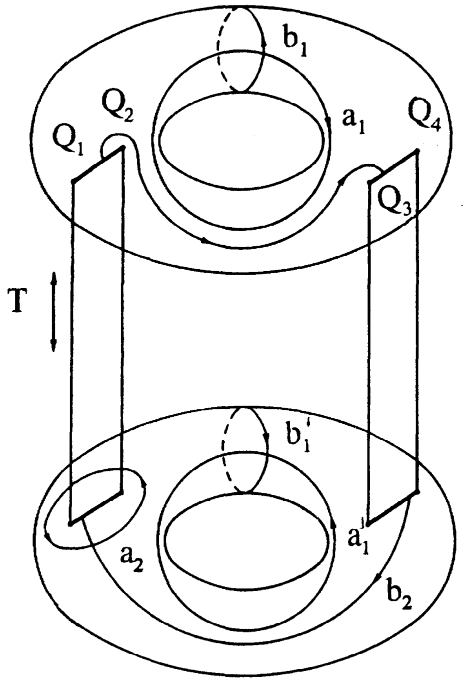

Fig. 7.2. Homology basis of $H(X_3, \mathbb{Z})$, when X_3 is a Riemann surface that doubly covers a torus

$$w_\nu = \omega_\nu + \omega'_\nu, \quad 1 \le \nu \le g_0; \quad w_i = \omega_i, \quad g_0 + 1 \le i \le g_0 + n - 1$$

give $g_0 + n - 1$ linearly independent normalized Prym differentials

$$v_\nu(P) = v_\nu(TP), \quad w_\nu(P) = -w_\nu(TP) \quad .$$

From (7.13.1) it follows that the period matrix B_1 of the Riemann surface Xg_1 has the form

$$B_1 = \begin{pmatrix} (\Pi_{\nu\lambda} + B^0_{\nu\lambda})/2 & \Pi_{\nu j} & (\Pi_{\nu\lambda} - B^0_{\nu\lambda})/2 \\ \Pi_{i\nu} & 2\Pi_{ij} & \Pi_{i\nu} \\ (\Pi_{\nu\lambda} - B^0_{\nu\lambda})/2 & \Pi_{ij} & (\Pi_{\nu\lambda} + B^0_{\nu\lambda})/2 \end{pmatrix} \quad , \tag{7.13.2}$$

where $1 \le \lambda,\ \nu \le g_0,\ g_0+1 < i,\ j \le g_0+n-1,\ B^0$ is the period matrix of the Riemann surface Xg_0, Π is the symmetric Prym $((g_0+n-1)\times(g_0+n-1))$-matrix

$$\Pi = \begin{pmatrix} \Pi_{\nu\lambda} & \Pi_{\nu j} \\ \Pi_{i\lambda} & \Pi_{ij} \end{pmatrix} = \begin{pmatrix} \int_{b_\lambda} w_\nu & 1/2\int_{b_j} w_\nu \\ \int_{b_\lambda} w_i & 1/2\int_{b_j} w_i \end{pmatrix} \quad . \tag{7.13.3}$$

We denote by $z = (z_1|z_2|z_3)$ a g_1-dimensional vector where z_1 and z_3 are g-dimensional vectors, and z_2 is an $(n-1)$-dimensional vector; $z' = (z'_1|z'_2) \in \mathbb{C}^{g_0+n-1}$, $z'_1 \in \mathbb{C}^{g_0}$, $z'_2 \in \mathbb{C}^{n-1}$.

Theorem 7.26. Let the matrix B_1 be given by (7.13.2). Then the following equation holds:

$$\begin{aligned} \theta((z_1|z_2|z_3); B_1) = \sum_{\delta\in(1/2)\mathbb{Z}^{g_0}/2\mathbb{Z}^{g_0}} \theta\begin{bmatrix}(\delta|0)\\0\end{bmatrix}((z_1 \\ + z_3|z_2); 2\Pi)\theta\begin{bmatrix}\delta\\0\end{bmatrix}(z_1 - z_3); 2B^0) \quad , \end{aligned} \tag{7.13.4}$$

where the summation is over all g_0-dimensional vectors with components 0 and 1/2, the first co-factor under the summation sign is a $(g_0 + n - 1)$-dimensional theta function, and the second co-factor is a g_0-dimensional theta function.

The proof of this theorem is given in [7.5] (see also the appendix in [7.3 b]). If $\pi : X \to X/\langle T\rangle$ is a cyclic unramified cover then there exists an expansion [7.5] similar to (7.13.4).

7.14 Example: 3-Gap Solution of the Sine-Gordon Equation in Terms of Elliptic Functions

We consider the simplest curve X_2 to which Theorem 7.26 is applicable; this is the curve of genus two,

$$\mu^2 = (\lambda^2 - e_1^2)(\lambda^2 - e_2^2)(\lambda^2 - e_3^2) \quad , \tag{7.14.1}$$

which is equipped by the homology basis given in Fig. 7.3. We consider the involution that acts on the curve (7.14.1). The involution T does not permute the sheets; its fixed points are over ∞ on both sheets, i.e., $n = 1$. The curve $X_1 = X_2/T$ (Fig. 7.4) is given by

$$\widetilde{\mu}^2 = \xi(\xi - e_1^2)(\xi - e_2^2)(\xi - e_3^2) \quad . \tag{7.14.2}$$

The normalized holomorphic differentials on X_2 and $\widetilde{X}_1$ are

$$u_1 = \frac{-c_1\lambda + c_2}{\mu(\lambda)}d\lambda, \quad u_1' = \frac{-c_1\lambda - c_2}{\mu(\lambda)}d\lambda \quad ,$$
$$v = u_1 - u_1' = \frac{2c_2 d\lambda}{\mu(\lambda)} = \frac{c_2 d\xi}{\widetilde{\mu}(\xi)} \quad .$$

The normalized Prym differential

$$w = u_1 + u_1' = -\frac{2c_1\lambda d\lambda}{\mu(\lambda)} = -\frac{c_1 d\eta}{\widetilde{\mu}(\eta)}$$

is also elliptic; it is given on the curve $\widetilde{\widetilde{X}}_1$ (Fig. 7.5) and defined by

$$\widetilde{\widetilde{\mu}}^2 = (\eta - e_1^2)(\eta - e_2^2)(\eta - e_3^2) \quad . \tag{7.14.3}$$

The constants c_1 and c_2 are determined from the normalization $\int_a v = 1$, $\int_a w = 1$, where the integration is over the a-cycles of Riemann surface $\widetilde{X}_1$ and $\widetilde{\widetilde{X}}_1$, respectively; besides, $B^0 = \int_b v$, $\Pi = \int_b w$. According to (7.13.4) the theta function of the curve (7.14.1) is represented by one-dimensional theta functions

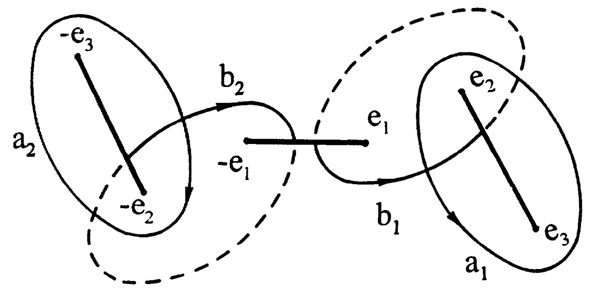

Fig. 7.3. Homology basis of the curve (7.14.1)

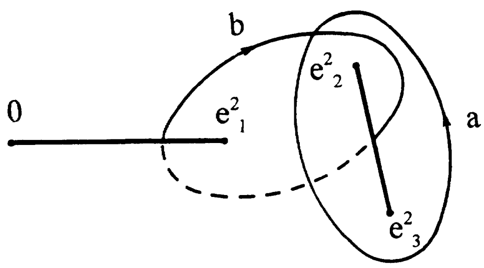

Fig. 7.4. Homology basis of the curve (7.14.2)

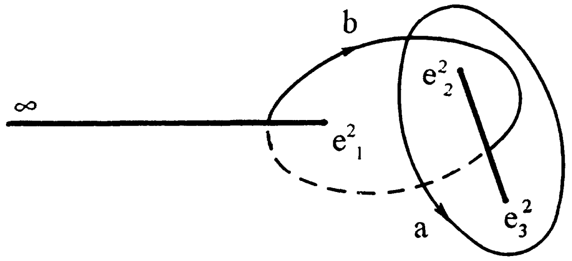

Fig. 7.5. Homology basis of the curve (7.14.3)

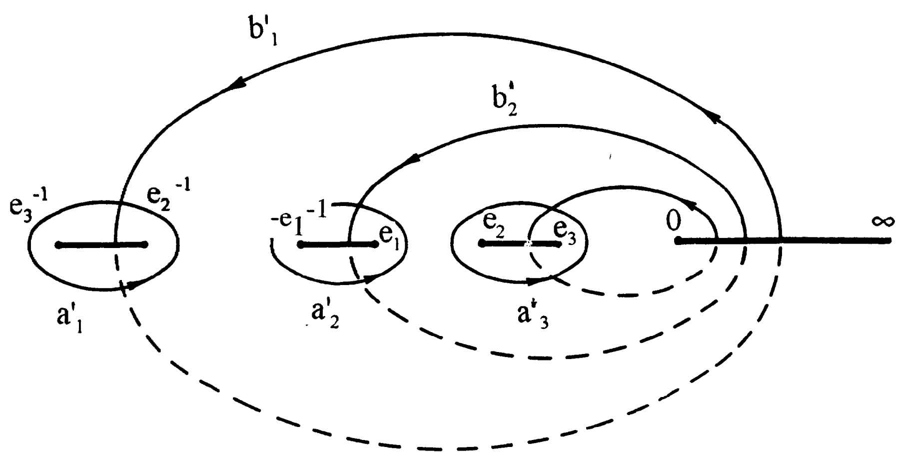

Fig. 7.6. Homology basis of the curve (7.14.5)

$$
\begin{aligned}
\theta((z_1|z_2); B) =& \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_1 + z_2; 2\Pi) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_1 - z_2; 2B^0) \\
&+ \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z_1 + z_2; 2\Pi) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_1 - z_2; 2B) \quad .
\end{aligned} \tag{7.14.4}
$$

We now consider the curve X_3 of genus $g = 3$ (Fig. 7.6)

$$
\mu^2 = \lambda(\lambda - e_1)(\lambda - e_1^{-1})(\lambda - e_2)(\lambda - e_2^{-1})(\lambda - e_3)(\lambda - e_3^{-1}) \quad . \tag{7.14.5}
$$

It has the involution $T_1 : \lambda \to \lambda^{-1}$. Under the action of this involution the homology basis changes according to (7.11.1), where

$$
T_1 b' = S_1' b', \quad c_1' = b_1' = 0, \quad T_1 a' = (S_1'^T)^{-1} a' \quad ,
$$

$$
S_1' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad .
$$

Here $b' = (b_1', b_2', b_3')$ stands for the vector whose components are the b-cycles illustrated in Fig. 7.6. According to (7.11.2), we can already get a restriction on the period matrix B of the curve (7.14.5), but it is straightforward to go over to another canonical basis $b = \Phi b'$, $a = (\Phi^T)^{-1} a'$,

$$
\Phi = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad . \tag{7.14.6}
$$

Under the action of the involution T_1, this new basis $(a, b) \in H_1(X_3, \mathbb{Z})$ is transformed by the matrix S_1, related to S_1' by the similarity transformation $S_1 = \Phi S_1' \Phi^{-1}$, $T_1 b = S_1 b$, $T_1 a = (S_1^T)^{-1} a$, where

$$
S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{7.14.7}
$$

With the restriction (7.11.2), the period matrix calculated in the basis (a, b) has the form

$$
B = \begin{pmatrix} \tau_3 & \tau_2 & 0 \\ \tau_2 & \tau_4 & \tau_1 \\ 0 & \tau_1 & 2\tau_1 \end{pmatrix} = \begin{pmatrix} \tau_3 & \tau_2 & 0 \\ \tau_2 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & & \end{pmatrix} \quad . \tag{7.14.8}
$$

The last column of this matrix satisfies the conditions of Theorem 7.25 ($\nu = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 1$), such that

$$
\begin{aligned}
\theta(z; B) =& \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (z_1, 2z_2 - z_3; A) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_3; 2\tau_1) \\
&+ \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (z_1, 2z_2 - z_3; A) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z_3; 2\tau_1) \quad .
\end{aligned} \tag{7.14.9}
$$

If the curve (7.14.5) also has the involution

$$T_2 : \lambda \longrightarrow \frac{\lambda - e_1}{e_1\lambda - 1} \tag{7.14.10}$$

with the parameters e_1, e_2, e_3 being related by $e_2 - e_1 = e_3(e_1e_2 - 1)$ we have, following the argument used for the involution T_1,

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_3 = 2\tau_2 \quad . \tag{7.14.11}$$

We thus conclude that according to Theorem 7.25 (in this case $\nu = 0$, $m_1 = m_2 = 1$, and it is necessary to replace the last column with the first one in the condition of the Theorem), the two-dimensional theta function in (7.14.9) is just a sum of products of one-dimensional ones[3]

$$\begin{aligned} \theta(z; B) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_1; 2\tau_2)\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (2z_2 - z_1 - z_3; 4\tau_4 - 2\tau_1 - 2\tau_2) \\ \times \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_3; 2\tau_1) + \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z_1; 2\tau_2) \\ \times \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z_3; 2\tau_1)\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (2z_2 - z_1 - z_3; 4\tau_4 - 2\tau_2 - 2\tau_1) \quad . \end{aligned} \tag{7.14.12}$$

In this case all basic holomorphic differentials are reduced to elliptic ones, and the quantities τ_i are expressed through elliptic integrals. In [7.3 c] the curve conformally equivalent to the curve (7.14.5) with the involution (7.14.10) is discussed.

We apply these results in order to derive a solution to the sine-Gordon equation (7.9.1) in terms of elliptic functions. For this purpose we represent the curve $X_3 = (\mu, \lambda)$,

$$\mu^2 = \prod_{j=1}^{6} \lambda(\lambda - E_j) \tag{7.14.13}$$

as in (7.14.5), i.e., we require that the branch point projections $E_1, \ldots, E_6$ satisfy the conditions

$$\begin{aligned} E_1 = E_2^{-1} = e_1, \quad E_3 = E_4^{-1} = e_2 \quad , \\ E_5 = E_6^{-1} = e_3; \quad e_1, e_2, e_3 \in \mathbb{C} \quad . \end{aligned} \tag{7.14.14}$$

Since the B-matrix of this curve is calculated (7.14.8) to derive a reduced solution to (7.9.1), it is necessary to find the vectors $U^{\pm}$, D, see (7.9.2), with the vector D being defined via the reality condition for the solution. We note that since the involution T does not change the sheets X_3, the following Abelian differentials of second type are equal:

[3] A theta function with non-zero characteristics can be represented in a similar way [7.3 c].

$$u^{(1)}_{T_1P} = u^{(1)}_P, \quad u^{(2)}_{T_1P} = -u^{(2)}_P \quad . \tag{7.14.15}$$

From (7.14.15, 7) and (7.9.3) it follows that

$$\boldsymbol{U}^+ = T_1\boldsymbol{U}^+ = (U_1^+, U_2^+, 0), \quad \boldsymbol{U}^- = -T_1\boldsymbol{U}^- = (0, U_2^-, 2U_2^-) \quad . \tag{7.14.16}$$

From this and the expansion (7.14.9) we get

Proposition 7.27. ([7.3 c]) If the branch point projections $E_1, \dots, E_6$ for the curve (7.14.13) satisfy the conditions (7.14.14), the solution (7.9.2) of the equation (7.9.1) has the form

$$\begin{aligned} \theta(x,t) = 2\mathrm{i} \log \Bigg\{ & \Bigg[\theta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y_1, y_2; A) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y_3; 2\tau_1) \\ & - \theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y_1, y_2; A)\, \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (y_3; 2\tau_1) \Bigg] \\ & \times \Bigg[\theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y_1, y_2; A) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y_3; 2\tau_1) \\ & + \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y_1, y_2; A) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (y_3; 2\tau_1) \Bigg]^{-1} \Bigg\} \quad , \end{aligned} \tag{7.14.17}$$

where the parameter τ_1 and the 2×2 matrix A are defined by (7.14.8) and

$$y_1 = \frac{U_1^+}{2\pi} x + D_1, \quad y_2 = \frac{U_2^+}{\pi} x + 2D_2 - D_3, \quad y_3 = \frac{U_2^-}{\pi} t + D_3 \quad .$$

The solution (7.14.17) is a 6-parametric (parameters $e_1, e_2, e_3 \in \mathbb{C}$, $\boldsymbol{W} \in \mathbb{C}$) family of solutions doubly periodic in t (replacement $x \to \mathrm{i}t$, $t \to \mathrm{i}x$ produces a solution doubly periodic in x). We note further that the general finite-gap solutions of genus g to the equation (7.9.1) are parameterized by $3g - 1$ independent parameters ($2g - 1$ corresponds to branch points and $\boldsymbol{D} \in \mathbb{C}^g$); the periodicity condition leads additionally (the period is not fixed) to a $g - 1$ restriction. So, the family of solutions (7.14.17) that we have constructed has the same number of independent parameters as the general periodic solution of genus 3.

To isolate real solutions we use the results of Sect. 4.3. It turns out that in the present case, when $e_1, e_2, e_3 \in \mathbb{R}$, $\boldsymbol{\Delta} = (1, 1, 0)$, $\mathrm{Re}\, \boldsymbol{U}^+ = \mathrm{Re}\, \boldsymbol{V}^- = 0$, $B = -\overline{B}$, there are four components of the real solutions $\mathrm{Re}\, \boldsymbol{D} = \mathrm{Re}(D_1, D_2, D_3) = \{(1/4, 1/4, 0), (3/4, 1/4, 0), (1/4, 1/4, 1/2), (3/4, 1/4, 1/2)\}$ that are non-trivially different from one another. Furthermore, (7.14.17) describes a real solution periodic in t. Equation (7.9.1) is invariant under the transformation $x \to t$, $t \to x$, $\varphi \to \varphi + \pi$, by means of which a real solution periodic in x is derived from (7.14.17). Using the results of Sect. 4.3, we can also distinguish real solutions for the surface (7.14.5) with complex branch points.

If the curve X_3 has a dihedral group of automorphisms, $T_1^2 = T_2^2 = 1$, $T_1T_2 = T_2T_1$ (7.14.10), then the initial three-dimensional theta function is expressed in terms of one-dimensional ones by (7.14.12), so that we have from (7.14.17) a solution in terms of Jacobi theta functions

$$
\begin{aligned}
u(x,t) = {} & 2\mathrm{i}\log \\
& \times\left\{\left[\theta\begin{bmatrix}0\\1\end{bmatrix}(y_1;2\tau_2)\theta\begin{bmatrix}0\\1\end{bmatrix}(y_4;4\tau_4-2\tau_1-2\tau_2)\theta\begin{bmatrix}0\\0\end{bmatrix}(y_3;2\tau_1)\right.\right. \\
& \left.+\theta\begin{bmatrix}1\\1\end{bmatrix}(y_1;2\tau_2)\theta\begin{bmatrix}1\\1\end{bmatrix}(y_4;4\tau_4-2\tau_1-2\tau_2)\theta\begin{bmatrix}1\\0\end{bmatrix}(y_3;2\tau_1)\right] \\
& \times\left[\theta\begin{bmatrix}0\\0\end{bmatrix}(y_1;2\tau_2)\theta\begin{bmatrix}0\\0\end{bmatrix}(y_4;4\tau_4-2\tau_1-2\tau_2)\theta\begin{bmatrix}0\\0\end{bmatrix}(y_3;2\tau_1)\right. \\
& \left.\left.+\theta\begin{bmatrix}1\\0\end{bmatrix}(y_1;2\tau_2)\theta\begin{bmatrix}1\\0\end{bmatrix}(y_4;4\tau_4-2\tau_1-2\tau_2)\theta\begin{bmatrix}1\\0\end{bmatrix}(y_3;2\tau_1)\right]^{-1}\right\} \quad ,
\end{aligned}
\tag{7.14.18}
$$

where

$$
y_4 = y_2 - y_1 = \frac{1}{2\pi}(2U_2^+ - U_1^+)x + 2D_2 - D_3 - D_1 \quad .
$$

We note that the existence of involution T_2 does not imply restrictions on the vectors $\boldsymbol{U}$ and $\boldsymbol{V}$, similar to (7.14.16), because the singularities of differentials $u_P^{(i)}$ are not invariant with respect to this involution.

