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Integrable Discrete Nets in Grassmannians

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Abstract. We consider discrete nets in Grassmannians $\mathbb{G}^d_r$, which generalize Q-nets (maps $\mathbb{Z}^N \to \mathbb{P}^d$ with planar elementary quadrilaterals) and Darboux nets ($\mathbb{P}^d$-valued maps defined on the edges of $\mathbb{Z}^N$ such that quadruples of points corresponding to elementary squares are all collinear). We give a geometric proof of integrability (multidimensional consistency) of these novel nets, and show that they are analytically described by the noncommutative discrete Darboux system.

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1. Introduction

One of the central notions in discrete differential geometry constitute discrete nets, that is, maps $\mathbb{Z}^3 \to \mathbb{P}^d$ specified by certain geometric properties. Their study was initiated by Sauer [11], while their appearance in the modern theory of integrable systems is connected with the work of Bobenko and Pinkall [1,2] and of Doliwa and Santini [9]. A systematic exposition of discrete differential geometry, including detailed bibliographical and historical remarks, is given in the monograph [4]. In many aspects, the discrete differential geometry of parametrized surfaces and coordinate systems turns out to be more transparent and fundamental than the classical (smooth) differential geometry, since the transformations of discrete surfaces possess the same geometric properties and therefore are described by the same equations as the surfaces themselves. This leads to the notion of multidimensional consistency, which can be seen as the fundamental geometric definition of integrability in the discrete context that yields standard integrability structures of both discrete and continuous systems, such as Bäcklund and Darboux transformations, zero curvature representations, hierarchies of commuting flows, etc.

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In this note, we present a generalization of two classes of multidimensional nets, Q-nets (or discrete conjugate nets) and Darboux nets, to the maps with values in the Grassmannian $\mathbb{G}_r^d$ instead of $\mathbb{P}^d$. The basic idea underlying this work goes back to Grassmann and Plücker and consists in regarding more complicated objects than just points (such as lines, spheres, multidimensional planes, contact elements, etc.) as elementary objects of certain geometries. Such objects are then represented as points belonging to some auxiliary projective spaces or to certain varieties in these spaces. In the framework of discrete differential geometry, one can assign such objects to the sites of the lattice $\mathbb{Z}^N$ and impose certain geometric conditions to characterize interesting classes of multidimensional nets. Several such classes have been introduced in the literature, for instance:

- discrete line congruences [10], which are nets in the set of lines in $\mathbb{P}^d$ subject to the condition that any two neighboring lines intersect (are coplanar);
- discrete W-congruences [6], which are nets in the set of lines in $\mathbb{P}^3$ such that four lines corresponding to the vertices of every elementary square of $\mathbb{Z}^N$ belong to a regulus. If one represents the lines in $\mathbb{P}^3$ by points of the Plücker quadric in $\mathbb{P}(\mathbb{R}^{4,2})$, then this condition is equivalent to the planarity of elementary quadrilaterals;
- discrete R-congruences of spheres [3,7], which are nets in the set of oriented spheres in $\mathbb{R}^3$ that in the framework of Lie geometry are represented by points of the Lie quadric in $\mathbb{P}(\mathbb{R}^{3,3})$. Again, the defining condition is the planarity of all elementary quadrilaterals of the net;
- principal contact element nets [3], which are nets in the set of contact elements in $\mathbb{R}^3$ such that any two neighboring contact elements share a common oriented sphere. In the framework of Lie geometry, such nets are represented by isotropic line congruences.

In the present work, we study two related classes of multidimensional nets in Grassmannians $\mathbb{G}_r^d$, which generalize Q-nets (nets in $\mathbb{P}^d$ with planar elementary quadrilaterals) and the so-called Darboux nets introduced in [12].