Under the conformal transformation

$$
\zeta = \frac{\lambda+1}{\lambda-1}\sqrt{\frac{1-e_1}{1+e_1}} \quad , \tag{7.4.19}
$$

the curve (7.14.5) changes to the curve

$$
\mu^2 = (\zeta^2-u^2)(\zeta^2-v^2)(\zeta^2-u^{-2})(\zeta^2-v^{-2}) \quad , \tag{7.14.20}
$$

whose involutions T_1 and T_2 are very simple:

$$
T_1 : (\mu,\lambda) \to (\mu,-\zeta), \quad T_2 : (\mu,\lambda) \to (\mu\zeta^{-3},\zeta^{-1}) \quad . \tag{7.14.21}
$$

The solution of (7.9.1) that corresponds to the Riemann surface (7.14.20) is constructed in [7.3 c, d]. Since the curves (7.14.5, 21) are conformally equivalent by (7.14.19), the solutions constructed in [7.3 c, d] are the same as in (7.14.18). Therefore, here we do not give the expressions in terms of the branch points X_3 and complete elliptic integrals for the constants in the solution (7.14.18). For the curve (7.14.20), these are presented in [7.3 d].

To conclude, we note that the approach of Sects. 7.11-14 made it possible to derive new solutions in terms of elliptic functions for a number of soliton equations involving non-hyperelliptic curves [7.3 f-h].

Appendix 7.1 Relations Between Theta Constants for $g = 2$

Here we give three groups of formulas which are a consequence of the Riemann theta formula (2.4.1) for theta constants when $g = 2$. These are the relations between the fourth powers of even theta constants (7.A.1), the relations between the squares of even theta constants (7.A.2) and Rosenhain formulas (7.A.3). In (7.A.1, 2) each of the rows at the right gives the characteristic equal to a sum of the characteristics that make up the term

$$\begin{aligned}
\theta^4\begin{bmatrix}0\,0\\0\,0\end{bmatrix} - \theta^4\begin{bmatrix}0\,0\\1\,1\end{bmatrix} &= \theta^4\begin{bmatrix}1\,0\\0\,1\end{bmatrix} + \theta^4\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\\
&= \theta^4\begin{bmatrix}1\,0\\0\,0\end{bmatrix} + \theta^4\begin{bmatrix}0\,1\\1\,0\end{bmatrix},\\
\theta^4\begin{bmatrix}0\,0\\0\,0\end{bmatrix} - \theta^4\begin{bmatrix}1\,1\\0\,0\end{bmatrix} &= \theta^4\begin{bmatrix}0\,1\\1\,0\end{bmatrix} + \theta^4\begin{bmatrix}0\,0\\0\,1\end{bmatrix}\\
&= \theta^4\begin{bmatrix}0\,0\\1\,0\end{bmatrix} + \theta^4\begin{bmatrix}1\,0\\0\,1\end{bmatrix},\\
\theta^4\begin{bmatrix}0\,0\\0\,0\end{bmatrix} - \theta^4\begin{bmatrix}1\,1\\1\,1\end{bmatrix} &= \theta^4\begin{bmatrix}1\,0\\0\,0\end{bmatrix} + \theta^4\begin{bmatrix}0\,0\\1\,0\end{bmatrix}\\
&= \theta^4\begin{bmatrix}0\,1\\0\,0\end{bmatrix} + \theta^4\begin{bmatrix}0\,0\\0\,1\end{bmatrix},
\end{aligned}\tag{7.A.1}$$

$$\begin{aligned}
\theta^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix} &= \theta^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\\
&+ \theta^2\begin{bmatrix}0\,0\\0\,1\end{bmatrix}\theta^2\begin{bmatrix}1\,0\\0\,1\end{bmatrix}, \quad \left(\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix} &= \theta^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\\
&+ \theta^2\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\theta^2\begin{bmatrix}0\,0\\1\,0\end{bmatrix}, \quad \left(\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix} &= \theta^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\\
&+ \theta^2\begin{bmatrix}0\,0\\1\,1\end{bmatrix}\theta^2\begin{bmatrix}1\,1\\1\,1\end{bmatrix}, \quad \left(\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}0\,0\\1\,0\end{bmatrix} &= \theta^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\theta^2\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\\
&+ \theta^2\begin{bmatrix}0\,0\\0\,1\end{bmatrix}\theta^2\begin{bmatrix}0\,0\\1\,1\end{bmatrix}, \quad \left(\begin{bmatrix}0\,0\\1\,0\end{bmatrix}\right);
\end{aligned}$$

$$
\begin{aligned}
\theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix} &= \theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}, \quad \left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix} &= \theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad \left(\begin{bmatrix}0&1\\1&0\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\0&0\end{bmatrix} &= = \theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}, \quad \left(\begin{bmatrix}1&1\\1&0\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix} &= \theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}, \quad \left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix} &= \theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&0\\1&0\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}, \quad \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix} &= \theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}, \quad \left(\begin{bmatrix}0&1\\0&1\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix} &= \theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta^2\begin{bmatrix}0&1\\0&1\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}, \quad \left(\begin{bmatrix}1&1\\0&1\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix} &= \theta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}, \quad \left(\begin{bmatrix}0&0\\1&1\end{bmatrix}\right);\\
\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix} &= \theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\\
&\quad+\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta^2\begin{bmatrix}0&0\\1&0\end{bmatrix}, \quad \left(\begin{bmatrix}1&0\\1&1\end{bmatrix}\right);
\end{aligned}
\tag{7.A.2}
$$

$$\begin{aligned}\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta^2\begin{bmatrix}0&1\\0&0\end{bmatrix} &= \theta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\\ &\quad+\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}, \qquad \left(\begin{bmatrix}0&1\\1&1\end{bmatrix}\right);\\ \theta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta^2\begin{bmatrix}0&0\\1&1\end{bmatrix} &= \theta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\\ &\quad+\theta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}, \qquad \left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right).\end{aligned}$$

We introduce the notation $D([\varepsilon],[\delta]) = \theta_1[\varepsilon]\theta_2[\delta] - \theta_2[\varepsilon]\theta_1[\delta]$. Then the Rosenhain formulas are valid

$$\begin{aligned}
&4D\left(\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}, && \left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix}\right) = \theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}, && \left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&1\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}, && \left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}, && \left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}, && \left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}, && \left(\begin{bmatrix}0&1\\1&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}, && \left(\begin{bmatrix}1&1\\1&0\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}1&0\\1&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}, && \left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right); \quad (7.A.3)\\
&4D\left(\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}, && \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\1&1\end{bmatrix}\right) = \theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}, && \left(\begin{bmatrix}0&1\\0&1\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&1\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}, && \left(\begin{bmatrix}1&1\\0&1\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}, && \left(\begin{bmatrix}0&0\\1&1\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&1\\1&0\end{bmatrix}, && \left(\begin{bmatrix}1&0\\1&1\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}1&1\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}, && \left(\begin{bmatrix}0&1\\1&1\end{bmatrix}\right);\\
&4D\left(\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\right) = \theta\begin{bmatrix}0&1\\0&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}, && \left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right).
\end{aligned}$$

Appendix 7.2 Addition Formulas for Second-Order Theta Functions at $g = 2$

Here we give the expanded forms of (2.6.8) for $g = 2$, which for instance, were used in deriving a covering in Sect. 7.6. As before, we introduce the notation $\hat{\theta}[\varepsilon](z) = \theta[\varepsilon](z; 2B)$.

$$
\begin{aligned}
\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z) &= \hat{\theta}^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z) + \hat{\theta}^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\\
&\quad + \hat{\theta}^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z) + \hat{\theta}^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,0\\1\,1\end{bmatrix}\theta\begin{bmatrix}0\,0\\1\,1\end{bmatrix}(z) &= \hat{\theta}^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z) + \hat{\theta}^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\\
&\quad - \hat{\theta}^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z) - \hat{\theta}^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z) &= \hat{\theta}^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z) - \hat{\theta}^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\\
&\quad - \hat{\theta}^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z) + \hat{\theta}^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,0\\0\,1\end{bmatrix}\theta\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z) &= \hat{\theta}^2\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z) - \hat{\theta}^2\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\\
&\quad + \hat{\theta}^2\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z) - \hat{\theta}^2\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z).
\end{aligned}
\tag{7.A.4}
$$

$$
\begin{aligned}
\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\\
&\quad + 2\hat{\theta}\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix}\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\\
&\quad - 2\hat{\theta}\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\\
&\quad + 2\hat{\theta}\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z),
\end{aligned}
$$

$$\begin{aligned}
\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\\
&\quad -2\hat{\theta}\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\\
&\quad +2\hat{\theta}\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,0\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\\
&\quad -2\hat{\theta}\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,1\\0\,0\end{bmatrix}(z).
\end{aligned} \tag{7.A.5}$$

$$\begin{aligned}
\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\theta\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\\
&\quad +2\hat{\theta}\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix}\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\\
&\quad -2\hat{\theta}\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\theta\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z)\\
&\quad +2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\theta\begin{bmatrix}0\,1\\1\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z)\\
&\quad -2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\\
&\quad +2\hat{\theta}\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}\theta\begin{bmatrix}1\,0\\1\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\\
&\quad -2\hat{\theta}\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\hat{\theta}\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z).
\end{aligned} \tag{7.A.6}$$

$$\begin{aligned}
\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}\theta\begin{bmatrix}0\,0\\1\,1\end{bmatrix}(z) &= \hat{\theta}^2\begin{bmatrix}0\,0\\1\,1\end{bmatrix}(z)+\hat{\theta}^2\begin{bmatrix}1\,1\\1\,1\end{bmatrix}(z)\\
&\quad+\hat{\theta}^2\begin{bmatrix}1\,0\\1\,1\end{bmatrix}(z)+\hat{\theta}^2\begin{bmatrix}0\,1\\1\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}\theta\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z) &= \hat{\theta}^2\begin{bmatrix}0\,0\\1\,1\end{bmatrix}(z)-\hat{\theta}^2\begin{bmatrix}1\,1\\1\,1\end{bmatrix}(z)\\
&\quad-\hat{\theta}^2\begin{bmatrix}1\,0\\1\,1\end{bmatrix}(z)+\hat{\theta}^2\begin{bmatrix}0\,1\\1\,1\end{bmatrix}(z).
\end{aligned}\tag{7.A.7}$$

$$\begin{aligned}
\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\theta_k\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z)\\
&\quad+2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix}\theta_k\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\\
&\quad-2\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\theta_k\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\\
&\quad+2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z), \qquad k=1,2\ .\\
\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\theta_k\begin{bmatrix}0\,1\\1\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}0\,1\\0\,1\end{bmatrix}(z)\\
&\quad-2\hat{\theta}\begin{bmatrix}1\,0\\0\,1\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,1\\0\,1\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\theta_k\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\\
&\quad+2\hat{\theta}\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z),\\
\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}\theta_k\begin{bmatrix}1\,0\\1\,1\end{bmatrix}(z) &= 2\hat{\theta}\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\\
&\quad-2\hat{\theta}\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z)\hat{\theta}_k\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z),
\end{aligned}\tag{7.A.8}$$

Appendix 7.3 Theta Constants of 2- and 4-Sheeted Coverings Over a Torus

Here we use the formulas given in Appendix 7.2, to construct reduced theta constants for a 2-sheeted covering of genus 2 (item b);

a) Let $B = \begin{pmatrix} B_{11} & \pi\mathrm{i} \\ \pi\mathrm{i} & B_{22} \end{pmatrix}$ and $\vartheta_j = \vartheta_j\left(0, \frac{B_{11}}{\pi\mathrm{i}}\right)$, $\widetilde{\vartheta}_j = \vartheta_j\left(0, \frac{B_{22}}{\pi\mathrm{i}}\right)$, $j = 2,3,4$.
Then

$$
\begin{aligned}
&\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix} = \theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix} = (2\vartheta_2\vartheta_3\widetilde{\vartheta}_3\widetilde{\vartheta}_4)^{1/2},\\
&\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix} = \theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix} = (2\vartheta_3\vartheta_4\widetilde{\vartheta}_2\widetilde{\vartheta}_3)^{1/2},\\
&\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix} = \mathrm{i}\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix} = (2\vartheta_2\vartheta_4\widetilde{\vartheta}_2\widetilde{\vartheta}_4)^{1/2},\\
&\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 + \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},\\
&\theta\begin{bmatrix}0\,0\\1\,1\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 - \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},\\
&\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 - \vartheta_2^2\widetilde{\vartheta}_4^2 + \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},\\
&\theta\begin{bmatrix}0\,0\\0\,1\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},
\end{aligned}
\tag{7.A.9}
$$

$$
\begin{aligned}
&\theta_1\begin{bmatrix}1\,1\\1\,0\end{bmatrix} = \frac{\mathrm{i}}{2}\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix}\vartheta_3^2, \quad \theta_2\begin{bmatrix}1\,1\\1\,0\end{bmatrix} = -\frac{1}{2}\theta\begin{bmatrix}1\,1\\1\,1\end{bmatrix}\widetilde{\vartheta}_3^2,\\
&\theta_1\begin{bmatrix}1\,1\\0\,1\end{bmatrix} = -\frac{1}{2}\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\vartheta_3^2, \quad \theta_2\begin{bmatrix}1\,1\\0\,1\end{bmatrix} = \frac{\mathrm{i}}{2}\theta\begin{bmatrix}1\,1\\0\,0\end{bmatrix}\widetilde{\vartheta}_3^2,\\
&\theta_1\begin{bmatrix}0\,1\\0\,1\end{bmatrix} = -\frac{1}{2}\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\vartheta_2^2, \quad \theta_2\begin{bmatrix}0\,1\\0\,1\end{bmatrix} = \frac{\mathrm{i}}{2}\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}\widetilde{\vartheta}_4^2,\\
&\theta_1\begin{bmatrix}0\,1\\1\,1\end{bmatrix} = \frac{1}{2}\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\vartheta_2^2, \quad \theta_2\begin{bmatrix}0\,1\\1\,1\end{bmatrix} = \frac{\mathrm{i}}{2}\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}\widetilde{\vartheta}_4^2,\\
&\theta_1\begin{bmatrix}1\,0\\1\,1\end{bmatrix} = \frac{\mathrm{i}}{2}\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}\vartheta_4^2, \quad \theta_2\begin{bmatrix}1\,0\\1\,1\end{bmatrix} = \frac{1}{2}\theta\begin{bmatrix}1\,0\\0\,1\end{bmatrix}\widetilde{\vartheta}_2^2,\\
&\theta_1\begin{bmatrix}1\,0\\1\,0\end{bmatrix} = \frac{\mathrm{i}}{2}\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\vartheta_4^2, \quad \theta_2\begin{bmatrix}1\,0\\1\,0\end{bmatrix} = \frac{1}{2}\theta\begin{bmatrix}1\,0\\0\,0\end{bmatrix}\widetilde{\vartheta}_2^2.
\end{aligned}
\tag{7.A.10}
$$

b) Let $B = \begin{pmatrix} B_{11} & \pi\mathrm{i}/2 \\ \pi\mathrm{i}/2 & B_{22} \end{pmatrix}$ $X = \vartheta_3\widetilde{\vartheta}_3$, $Y = \vartheta_2\widetilde{\vartheta}_4$, $Z = \vartheta_4\widetilde{\vartheta}_2$, $\vartheta_k = \vartheta_k(0; 2B_{11}/\pi\mathrm{i})$, $\widetilde{\vartheta}_k = \vartheta_k(0; 2B_{22}/\pi\mathrm{i})$, $k = 2,3,4$, $A = -X^2 + Y^2 + Z^2$, $B = X^2 - Y^2 + Z^2$, $C = X^2 + Y^2 - Z^2$, $D = A + B + C$. Then the following formulas hold:

$$
\begin{aligned}
&\theta\begin{bmatrix} 0\ 0 \\ 0\ 0 \end{bmatrix} = X + Y + Z, \quad \theta\begin{bmatrix} 0\ 0 \\ 0\ 1 \end{bmatrix} = X + Y - Z, \\
&\theta\begin{bmatrix} 0\ 0 \\ 1\ 0 \end{bmatrix} = X - Y + Z, \quad \theta\begin{bmatrix} 0\ 0 \\ 1\ 1 \end{bmatrix} = X - Y - Z, \\
&\theta^2\begin{bmatrix} 1\ 0 \\ 0\ 0 \end{bmatrix} = 2^{3/2}(XY)^{1/2}(D^{1/2} + 2^{1/2}Z), \\
&\theta^2\begin{bmatrix} 1\ 0 \\ 0\ 1 \end{bmatrix} = 2^{3/2}(XY)^{1/2}(D^{1/2} - 2^{1/2}Z), \\
&\theta^2\begin{bmatrix} 0\ 1 \\ 0\ 0 \end{bmatrix} = 2^{3/2}(XZ)^{1/2}(D^{1/2} + 2^{1/2}Y), \\
&\theta^2\begin{bmatrix} 0\ 1 \\ 1\ 0 \end{bmatrix} = 2^{3/2}(XZ)^{1/2}(D^{1/2} - 2^{1/2}Y), \\
&\theta^2\begin{bmatrix} 1\ 1 \\ 0\ 0 \end{bmatrix} = 2^{3/2}(YZ)^{1/2}(D^{1/2} + 2^{1/2}X), \\
&\theta^2\begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix} = 2^{3/2}(YZ)^{1/2}(D^{1/2} - 2^{1/2}X).
\end{aligned}
\tag{7.A.11}
$$

$$
\begin{aligned}
\theta_1\begin{bmatrix} 1\ 0 \\ 1\ 0 \end{bmatrix} &= \frac{\mathrm{i}}{2}(2XY)^{1/4}(\vartheta_4^2 C^{1/2} + 2^{1/2}\vartheta_3^2 Z) \\
&\qquad \times (D^{1/2} + 2^{1/2}Z)^{-1/2}, \\
\theta_2\begin{bmatrix} 1\ 0 \\ 1\ 0 \end{bmatrix} &= -\frac{1}{2}(2XY)^{1/4}(\widetilde{\vartheta}_2^2 C^{1/2} + 2^{1/2}\widetilde{\vartheta}_3^2 Z) \\
&\qquad \times (D^{1/2} + 2^{1/2}Z)^{-1/2}, \\
\theta_1\begin{bmatrix} 1\ 0 \\ 1\ 1 \end{bmatrix} &= \frac{\mathrm{i}}{2}(2XY)^{1/4}(\vartheta_4^2 C^{1/2} - 2^{1/2}\vartheta_3^2 Z) \\
&\qquad \times (D^{1/2} - 2^{1/2}Z)^{-1/2}, \\
\theta_2\begin{bmatrix} 1\ 0 \\ 1\ 1 \end{bmatrix} &= -\frac{1}{2}(2XY)^{1/4}(\widetilde{\vartheta}_2^2 C^{1/2} - 2^{1/2}\widetilde{\vartheta}_3^2 Z) \\
&\qquad \times (D^{1/2} - 2^{1/2}Z)^{-1/2},
\end{aligned}
$$

$$
\begin{aligned}
\theta_1\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} &= \frac{\mathrm{i}}{2}(2XZ)^{1/4}(\vartheta_2^2 B^{1/2} + 2^{1/2}\vartheta_3^2 Y) \\
&\qquad \times (D^{1/2} + 2^{1/2}Y)^{-1/2}, \\
\theta_2\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} &= -\frac{1}{2}(2XZ)^{1/4}(\tilde{\vartheta}_4^2 B^{1/2} + 2^{1/2}\tilde{\vartheta}_3^2 Y) \\
&\qquad \times (D^{1/2} + 2^{1/2}Y)^{-1/2}, \\
\theta_1\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} &= \frac{\mathrm{i}}{2}(2XZ)^{1/4}(\vartheta_2^2 B^{1/2} - 2^{1/2}\vartheta_3^2 Y) \\
&\qquad \times (D^{1/2} - 2^{1/2}Y)^{-1/2}, \\
\theta_2\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} &= -\frac{1}{2}(2XZ)^{1/4}(\tilde{\vartheta}_4^2 B^{1/2} - 2^{1/2}\tilde{\vartheta}_3^2 Y) \\
&\qquad \times (D^{1/2} - 2^{1/2}Y)^{-1/2}, \\
\theta_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} &= \frac{\mathrm{i}}{2}(2ZY)^{1/4}(\vartheta_3^2 C^{1/2} + 2^{1/2}\vartheta_2^2 X) \\
&\qquad \times (D^{1/2} + 2^{1/2}X)^{-1/2}, \\
\theta_2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} &= -\frac{1}{2}(2ZY)^{1/4}(\tilde{\vartheta}_3^2 C^{1/2} + 2^{1/2}\tilde{\vartheta}_4^2 X) \\
&\qquad \times (D^{1/2} + 2^{1/2}X)^{-1/2}, \\
\theta_1\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} &= -\frac{\mathrm{i}}{2}(2ZY)^{1/4}(\vartheta_3^2 C^{1/2} - 2^{1/2}\vartheta_2^2 X) \\
&\qquad \times (D^{1/2} - 2^{1/2}X)^{-1/2}, \\
\theta_2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} &= \frac{1}{2}(2ZY)^{1/4}(\tilde{\vartheta}_3^2 C^{1/2} - 2^{1/2}\tilde{\vartheta}_2^2 X) \\
&\qquad \times (D^{1/2} - 2^{1/2}X)^{-1/2}.
\end{aligned}
\tag{7.A.12}
$$

8. An Application: The Peierls-Fröhlich Problem and Finite-Gap Potentials

One of the remarkable applications of algebro-geometric methods in solid-state physics is the construction of an exact solution to the Peierls problem. This problem, arising from a number of fundamental problems in solid-state physics, reduces to finding a self-consistent state of the lattice and conduction electrons. According to the qualitative arguments proposed by *Peierls* [8.1], the specific feature of the self-consistent state in point is that the lattice-induced potential generates a gap in the energy spectrum of the conduction electrons at an energy equal to the Fermi one. We can, therefore, hypothesize that the exact solution to the Peierls problem in the jellium model would be a single-band potential. The hypothesis was proven in 1980 [8.2, 3].

In Sect. 8.1 we present the foundations of the theory of finite-gap potentials. In Sect. 8.2, an exact solution to the Peierls problem is constructed. In Sect. 8.3 the theory of the Fröhlich conductivity created by a uniform motion of the Peierls state is presented and in Sect. 8.4 some final remarks are made.

8.1 Finite-Gap Potentials

We recall that the function $u(x)$ taken from a Banach space $C_b(\mathbb{R})$ of continuous bounded functions is called almost periodic if the set $\{T_x u(\cdot),\ x \in \mathbb{R}\}$, where $T_x u(\cdot) = u(\cdot + x)$, is relatively compact in $C_b(\mathbb{R})$. A closure Ω of this set is known to be compact in a metrizable Abelian group. A normalized Haar measure μ on the set Ω turns out to be T_x-invariant and ergodic. Thus, each almost-periodic function generates a probability space (Ω, μ, T_x). The operation of averaging on this space is given by

$$Mf(u) = \lim_{x\to\infty} \frac{1}{x} \int_0^x f(T_x u)dx = \int_\Omega f(u)\mu(du). \tag{8.1.1}$$

By means of the differential expression

$$L(u) = -\partial_x^2 + u(x)$$

we define for $u \in \Omega$ a Schrödinger operator in $\mathcal{L}^2(\mathbb{R})$, which is essentially self-adjoint. Let $\lambda \in \mathbb{C}$ and $c(\lambda, x),\ s(\lambda, x)$ denote solutions to the equation $L\varphi = \lambda\varphi$

with the initial data $c(\lambda,0) = 1$, $c'(\lambda,0) = 0$, $s(\lambda,0) = 0$, $s'(\lambda,0) = 1$. The functions $c(\lambda,x)$, $s(\lambda,x)$ are continuous with respect to a totality of variables (λ,x) and integral ones of order $1/2$ with respect to λ. The limits

$$w_\pm(\lambda,u(\cdot)) = \mp \lim_{x\to\infty} c(\lambda,x)/s(\lambda,x)$$

exist and are called the Weyl functions. Besides these, we shall also employ the functions $w_\pm(\lambda,T_x u(\cdot))$. In what follows we shall use simpler notations such as $w_\pm(\lambda)$, $w_\pm(\lambda,x)$ instead of $w_\pm(\lambda,u(\cdot))$, $w_\pm(\lambda,T_x u(\cdot))$, accordingly. It is possible to show that the Weyl functions are holomorphic in $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda > 0\}$, map $\mathbb{C}_+ \to \mathbb{C}_+$, and their zeros or poles on the real axis can only be simple ones [8.4], if there are any. We use the Weyl functions to define the functions

$$\psi(\lambda,x) = c(\lambda,x) \pm w_\pm(\lambda)s(\lambda,x) = \exp\Big(\pm \int_0^x w_\pm(\lambda,y)dy\Big) \tag{8.1.2}$$

which, for each $\lambda \in \mathbb{C}_+$, belong to $\mathcal{L}^2(\mathbb{R}_\pm)$, where $\mathbb{R}_+ = [0,\infty)$, and $\mathbb{R}_- = (-\infty,0)$, respectively. By definition, the functions $\psi_\pm(\lambda,x)$ satisfy the equation

$$-\partial_x^2\psi_\pm(\lambda,x) + (u(x)-\lambda)\psi_\pm(\lambda,x) = 0 \quad . \tag{8.1.3}$$

By substituting (8.1.2) into this equation, we have the following equation for the Weyl functions:

$$\pm\partial_x w_\pm(\lambda,x) + w_\pm^2(\lambda,x) + \lambda - u(x) = 0. \tag{8.1.4}$$

The Green function for the operator $L(u)$ is

$$g(\lambda,x,y) = g(\lambda,y,x) = -\big(w_+(\lambda)+w_-(\lambda)\big)^{-1}\psi_+(\lambda,x)\psi_-(\lambda,y) \tag{8.1.5}$$

when $x \geq y$. Using the above expression, it is easy to establish the following properties of Green function [8.1.5]:

$$w_+(\lambda,x) - w_-(\lambda,x) = \partial_x \log g(\lambda,x,x,T_x u) \tag{8.1.6}$$

$$\begin{aligned} &g(\lambda,x,x,u) + \partial_\lambda[2g(\lambda,x,x,u)]^{-1} \\ &= -\frac{1}{2}\partial_x\left\{ g(\lambda,0,0,u)\left[\int_0^\infty \psi_+^2(\lambda,y)dy - \int_{-\infty}^0 \psi_-^2(\lambda,y)dy\right]\right\}. \end{aligned} \tag{8.1.7}$$

Following [8.6], we define the Floquet index

$$f(\lambda) = (1/2)M\big(w_+(\lambda)+w_-(\lambda)\big) = -(1/2)M\big(g(\lambda,0,0)^{-1}\big) \quad . \tag{8.1.8}$$

Let $\lambda = \xi + \mathrm{i}\eta$. The Floquet exponent has a finite limit a.e. at $\xi \in \mathbb{R}$, $\eta \downarrow 0$

$$f(\xi+\mathrm{i}0) = -l(\xi) + \mathrm{i}\pi n(\xi) \quad , \tag{8.1.9}$$

where $l(\xi)$ is the Lyapunov exponent, and $n(\xi)$ is the number of states that can be defined, using the functions $c(\lambda,x)$, $s(\lambda,x)$, as follows:

$$
\begin{aligned}
l(\xi) &= \lim_{x\to\pm\infty} \pm\frac{1}{2x}\log\left\{c(\xi,x)^2 + c'(\xi,x)^2\right\} = \\
&= \lim_{x\to\pm\infty} \pm\frac{1}{2x}\log\left\{s(\xi,x)^2 + s'(\xi,x)^2\right\} \quad , \\
n(\xi) &= \lim_{x\to\pm\infty} \pm\frac{1}{\pi x}\arg\left\{c(\xi,x) - \mathrm{i}c'(\xi,x)\right\} = \\
&= \lim_{x\to\pm\infty} \pm\frac{1}{\pi x}\arg\left\{s(\xi,x) - \mathrm{i}s'(\xi,x)\right\} \quad .
\end{aligned}
$$

Here the expression for $l(\xi)$ follows from the conventional definition of the Lyapunov exponent, and the expression for $n(\xi)$ from the oscillation theorem.

We give an outline of the proof for (8.1.9). The function $r(\lambda,x) = c(\lambda,x) + \mathrm{i}c'(\lambda,x)$ is non-zero at $\lambda \in \overline{\mathbb{C}_+} = \{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \geq 0\}$, so that $\log r(\lambda,x)$ is a continuous function of x and λ in $\mathbb{C}_+ \times \mathbb{R}$ and a holomorphic function of λ in $\mathbb{C}_+$. Then, for $\lambda = \xi + \mathrm{i}\eta$, $\eta \downarrow 0$

$$
\begin{aligned}
\lim_{x\to\infty}\frac{1}{x}\log r(\xi,x) = \lim_{x\to\infty}\frac{1}{2x}&\log\{c(\xi,x)^2 + c'(\xi,x)^2\} + \\
&+ \mathrm{i}\lim_{x\to\infty}\frac{1}{x}\arg\{c(\xi,x) + \mathrm{i}c'(\xi,x)\} = l(\xi) - \mathrm{i}\pi n(\xi) \quad .
\end{aligned}
$$

On the other hand, we have, according to (8.1.2),

$$
c(\lambda,x) = \frac{w_-(\lambda)}{w_+(\lambda) + w_-(\lambda)}\psi_+(\lambda,x) + \frac{w_+(\lambda)}{w_+(\lambda) + w_-(\lambda)}\psi_-(\lambda,x) \quad .
$$

Since the first term tends exponentially to zero when $x \to +\infty$ and $\lambda \in \mathbb{C}_+$, we have

$$
\begin{aligned}
\lim_{x\to\infty}\frac{1}{x}\log r(\xi,x) &= \lim_{x\to\infty}\frac{1}{x}\log\{\psi_-(\xi,x) + \mathrm{i}\psi_-'(\xi,x)\} \\
&= -\lim_{x\to\infty}\frac{1}{x}\int_0^x w_-(\xi + \mathrm{i}0, y)dy \\
&= -\int_\Omega w_-(\xi + i0, T_y u)\mu(du) = -f(\xi + \mathrm{i}0) \quad .
\end{aligned}
$$

Here the last equality follows from the equality $Mw_+(\lambda) = Mw_-(\lambda)$, which is obtained by averaging from (8.1.6). By equating the two expressions for $\lim_{x\to\infty}\frac{1}{x}\log r(\xi,x)$, we have (8.1.9).

Being real and imaginary parts of the boundary value of the function holomorphic in $\mathbb{C}_+$, the Lyapunov exponent and the number of states are related by an equality which is called the Thouless-Herbert-Jones formula:

$$
\begin{aligned}
&l(\xi) - l_0(\xi) = \int \log|\xi - \zeta|\big(n(d\zeta) - n_0(d\zeta)\big), \\
&l_0(\xi) = \max(0, -\xi)^{1/2}, \ n_0(\xi) = \max(0,\xi)^{1/2} \quad .
\end{aligned}
\tag{8.1.10}
$$

The number of states, $n(\xi)$, determines the spectrum $\Sigma(u)$ of the operator $L(u)$: for a.e. $u \in \Omega$ $\Sigma(u) = \mathrm{supp}(dn)$ [8.7]. The Lyapunov exponent $l(\xi)$ determines an absolutely continuous spectrum $\Sigma_{a.c.}(u) = \overline{\{\xi \in \mathbb{R} \mid l(\xi) = 0\}}$, where the closure is with respect to Lebesgue measure [8.5, 7]. The last proposition may be formulated in a stricter way.

Theorem 8.1. [8.5] The Lyapunov exponent $l(\xi) = 0$ a.e. on a Borel set $A \in \mathbb{R}$ with a positive Lebesgue measure if and only if the density of an absolutely continuous spectrum is positive a.e.on A. Under the same condition

$$w_+(\xi + \mathrm{i}0, x) + \overline{w_-(\xi + \mathrm{i}0, x)} = 0 \tag{8.1.11}$$

or, equivalently, $\mathrm{Re}\, g(\xi + \mathrm{i}0, x, x) = 0$ a.e. on A. The last equality is called the reflectionless property of the potential.

We now give the proof of this property. By separating the real and imaginary parts of (8.1.4), it is easy to show that

$$-2\mathrm{Re}\, w_\pm(\lambda, x) = \pm\partial_x \log \mathrm{Im}\, w_\pm(\lambda, x) + \mathrm{Im}\,\lambda/\mathrm{Im}\, w_\pm(\lambda, x) \quad .$$

By averaging this equality, we come to an expression such as follows

$$-\mathrm{Re}\, f(\lambda)/\mathrm{Im}\,\lambda = (1/2)M\big(1/\mathrm{Im}\, w_\pm(\lambda, x)\big) \quad .$$

Equality (8.1.7), when averaged, takes the form

$$df/d\lambda = M(g(\lambda), 0, 0, u) = -(w_\pm(\lambda, x) + w_-(\lambda, x))^{-1} \quad .$$

By adding the last two equalities, we have

$$\begin{aligned}
-4\left(\frac{\mathrm{Re}\, f}{\mathrm{Im}\,\lambda} + \mathrm{Im}\,\frac{df}{d\lambda}\right) &= M\left\{\frac{1}{\mathrm{Im}\, w_+(\lambda, x)} + \frac{1}{\mathrm{Im}\, w_-(\lambda, x)}\right.\\
&\qquad \left. + 4\mathrm{Im}\,\frac{1}{w_+(\lambda, x) + w_-(\lambda, x)}\right\}\\
&= M\left\{\left[\frac{1}{\mathrm{Im}\, w_+(\lambda, x)} + \frac{1}{\mathrm{Im}\, w_-(\lambda, x)}\right]\right.\\
&\qquad \times \frac{[\mathrm{Re}\,(w_+(\lambda, x) + w_-(\lambda, x)]^2}{|w_+(\lambda, x) + w_-(\lambda, x)|^2}\\
&\qquad \left. + \frac{[\mathrm{Im}\,(w_+(\lambda, x) - w_-(\lambda, x)]^2}{|w_+(\lambda, x) + w_-(\lambda, x)|^2}\right\} \quad .
\end{aligned} \tag{8.1.12}$$

If $l(\xi) = -\mathrm{Re}\, f(\xi + \mathrm{i}0) = 0$, then

$$\lim_{\eta\to 0}\frac{\mathrm{Re}\, f(\xi+\mathrm{i}\eta)}{\eta}=\lim_{\eta\to 0}\frac{\mathrm{Re}\, f(\xi+\mathrm{i}\eta)-\mathrm{Re}\, f(\xi+\mathrm{i}0)}{\eta}=$$
$$\partial_\eta \mathrm{Re}\, f(\xi+\mathrm{i}\eta)\Big|_{\eta=0}=-\mathrm{Im}\,\frac{df}{d\lambda}\Big|_{\lambda=\xi+\mathrm{i}0}\quad .$$

Therefore, when $\eta\downarrow 0$, $(\mathrm{Re}\, f/\mathrm{Im}\,\lambda)+\mathrm{Im}\,(df/d\lambda)\to 0$. As a result, (8.1.12) yields, with $\eta\downarrow 0$

$$\mathrm{Re}\,(w_+(\xi+\mathrm{i}0,x)+w_-(\xi+\mathrm{i}0,x))=0\quad ,$$
$$\mathrm{Im}\,(w_+(\xi+\mathrm{i}0,x)-w_-(\xi+\mathrm{i}0,x))=0\quad ,$$

or $w_+(\xi+\mathrm{i}0,x)+\overline{w_-(\xi+\mathrm{i}0,x)}=0$.

We now proceed to discuss finite-gap potentials.

Definition. *The almost-periodic function $u(x)$ is called a finite-gap potential if the spectrum of the Schrödinger operator $L(u)=-\partial_x^2+u(x)$ is a union of the finite set of segments of a Lebesgue (double absolutely continuous) spectrum.*

Starting directly from this definition, we derive an explicit expression and the basic properties of a finite-gap potential.

Theorem 8.2. For a g–band Lebesgue spectrum $\Sigma=[E_1,E_2]\cup\ldots\cup[E_{2g+1},\infty)$, the potential $u(x)$ and the eigenfunction $\psi(P,x)$ of the Schrödinger operator $L(u)=-\partial_x^2+u(x)$ are expressed by

$$u(x)=-2\partial_x^2\log\theta(\mathrm{i}\boldsymbol{U}x-\boldsymbol{A}(D)-\boldsymbol{K};B)+\mathrm{const}\tag{8.1.13}$$

$$\psi(P,x)=\frac{\theta(\mathrm{i}\boldsymbol{U}x+\boldsymbol{A}(P)-\boldsymbol{A}(D)-\boldsymbol{K};B)}{\theta(\boldsymbol{A}(P)-\boldsymbol{A}(D)-\boldsymbol{K};B)}\exp\left(\mathrm{i}x\int_\infty^P\Omega\right)\quad .\tag{8.1.14}$$

Here P is the point of a hyperelliptic Riemann surface X, defined by the equation $\mu^2=\prod_{i=1}^{2g+1}(\lambda-E_i)$, ω is the vector of normalized holomorphic differentials, B is the matrix of their periods, $\boldsymbol{A}(P)=\int_\infty^P\omega$ is an Abelian mapping, Ω is a normalized Abelian differential of the second kind, which at infinity has a second-order pole with the principal part $\zeta^{-2}d\zeta$, where ζ is a local variable; $\boldsymbol{U}$ is the vector of the periods of the differential Ω, D is a non-special divisor, and $\boldsymbol{K}$ is the vector of Riemann constants, see Sect. 3.5.

Proof. The function $w_+(\lambda,x)+w_-(\lambda,x)$ as a sum of the Weyl functions, is holomorphic in $\mathbb{C}_+$ and maps $\mathbb{C}_+$ into $\mathbb{C}_+$. For a finite-gap potential this function (according to Theorem 8.1) takes, on the real axis, imaginary values on the segments of the spectrum Σ, and real values on additional intervals $\mathbf{R}\setminus\Sigma$. The latter property makes it possible to continue the function $w_+(\lambda,x)+w_-(\lambda,x)$ to the whole complex plane by employing the Schwartz reflection method. As a result, for finite-gap potential this function turns out to be holomorphic on a hyperelliptic Riemann surface X, given by the equation $\mu^2 = \prod_{i=1}^{2g+1}(\lambda - E_i)$. Considering that, according to (8.1.4),

$$w_\pm(\lambda, x) = \mathrm{i}\sqrt{\lambda} + \mathrm{i}u(x)/2\sqrt{\lambda} + o(|\lambda|^{-1/2}) \tag{8.1.15}$$

with $\lambda \to \infty$, it is easy to see that

$$w_+(\lambda, x) + w_-(\lambda, x) = 2\mathrm{i}\sqrt{P(\lambda)}/S(\lambda, x) \tag{8.1.16}$$

$$P(\lambda) = \prod_{i=1}^{2g+1}(\lambda - E_i), \; S(\lambda, x) = \prod_{k=1}^{g}(\lambda - \lambda_k(x)) \quad . \tag{8.1.17}$$

The function $w_+(\lambda, x) - w_-(\lambda, x)$, as a difference of Weyl functions, is also holomorphic in $\mathbb{C}_+$ and maps $\mathbb{C}_+$ into $\mathbb{C}_+$. For a finite-gap potential this function (according to Theorem 8.1) takes real values on the real axis and, as a result, can be continued to the whole complex plane, using the Schwarz reflection method. Thus, this function turns out to be a rational function of λ. An explicit formula for the function under consideration follows from (8.1.4):

$$w_+(\lambda, x) - w_-(\lambda, x) = -\partial_x \log(w_+(\lambda, x) + w_-(\lambda, x)) = \partial_x \log S(\lambda, x) \quad .$$

The formulas derived above enables us to write down explicit expressions for the Weyl functions:

$$w_\pm(\lambda, x) = \pm\frac{1}{2}\frac{\partial_x S(\lambda, x)}{S(\lambda, x)} + \mathrm{i}\frac{\sqrt{P(\lambda)}}{S(\lambda, x)} \quad . \tag{8.1.18}$$

This formula implies that $\pm w_\pm(\lambda, x)$ are the two branches of a meromorphic function on the hyperelliptic Riemann surface X. Taking into account the number and order of the poles, it is easy to express this meromorphic function through the Riemann theta function θ introduced in Sect.2:

$$w(P, x) = \partial_x \log \frac{\theta(\mathrm{i}x\boldsymbol{U} + \boldsymbol{A}(P) - \boldsymbol{A}(D) - \boldsymbol{K}, B)}{\theta(\boldsymbol{A}(P) - \boldsymbol{A}(D) - \boldsymbol{K}, B)} + \mathrm{i}\int_\infty^P \Omega \quad . \tag{8.1.19}$$

All the notations used in the above formula were explained when the Theorem was formulated. Taking into account the asymptotic expression (8.1.15) for the function $w(P, x)$, we get a formula for a finite-gap potential such as

$$\begin{aligned} u(x) = & -2\mathrm{i}\,\mathrm{res}\big|_{P=\infty}\lambda(P)w(P, x) \\ = & -2\mathrm{i}\partial_x \mathrm{res}\big|_{P=\infty}\Big\{\lambda(P) \\ & \times \log\theta(\mathrm{i}x\boldsymbol{U} + \boldsymbol{A}(P) - \boldsymbol{A}(D) - \boldsymbol{K}, B)\Big\} + \mathrm{const} \\ = & -2\partial_x^2 \log\theta(\mathrm{i}x\boldsymbol{U} - \boldsymbol{A}(D) - \boldsymbol{K}, B) + \mathrm{const} \quad . \end{aligned} \tag{8.1.20}$$

When deriving the last equality, we used the fact that $\boldsymbol{A}(P) = \zeta\boldsymbol{U} + O(\zeta^3)$, where $\zeta = \lambda^{-1/2}$ is a local parameter in the neighbourhood of ∞. Finally, taking into account that $\psi(P,x)=\exp\left(\int_0^x w(P,y)dy\right)$, we obtain an expression for the eigenfunction, i.e.,

$$\psi(P,x)=\frac{\theta(\mathrm{i}x\boldsymbol{U}+\boldsymbol{A}(P)-\boldsymbol{A}(D)-\boldsymbol{K},B)}{\theta(\boldsymbol{A}(P)-\boldsymbol{A}(D)-\boldsymbol{K},B)}\exp\left(\mathrm{i}x\int_{\infty}^{P}\Omega\right) \quad .$$

The proof is complete.