It turns out that Grassmannian Q-nets can be analytically described by a noncommutative version of the so-called discrete Darboux system, which was introduced, without a geometric interpretation, in [5]. Our present investigations provide also a geometric meaning for the abstract Q-nets in a projective space over a noncommutative ring, considered in [8]. More precisely, we demonstrate that equations of abstract Q-nets in a projective space over the matrix ring $\mathrm{Mat}(r+1, r+1)$ can be interpreted as the analytical description of the Grassmannian Q-nets in the suitable parametrization. The fact that the equations of Q-nets are considered over a ring rather than over a field is not very essential in this context, since the very notion of Q-nets is related to subspaces in general position, and an accident degeneration of some coefficients is treated as a singularity of the discrete mapping. A much more important circumstance is the noncommutativity of the matrix ring, which is equivalent to the absence of Pappus theorem in the geometries over this ring.

The main results of the paper are Theorems 1 and 2 on the multidimensional consistency of Grassmannian Q-nets and Theorem 3 on the equations for integrable evolution of the discrete rotation coefficients.

2. Multidimensional Consistency of Grassmannian Q-Nets

Recall that the Grassmannian $\mathbb{G}_r^d$ is defined as the variety of r-planes in $\mathbb{P}^d$. It can be also described as the variety of $(r+1)$-dimensional vector subspaces of the $(d+1)$-dimensional vector space $\mathbb{R}^{d+1}$. In the latter realization, the Grassmannian is alternatively denoted by G_{r+1}^{d+1}. In what follows, the term "dimension" is used in the projective sense.

DEFINITION 1. (*Grassmannian Q-net*) A map $\mathbb{Z}^N \to \mathbb{G}_r^d$, $N \geq 2, d > 3r+2$ is called an N-dimensional Grassmannian Q-net of rank r if for every elementary square of $\mathbb{Z}^N$ the four r-planes in $\mathbb{P}^d$ corresponding to its vertices belong to some $(3r+2)$-plane.

Note that three generic r-planes in $\mathbb{P}^d$ span a $(3r+2)$-plane. Therefore, the meaning of Definition 1 is that if any three of the r-planes corresponding to an elementary cell are chosen in the general position, then the last one belongs to the $(3r+2)$-plane spanned by the first three.

EXAMPLE 1. In the case of rank $r=0$, Definition 1 requires that four points corresponding to any elementary square of $\mathbb{Z}^N$ be coplanar. Thus, we arrive at the notion of usual Q-nets introduced in [11] for $N=2$ and in [9] for $N \geq 3$.

EXAMPLE 2. Q-nets of rank $r=1$ are built of projective lines assigned to vertices of the lattice $\mathbb{Z}^N$, and Definition 1 requires that four lines corresponding to any elementary square lie in a 5-plane.

The main properties of usual Q-nets, which will be generalized now to the Grassmannian context, are the following (see a detailed account in [4,9]):

- Within an elementary cube of $\mathbb{Z}^3$, the points assigned arbitrarily to any seven vertices determine the point assigned to the eighth vertex uniquely. This can be expressed by saying that Q-nets are described by a *discrete 3D system*.
- This 3D system can be imposed on all 3D faces of an elementary cube of any dimension $N \geq 4$. This property is called the *multidimensional consistency* of the corresponding 3D system and follows for any $N \geq 4$ from the 4D consistency. The multidimensional consistency is treated as the integrability of the corresponding 3D system.