We note that this theorem was first proved in [8.8] for a periodic finite-gap potential only. The proof given above is free of this limitation and valid for an arbitrary finite-gap potential which, predominantly, is almost-periodic.

Let us now establish the necessary and sufficient conditions for the function $u(x)$ to be a finite-gap potential.

Theorem 8.3. For the function $u(x)$ to be a finite-gap potential, it is necessary and sufficient that it satisfies an ordinary differential equation of order $2g$:

$$\sum_{m=-1}^{g} c_m\frac{\delta I_m}{\delta u}=0, \tag{8.1.21}$$

where the c_m are constants and the I_m are average values of the polynomials of the function $u(x)$ and its derivatives up to order m included; they are determined recurrently as coefficients of the asymptotic expansion of the number of states, $n(\xi)$, for $\xi\to\infty$,

$$n(\xi)\sim\frac{1}{\pi}\left(k-\sum_{m=0}^{\infty}\frac{I_{m-1}}{(2k)^{2m+1}}\right),\ k=\xi^{1/2} \quad . \tag{8.1.22}$$

Specifically,

$$\begin{aligned}&I_{-1}=Mu,\ I_0=Mu^2,\ I_1=M(2u^3+u'^2) \quad ,\\ &I_2=M(5u^4+10uu'^2+u''^2) \quad ,\end{aligned} \tag{8.1.23}$$

while the differential equations for 0-, 1-, 2-band potentials have the form

$$\begin{aligned}&2c_0u+c_{-1}=0,\ -2c_1(u''-3u^2)+2c_0u+c_{-1}=0 \quad ,\\ &2c_2(u^{\mathrm{iv}}-10u''u-5u'^2+10u^3)-\\ &-2c_1(u''-3u^2)+2c_0u+c_{-1}=0 \quad .\end{aligned} \tag{8.1.24}$$

Proof. Necessity [8.9, 10]: For the function $w_+(\xi,x)$ the following Riccati differential equation (8.1.4) is valid:

$$\partial_x w_+(\xi,x)+w_+(\xi,x)^2+\xi-u(x)=0 \quad .$$

It is easy to establish from the equation that the real part $w_+(\xi,x)$ is a total derivative, i.e.,

$$\operatorname{Re} w_+(\xi, x) = (1/2)\partial_x \log \operatorname{Im} w_+(\xi, x) \quad .$$

From the same equation it follows that for $\xi \to \infty$ the following asymptotic expansion is valid:

$$\operatorname{Im} w_+(\xi, x) \sim k - \sum_{n=0}^{\infty} \frac{w_{n-1}(x)}{(2k)^{2n+1}}, \; k = \xi^{1/2} \quad ,$$

whose coefficients $w_{n-1}(x)$ are easily determined recurrently. They are polynomials of the potential $u(x)$ and its derivatives; specifically,

$$\begin{aligned} &w_{-1} = u, \; w_0 = u^2 - u'' \; w_1 = 2u^3 - 5u'^2 - 6uu'' + u^{\mathrm{iv}} \quad , \\ &\quad w_2 = 5u^4 - 50uu'^2 - 30u^2u'' + 19u''^2 + 28u'u''' + 10uu^{\mathrm{iv}} - u^{\mathrm{vi}} \quad . \end{aligned}$$

For our further discussion we need the expression of the number of states, $n(\xi)$, in terms of the function $w_+(\xi, x)$, i.e.,

$$n(\xi) = (1/\pi)M w_+(\xi, x) = (1/\pi)M \operatorname{Im} w_+(\xi, x) \quad . \tag{8.1.25}$$

When deriving the last equation, we took into account the fact that $\operatorname{Re} w_+(\xi, x)$ is a total derivative and, consequently, $M\operatorname{Re} w_+(\xi, x) = 0$. Using the asymptotic expansion for $\operatorname{Im} w_+(\xi, x)$, we obtain an asymptotic expansion for the number of states

$$n(\xi) \sim \frac{1}{\pi}\left(k - \sum_{n=0}^{\infty} \frac{I_{n-1}}{(2k)^{2n+1}}\right), \; k = \xi^{1/2} \quad .$$

Here

$$I_n = M w_n$$

and, more specifically,

$$\begin{aligned} &I_{-1} = Mu, \; I_0 = Mu^2, \; I_1 = M(2u^3 + u'^2) \quad , \\ &\quad I_2 = M(5u^4 + 10uu'^2 + u''^2) \quad . \end{aligned}$$

A simple consequence of the above expression is the asymptotic expansion for variations in the number of states

$$\frac{\delta n}{\delta u} \sim \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2k)^{2n+1}} \frac{\delta I_{n-1}}{\delta u} \quad , \tag{8.1.26}$$

where

$$\frac{\delta I_r}{\delta u} = \frac{\delta w_r}{\delta u} + \sum_{s=1}^{r} (-1)^s \frac{d^s}{dx^s} \frac{\partial w_r}{\partial u^{(s)}} \quad .$$

We now consider the variation of the number of states of the Schrödinger operator with an arbitrary quasi-periodic potential with respect to such variations of the potential that do not affect the set of frequencies of the potential and show that for this case the following formula is valid:

$$\frac{\delta n}{\delta u} = -\frac{1}{2\pi}\frac{1}{\operatorname{Im} w_+(\xi, x)}. \tag{8.1.27}$$

Indeed, by varying the Riccati equation given above, we have

$$\partial_x \delta w_+(\xi, x) + 2w_+(\xi, x)\delta w_+(\xi, x) - \delta u(x) = 0 \quad .$$

The solution of this equation with initial condition $\delta w_+(\xi, 0) = 0$ is as follows:

$$\delta w_+(\xi, x) = \int_0^x dx'\delta u(x')\exp\left[-\mathrm{i}\int_{x'}^x dx''2w_+(\xi, x'')\right] \quad .$$

The imaginary part of the solution is

$$\begin{aligned}\delta\operatorname{Im} w_+(\xi, x) = &-\operatorname{Im} w_+(\xi, x)\int_0^x dx'\frac{\delta u(x')}{\operatorname{Im} w_+(\xi, x')}\\ &\times \sin\left[\int_{x'}^x dx''2\operatorname{Im} w_+(\xi, x'')\right] \quad .\end{aligned}$$

The corresponding variation of the number of states has the form

$$\begin{aligned}\delta n &= (1/\pi)M\delta\operatorname{Im} w_+(\xi, x)\\ &= -\lim_{l\to\infty}\frac{1}{l}\int_0^l dx\frac{\delta u(x)}{2\pi\operatorname{Im} w_+(\xi, x)}\left\{1 - \cos\left[\int_{x'}^l dx'2\operatorname{Im} w_+(\xi, x')\right]\right\}\end{aligned}$$

which is the sum of two terms. The second term vanishes if the following two conditions are met:
(1) the variation of the potential $\delta u(x)$ is a quasi-periodic function with the same set of frequencies as that of the potential $u(x)$;
(2) $2M\operatorname{Im} w_+(\xi, x) \notin \mathcal{M}u$, $\mathcal{M}u$ is the modulus of frequencies for the potential $u(x)$; [in other words, the generalized *Bragg-Wolff conditions* [8.11] are not fulfilled].

When the second term is absent, the above expression yields formula (8.1.27). It is possible to say that this formula for a variational derivative holds for a certain class of variations in quasi-periodic potentials that affect the coefficients, but not the frequencies of a Fourier series corresponding to the potential.

Let $u(x)$ be a finite-gap potential. It then follows from Theorem 8.2 that

$$\operatorname{Im} w_+(\xi, x) = \sqrt{P(\xi)}/S(\xi, x) \quad .$$

By substituting this relation into (8.1.27), we obtain the following expression for the variational derivative of a finite-gap potential:

$$\frac{\delta n}{\delta u} = -\frac{1}{2\pi}\frac{S(\xi, x)}{\sqrt{P(\xi)}} \tag{8.1.28}$$

Let us expand this expression in an asymptotic series of $k = \xi^{1/2}$. Among the coefficients of this series there are $g+1$ linearly independent ones only, because all of them are expressed linearly in terms of $g+1$ coefficients of the polynomial $S(\xi, x)$. On the other hand, because the coefficients of this asymptotic series are, according to (8.1.26), proportional to $\delta I_m/\delta u$, the variational derivatives $\delta I_m/\delta u$, $m = -1, 0, \dots, g$, for a finite-gap potential should be linearly dependent, i.e.,

$$\sum_{m=-1}^{g} c_m \frac{\delta I_m}{\delta u} = 0 \quad .$$

This is the desired result for the *necessity direction* of the proof.

By comparing the asymptotic expansions (8.1.26,28), it is possible to derive the so-called "trace formulas"

$$\begin{aligned} \frac{1}{2}\frac{\delta I_0}{\delta u} &= u = -2\sum_{j=1}^{g}\lambda_j + \sum_{i=1}^{2g+1} E_i \quad , \\ \frac{1}{2}\frac{\delta I_1}{\delta u} &= -u'' + 3u^2 = 8\sum_{j<k}\lambda_j\lambda_k - 4\left(\sum_i E_i\right)^2, \text{etc} \quad . \end{aligned} \tag{8.1.29}$$

Sufficiency [8.12]. Let the following ordinary differential equation of order $2g+1$ be valid:

$$\sum_{m=-1}^{g} c_m \frac{\delta I_m}{\delta u} = 0 \quad .$$

This equation can always be represented as

$$[L, A_g] = 0 \tag{8.1.30}$$

where $L = -\partial_x^2 + u(x)$ is the Schrödinger operator, and A_g is an ordinary differential operator of order $2g+1$, the coefficients of which are dependent on the potential $u(x)$ and its derivatives. The operator A_g is actually defined by the commutational equation above. Since the operators L and A_g are commuting, they have the same set of eigenfunctions ψ and can simultaneously be transformed to a canonical form. We choose the functions $c(\lambda, x)$, $s(\lambda, x)$ introduced above as a basis in the 2-dimensional space of solutions to the equation $L\psi = \lambda\psi$. In this basis the differential operator A_g is given by a matrix M_g of order 2×2, whose matrix elements are polynomials of the potential $u(x)$ and its derivatives and of the spectral parameter λ. From linear algebra it is known that if the matrix M depends polynomially on the parameter λ then its eigenvalues and eigenvectors are meromorphic functions of λ on a Riemann surface defined by the equation

$$\det(\mu E - M_g) = 0 \quad .$$

Because, in the case under consideration, the matrix M_g is a 2×2 matrix and $\operatorname{Tr} M_g = 0$ due to the anti-symmetric character of the operator A_g we have

$$\det(\mu E - M_g) = \mu^2 + \det M_g = 0 \quad .$$

Thus $\det M_g$ is a polynomial $P(\lambda)$ of order $2g + 1$ in λ, so that the above equations define a two-sheeted Riemann surface with branch points at $2g + 1$ zeroes of the polynomial $P(\lambda)$. These branch points are boundaries of the bands of the spectrum of the operator L.

We conclude our discussion of finite-gap potentials. A survey of the theory of finite-dimensional potentials is given in Chap. 3 and [8.13].

8.2 The Peierls Problem

The problem of finding a self-consistent state of electrons and ions in a one-dimensional metal in the jellium model can be reduced [8.1] to that of minimizing the functional of a free energy

$$\mathcal{F} = \mu \mathcal{N} - \int d\varepsilon \, n\big(\varepsilon, u(\cdot)\big) f(\varepsilon) + (\kappa/2)\langle u^2 \rangle \tag{8.2.1}$$

with the additional condition

$$\langle u \rangle = 0 \quad . \tag{8.2.2}$$

In this section $\langle u \rangle$ denotes the average value of the function $u(x)$.

Here we have used following notations:

$$\varepsilon = (2m/\hbar^2)E, \ u = (2m/\hbar^2)\alpha w_x, \ \kappa = (\hbar^2/2m\alpha)\rho s^2 \quad , \tag{8.2.3}$$

where $\hbar$ is the Planck constant, x is a space coordinate, m is a mass of electron, E is an electron energy, $w(x)$ is a lattice displacement, ρ is a lattice density, s is a sound velocity, α is an electron-phonon interaction constant.

The first two terms in (8.2.1) represent the free energy of an electron gas in the potential $u(x)$, and the third term the elastic energy of a lattice. The notations used are as follows: μ is the chemical potential, $\mathcal{N}$ is the electron density, $n\big(\varepsilon, u(\cdot)\big)$ is the number of states with energy less than ε in the potential $u(x)$, referred to a unit of length; $f(\varepsilon)$ is the Fermi function, κ is the elastic constant.

To determine the form of a solution to the variational problem stated, we consider the change in the free energy (8.2.1) when the potential $u = 0$ is replaced by a small band potential $u(x)$ that leads to the appearance of gaps in the energy spectrum of electrons. According to perturbation theory, the boundaries of a gap formed at energy ε are located symmetrically with respect to ε. Consequently, if the states of the spectrum on both sides of the gap are equally occupied by electrons, the same numbers of electrons on both sides of the gap will decrease and increase their energy by the same amount, so that the change in the electron part of the free energy, related to the gap formation, becomes zero. Because the lattice part of a free energy, proportional to $\langle u^2 \rangle$, rises simultaneously, the formation of a gap is energetically disadvantageous under the conditions described above. If, however, the states situated under the gap are occupied more than the states above it, then the electron part of a free energy will decrease when the gap is formed, and this decrease may be greater than the corresponding increase in the lattice part of a free energy. As a result, the formation of a gap is energetically advantageous under the conditions in point. The form of the Fermi function that defines the electron occupation of the states of the spectrum shows that a gap can occur at $\varepsilon = \mu$ only. The above heuristic considerations put forward by *Peierls* [8.1] result in the following hypothesis:

The minimum of the free energy (8.2.1) is attained, with additional condition (8.2.2), on a single-band potential, i.e., on the potential which generates not more than one gap in the energy spectrum of conduction electrons.

The proof of this hypothesis is the main result of the present section. To prove it, we consider the free energy (8.2.1) on the set of quasi-periodic potentials. We choose such variations of the potential that its periods remain unchanged. By setting the variation derivative of free energy (8.2.1) with respect to the above class of potential variations equal to zero, we have the following equation:

$$-\int d\varepsilon f(\varepsilon)\delta n\big(\varepsilon, u(\cdot)\big)/\delta u + \kappa u + \lambda = 0 \quad . \tag{8.2.4}$$

Here λ is the Lagrange multiplier used to take condition (8.2.2) into account. This equation implies that the variational derivative of the number of states with respect to the class of variations chosen is a linear function of the potential. According to Theorem 8.3, this is a necessary and sufficient condition for the potential to be a single-band one.

We present the necessary data related to a single-band potential [8.13]. The single-band potential is an elliptic function that can be expressed explicitly via the Weierstrass $\wp$-function or the Jacobi theta function:

$$\begin{aligned} u(x) &= -2\big(\wp(\mathrm{i}x+\omega) - \langle \wp(\mathrm{i}x+\omega)\rangle\big) \\ &= -2\partial_x^2 \log \vartheta_4(L^{-1}x + \mathrm{const}, q) \quad , \end{aligned} \tag{8.2.5}$$

where $\langle \wp(ix+\omega)\rangle = -\eta'/\omega'$, $L = -i2\omega'$.

For all the quantities, except for the spectrum boundaries, we use below the conventional notations adopted in the theory of elliptic functions [8.14]. For the boundaries of the spectrum ε_1, ε_2, ε_3 we adopt the following notations and ordering:

$$\varepsilon_1 \le \varepsilon_2 \le \varepsilon_3 \quad .$$

Following (8.2.5), the number of states of a single-band potential is expressed in terms of the potential

$$n(\varepsilon) = (1/\pi)\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}\langle(\varepsilon-\gamma(x))^{-1}\rangle$$
$$\gamma(x) = (1/2)(\varepsilon_1+\varepsilon_2+\varepsilon_3-u(x))$$

Using the explicit expression for a single-band potential, we then obtain a parametric representation for the number of states

$$\begin{aligned} n(z) &= (1/\pi)[\zeta(z) - z(\eta'/\omega')], \quad \varepsilon = \wp(z), \quad \varepsilon_1 \le \varepsilon \le \varepsilon_2 \\ n(z) &= -(1/\pi)[\zeta(z) - z(\eta'/\omega')] + 2/L, \quad \varepsilon = \wp(z), \quad \varepsilon_3 \le \varepsilon < \infty \end{aligned} \tag{8.2.6}$$

The wave function of a quantum particle with energy ε in a single-band potential is as follows [8.14]:

$$\begin{aligned} &\psi_{\pm}(x,z) = \frac{\sigma(\mathrm{i}x_0+\omega)\sigma(\mathrm{i}x+\omega\mp z)}{\sigma(\mathrm{i}x+\omega)\sigma(\mathrm{i}x_0+\omega\mp z)}\exp[\pm\mathrm{i}(x-x_0)\zeta(z)] \quad , \\ &\varepsilon = \wp(z) \quad . \end{aligned} \tag{8.2.7}$$

This formula, valid inside energy zones, assigns to each energy value two linearly independent wave functions that correspond to two sheets of a Riemann surface given by the equation $\mu^2 = P(\varepsilon) = (\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)$. Sometimes it is appropriate to use for the wave function the following form corresponding to (8.1.2)

$$\begin{aligned} &\psi(x,\varepsilon) = [\langle\chi^{-1}(x,\varepsilon)\rangle\chi(x,\varepsilon)]^{-1/2}\exp\Big(\mathrm{i}\int^x dx\,\chi(x,\varepsilon)\Big) \quad , \\ &\chi(x,\varepsilon) = [P(\varepsilon)]^{1/2}\big(\varepsilon-\gamma(x)\big)^{-1} \quad , \\ &P(\varepsilon) = (\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3) \quad , \\ &\gamma(x) = (1/2)\big(\varepsilon_1+\varepsilon_2+\varepsilon_3-u(x)\big) \quad . \end{aligned} \tag{8.2.8}$$

In such a form the wave function is normalized by the condition $\langle|\psi|^2\rangle = 1$.

According to (8.2.5), the single-band potential is defined by three parameters such as ε_1, ε_2, ε_3 or $\langle u\rangle$, ω, ω'. If we are interested in relative but not absolute values of the potential, it is sufficient to have two parameters such as $\varepsilon_2-\varepsilon_1$, $\varepsilon_3-\varepsilon_1$, or ω, ω'.

Let us now derive equations that determine the potential parameters. To obtain the first equation, we subtract from the variational equation (8.2.4) the same equation averaged over the period. As a result, we have

$$-\int d\varepsilon\, f(\varepsilon)\left(\frac{\delta n}{\delta u} - \left\langle\frac{\delta n}{\delta u}\right\rangle\right) + \kappa(u-\langle u\rangle) = 0 \quad .$$

The variation in the number of states of a single-band potential with respect to the variations leaving the period of the potential unchanged is, according to (8.1.28),

$$\frac{\delta n}{\delta u} = -\frac{1}{2\pi}\frac{\varepsilon-\gamma(x)}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} \quad ,$$

By substituting this expression into the previous one, we have the first equation for determining the potential parameter:

$$(u - \langle u\rangle)\left(4\pi\kappa + \int d\varepsilon \frac{f(\varepsilon)}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}}\right) = 0 \quad . \tag{8.2.9}$$

We note at once that this equation has two solutions. The first is

$$u = \langle u\rangle = 0 \quad , \tag{8.2.10}$$

implying that a single-band potential degenerates into a zero-band one. The second is

$$\int d\varepsilon \frac{f(\varepsilon)}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} + 4\pi\kappa = 0 \quad . \tag{8.2.11}$$

Equation (8.2.9) is given by the variation of the free energy with respect to potential variations that leave the potential period unchanged. To derive the second equation, we consider the change in free energy when the potential periods change. For this purpose we make use of the homogeneity property of an elliptic Weierstrass function:

$$\wp(\nu x \mid \nu\omega, \nu\omega') = \nu^{-2}\wp(x \mid \omega, \omega') \,, \quad \text{where } \nu \text{ is the real parameter} \quad .$$

Since the single-band potential is expressed via the Weierstrass function, we have the similarity property of a single-band potential

$$u(\nu x, \nu^{-2}\varepsilon_1, \nu^{-2}\varepsilon_2, \nu^{-2}\varepsilon_3) = \nu^{-2}u(x, \varepsilon_1, \varepsilon_2, \varepsilon_3) \quad .$$

It is easy to see that, under the similarity transformations of a single-band potential, the free energy is transformed as follows:

$$\begin{aligned}\mathcal{F}(\nu^{-2}\varepsilon_1, \nu^{-2}\varepsilon_2, \nu^{-2}\varepsilon_3) = (\mu - \nu^{-2}\langle u\rangle)\mathcal{N} - \nu^3 \int d\varepsilon\, n f(\nu^{-2}\varepsilon - \mu) \\ + \nu^4(\kappa/2)(\langle u^2\rangle - \langle u\rangle^2) \quad .\end{aligned}$$

By differentiating this expression with respect to ν at the value $\nu = 1$ and setting the obtained derivative equal to zero, we have the second equation for determining the potential parameters

$$\int d\varepsilon\, f\left[n - 2(\varepsilon - \langle u\rangle)(dn/d\varepsilon)\right] = 2\kappa\left(\langle u^2\rangle - \langle u\rangle^2\right) \quad . \tag{8.2.12}$$

Using (8.2.11), this equation can be represented as

$$\int d\varepsilon\, f\left[n - 2(\varepsilon - \langle u\rangle)\frac{dn}{d\varepsilon} + \frac{1}{2\pi}\frac{\langle u^2\rangle - \langle u\rangle^2}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}}\right] = 0 \quad .$$

By integrating the last equation by parts, it is possible to represent the second equation for the potential parameters as follows:

$$\int d\varepsilon\, g(\varepsilon)(df/d\varepsilon) = 0 \quad ,$$

$$g(\varepsilon) = \int_{\varepsilon_1}^{\varepsilon} d\varepsilon \left[n - 2(\varepsilon - \langle u \rangle)\frac{dn}{d\varepsilon} + \frac{1}{2\pi}\frac{\langle u^2 \rangle - \langle u \rangle^2}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} \right] \quad . \tag{8.2.13}$$

We present the basic properties of the function $g(\varepsilon)$ which will be used in what follows:

(1.) Function $g(\varepsilon)$ is continuous on the interval (ε_1, ∞) and differentiable on it everywhere, except for the points ε_2, ε_3, where it has a discontinuity in the derivative.

(2.) $g(\varepsilon_1) = g(+\infty) = 0$.

(3.) For $\varepsilon_1 \le \varepsilon \le \varepsilon_2$

$$g(\varepsilon) = \frac{1}{\pi}\int_{\varepsilon_1}^{\varepsilon} d\varepsilon \frac{1}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} \frac{(\gamma-\varepsilon_1)(\gamma-\varepsilon_2)(\gamma-\varepsilon_3)}{\varepsilon-\gamma} < 0 \quad .$$

This inequality follows from the fact that, by definition, $\varepsilon_2 \le \gamma(x) \le \varepsilon_3$. For $\varepsilon_2 \le \varepsilon \le \varepsilon_3$

$$g(\varepsilon) = g(\varepsilon_2) + \frac{\varepsilon-\varepsilon_2}{\varepsilon_3-\varepsilon_2}[g(\varepsilon_3) - g(\varepsilon_2)] \quad .$$

For $\varepsilon_3 \le \varepsilon < \infty$

$$g(\varepsilon) = \frac{1}{\pi}\int_{\varepsilon}^{\infty} d\varepsilon \frac{1}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} \frac{(\gamma-\varepsilon_1)(\gamma-\varepsilon_2)(\gamma-\varepsilon_3)}{\gamma-\varepsilon} > 0 \quad .$$

Thus, the function $g(\varepsilon)$ becomes zero at some point of the interval $[\varepsilon_2, \varepsilon_3]$, too.

(4.) Let $\varepsilon_1 = -2a(1-\rho^2/2)$, $\varepsilon_2 = a(1-2\rho^2)$, $\varepsilon_3 = a(1+\rho^2)$. Then, as $\rho \to 0$

$$g(\varepsilon_3) = -g(\varepsilon_2) = \frac{(3a)^{1/2}}{2\pi}\rho^2[1 + O(\rho^2 \log 4/\rho)] \quad .$$

Consequently, as $\rho \to 0$, the function $g(\varepsilon)$ becomes zero for $\varepsilon = (\varepsilon_3 + \varepsilon_2)/2$.

The proof of these propositions is too cumbersome to be given here.

Now we first analyze the present solution to the Peierls problem at absolute zero. Since $df/d\varepsilon = -\delta(\varepsilon-\mu)$, where $\delta(x)$ means the Dirac function, the equation (8.2.13) takes the form $g(\mu) = 0$. This means (considering the third property of the function $g(\varepsilon)$) that $\varepsilon_2 \le \mu \le \varepsilon_3$. The latter is equivalent to $\mathcal{N} = \mathcal{N}^*$, where $\mathcal{N}^*$ is the number of states in the conduction band $\varepsilon_1 \le \varepsilon \le \varepsilon_2$. Since for the periodic potential (and the single-band potential belongs to this class) $\mathcal{N}^* L = 1$, where L is the potential period, the equation (8.2.13) can be represented as $\mathcal{N} L = 1$. This equation is called the Peierls equation. If $\mathcal{N}$ and L in this equation are expressed in terms of the Fermi momentum k_F and the reciprocal lattice vector

b, according to the equalities $\mathcal{N} = k_F/\pi$, $L = 2\pi/b$, then the Peierls equation can be written in the conventional form $b = 2k_F$. The single-band potential period is $L = -\mathrm{i}2\omega'$, so that the Peierls equation also has the form

$$\omega' = \mathrm{i}/2\mathcal{N} \quad . \tag{8.2.14}$$

At absolute zero (8.2.11) is readily written as

$$\omega = 2\pi\kappa \quad . \tag{8.2.15}$$

According to formulas available in the theory of elliptic functions,

$$\begin{aligned} \varepsilon_3 - \varepsilon_1 &= (\pi/2\omega)^2 \vartheta_3^4(0,q) \quad , \\ \varepsilon_2 - \varepsilon_1 &= (\pi/2\omega)^2 \vartheta_2^4(0,q) \quad , \\ \varepsilon_3 - \varepsilon_2 &= (\pi/2\omega)^2 \vartheta_4^4(0,q) \quad , \end{aligned} \tag{8.2.16}$$

where $q = \exp(\mathrm{i}\pi\omega'/\omega)$. These formulas express the single-band potential parameters in terms of electron density and elastic lattice constant. As a result, the potential and the corresponding number of states and the wave functions are completely determined. This allows us to calculate any characteristics of the system as, for example, the free energy:

$$\begin{aligned} \mathcal{F} &= \int_{\varepsilon_1}^{\varepsilon_2} dn(\varepsilon - \langle u \rangle) + (\kappa/2)\left(\langle u^2 \rangle - \langle u \rangle^2\right) \\ &= -\frac{\mathcal{N}}{48\pi^2\kappa^2}\left[1 + \frac{1}{24\pi^2\kappa\mathcal{N}}\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)}\right]\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)} \\ &\qquad + \frac{\mathcal{N}^2}{2\kappa}\left[1 + \frac{1}{24\pi^2\kappa\mathcal{N}}\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)}\right]^2 \quad , \end{aligned} \tag{8.2.17}$$

where $q = \exp(-1/4\kappa\mathcal{N})$.

Let us treat the limiting cases.

Let $q \to 0$; this corresponds to small electron density. Then,

$$\begin{aligned} \varepsilon_3 - \varepsilon_1 &= (1/16\kappa^2)(1 + O(q)) \quad , \\ \varepsilon_2 - \varepsilon_1 &= (1/\kappa^2)(1 + O(q^2)) \quad . \end{aligned} \tag{8.2.18}$$

The potential turns out to be a set of potential wells such as

$$u(x) = \text{const} - 1/8\kappa^2 \cosh^2(x/4\kappa) \quad . \tag{8.2.19}$$

They are independent of electron density. Each electron creates a potential well itself and occupies the only energy level existing in the well. This autolocalized state of an electron is referred to as a soliton (polaron) in physics. The energy level width dependent on electron density is determined by electron tunneling from one potential well to another. The free energy is

$$\mathcal{F} = -(\mathcal{N}/48\kappa^2)\left(1 - 6q^2 + O(q^4)\right) + \mathcal{N}^2/2\kappa \quad . \tag{8.2.20}$$

In this expression, the first term is proportional to soliton density and the proportionality coefficient is the soliton binding energy. The second term is proportional to a squared density of solitons and describes their mutual repulsion.

Let $q \to 1$; this corresponds to great electron density. Then,

$$\begin{aligned} \varepsilon_3 - \varepsilon_1 &= \pi^2 \mathcal{N}^2 (1 + O(\dot{q})) \quad , \\ \varepsilon_3 - \varepsilon_2 &= \pi^2 \mathcal{N}^2 16 \dot{q} \left(1 + O(\dot{q})^2\right) \quad , \end{aligned} \tag{8.2.21}$$

where $\dot{q} = \exp(-4\pi^2 \kappa \mathcal{N})$. The potential

$$u(x) = \text{const} + 16\pi^2 \mathcal{N}^2 \dot{q} \cos(2\pi \mathcal{N} x + \varphi) \tag{8.2.22}$$

depends on the electron density and is therefore generated by all electrons collectively. This limiting case was studied earlier by Fröhlich and designated by him as a charge-density wave [8.15]. The free energy is

$$\mathcal{F} = (1/3)\pi^2 \mathcal{N}^3 \left(1 - 6\dot{q}^2 + O(\dot{q}^4)\right) \quad . \tag{8.2.23}$$

The first term in this expression is the energy of free electrons. The second term describes the decrease in electron energy when the gap is formed and is proportional to a squared gap.

Thus, the charge-density waves and solitons (polarons) are limiting cases of the Peierls self-consistent state.

Let us clarify this statement more carefully. Using well-known results of the theory of elliptic functions it is easy to show

$$\begin{aligned} \wp(\mathrm{i}x + \omega) - \langle \wp(\mathrm{i}x + \omega) \rangle &= \frac{\mathrm{i}\pi}{2\omega\omega'} + \frac{d^2}{dx^2} \log \vartheta_2 \left(\frac{\mathrm{i}x}{2\omega} \middle| \tau \right) \\ &= \frac{\mathrm{i}\pi}{2\omega\omega'} + \frac{\pi^2}{4\omega^2} \sum_{s=-\infty}^{\infty} \cosh^{-2} \left[\frac{\pi}{2\omega} (x - 2\mathrm{i}\omega' s) \right] \end{aligned} \tag{8.2.24}$$

to hold. With the help of the above equality we get from (8.2.5) the following expression for the potential $u(x)$:

$$u(x) = \frac{\mathrm{i}\pi}{\omega\omega'} + \frac{\pi^2}{2\omega^2} \sum_{s=-\infty}^{\infty} \cosh^{-2} \left[\frac{\pi}{2\omega} (x - 2\mathrm{i}\omega' s) \right] \quad . \tag{8.2.25}$$

According to this expression the potential $u(x)$ represents a lattice made up of potential wells of the form $-A \cosh^{-2} Bx$, each well separated from the other by the period $L = -\mathrm{i}2\omega'$. The potential well $-A \cosh^{-2} Bx$ is well known (see, e.g., [8.16]) to contain only a finite number of bound states equal to $\left[(1/2)\{(1 + 4A/B^2)^{1/2} - 1\}\right]$, where $[x]$ means here an integer part of a number x. In our case a potential well contains only one bound state. Thus, if $-\mathrm{i}\pi\tau = -\mathrm{i}\pi\omega'/\omega = 1/4\kappa\mathcal{N} \to \infty$ and the separation $-2\mathrm{i}\omega'$ between the potential wells is larger than their width, the Peierls state is (almost) a soliton lattice, i.e., a lattice of potential wells each of which contains only one electron in a unique bound state (we do not take into account the electron spin). If $-i\pi\tau \to 0$, the potential wells composing the lattice are overlapping to such an extent that an energy level is transformed into an energy band. In this case it is reasonable to rewrite (8.2.24) by means of the Jacobi imaginary transformation of the right-hand-side of the equality,

$$\begin{aligned}\wp(ix+\omega) - \langle \wp(ix+\omega)\rangle &= \frac{d^2}{dx^2}\log\theta_4\left(\frac{ix}{2\omega'}\Big|\tau\right)\\ &= \frac{i\pi}{\omega\omega'}\sum_{s=1}^{\infty}\frac{i\pi s/\tau}{\cosh(i\pi s/\tau)}\cos(i\pi sx/\omega')\ .\end{aligned} \tag{8.2.26}$$

Taking this expression into account we obtain with the help of (8.2.5) for the potential $u(x)$ the expression

$$u(x) = -\frac{2i\pi}{\omega\omega'}\sum_{s=1}^{\infty}\frac{i\pi s/\tau}{\cosh(i\pi s/\tau)}\cos(i\pi sx/\omega')\quad . \tag{8.2.27}$$

This supports the picture of the Peierls state in the limiting case $-i\pi\tau \to \infty$ as a superposition of charge-density waves. In the language of mathematics a transition from one physical picture to another one corresponds to an imaginary Jacobi transformation. According to the theory of elliptic functions, under this transformation the equality $(\log q)(\log \dot{q}) = \pi^2$ is valid. Therefore a transition from one form of definition to another takes place at $q = \dot{q} = \exp(-\pi)$, i.e., at $\tau = i$ or at

$$4\pi\kappa\mathcal{N} = 1\quad .$$

To determine the temperature dependence of potential parameters, it is necessary to solve the set of equations (8.2.11, 13). We shall calculate the critical temperature T_p at which a single-band potential occurs. For this purpose we analyse the above set of equations under the assumption that the gap width is zero, $\varepsilon_2 = \varepsilon_3$. As long as the function $df/d\varepsilon$ is symmetric with respect to $E = \mu$, and the function $g(\varepsilon)$ is, in view of property 4, anti-symmetric with respect to $\varepsilon = (\varepsilon_3 + \varepsilon_2)/2$ while $\varepsilon_3 - \varepsilon_2 \to 0$, it follows from (8.2.12) that

$$\varepsilon_3 = \varepsilon_2 = \mu\quad . \tag{8.2.28}$$

We multiply the equality

$$\int d\varepsilon \frac{1}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} = 0$$

by $f(\varepsilon_2)$ and subtract it from (8.2.11). Setting in the above expression $\varepsilon_3 = \varepsilon_2 = \mu$, we find

$$\begin{aligned}8\pi\kappa T_p^{1/2} &= \int_{z_1}^{\infty} dz\frac{\tanh(z/2)}{z\sqrt{z-z_1}},\\ z = (\varepsilon-\mu)/T_p, &\quad z_1 = (\varepsilon_1-\mu)/T_p\quad .\end{aligned} \tag{8.2.29}$$

Let $\mu - \varepsilon_1 \gg T_p$. By partial integration in (8.2.29) we have

$$8\pi\kappa T_p^{1/2} = -\frac{1}{|z_1|^{1/2}} \int_{z_1}^{\infty} \frac{dz}{\cosh^2(z/2)} \log \left| \frac{(z+|z_1|)^{1/2} - |z_1|^{1/2}}{(z+|z_1|)^{1/2} + |z_1|^{1/2}} \right| \quad .$$

Expanding the integrand in a series of $z/|z_1|$ yields an asymptotic series such that

$$\begin{aligned} 8\pi\kappa T_p^{1/2} &\simeq -\frac{1}{2|z_1|^{1/2}} \int_{-\infty}^{\infty} \frac{dz}{\cosh^2(z/2)} \\ &\quad \times \left[\log \left| \frac{z}{4|z_1|} \right| + \frac{3}{16} \frac{z^2}{|z_1|^2} + O\left(\frac{z^4}{|z_1|^4}\right) \right] \\ &= \frac{1}{|z_1|^{1/2}} \left[2\log \frac{8\gamma|z_1|}{\pi} - \frac{\pi^2}{8|z_1|^2} + O(|z_1|^4) \right] \quad , \end{aligned}$$

where $\gamma = \exp C$, C is the Euler constant. When the previous notation is used, the resulting equation reads

$$4\pi\kappa|\varepsilon_1 - \mu|^{1/2} = \log \frac{8\gamma|\varepsilon_1 - \mu|}{T_p} - \frac{\pi^2}{16} \frac{T_p}{|\varepsilon_1 - \mu|^2} + O\left(\frac{T_p^4}{|\varepsilon_1 - \mu|^4}\right)$$

Restricting ourselves to the first term on the right-hand-side of this equation, we have the standard expression

$$T_p = (8\gamma/\pi)|\varepsilon_1 - \mu| \exp\left[-4\pi\kappa|\varepsilon_1 - \mu|^{1/2}\right] \quad . \tag{8.2.30}$$

This expression can be represented as $T_p = (\gamma/\pi)\Delta$, where Δ is the half-width of a gap at zero temperature. This is easily seen by comparing (8.2.21) and (8.2.30).

8.3 Theory of the Fröhlich Conductivity

A conductivity created by a uniform motion of the Peierls state is known as the Fröhlich conductivity, since Fröhlich was the first to investigate this new type of conductivity in a limit case of charge-density wave in 1954 [8.15]. The existence of Fröhlich conductivity was confirmed about 10-15 years ago by experiments in quasi-one-dimensional conductors. Here we build a rigorous theory of the Fröhlich conductivity for a general case, obtain an exact expression for an effective mass and investigate current oscillations. Further we closely follow [8.17].

The theory of the Fröhlich conductivity of a one-dimensional conductor is developed here similar to the previous Sect. 8.2. and in line with Fröhlich, according to whom electron wave functions $\psi(x,t,E)$ and lattice displacements $w(x,t)$ in a uniformly moving Peierls state are defined as extremals of an energy $\mathcal{E}$ of the system with a given number of electrons $\mathcal{N}$ and a momentum $\mathcal{P}$ with

$$\mathcal{E} = \sum \left\langle \frac{\hbar^2}{2m}|\psi_x|^2 + \alpha w_x |\psi|^2 \right\rangle + \frac{1}{2}\rho\langle w_t^2 + s^2 w_x^2\rangle \tag{8.3.1}$$

$$\mathcal{N} = \sum \langle |\psi|^2 \rangle \tag{8.3.2}$$

$$\mathcal{P} = \sum \frac{\hbar}{2i}\langle \bar{\psi}\psi_x - \bar{\psi}_x\psi\rangle - \rho\langle w_t w_x\rangle \tag{8.3.3}$$

The following notations are used: $\hbar$ is the Planck constant, x is the space coordinate, t is the time, m is the electron mass, E is the electron energy, ρ is the lattice density, s is the sound velocity, α is the electron-phonon interaction constant,

$$\langle A \rangle = \lim_{l\to\infty} \frac{1}{l}\int_0^l A(x)\,dx$$

The summation in (8.3.1 - 3) is taken over all occupied electron states. We will consider our system at the temperature of absolute zero and therefore we will take the summation from the ground state up to ones with a Fermi energy. Thus the problem of Fröhlich conductivity reduces effectively to the determination of a minimum for the functional

$$\mathcal{F} = \mathcal{E} - \mu\mathcal{N} - v\mathcal{P} \tag{8.3.4}$$

where the chemical potential μ and the speed of motion v are the Lagrange multipliers. It is worthwhile to note that the functional (8.2.1) coincides at the temperature $T = 0$ with the functional (8.3.4) at the momentum $\mathcal{P} = 0$.