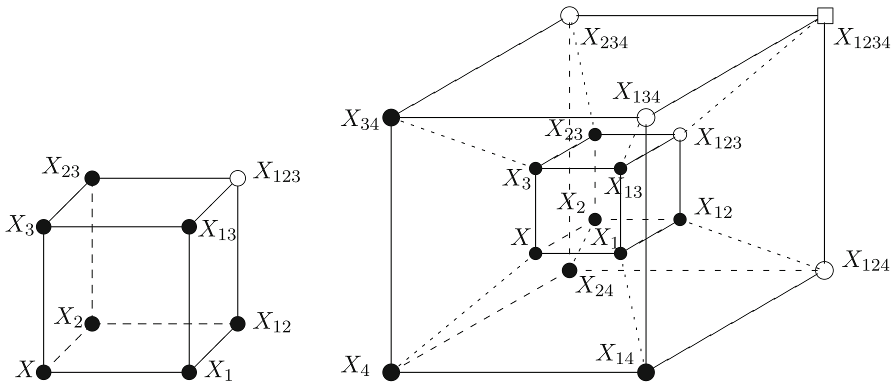

Figure 1. The combinatorial meaning of a discrete 3D system and its 4D consistency. *Black circles mark* the initial data within an elementary cube; *white circles mark* the vertices uniquely determined by the initial data; *white square marks* the vertex where the consistency condition appears. In Grassmannian Q-nets, the vertices carry r-planes; two r-planes corresponding to an edge span a $(2r+1)$-plane; four r-planes assigned to the vertices of an elementary square span a $(3r+2)$-planes; eight r-planes assigned to the vertices of an elementary 3-cube span a $(4r+3)$-plane in $\mathbb{P}^d$.

These properties are illustrated in Figure 1. In this figure and everywhere else, we use the notation X_i for the shift of the ith argument of a function X on $\mathbb{Z}^N$, that is, for $X(n_1,\ldots,n_i+1,\ldots)$. It is clear that the order of the subscripts does not matter, $X_{ij}=X_{ji}$.

THEOREM 1. (Grassmannian Q-nets are described by a discrete 3D system) *Let seven r-planes* $X, X_i, X_{ij}\in\mathbb{G}_r^d, 1\le i\ne j\le 3, d\ge 4r+3$, *be given such that*

$$\dim\operatorname{span}(X, X_i, X_j, X_{ij}) = 3r+2$$

for each pair of indices ij, but with no other degeneracies. Then there exists a unique r-plane $X_{123}\in\mathbb{G}_r^d$, *such that the conditions*

$$\dim\operatorname{span}(X_i, X_{ij}, X_{ik}, X_{123}) = 3r+2$$

are fulfilled as well.

Proof. The general position condition yields that the projective plane $V=\operatorname{span}(X, X_1, X_2, X_3)$ is of dimension $4r+3$. The assumptions of the theorem imply that the r-planes X_{ij} are contained in the corresponding $(3r+2)$-planes $\operatorname{span}(X, X_i, X_j)$ and, therefore, are also contained in V. In the case of the general position, the planes spanned by X_i, X_{ij}, X_{ik} are also $(3r+2)$-dimensional. The r-plane X_{123}, if exists, must lie in the intersection of three such $(3r+2)$-planes. In the $(4r+3)$-dimensional space V, the dimension of a pairwise intersection is $2(3r+2)-(4r+3)=2r+1$ and, therefore, the dimension of the triple intersection is $(4r+3)-3(3r+2)+3(2r+1)=r$, as required. $\square$

THEOREM 2. (Multidimensional consistency of Grassmannian Q-nets) *The 3D system governing Grassmannian Q-nets is 4D-consistent, and therefore, N-dimensionally consistent for all $N \geq 4$.*

Proof. One has to show that the four r-planes,

$$\mathrm{span}(X_{12}, X_{123}, X_{124}) \cap \mathrm{span}(X_{13}, X_{123}, X_{124}) \cap \mathrm{span}(X_{14}, X_{124}, X_{134}),$$

and the three others obtained by cyclic shifts of indices coincide. Thus, we have to prove that the six $(3r+2)$-planes $\mathrm{span}(X_{ij}, X_{ijk}, X_{ij\ell})$ intersect along a r-plane. We assume that the ambient space $\mathbb{P}^d$ has dimension $d \geq 5r+4$. Then, in general position, the plane $\mathrm{span}(X, X_1, X_2, X_3, X_4)$, which contains all elements of our construction, is of dimension $5r+4$. It is easy to understand that the $(3r+2)$-plane $\mathrm{span}(X_{ij}, X_{ijk}, X_{ij\ell})$ is the intersection of two $(4r+3)$-planes $V_i = \mathrm{span}(X_i, X_{ij}, X_{ik}, X_{i\ell})$ and $V_j = \mathrm{span}(X_j, X_{ij}, X_{jk}, X_{j\ell})$. Indeed, the plane V_i contains also X_{ijk}, $X_{ij\ell}$ and $X_{ik\ell}$. Therefore, both V_i and V_j contain the three r-planes X_{ij}, X_{ijk} and $X_{ij\ell}$, which determine the $(3r+2)$-plane $\mathrm{span}(X_{ij}, X_{ijk}, X_{ij\ell})$. Now the intersection in question can be alternatively described as the intersection of the four $(4r+3)$-planes, V_1, V_2, V_3 and V_4, of one and the same $(5r+4)$-dimensional space. This intersection is generically an r-plane. □