Before considering the minimization problem it is appropriate to go over from the laboratory frame of reference K to the frame of reference K' moving with the velocity v. The well-known Galilean transformation is accomplished by

$$\begin{aligned}
&x = x' + vt', \quad t = t' \quad , \\
&E = E' + P'v + (1/2)mv^2, \quad P = P' + mv \quad , \\
&\psi(x,t) = \psi'(x',t')\exp\left[(\mathrm{i}/\pi)\left(mvx - (1/2)mv^2 t\right)\right] \quad , \\
&w(x,t) = w(x',t') \quad .
\end{aligned} \tag{8.3.5}$$

Here and below all quantities in the moving frame of reference K' are marked with a prime. The Galilean transformation brings the functional $\mathcal{F}$ into the form

$$\begin{aligned}
\mathcal{F} = &\sum \left\langle \frac{\hbar^2}{2m}|\psi'_{x'}|^2 + \alpha w'_{x'}|\psi'|^2 \right\rangle \\
&- \left(\mu + \frac{mv^2}{2}\right)\sum\langle|\psi'|^2\rangle + \frac{1}{2}\rho\,\left\langle (w'_{t'})^2 + s^2(1 - v^2/s^2)(w'_{x'})^2\right\rangle \quad .
\end{aligned} \tag{8.3.6}$$

Variation of the functional yields the Euler-Lagrange equations for wave functions and lattice displacements:

$$
\begin{aligned}
&-\frac{\hbar^2}{2m}\psi'_{x'x'} + \alpha w'_{x'}\psi = -\mathrm{i}\hbar\psi'_{t'} \quad , \\
&w'_{t't'} - s^2\left(1-\frac{v^2}{s^2}\right) w'_{x'x'} = \frac{\alpha}{\rho}\left(\sum|\psi'|^2\right)_{x'} \quad .
\end{aligned}
\tag{8.3.7}
$$

Further, we consider the stationary case with $w'(x',t') = w'(x')$ and $\psi'(x',t') = \psi'(x')\exp[(\mathrm{i}/\hbar)E't']$. In this case the system (8.3.7) acquires the simple form

$$
\begin{aligned}
&-\frac{\hbar^2}{2m}\psi'_{x'x'} + \alpha w'_{x'}\psi = E'\psi' \quad , \\
&-s^2\left(1-\frac{v^2}{s^2}\right) w'_{x'} = \frac{\alpha}{p}\sum\left(|\psi'|^2 - \langle|\psi'|^2\rangle\right) \quad .
\end{aligned}
\tag{8.3.8}
$$

To simplify the subsequent calculations we shall rewrite the above system of equations:

$$
-\psi'_{x'x'} + u\psi' = \varepsilon\psi', \quad -\kappa u = \sum(|\psi'|^2 - \langle|\psi'|^2\rangle) \quad , \tag{8.3.9}
$$

where we introduced the new notations

$$
\begin{aligned}
&\varepsilon = (2m/\hbar^2)E \ , \quad u = (2m/\hbar^2)\alpha w'_{x'} \quad , \\
&\kappa = (\hbar^2/2m\alpha^2)\rho s^2(1 - v^2/s^2) \quad ,
\end{aligned}
\tag{8.3.10}
$$

to be compared with (8.2.3). Solving the system (8.3.9) we find electron wave functions and lattice displacements in the moving Peierls state similar to Sect. 8.2,

$$
u(x) = -2\big(\wp(ix+\omega) - \langle\wp(ix+\omega)\rangle\big) \quad , \tag{8.3.11}
$$

where $\wp(z)$ is the Weierstrass elliptic function. The half-periods of the Weierstrass function satisfy the relations

$$
\omega' = \mathrm{i}/2\mathcal{N}, \quad \omega = 2\pi\kappa \quad . \tag{8.3.12}
$$

The first relation connecting the period of the potential $u(x)$ and the electron density $\mathcal{N}$ is named after Peierls. The second relation appeared for the first time in a paper by Fröhlich and thus received his name. The electron wave function is given by

$$
\begin{aligned}
&\psi(x',\varepsilon) = [\langle\chi^{-1}(x',\varepsilon)\rangle\,\chi(x',\varepsilon)]^{-1/2}\exp\left(\mathrm{i}\int^{x'} dx\,\chi(x,\varepsilon)\right) \quad , \\
&\chi(x',\varepsilon) = [P(\varepsilon)]^{1/2}\left(\varepsilon - \gamma(x')\right)^{-1} \quad , \\
&P(\varepsilon) = (\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3) \quad , \\
&\gamma(x') = (1/2)\left(\varepsilon_1+\varepsilon_2+\varepsilon_3 - u(x')\right) \quad ,
\end{aligned}
\tag{8.3.13}
$$

where ε_1, ε_2, ε_3 are boundaries of the spectrum. Note that according to (8.3.13) the wave function is normalized, $\langle|\psi|^2\rangle = 1$.

Formally the expressions (8.3.11, 12) coincide completely with analogous expressions obtained in Sect. 8.2. at $T = 0$. Nevertheless there is an important difference. It consists in the definition (8.3.10) for the parameter κ that contains in comparison with a similar parameter of the previous Sect. 8.2. an important multiplier $(1 - v^2/s^2)$ that takes into account a dependence on a velocity. Thus a dimensionless parameter $\kappa\mathcal{N}$ for the moving Peierls state is less than that for the static one. In a physical language it means that the moving Peierls state is more similar to a soliton lattice than the static one.

In order to obtain an expression for an effective mass M of the Peierls state let us write down a relation between momenta (and between energies also) of this state in laboratory and movable frames of reference. It is easy to do with the Galilean transformations

$$\begin{aligned}\mathcal{P} = \mathcal{P}' + \mathcal{N}Mv = \sum \frac{\hbar}{2\mathrm{i}}\langle \bar{\psi}'\psi'_{x'} - \bar{\psi}'_{x'}\psi'\rangle - \rho\langle w'_{t'}w'_{x'}\rangle \\ + v\left(m\sum\langle|\psi'|^2\rangle + \rho\langle(w'_{x'})^2\rangle\right)\end{aligned} \tag{8.3.14}$$

$$\begin{aligned}\mathcal{E} &= \mathcal{E}' + v\mathcal{P}' + (1/2)\mathcal{N}Mv^2 \\ &= \sum\left\langle \frac{\hbar^2}{2m}|\psi'_{x'}|^2 + \alpha w'_{x'}|\psi'|^2\right\rangle \\ &\quad + (1/2)\rho\langle(w'_{t'})^2 + s^2(w'_{x'})^2\rangle + v\sum\frac{\hbar}{2i}\langle\bar{\psi}'\psi'_{x'} - \bar{\psi}'_{x'}\psi'\rangle \\ &\quad + (1/2)v^2\left(m\sum\langle|\psi'|^2\rangle + \rho\langle(w'_{x'})^2\rangle\right) \quad .\end{aligned} \tag{8.3.15}$$

Taking $\langle|\psi|^2\rangle = 1$ into account we get from the last formulas the following expression for an effective mass M:

$$M = m\left(1 + \frac{\rho}{\mathcal{N}m}\langle(w'_{x'})^2\rangle\right) = m\left(1 + \frac{\rho}{\mathcal{N}m}\left(\frac{\hbar^2}{2m\alpha}\right)^2\langle u^2\rangle\right) \tag{8.3.16}$$

The quantity M is called an effective mass since according to relations (8.3.14), (8.3.15) at $\mathcal{P}' = 0$ it is a proportionality coefficient between a momentum $\mathcal{P}$ in a laboratory frame of reference and a velocity v and also it is a coefficient of a quadratic with respect to velocity additional term to energy $\mathcal{E}$ in a laboratory frame of reference. We call M an effective mass although it depends on a velocity in fact and strictly speaking we should consider as an effective mass a quantity $M_{\mathrm{eff}} = \lim_{v\to 0} M$.

Note also that according to (8.3.16) the effective mass M is always larger than the electron mass m since it includes an accompanying lattice distortion $(\rho/\mathcal{N}m)\langle(w'_{x'})^2\rangle$. This lattice distortion and an appropriate part of an effective mass vanishes in the transition from the soliton limit to the charge-density wave limit, i.e., when the dimensionless parameter $\kappa\mathcal{N}$ grows.

Substituting (8.3.11) into (8.3.16) we get the following formula for the ratio of the effective mass M to the electron mass m:

$$\frac{M}{m} = 1 + \frac{\rho}{2\mathcal{N}m}\left(\frac{\pi^2\hbar^2}{12m\alpha\omega^2}\right)^2 \left\{\vartheta_2^8(0,q) + \vartheta_3^8(0,q) + \vartheta_4^8(0,q) \right.$$

$$\left. - 2\left[\frac{6\mathrm{i}}{\pi\tau} + \frac{\vartheta_1'''(0,q)}{\pi^2\vartheta_1'(0,q)}\right]^2\right\} = 1 + \frac{\rho}{2\mathcal{N}m}\left(\frac{\pi^2\hbar^2}{12m\alpha\omega'^2}\right)^2 \tag{8.3.17}$$

$$\times\left\{\vartheta_2^8(0,\dot{q}) + \vartheta_3^8(0,\dot{q}) + \vartheta_4^8(0,\dot{q}) - 2\left[\frac{\vartheta_1'''(0,\dot{q})}{\pi^2\vartheta_1'(0,\dot{q})}\right]^2\right\} \quad .$$

Here $\vartheta_\alpha(0,q)$ are theta-constants and $q = \exp(\mathrm{i}\pi\omega'/\omega)$, $\dot{q} = \exp(-\mathrm{i}\pi\omega/\omega')$. Substituting into the obtained formula the halfperiods ω, ω' defined by (8.3.12) we get an expression for the effective mass for the Peierls state in terms of the physical parameters $\mathcal{N}$, κ. Since, according to (8.3.10), the parameter κ depends on the velocity v so does the effective mass. The two expressions given above for the effective mass are both suitable to study one of the two limiting cases according to the value of the dimensionless parameter $\kappa\mathcal{N}$. These expressions transform into each other by means of the imaginary Jacobi transformation. Let us now consider the limiting cases.

1) The soliton limit, $\kappa\mathcal{N} = \mathrm{i}/4\pi\tau \to 0$. In this case $q = \exp(-1/4\pi^2\kappa\mathcal{N}) \to 0$ also. Writing the theta-constants in (8.3.17) in terms of the Taylor series we get

$$\frac{M}{m} = 1 + \frac{1}{6}\frac{\alpha^4}{\hbar^2\rho^2 s^6(1 - v^2/s^2)^3} \quad . \tag{8.3.18}$$

This is a well-known formula for the effective soliton mass.

2) The charge-density wave limit, $\kappa\mathcal{N} = \mathrm{i}/4\pi\tau \to \infty$. In this case $\dot{q} = \exp(-4\pi^2\kappa\mathcal{N}) \to 0$. In accordance with the formula (8.3.17) the charge-density wave effective mass is

$$\frac{M}{m} = 1 + 2^5\pi^4\frac{\rho\mathcal{N}^3\hbar^4}{m^3\alpha^2}\exp(-8\pi^2\kappa\mathcal{N}) = 1 + \frac{\rho}{2\mathcal{N}m}\left(\frac{\Delta}{\alpha}\right)^2 \quad . \tag{8.3.19}$$

Here $\Delta = (8\pi^2\hbar^2\mathcal{N}^2/m)\exp(-4\pi^2\kappa\mathcal{N})$ is a width of a spectral gap. This expression for the charge-density wave effective mass coincides with a result of *Fröhlich* [8.15], who studied a uniform movement of the Peierls state in such a limiting case. From (8.3.19) it follows that $M/m \to 1$ at $\kappa\mathcal{N} \to \infty$.

Using our expression for the effective mass it is not difficult to get an estimate for a conductivity σ of quasi-one-dimensional conductor, e.g. by means of the Drude-Lorentz formula $\sigma = e^2\mathcal{N}\tau/M$, where e is an elementary electric charge and τ is a relaxation time.

The mean current I, created by a uniform movement of the Peierls state with a velocity v is defined by

$$I = \langle J \rangle \tag{8.3.20}$$

where the expression for the microscopic current J has the well-known form

$$\begin{aligned} J(x) &= \frac{e}{m}\frac{\hbar}{2\mathrm{i}}\sum(\bar{\psi}\psi_x - \bar{\psi}_x\psi) \\ &= \frac{e}{m}\frac{\hbar}{2\mathrm{i}}\sum(\bar{\psi}'\psi'_{x'} - \bar{\psi}'_{x'}\psi') + ev\sum|\psi'|^2 \quad . \end{aligned} \tag{8.3.21}$$

For the following considerations we assume that the states in the conductivity band are occupied, i.e.,

$$\frac{\hbar}{2\mathrm{i}}\sum(\bar{\psi}'\psi'_{x'} - \bar{\psi}'_{x'}\psi') = 0 \quad .$$

In this case the mean current has the well-known expression

$$I = ev\sum\langle|\psi'|^2\rangle = ev\mathcal{N} \quad . \tag{8.3.22}$$

For the microscopic current we get

$$\begin{aligned} J(x') &= ev\sum|\psi'|^2 = ev\left[\mathcal{N} - \sum(|\psi'|^2 - \langle|\psi'|^2\rangle)\right] \\ &= ev\big(\mathcal{N} - \kappa u(x')\big) \\ &= I\left\{1 + 2\frac{\kappa}{\mathcal{N}}\left[\wp(\mathrm{i}x' + \omega) - \langle\wp(\mathrm{i}x' + \omega)\rangle\right]\right\} \quad . \end{aligned} \tag{8.3.23}$$

When we measure the microscopic current at a fixed point $x = 0$ in the laboratory frame of reference we obtain $x' = -vt$; therefore the microscopic current appears to be a periodic function of time

$$\begin{aligned} J(t) &= I\left\{1 + 2\frac{\omega\omega'}{\mathrm{i}\pi}\left[\wp(\mathrm{i}vt + \omega) + \frac{\eta'}{\omega'}\right]\right\} \\ &= I\frac{\pi\omega'}{2\mathrm{i}\omega}\sum_{s=-\infty}^{\infty}\cosh^{-2}\left(\frac{\pi vt}{2\omega} + \mathrm{i}\pi\tau s\right) \\ &= I\left[1 + 2\sum_{s=1}^{\infty}\frac{\mathrm{i}\pi s/\tau}{\sinh(\mathrm{i}\pi s/\tau)}\cos(\mathrm{i}\pi svt/\omega')\right] \quad . \end{aligned} \tag{8.3.24}$$

For the period T and the frequency ν of the microscopic current we get from (8.3.24) the expressions

$$T = e/I, \quad \nu = 2\pi I/e \quad . \tag{8.3.25}$$

We have given several expressions for the microscopic current. The second expression is convenient for the study of the soliton limiting case and the third one is suitable to investigate the charge-density wave limiting case. Let us now consider the expressions for $J(t)$ in these limiting cases.

1) The soliton limiting case, $\kappa\mathcal{N} = \mathrm{i}/4\pi\tau \to 0$. For this case

$$J(t) = I\sum_{s=-\infty}^{\infty}\exp(\mathrm{i}s\nu t) = 2\pi I\sum_{s=-\infty}^{\infty}\delta(\nu t - 2\pi s) \quad , \tag{8.3.26}$$

i.e., the current is just a periodic train of 2π-pulses. This result is natural since in the soliton limiting case the Peierls state is just a lattice of potential wells each containing one electron (we do not take into account the electron spin).

2) The charge-density wave limiting case, $\kappa\mathcal{N} = i/4\pi\tau \to \infty$. For this case

$$J(t) = I\left[1 + \frac{8\pi^2\kappa\mathcal{N}}{\sinh(4\pi^2\kappa\mathcal{N})}\cos(\nu t)\right] \quad , \tag{8.3.27}$$

i.e., the current has only one harmonic.

Let us discuss these results. The current oscillations (or depending on experimental situation, the voltage oscillations) with a frequency proportional to an average current that follows from formulas (8.3.24, 25) have been discovered in quasi-one-dimensional conductors about ten years ago [8.18,19]. Commonly they are designated as "narrow band noise" but it seems more reasonable to call them "current oscillations" as it has been proposed by the author of a review article [8.19]. A lot of harmonics of current oscillations are often observed in experiments. This finds a natural explanation via (8.3.24) and can hardly be explained by the Fröhlich theory [8.15] that takes into account only one harmonic of the lattice distortion in the Peierls state.

In conclusion we note that the papers [8.2, 3, 17, 20-22] allow to describe from a unified point of view all equilibrium properties of quasi-one-dimensional conductors with the Peierls state and the properties of the Fröhlich conductivity created by a uniform motion of this state.

8.4 Conclusion

In this section we discuss the development and generalization of the studies dealt with in preceding sections.

First of all, it has to be mentioned that the authors of [8.20, 21] succeeded in calculating the temperature dependence of the boundaries of the spectrum of a single-band potential which is an extremal of the Peierls thermodynamic functional. The results obtained there were used to give a classification of quasi-one-dimensional conductors, related to the dimensionless quantity $\kappa = (h^2\mu/2m)^{1/2}\hbar\omega/\lambda^2$, where μ is the chemical potential, ω is the frequency of acoustic phonons, λ is the electron-phonon interaction constant. If $\kappa > \kappa_c$, the quasi-one-dimensional conductor is a conductor of the charge-density wave type, and if $\kappa < \kappa_c$, it is a conductor of the soliton (polaron) type. Analytical calculations give $\kappa_c = 0.1326$. Analytical expressions in good agreement with the calculations were obtained for the energy and temperature values at which a gap ($\kappa > \kappa_c$) or a discrete level ($\kappa < \kappa_c$) appear in the spectrum. In [8.22], it was shown that the equations arising from the variation of Peierls thermodynamic functional are actually the self-consistency conditions that result from applying the approximating Hamiltonian method to the Fröhlich Hamiltonian [8.23]. Consequently, the method proposed in the preceding sections enables us to find exact solutions of the self-consistency equations in the approximating Hamiltonian method. Establishing this fact (which is well-studied within the framework of the Peierls problem, but in fact has a much wider range of applications) appears to be very important.

In [8.24, 25], the generalization of the Peierls problem, discussed in the preceding sections, to the case of a discrete Schrödinger operator is treated. The generalization of the Peierls thermodynamic potential to the case when its extremals are many-band potentials is proposed in [8.3]. The variations of such a thermodynamic potential were discussed in [8.26].

Different aspects of the Peierls problem are now being studied in an intensive way, so that the list of references enclosed in this section is, regrettably, far from complete.

It is to be emphasized that the Peierls problem has, besides its direct application in solid-state physics, a more general significance, because its solution leads us to the necessity of considering the deformation of Riemann surfaces and different fields on them, of developing a spectral theory for the Schrödinger operator with an almost periodic potential, of formulating a theory of infinite-dimensional integrable Hamiltonian systems, of developing the theory of the Kac-Moody algebras and loop groups etc.

In conclusion, we make some general comments concerning the role and importance of finite-gap potentials in solid-state physics. The simplest periodic potential is commonly represented by quantum physics text books as a potential consisting of rectangular wells (the Kronig-Penny model). However, to determine the boundaries of the spectrum that corresponds to this potential, it is necessary to solve transcendental equations, and, moreover, the appropriate eigenfunctions are so cumbersome that it is very difficult to use them to calculate the matrix elements of any observables. Finite-gap potentials are free of these limitations and, because of that only these potentials should play the same role in solid-state physics as does the Kepler problem in atomic theory. We note that any periodic potential generally having an infinite number of bands (similar to the Mathieu problem) can be approximated by a finite-gap potential (provided narrow enough gaps in the spectrum are disregarded) [8.27]. The matrix elements of any observables on the eigenfunctions that correspond to finite-gap potentials can easily be calculated analytically by the residue method. Therefore, finite-gap potentials are important in solid-state physics. Finite-gap potentials have recently been applied in the way indicated above to solve some specific problems in solid-state physics [8.28-32].

References

Chapter 1

1.1 B.A. Dubrovin, V.B. Matveev, S.P. Novikov: Usp. Math. Nauk **31** (1), 55–136 (1976) in *London Math. Soc. Lecture Notes Ser.*, **60** (Cambridge University Press, Cambridge 1981)
1.2 V.B. Matveev: *Abelian Functions and Solitons*, Preprint No. 373, (University of Wroclaw, Wroclaw 1976)
1.3 C.S. Gardner, J.M. Green, M.D. Kruskal, R.M. Miura: Phys. Rev. Lett., **19**, 1095–1098 (1976)
1.4 P.D. Lax: Comm. Pure Appl. Math. **21** (2), 467–490 (1968)
1.5 S.P. Novikov, V.E. Zakharov, S.V. Manakov, L.V. Pitaevski: *Soliton Theory: The Inverse Scattering Method* (Plenum, New York 1984) [the Russian original appeared in 1980 with Nauka (Moscow)]
1.6 V.B. Matveev, M.A. Salle: *Darboux Transformations and Solitons*, Springer Ser. Nonlinear Dynamics (Springer, Berlin, Heidelberg 1991)
1.7 S.P. Novikov: Funkt. Anal. Pril. **8** (3), 54–66 (1974)
1.8 V.B. Matveev: Usp. Mat. Nauk, **30** (6), 201–203 (1975)
1.9 P.D. Lax: Comm. Pure Appl. Math. **28**, 141–188 (1975)
1.10 P.D. Lax: Lect. in Appl. Math. **15**, 85–96 (1974)
1.11 V.A. Marchenko: Dokl. Akad. Nauk SSSR **217**, 276–279 (1974)
1.12 N.I. Akhiezer: Dokl. Akad. Nauk SSSR **2**, 687–690 (1961)
1.13 B.A. Dubrovin: Funkt. Anal. Pril. **9** (3), 41–75 (1975)
1.14 B.A. Dubrovin: Funkt. Anal. Pril. **9** (1), 63–62 (1975)
1.15 A.R. Its, V.B. Matveev: Teor. Mat. Fiz. **23** (1), 51–67 (1975)
1.16 A.R. Its, V.B. Matveev: Funkt. Anal. Pril. **9** (1), 69–70 (1975)
1.17 H.P. McKean, P. van Moerbeke: Invent. Math. **30**, 217–274 (1975)
1.18 A.R. Its, V.B. Matveev: in *Problems in Mathematical Physics*, Vol. **8**, ed. by M.Sh. Birman, (Leningrad University Press, Leningrad 1976) [in Russian]
1.19 A.R. Its, V.P. Kotlyarov: Dokl. Ukr. Akad. Nauk, ser. A, 11 (1976)
1.20 H.P. McKean, E. Trubowitz: Comm. Pure Appl. Math. **29**, 143–226 (1977)
1.21 I.M. Krichever: Dokl. Akad. Nauk SSSR **227** (2), 291–294 (1976)
1.22 I.M. Krichever: Funkt. Anal. Pril. **11** (1), 15–31 (1977)
1.23 I.M. Krichever: Usp. Mat. Nauk **32** (6), 183–208 (1977) Reprinted in *London Math. Soc. Lecture Notes Ser.*, Vol. **60** (Cambridge University Press, Cambridge 1981)
1.24 H.M. Farkas, I. Kra: *Riemann Surfaces* (Springer, Berlin, Heidelberg 1980)
1.25 J.D. Fay: *Theta-Functions on Riemann Surfaces*, Lect.Notes Math., Vol.**352** (Springer, Berlin, Heidelberg 1973)
1.26 D. Mumford: Tata Lectures on Theta, vol. I,II (Birkhäuser, Boston, Basel, Stuttgart 1983-1984)
1.27 B.A. Dubrovin, S.M. Natanzon: Izv. Akad. Nauk SSSR, ser. mat. **52**, 267 (1988)
1.28 V.E. Zakharov, A.B. Shabat: Zh. Eksp. Teor. Fiz. **61** (1), 118–134 (1971)
1.29 M.J. Ablowitz, D.J. Kaup, A. Newell, H. Segur: Phys. Rev. Lett. **30**, 1262–1264 (1973)
1.30 V.E. Zakharov, L.A. Takhtajan, L.D. Faddeev: Dokl. Akad. Nauk SSSR **219**, 1334–1337 (1974)
1.31 L.A. Takhtajan: Zh. Eksp. Teor. Fiz. **66**, 476–489 (1974)

1.32 V.A.Kozel, V.P. Kotlyarov: Dokl. Ukr. Akad. Nauk, ser. A **10**, 878–880 (1976)
1.33 D. Forest, J. McLaughlin: J. Math. Phys. **23** (7), 1428–1440 (1982)
1.34 E.D. Belokolos, V.Z. Enol'skii: Usp. Math. Nauk **37** (4), 89 (1982)
1.35 B.A. Dubrovin, S.G. Natanzon: Funkt. Anal. Pril. **16** (1), 27–43 (1982)
1.36 B.A. Dubrovin, S.P. Novikov: Dokl. Akad. Nauk SSSR **267** (6), 1295–1300 (1982)
1.37 S.P. Novikov: Zap. Semin. LOMI **133**, 177–196 (1984)
1.38 B.A. Dubrovin: Usp. Mat. Nauk **36** (2), 12–78 (1980)
1.39 A. Nakamura, J. Matsuno: J. Phys. Soc. Jap. **47**, 1701–1708 (1979)
1.40 A. Nakamura, J. Matsuno: J. Phys. Soc. Jap. **48**, 1365–1370 (1980)
1.41 J.P. Boyd: J. Math. Phys. **12**, 3390–3423 (1984)
1.42 H. Segur, A. Finkel: Stud. Appl. Math. **73**, 183–220 (1985)
1.43 A.R. Osborne, E. Segre: *Numerical Solutions of the KdV-Equation Using the Periodic Scattering Transform*, Preprint (Inst. of Cosmogeophysics, Torino 1989)
1.44 J. Hammack, N. Schriffner, H. Segur: J. Fluid Mech. **209**, 567–589 (1989)
1.45 A.I. Bobenko: Dokl. Akad. Nauk SSSR **295** (2), 268–272 (1987)
1.46 A.I. Bobenko: *Uniformization and Finite-Gap Integration*, Preprint LOMI P-10-86 (1986)
1.47 A.I. Bobenko, L.A. Bordag: J. Phys. A: Math. Gen. **22**, 1259–1274 (1989)
1.48 A.I. Bobenko, D.A. Kubensky: Teor. Mat. Fiz. **72**, 352–360 (1987)
1.49 S.V. Kowalewski: Acta Math. **12**, 177–232 (1889)
1.50 F. Kötter: Acta Math. **17**, 209–265 (1893)
1.51 F. Kötter: J. reine und angew. Math. **109**, 51–81 (1892)
1.52 F. Kötter: Königlich Preuss. Akad. Wiss. Berlin **6**, 79–87 (1900)
1.53 A.I. Bobenko, A.G. Reyman, M.A. Semenov-Tian-Shansky: Comm. Math. Phys. **122**, 321–354 (1989)
1.54 A.I. Bobenko, V.B. Kuznetsov: J. Phys. A: Math. Gen. **21**, 1999–2006 (1988)
1.55 A.I. Bobenko: *Solutions of Classical Tops via the Inverse Scattering Method* Preprint LOMI P-10-87 (1987) [in Russian]
1.56 M. Adler, P. van Moerbeke: *A Systematic Approach Towards Solving Integrable Systems* (to be published)
1.57 Letter N9 of S. Kowalewski to G. Mittag-Leffler in *Correspondence between S. Kowalewski and G. Mittag-Leffler*, Nauchnoe Nasledstvo Vol. **7** (Nauka, Moscow 1984) [in Russian]
1.58 E. Horozov, P. van Moerbeke: Commun. Pure Appl. Math. **42**, 357–408 (1989)
1.59 A. Lesfari: Ann. Scient. Ec. Norm. Sup., 4 serie, **21**, 193–223 (1988)
1.60 C. Bechlivandis, P. van Moerbeke: Comm. Math. Phys. **110**, 317–324 (1987)
1.61 L. Haine, E. Horozov: Physica **29D**, 173–180 (1987)
1.62 M. Adler, P. van Moerbeke: Commun. Math. Phys. **113**, 659–700 (1988)
1.63 E.D. Belokolos, V.Z. Enol'skii: Usp. Math. Nauk **37** (4), 89–90 (1982)
1.64 E.D. Belokolos, V.Z. Enol'skii: Teor. Mat. Fiz. **53**, 271–282 (1982)
1.65 M.V. Babich, A.I. Bobenko, V.B. Matveev: Dokl. Akad. Nauk SSSR, **72**, 13–17 (1983)
1.66 M.V. Babich, A.I. Bobenko, V.B. Matveev: Izv. Akad. Nauk., ser. mat. **49**, 511–529 (1985)
1.67 E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, V.B. Matveev: Usp. Mat. Nauk **41** (2), 3–42 (1986)
1.68 J.Igusa: Bull. of the AMS **6**, 161–174 (1982)
1.69 J.L.Verdier: *New Elliptic Solutions*: Vol. dedicated to the 60th. birthday of Prof. Grothendieck (Birkhaäuser, Boston 1991)
1.70 G. van der Geer: Mathem. Ann. **260**, 317–350 (1982)
1.71 A. Treibich: Duke Math. J. **53:33**, 611–620 (1989)
1.72 E.D. Belokolos: Teor. Mat. Fiz. **45**, 268–275 (1980)
1.73 E.D. Belokolos: Teor. Mat. Fiz. **48**, 60–69 (1981)
1.74 A.G. Reyman, M.A. Semenov-Tian-Shansky: Funkt. Anal. Pril. **22**, (2), 87-88 (1988)

Chapter 2

2.1 A. Hurwitz, R. Courant: *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* (Springer, Berlin, Heidelberg 1964)
2.2 O. Forster: *Riemannsche Flächen* (Springer, Berlin, Heidelberg 1977)
2.3 H.M. Farkas, I. Kra: *Riemann Surfaces* (Springer, Berlin, Heidelberg 1980)
2.4 A. Krazer: *Lehrbuch der Thetafunktionen* (Teubner, Leipzig 1903)
2.5 A. Krazer, W. Wirtinger: *Abelsche Funktionen und allgemeine Thetafunktionen* i *Encyklopädie der Mathematischen Wissenschaften*, Band II (2), Heft 7 (Teubner, Leipzig 1915)
2.6 J. Igusa: *Theta Functions*, Grund. Mat. Wiss., Vol. 194 (Springer, Berlin, Heidelberg 1972)
2.7 J.D. Fay: *Theta Functions on Riemann Surfaces*, Lect. Notes Math., Vol. **352** (Springer, Berlin, Heidelberg 1973)
2.8 D. Mumford: *Tata Lectures on Theta Functions*, Vols. I, II (Birkhäuser, Boston, Basel, Stuttgart 1983, 1984)
2.9 P. Griffith, J. Harris: *Principles of Algebraic Geometry* (Wiley, New York 1978)
2.10 H. Bateman, A. Erdelyi: *Higher Transcendental Functions*, Vol. 3 (McGraw-Hill, New York 1955)
2.11 J. Igusa: Bull. AMS **6**(2), 161–174 (1982)
2.12 J.D. Fay: Nachr. Akad. Wiss. Göttingen. II. Mathematisch-Physikalische Klasse **5**, 61–75 (1979)
2.13 S. Koizumi: Amer. J. Math. **98**, 865–889 (1976)
2.14 J.D. Fay: Duke Math. **51**, 109–132 (1984)

Chapter 3

3.1 J.D. Fay: *Theta-Functions on Riemann Surfaces*, Lect. Notes Math., Vol. 352 (Springer, Berlin, Heidelberg 1973)
3.2 B.A. Dubrovin, S.M. Natanzon: Izv. Akad. Nauk SSSR, ser. mat. **52** 267–286 (1988) [English transl.: Math. USSR Izvestiya **32** 269–288 (1989)]
3.3 I.M. Krichever: Usp. Mat. Nauk **44** (2), 121–184 (1989) [English transl.: Russ. Math. Surveys **44** (2), 145–225 (1989)]
3.4 V.B. Matveev, A.O. Smirnov: Lett. Math. Phys. **14**, 25–31 (1987)
3.5 A. Kuribajashi, K. Komiya: *On Weierstrass Points and Automorphism of Curves of Genus Three*, Lect. Notes Math., Vol. 732, 252–299 (Springer, Berlin, Heidelberg 1979)
3.6 E. Titchmarsh: Eigenfunction expansions, **2**
3.7 V.A. Marchenko: *Sturm–Liouville Operators and Their Applications* (Naukova Dumka, Kiev 1977)
3.8 E.A. Coddington, N. Levinson: *Theory of Ordinary Differential Equations* (McGraw-Hill, New York 1955)
3.9 A.R. Its, V.B. Matveev: Theor. Mat. Fiz. **23** (1), 51–67 (1975)
3.10 C. Hermite: *Oeuvres*, Vol. 3 (Gauthier-Villars, Paris 1912)
3.11 I.M. Gelfand: Usp. Mat. Nauk **11** (1), 191–198 (1956) [English transl.: Russ. Math. Surveys **11** (1) (1956)]
3.12 L.A. Dikii: Izv. Akad. Nauk SSSR, ser. mat. **19**, 187–200 (1955)
3.13 N.I. Akhiezer: Dokl. Akad. Nauk SSSR **141**, 263–266 (1961) [English transl.: Soviet Math. Dokl. **2**, 1409–1412 (1961)]
3.14 J. Drach: Compt. Rend. Acad. Sci. **168** (7), 337–340 (1919)
3.15 F. Ehlers, H. Knörrer: Comm. Math. Helv. **57**, 1–10 (1980)

Chapter 4

4.1 V.E. Zakharov, A.B. Shabat: J. Exp. Theor. Phys. **61** (1), 118–134 (1971)
4.2 M.J. Ablowitz, D.J. Kaup, A.S. Newell, H. Segur: Phys. Rev. Lett. **30** (25), 1262–1264 (1973)
4.3 L.A. Takhtajan: J. Exp. Theor. Phys. **66** (2), 476–489 (1974)
4.4 V.E. Zakharov, L.A. Takhtajan, L.D. Faddeev: Dokl. Akad. Nauk SSSR **216** (6), 1334–1337 (1974)
4.5 I.M. Krichever: *Nonlinear Equations and Elliptic Curves* in *Modern Problems of Mathematics*, Vol.23 (VINITI, Moscow 1983) 79–136
4.6 A.R. Its: *Connection Between Soliton and Finite-Gap Solutions to the NS Equation* in *Problems of Mathematical Physics*, Vol.10 (Leningrad University Press, Leningrad 1983) 118–136
4.7 N.N. Akhmedjeva, V.M. Eleonsky, N.E. Kulagin: J. Exp. Theor. Phys. **89**, 1542–1551 (1985)
4.8 H. Bateman, A. Erdelyi: *Higher Transcendental Functions*, Vol. 3 (McGraw-Hill, New York 1955)

Chapter 5

5.1 A.I. Bobenko: *Uniformization and Finite-Gap Integration*, Preprint LOMI P-10-86, Leningrad (1986)
5.2 A.I. Bobenko: Dokl. Akad. Nauk SSSR **295**, 268–272 (1987) [English transl.: Soviet Math. Dokl. **36**, 38–42 (1988)]
5.3 A.I. Bobenko, D.A. Kubensky: Teor. Mat. Fiz. **72**, 352–360 (1987) [English transl.: Theor. Math. Phys. **72**, 929 (1987)]
5.4 A.I. Bobenko, L.A. Bordag: Zap. LOMI **165**, 31–41 (1987) [English transl.: J. Sov. Math. **50** (6), 1951–1959 (1990)]
5.5 A.I. Bobenko, L.A. Bordag: J. Phys. A: Math. Gen. **22**, 1259–1274 (1989)
5.6 F. Schottky: J. reine angew. Math. **101**, 227–272 (1887)
5.7 A. Marden: *Schottky Groups and Circles*, in *Contrib. to Analysis* (Academic Press, New York, London 1974) pp. 205–224
5.8 B. Maskit: J. Analyse Math. **19**, 227–230 (1967)
5.9 V. Chukrow: Ann. Math. **88**, 47–61 (1968)
5.10 L. Ford: *Automorphic Functions* (McGraw–Hill, New York 1929)
5.11 W. Burnside: Proc. London Math. Soc. **23**, 49–88 (1892)
5.12 H.F. Baker: *Abel's Theorem and the Allied Theory Including the Theory of Theta Functions* (Cambridge Univ. Press, Cambridge 1897)
5.13 H. Sato: Duke Math. J. **57**, 205–209 (1988)
5.14 P.J. Myrberg: Ann. Acad. Sci. Fennicae **9A**, 1–75 (1916)
5.15 T. Akaza: Nagoya Math. J. **29**, 145–162 (1967)
5.16 R. Fricke, F. Klein: *Vorlesungen über die Theorie der automorphen Funktionen* (Johnson, New York 1965)
5.17 R.S. Phillips, P. Sarnak: Acta Math. **155**, 173–241 (1985)
5.18 L. Keen: Acta Math. **115**, 1–16 (1966)
5.19 S.M. Natanzon: Trudy Mosk. Mat. O-va **37**, 219–253 (1978) [in Russian]
5.20 J.P. Boyd: J. Math. Phys. **12**, 3390–3423 (1984)
5.21 H. Segur, A. Finkel: Stud. Appl. Math. **73**, 183–220 (1985)
5.22 S.P. Novikov: Zap. LOMI **133**, 177–196 (1984) [English transl.: J. Soviet Math. **31**, 3373–3387 (1985)]
5.23 A.F. Berdon: *The Geometry of Discrete Groups* (Springer, Berlin, Heidelberg 1983)
5.24 L. Keen: Ann. of Math. **84**, 404–420 (1966)
5.25 L. Keen: Ann. of Math. Studies **66**, 205–224 (1971)

5.26 H. Zieschang: *Finite Groups of Mapping Classes of Surfaces*, Lect. Notes Math., Vol. 875 (Springer, Berlin, Heidelberg 1981)
5.27 H. Sato: Tohoku Math. J. **40**, 51–75 (1988)
5.28 S.M. Natanzon: Usp. Mat. Nauk **27**, 145–160 (1972) [English transl.: Russ. Math. Surveys **27**, 161–178 (1972)]
5.29 R. Bowen: Publ. Math. IHES **50**, 11-25 (1979)

Chapter 6

6.1 A.I. Bobenko: *Solutions of Classical Tops via the Inverse Scattering Method*, Preprint LOMI, P-10-87, Leningrad (1987) [in Russian]
6.2 A.I. Bobenko, A.G. Reyman, M.A. Semenov-Tian-Shansky: Commun. Math. Phys. **122**, 321-354 (1989)
6.3 A.I. Bobenko, V.B. Kuznetsov: J. Phys. A **21**, 1999-2006 (1988)
6.4 A.I. Bobenko: Funkt. Anal. Pril. **19** (1), 6-19 (1985) [English transl.: Functional Anal. Appl. **19**, 5-17 (1985)]
6.5 A.I. Bobenko: Funkt. Anal. Pril. **20** (1), 64-66 (1986) [English transl.: Functional Anal. Appl. **20**, 53-56 (1986)]
6.6 S.P. Novikov: Usp. Mat. Nauk **37** (5), 3-49 (1982) [English transl.: Russ. Math. Surveys **37** (5), 1-56 (1982)]
6.7 S.L. Ziglin: Funkt. Anal. Pril. **16** (3), 30-41 (1982), ibid. 17 (1), 8-23 (1983)
6.8 C. Clebsch: Math. Annal. **3**, 238-257 (1871)
6.9 V.A. Steklov: *On the Motion of a Rigid Body in a Liquid*, (Kharkov 1893) [in Russian]
6.10 S.V. Manakov: Funkt. Anal. Pril. **10** (4), 93-94 (1976)
6.11 A.S. Mishchenko: Funkt. Anal. Pril. **4** (3), 73-77 (1970) L.A. Dikii: Funkt. Anal. Pril. **6** (4), 83-84 (1972)
6.12 V.A. Steklov: Annales de la Faculté des Sciences de Toulouse, ser. 3, 1, 145-256 (1909)
6.13 E.G. Kalnins, W. Miller (jr.), P. Winternitz: SIAM Journ. Appl. Math. **30**, 630-664 (1976)
6.14 A.P. Veselov: Dokl. Akad. Nauk SSSR **270**, 1298-1300 (1983)
6.15 O.I. Bogoyavlensky: Dokl. Akad. Nauk SSSR **268**, 277-280 (1983)
6.16 M. Adler, P. van Moerbeke: Proc. Natl. Acad. Sci. USA **91**, 4613-4616 (1984)
6.17 A.G. Reyman, M.A. Semenov-Tian-Shansky: Commun. Math. Phys. **105**, 461-472 (1986)
6.18 S.V. Kowalewski: Acta Math. **12**, 177-232 (1889)
6.19 Letter No 9 of S. Kowalewski to G. Mittag-Leffler in *Correspondence between S. Kowalewski and G. Mittag-Leffler*, Nauchnoe Nasledstvo Vol. 7, (Nauka, Moscow 1984) [in Russian]
6.20 S. Kowalewski: *Scientific Papers* (Nauka, Moscow 1948) [in Russian]
6.21 M. Adler, P. van Moerbeke: *A Systematic Approach Towards Solving Integrable Systems* (to be published)
6.22 E. Horozov, P. van Moerbeke: Commun. Pure Appl. Math. **42**, 357-408 (1989)
6.23 A. Lesfari: Ann. Scient. Éc. Norm. Sup., 4 série, **21**, 193-223 (1988)
6.24 H. Yehia: Mech. Res. Commun. **13**, 169-172 (1986)
6.25 I.V. Komarov: Phys. Lett. **123** A, 14-15 (1987)
6.26 S.A. Chaplygin: *Collected Works*, Vol. 1, (Gostekhizdat, Moscow 1948), p. 118 [in Russian]
6.27 L.N. Sretensky: Vest. Mosk. Univ., ser. **1**, 3 (1960) [in Russian]
6.28 B.A. Dubrovin, I.M. Krichever, S.P. Novikov: *Integrable systems I* in *Modern Problems in Mathematics. Fundamental Developments*. Vol. 4, Itogi Nauki i Tekhniki (VINITI, Moscow 1985), *p.* 179-285 [in Russian]
6.29 M.A. Olshanetsky, A.M. Perelomov, A.G. Reyman, M.A. Semenov-Tian-Shansky: *Integrable systems II* in: *Modern Problems in Mathematics. Fundamental Developments*, Vol. 16, Itogi Nauki i Tekhniki (VINITI, Moscow 1987) 86-226 [in Russian]