3. Analytical Description: Noncommutative Q-Nets

Here, we give an analytical description of Grassmannian Q-nets. The corresponding integrable difference equations were introduced, without a geometric interpretation, in [5].

In the case of ordinary Q-nets (of rank $r=0$), the planarity condition is written in affine coordinates as

$$x_{ij} = x + a^{ij}(x_i - x) + a^{ji}(x_j - x), \tag{1}$$

where the scalar coefficients a^{ij}, a^{ji} are naturally assigned to the corresponding elementary squares of $\mathbb{Z}^N$ (parallel to the coordinate plane (ij)). Consistency of these equations around an elementary cube (Theorem 1) yields a mapping

$$(a^{12}, a^{21}, a^{13}, a^{31}, a^{23}, a^{32}) \mapsto (a_3^{12}, a_3^{21}, a_2^{13}, a_2^{31}, a_1^{23}, a_1^{32}).$$

This mapping can be well rewritten the form in terms of the so-called discrete rotation coefficients, see [9]. The same approach works in the case $r>0$ as well, with the only difference that now we have to assume that the coefficients a^{ij} are noncommutative.

In order to demonstrate this, we use the interpretation of the Grassmannian $\mathbb{G}_r^d$ as the variety G_{r+1}^{d+1} of all $(r+1)$-dimensional subspaces of the vector space $\mathbb{R}^{d+1}$. One can represent an $(r+1)$-dimensional subspace X of $\mathbb{R}^{d+1}$ by a $(r+1)\times(d+1)$ matrix x, whose rows contain vectors of some basis of X. The change of basis of

X corresponds to a left multiplication of x by an element of GL_{r+1}. Thus, one gets the isomorphism $G_{r+1}^{d+1} \equiv (\mathbb{R}^{d+1})^{r+1}/GL_{r+1}$.

The condition that the $(r+1)$-dimensional vector subspace X_{12} belongs to the $3(r+1)$-dimensional vector space spanned by X, X_1, X_2 is now expressed by an equation

$$x_{12} = ax + bx_1 + cx_2, \quad a, b, c \in \mathrm{Mat}(r+1, r+1).$$

The set of coefficients a, b, c is abundant since it contains $3(r+1)^2$ parameters, while $\dim G_{r+1}^{3r+3} = 2(r+1)^2$. In order to get rid of this abundance, we adopt an "affine" normalization of the representatives x, analogous to the case $r = 0$. Namely, the representative of a generic subspace can be chosen by applying the left multiplication with a suitable matrix, in the form,

$$x = \begin{pmatrix} x^{1,1} & \dots & x^{1,d-r} & 1 & \dots & 0 \\ \vdots & & \vdots & & \ddots & \\ x^{r+1,1} & \dots & x^{r+1,d-r} & 0 & \dots & 1 \end{pmatrix}, \tag{2}$$

with the unit matrix I in the last $r+1$ columns. Under this normalization, the coefficients in the equation $x_{12} = ax + bx_1 + cx_2$ obey the relation $I = a + b + c$, and we come to the equation of the form (1).