6.30 E.K. Sklyanin: Zap. LOMI **133**, 236-257 (1984) [English transl.: J. Sov. Math. **31**, 3417 (1985)]
6.31 C. Bechlivandis, P. van Moerbeke: Commun. Math. Phys. **110**, 317-324 (1987)
6.32 E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, V.B. Matveev: Usp. Mat. Nauk **41** (2), 3-42 (1986) [English transl.: Russ. Math. Surv. **41** (2), 1-49 (1986)]
6.33 I.V. Cherednik: Izv. Akad. Nauk SSSR, ser. mat. **47**, 384-406 (1983) [English transl.: Math. USSR Izvestiya. **22**, 357-377 (1984)]
6.34 E.K. Skylanin: *On the Complete Integrability of the Landau-Lifshitz Equation*, Preprint LOMI E-3-79, Leningrad (1979)
6.35 I.V. Cherednik: Yadernaya Fizika **33** (1), 278-281 (1981) [in Russian]
6.36 V.G. Baryakhtar, E.D. Belokolos, P.I. Holod: Preprint ITF-84-128P, Kiev (1984) [in Russian]
6.37 A.P. Veselov: Dokl. Akad. Nauk SSSR **270**, 1094-1097 (1983)
6.38 C. Neumann: Journ. reine u. angew. Math. **56**, 46-63 (1859)
6.39 J. Moser: *Various Aspects of Integrable Hamiltonian Systems* in *Dynamical Systems*, Prog. in Math. 8, (Birkhäuser, Boston 1980), 233-289
6.40 A.P. Veselov: Funkt. Anal. Pril. **14** (1), 48-50 (1980)
6.41 E.I. Horozov: C.R. Acad. Bulg. Sci. **37**, 277-280 (1984)
6.42 F. Kötter: J. reine u. angew. Math. **109**, 51-81 (1892)
6.43 L. Haine: Math. Annal. **263**, 435-472 (1983)
6.44 F. Kötter: Königlich Preußische Akad. Wissenschaften zu Berlin, **6**, 79-87 (1900)
6.45 L.N. Gavrilov: C.R. Acad. Bulg. Sci. **40**, 19 (1987)
6.46 A.P. Veselov: Dokl. Akad. Nauk SSSR **276**, 590-593 (1984)
6.47 F. Kötter: Acta Math. **17**, 209-264 (1893)
6.48 A.G. Reyman, M.A. Semenov-Tian-Shansky: Lett. Math. Phys. **14**, 55-62 (1987)
6.49 L. Haine, E. Horozov: Physica **29D**, 173-180 (1987)
6.50 M. Adler, P. van Moerbeke: Commun. Math. Phys. **113**, 659-700 (1988)
6.51 J. Fay: *Theta Functions on Riemann Surfaces*, Lect. Notes in Math., Vol. 352, (Springer, Berlin, Heidelberg 1973)
6.52 A.M. Perelomov: *Integrable Systems of Classical Mechanics and Lie Algebras*, Vol. 1, (Birkhäuser, Basel, Boston, Stuttgart 1990)
6.53 A.G. Reyman, M.A. Semenov-Tian-Shansky: Funkt. Anal. Pril. **22** (2), 87-88 (1988)

Chapter 7

7.1a E.D. Belokolos, V.Z. Enol'skii: Usp. Mat. Nauk **37**, 89 (1982) [English transl.: Russian Math. Surveys **37** 00 (1982)]
b E.D. Belokolos, V.Z. Enol'skii: Teoret. Mat. Fiz. **53** 271–282 (1982)
c V.Z. Enol'skii: Phys. Lett. **96**, 327–330 (1983)
d V.Z. Enol'skii: Dokl. Akad. Nauk SSSR **278**, 305–309 (1984) [English transl.: Sov. Math. Dokl. **30**, 394–397 (1984)]
e A.R. Its, V.Z. Enol'skii: Funkt. Anal. Priloz. **20** (1), 73–74 (1986)
f E.D. Belokolos, A.I. Bobenko, V.B. Matveev, V.Z. Enol'skii: Usp. Mat. Nauk **41** (2), 3–42 (1986) [English transl.: Russ. Math. Surveys **41** (2), 1–49 (1986)]
g E.D.Belokolos, V.Z. Enol'skii: *Algebraically integrable nonlinear equations and Humbert surfaces* in *Plasma Theory and Nonlinear and Turbulent Processes in Physics* ed. by V.G. Bar'yakhtar et al. Vol. 1, (World Scientific, Singapore 1988) 20–56
h E.D. Belokolos, V.Z. Enol'skii: Funkt. Anal. Priloz. **23** (1), 75–76 (1989)
i E.D. Belokolos, V.Z. Enol'skii: Usp. Mat. Nauk **44** (5), 155–156 (1989) [English transl.: Russ. Math. Surveys **44** (5), 191–192 (1989)]
j E.D. Belokolos, V.Z. Enol'skii: *Reduction of Theta Functions and Elliptic Finite-Gap Potentials* in Acta Applicandae Mathematicae (1993)

k V.Z. Enol'skii, N.A. Kostov: *On the Geometry of Elliptic Solitons*, Acta Applicandae Mathematicae (1993)

7.2a A. Krazer: *Lehrbuch der Thetafunktionen* (Teubner, Leipzig 1903)

b A. Krazer, W.Wirtinger: *Abelsche Funktionen und allgemeine Thetafunktionen* in *Enzyklopädie der Mathematischen Wissenschaften*, Band **11** (2), Heft 7 (Teubner, Leipzig 1915) 603–873

7.3a M.B. Babich, A.I. Bobenko, V.B. Matveev: Dokl. Akad. Nauk SSSR **272**, 13–17 (1983) [English transl.: Sov. Math. – Dokl., **23**, 304–308 (1989)]

b M.B. Babich, A.I. Bobenko, V.B. Matveev: Izv. Akad. Nauk SSSR, ser. mat. **49**, 511–529 (1985) [English transl.: Math. USSR Izvestiya **26**, 479–496 (1986)]

c A.I. Bobenko: Funkt. Anal. Priloz. **18** (3), 74–75 (1984)

d M.B. Babich: Funkt. Anal. Priloz. **19** (3), 75–76 (1985)

e A.I. Bobenko: Funkt. Anal. Priloz. **19** (1), 6–19 (1985)

f V.B. Matveev, A.O. Smirnov: Dokl. Akad. Nauk SSSR **293**, 78–82 (1987)

g A.O. Smirnov: Mathem. Sborn. **133** (175), 382–391 (1987) [English transl.: Math. USSR Sbornik **61**, 379–388 (1988)]

h V.B. Matveev, A.O. Smirnov: *Symmetric Reductions of Riemann θ-functions and Their Applications to the Nonlinear Schrödinger Equation and the Bussinesq Equation* in *Problems of Mathematical Physics* Vol. 12 (Leningrad University Press, Leningrad 1987) 225–236 [in Russian]

7.4 I.M. Krichever (ed.): *Elliptic Solitons*, Acta Applicandae Mathematicae (1993)

7.5 J. Igusa: *Theta Functions*, Grundl. Math. Wiss., Vol. 194 (Springer, Berlin, Heidelberg 1972)

7.6 J.D. Fay: *Theta Functions on Riemann Surfaces*, Lect. Notes Math., Vol. 352 (Springer, Berlin, Heidelberg 1973)

7.7 P. Griffith, J. Harris: *Principles of Algebraic Geometry* (Wiley, New York 1978)

7.8 H.M. Farkas, I.Kra: *Riemann Surfaces* (Springer, New York 1980)

7.9 D. Mumford: *Tata Lectures on Theta*, Vols. I and II, (Birkhäuser, Berlin 1983, 1984)

7.10 J. Igusa: Bull. AMS **6**, 161–174 (1982)

7.11 H.H. Martens, Publ. Mathemàtique, **36**, 111-129 (1992)

7.12 F. Klein: *Gesammelte Mathematische Abhandlungen*, Band 3 (Teubner, Berlin 1923)

7.13 A. Kuribayashi, K. Komiya: *On Weierstrass Points and Automorphism of Curves of Genus Three*, Lect. Notes Math., Vol. 732 (Springer, Berlin, Heidelberg 1978) 252–299

7.14 R.D.M. Accola: *Riemann Surfaces, Theta-Functions and Abelian Automorphisms Groups*, Lect. Notes Math., Vol. 485 (Springer, Berlin, Heidelberg 1975)

7.15 R. Horiuchi: J. Math. Kyoto Univ. **19**, 487–524 (1979)

7.16 H. Poincaré: Amer. Math. **8**, 289–342 (1886)

7.17 S. Kowalewski: Acta Math. **4** , 393–414 (1884)

7.18 S. Koizumi: Amer. J. Math. **198**, 865–889 1976

7.19 G. van der Geer: Math. **260**, 317–350 (1982)

7.20 O. Bolza: Math. Ann. **28**, 447–496 (1887)

7.21 G. Rosenhain: *Abhandlung über die Funktionen zweier Variabler mit vier Perioden, Mem. des savants*, Bd. IX, 361–455 (1851)

7.22 B.A. Dubrovin: Usp. Mat. Nauk **32** (2), 11–80 (1981)

7.23 R. Sasaki: J. reine u. angew. Math. **345**, 11–121 (1983)

7.24 F. Hirzebruch, G. van der Geer: *Lectures on Hilbert Modular Surfaces*, (Presses de I'Universite de Montreal, Montreal 1981)

7.25 J. Igusa: Amer. J. Math. **86**, 392–412 (1964)

7.26 C. Hermite: *Sur un example de reduction arbitiennes aux functions elliptiques, Oeuvres de Charles Hermite*, Vol. III (Gauthier-Villars, Paris 1912) 249–261

7.27 M. Halpen: *Memorie sur la reduction des equations differentielles lineaires aux formes integrables* in *Mem. Acad. Sci. Inst. France*, Vol. 28 (1), 1–301 (1884)

7.28 E.L. Ince: Proc. Roy. Soc. Edinburgh **60**, 83–99 (1940)

7.29a J.L. Verdier: *New Elliptic Solitons, Algebraic Analysis*, Special Volume to the 60th Anniversary of Prof. Sato (Academic Press, New York, London 1988)
b A. Treibich, J.L. Verdier: Compt. Rendus Acad. Sci. Paris, **311**, 51-54 (1990)
c A. Treibich, J.L. Verdier: *Solitons elliptiques*, Special Volume to the 60th Anniversary of Prof. Grothendieck (Birkhäuser, Basel 1991)
7.30 A.R. Its, V.B. Matveev: Teor. Matem. Fiz. **23**, 51–67 (1975) [in Russian]
7.31 S.P. Novikov: Funkt. Anal. Pril. **8** (3), 54–66 (1974)
7.32a H. Airault, H. McKean, J. Moser: Comm. Pure Appl. Math. **30**, 95-125 (1977)
b D.V. Chudnovsky, G.V. Chudnovsky: Nuovo Cimento **40B**, 339–350 (1977)
c I.M. Krichever: Funkt. Anal. Pril. **12** (1), 76–78 (1978)
7.33 I.M. Krichever: Funkt. Anal. Pril. **14** (3), 45–54 (1980)
7.34 A.O. Smirnov: Math. Zametki **45** (6), 66–73 (1989)
7.35 S.P. Novikov, S.V. Manakov, L.P. Pitaevski, V.E. Zakharov: *Theory of Solitons* (Consultants Bureau, New York 1984)
7.36 B.A. Dubrovin, S.M. Natanzon: Funkt. Anal. Pril. **16** (1), 27–43 (1982)
7.37 S. Kowalewski: Acta Math. **14**, 81–93 (1889)
7.38a G.G. Appelrott: Mat. Sbornik **27**, 477–561 (1903) [in Russian]
b Yu. Arkhangelskii: *Analytical Dynamics of a Rigid Body* (Nauka, Moscow 1977) [in Russian]
7.39 A.M. Macbeth: Proc. of the London Math. Soc., third ser. **25**, 537–542 (1965)
7.40 H.E. Rauch, J. Lewittes: *The Riemann surfaces of Klein with 168 automorphism* in *Problems in Analysis* (Academic, New York 1970) 297–308
7.41 H. Poincaré: J. de Math. ser. 5, **9**, 139–212 (1903)
7.41 H. Baker: *Multiple Periodic Functions* (Cambridge University Press, Cambridge 1907)
7.43a V.B. Matveev, A.O. Smirnov: Mat. Zametki **46** (3), 31–39 (1989)
b V.B. Matveev, A.O. Smirnov: Lett. Math. Phys. **19**, 179–185 (1990)
7.44 P.Appell: Bull. Soc. Math. de France **10**, 59–67 (1882)

Chapter 8

8.1 R.E. Peierls: *Quantum Theory of Solids* (Clarendon, Oxford 1955)
8.2 E.D. Belokolos: Teor. Mat. Fiz. **45** (2), 268–275 (1980)
8.3 E.D. Belokolos: Teor. Mat. Fiz. **48** (1), 60–69 (1981)
8.4 E.A. Coddington, N. Levinson: *Theory of Ordinary Differential Equations* (McGraw-Hill, New York 1955)
8.5 S. Kotani: *Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators* (Taniguchi Symp. SA Katata 1982) 225–247
8.6 R. Johnson, J. Moser: Commun. Math. Phys. **84**, 403–438 (1982)
8.7 L.A. Pastur: Commun. Math. Phys. **75**, 107–196 (1980)
8.8 A.R. Its, V.B. Matveev: Teor. Mat. Fiz. **25** (1), 51–67 (1975)
8.9 B.A. Dubrovin: Funkt. Anal. Pril. **9** (3), 41–51 (1975)
8.10 E.D. Belokolos: *Mathematical foundations of the theory of solids with quasi-periodic structure*; Dissertation, Institute for Theor. Physics, Kiev (1982)
8.11 E.D. Belokolos: Teor. Mat. Fiz. **23** (3), 344–357 (1975)
8.12 S.P. Novikov: Funkt. Anal. Pril. **8** (3), 54–66 (1974)
8.13 S.P. Novikov, S.V. Manakov, L.P. Pitaevski, V.E. Zakharov: *Theory of Solitons* (Consultants Bureau, New York 1984)
8.14 H. Bateman, A. Erdelyi: *Higher Transcendental Functions*, Vol. 2 (McGraw–Hill, New York 1955)
8.15 H. Fröhlich: Proc. Roy. Soc. **A223**, 296–305 (1954)
8.16 L.D. Landau, E.M. Lifshitz: *Quantum Mechanics* (Nauka, Moscow 1974)

8.17 E.D. Belokolos: *Theory of the Fröhlich Conductivity*, Preprint, Kiev Inst. Theor. Phys. 91-6R (1991) p. 15 [in Russian]
8.18 *Electronic Properties of Inorganic Quasi-One-Dimensional Compounds*. Ed. by P. Monceau, Vols. 1, 2 (Reidel, Dordrecht 1985)
8.19 G. Grüner: Rev. Mod. Phys. **60**, 1129–1181 (1988)
8.20 E.D. Belokolos, I.M. Pershko: Phys. Lett. **A96** (3), 137–140 (1983)
8.21 E.D. Belokolos, I.M. Pershko: Teor. Mat. Fiz. **58** (2), 279–291 (1984)
8.22 E.D. Belokolos, D.Ya. Petrina: Dokl. Akad. Nauk **275**, 580–582 (1984)
8.23 N.N. Bogolyubov (jr.), I.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov, N.S. Tonchev: *Approximating Hamiltonian Method in Statistical Physics* (Sofia BAN, Sofia 1981)
8.24 I.M. Krichever: Dokl. Akad. Nauk **265** (5), 1054–1058 (1982) [Engl. transl.: Sov. Math. Dokl. **26**, 194–199 (1982)]
8.25 S.A. Brazovskii, I.E. Dzjaloshinskii, I.M. Krichever: J. Exp. Teor. Phys. **83** (1), 389–415 (1982)
8.26 I.M. Krichever: *Nonlinear equations and elliptic curves* in *Itogi nauki i tekhniki. Sovremen. probl. mat.* **23** (VINITI, Moscow 1983) 79–136 [in Russian]
8.27 V.A. Marchenko, I.V. Ostrovskii: Mat. Sbor. **97** (4), 540–606 (1975) [Engl. transl.: Math. USSR Sborn. **26**, 493–553 (1975)]
8.28 E.D. Belokolos, I.M. Pershko: Ukr. fiz. journ. **32** (9), 1737–1376 (1987)
8.29 E.D. Belokolos, I.M. Pershko: Teor. Mat. Fiz. **71** (2), 313–317 (1987)
8.30 V.N. Kostur: *Absorption of Electromagnetic Waves in a Single-Band Conductor*, Preprint, Kiev Inst. Theor. Phys. 83–165R (1983)
8.31 V.G. Baryakhtar, E.D. Belokolos, A.M. Korostil: Phys. Stat. Sol. (B) **169**, 105-114 (1992)
8.32 V.G. Baryakhtar, E.D. Belokolos, A.M. Korostil: Metalophysics **15** (1), 3-11 (1993) [in Russian]

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This book is an introduction to nonrelativistic scattering theory. The presentation is mathematically rigorous, but is accessible to upper level undergraduates in physics. The relationship between the scattering matrix and physical observables, i. e. transition probabilities, is discussed in detail. Among the emphasized topics are the stationary formulation of the scattering problem, the inverse scattering problem, dispersion relations, three-particle bound states and their scattering, collisions of particles with spin and polarization phenomena. The analytical properties of the scattering matrix are discussed. Problems round off this volume.

B. N. Zakhariev, A. A. Suzko

Direct and Inverse Problems

Potentials in Quantum Scattering

1990. XIII, 223 pp. 42 figs.
Softcover ISBN 3-540-52484-3

This textbook can almost be viewed as a "how-to" manual for solving quantum inverse problems, that is, for deriving the potential from spectra or scattering data and also, as somewhat of a quantum "picture book" which should enhance the reader's quantum intuition. The formal exposition of inverse methods is paralleled by a discussion of the direct problem. Differential and finite-difference equations are presented side by side. The common features and (dis)advantages of a variety of solution methods are analyzed. To foster a better understanding, the physical meaning of the mathematical quantities are discussed explicitly. Wave confinement in continuum bound states, resonance and collective tunneling, energy shifts and the spectral and phase equivalence of various interactions are some of the physical problems covered.

K. L. G. Heyde

The Nuclear Shell Model

1990. XII, 376 pp. 171 figs. (Springer Series in Nuclear and Particle Physics) Hardcover
ISBN 3-540-51581-X

This book evolved from a course in theoretical nuclear physics taught over seven years at the University of Gent and is thus well suited to and tested for lecture courses. The nuclear shell model is introduced from basic techniques such as angular momentum and tensor algebra. The material is developed from the beginning up to the present state-of-the-art calculations using self-consistent residual interactions. Problem sets and simple computer codes are included to facilitate a better acquaintance with the subject.
The appendices constitute an integral part of the text going into depth on a number of technical derivations to provide the reader with a detailed background facilitating active research. The book introduces the subject to advanced undergraduate and to graduate students providing them with knowledge and techniques for own research in this field. It is a highly useful prerequisite for lecturers teaching modern nuclear physics.

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