The calculation of the consistency conditions of Equation (1) remains rather simple in the noncommutative setup. One of three ways of getting vector x_{ijk} is

$$x_{ijk} = x_k + a_k^{ij}(x_{ik} - x_k) + a_k^{ji}(x_{jk} - x_k).$$

Note that after we substitute x_{ik} and x_{jk} from (1), the matrix x_i enters the right, hand side only once with the coefficient $a_k^{ij} a^{ik}$. Therefore, alternating of j and k yields the relation

$$a_k^{ij} a^{ik} = a_j^{ik} a^{ij}. \tag{3}$$

Analysis of relations (3) is based on the following statement.

LEMMA 1. (Integration of closed multiplicative matrix-valued one-form) *Let the GL_{r+1}-valued functions a^j be defined on edges of $\mathbb{Z}^N$ parallel to the jth coordinate axis (so that $a^j(n)$ is assigned to the edge $[n, n+e_j]$, where e_j is the unit vector of the jth coordinate axis). If a^j satisfies*

$$a^j(n+e_k)a^k(n) = a^k(n+e_j)a^j(n), \tag{4}$$

then there exists a GL_{r+1}-valued function h defined on vertices of $\mathbb{Z}^N$ such that

$$a^j(n) = h(n+e_j)h^{-1}(n), \quad 1 \le j \le N. \tag{5}$$

Proof. Prescribe $h(0)$ arbitrarily. In order to extend h to any point of $\mathbb{Z}^N$, connect it to 0 by a lattice path $(\mathfrak{e}_1, \ldots, \mathfrak{e}_M)$, where the endpoint of any edge $\mathfrak{e}_m$ coincides with the initial point of the edge $\mathfrak{e}_{m+1}$. Extend h along the path according to (5). This extension does not depend on the choice of the path. Indeed, any two lattice paths connecting any two points can be transformed into one another by means of elementary flips exchanging two edges $[n, n+e_j]$, $[n+e_j, n+e_j+e_k]$ to the two edges $[n, n+e_k]$, $[n+e_k, n+e_j+e_k]$. The value of h at the common points n and $n+e_j+e_k$ of such two paths remain unchanged under the flip, as follows from the "closedness condition" (4). □

Equation (3) together with Lemma 1 yield existence of matrices h^i (assigned to edges of $\mathbb{Z}^N$ parallel to the i-th coordinate axis) such that $a^{ij} = h^i_j (h^i)^{-1}$. They are called the *discrete Lamé coefficients*. Equation (1) takes the form

$$x_{ij} = x + h^i_j (h^i)^{-1}(x_i - x) + h^j_i (h^j)^{-1}(x_j - x),$$

cf. [5]. Let us introduce the new variable y^i (also assigned to edges parallel to the ith coordinate axis) by the formula $x_i - x = h^i y^i$. Then $x_{ij} = x + h^i_j y^i + h^j_i y^j$, and, on the other hand, $x_{ij} = x_j + h^i_j y^i_j = x + h^j y^j + h^i_j y^i_j$. This allows us to rewrite Equation (1) finally as

$$y^i_j = y^i - b^{ij} y^j. \tag{6}$$

The matrices

$$b^{ij} = (h^i_j)^{-1}(h^j_i - h^j), \tag{7}$$

(assigned to the elementary squares parallel to the (ij)th coordinate plane) are called the *discrete rotation coefficients*. The compatibility conditions in terms of these coefficients are perfectly simple. We have

$$y^i_{jk} = y^i + b^{ik} y^k + b^{ij}_k (y^j + b^{jk} y^k) = y^i + b^{ij} y^j + b^{ik}_j (y^k + b^{kj} y^j),$$

which leads to the coupled equations [5]

$$b^{ij}_k - b^{ik}_j b^{kj} = b^{ij}, \quad -b^{ij}_k b^{jk} + b^{ik}_j = b^{ik}.$$

These can be solved for b^{ij}_k to give an explicit map.

THEOREM 3. (Grassmannian Q-nets are described by the noncommutative discrete Darboux system) *Rotation coefficients $b^{ij} \in \mathrm{Mat}(r+1, r+1)$ of Q-nets in the Grassmannian $\mathbb{G}^d_r$ satisfy the noncommutative discrete Darboux system*

$$b^{ij}_k = (b^{ij} + b^{ik} b^{kj})(I - b^{jk} b^{kj})^{-1}, \quad k \neq i \neq j \neq k. \tag{8}$$

This map is multidimensionally consistent.

Consistency is a corollary of Theorem 2, but it is also not too difficult to prove it directly.

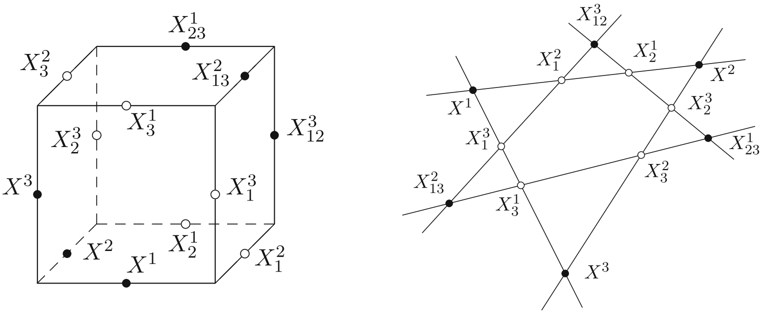

Figure 2. Combinatorics (*left*) and geometry (*right*) of an elementary cube of a Darboux net: *black circles mark* the initial data; *white circles mark* the data uniquely determined by the initial data. In Grassmannian Darboux nets, the edges carry r-planes; 4 r-planes corresponding to an elementary square span a $(2r+1)$-plane; 12 r-planes assigned to the edges of an elementary 3-cube span a $(3r+2)$-plane in $\mathbb{P}^d$.

4. Grassmannian Darboux Nets

DEFINITION 2. (*Grassmannian Darboux net*) A Grassmannian Darboux net (of rank r) is a map $E(\mathbb{Z}^N)\to\mathbb{G}^d_r$ defined on the edges of the regular square lattice, such that for every elementary quadrilateral of $\mathbb{Z}^N$ the four r-planes corresponding to its sides lie in a $(2r+1)$-plane.

In particular, for $r=0$ one arrives at the notion of Darboux nets introduced in [12]: the four points corresponding to the sides of every elementary square are required to be collinear (Figure 2).

We will denote by X^i the r-planes assigned to the edges of $\mathbb{Z}^N$ parallel to the ith coordinate axis; the subscripts will be still reserved for the shift operation.

To find an analytical description of Grassmannian Darboux nets, we continue to work with the "affine" representatives from G^{d+1}_{r+1} normalized as in (2). The defining property yields:

$$x^i_j = r^{ij}x^i + (I-r^{ij})x^j.$$

Hence,

$$x^i_{jk} = r^{ij}_k(r^{ik}x^i + (I-r^{ik})x^k) + (I-r^{ij}_k)(r^{jk}x^j + (I-r^{jk})x^k),$$

and therefore

$$r^{ij}_k r^{ik} = r^{ik}_j r^{ij}.$$

Comparing with (3) and using Lemma 1, we conclude that $r^{ij}=s^i_j(s^i)^{-1}$. Set $y^i=(s^i)^{-1}x^i$, then the linear problem takes the form (6) with the rotation coefficients

$$b^{ij} = ((s^i)^{-1} - (s^i_j)^{-1})s^j.$$

Thus, we come to the conclusion that Grassmannian Darboux nets are described by the same noncommutative discrete Darboux system (8) as Grassmannian Q-nets, with rotation coefficients $b^{ij} \in \mathrm{Mat}(r+1, r+1)$ defined by the last formula. Of course, this is not a coincidence, since Q-nets and Darboux nets are closely related. Indeed, considering an intersection of a Grassmannian Q-net in $\mathbb{P}^d$ with some plane Π of codimension $r+1$, one will find a Grassmannian Darboux net in Π. Conversely, any Grassmannian Darboux net can be extended (non-uniquely) to a Grassmannian Q-net. This is analogous to the case of ordinary nets (of rank $r=0$) explained in [4, p. 76].

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