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On the Integrability of Infinitesimal and Finite Deformations of Polyhedral Surfaces

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Abstract. It is established that there exists an intimate connection between isometric deformations of polyhedral surfaces and discrete integrable systems. In particular, Sauer's kinematic approach is adopted to show that second-order infinitesimal isometric deformations of discrete surfaces composed of planar quadrilaterals (discrete conjugate nets) are determined by the solutions of an integrable discrete version of Bianchi's classical equation governing finite isometric deformations of conjugate nets. Moreover, it is demonstrated that finite isometric deformations of discrete conjugate nets are completely encapsulated in the standard integrable discretization of a particular nonlinear σ-model subject to a constraint. The deformability of discrete Voss surfaces is thereby retrieved in a natural manner.

Keywords. Isometric deformations, polyhedral surfaces, integrable systems.

1. Introduction

The study of infinitesimal and finite deformations of both polyhedral and smooth surfaces has a long history. Various proofs of the fact that closed convex polyhedra are infinitesimally rigid are due to Cauchy [10] (1813), Dehn [13] (1916), Weyl [28] (1917) and Alexandrov [2] (1958). Analogous results for smooth surfaces were obtained by Liebmann [21] (1899) and Cohn-Vossen [12] (1936). Apart from their significance in differential geometry, isometric deformations of smooth surfaces also find diverse application in physics. For instance, it was observed by Blaschke [5] that the standard theory of shell membranes which are in equilibrium and not subjected to external forces may be set in correspondence with infinitesimal isometric deformations of surfaces. Thus, the geometric determination of such deformations corresponds to finding solutions of the equilibrium equations in membrane theory as set down and discussed by such luminaries as Beltrami, Clapeyron, Kirchhoff, Lagally, Lamé, Lecornu, Love and Rayleigh [22].

The main aim of the present paper is to show that there exists an intrinsic connection between isometric deformations of (open) quadrilateral surfaces and discrete integrable systems. Thus, the study of isometrically deformable quadrilateral surfaces is canonically embedded in the emerging field of *integrable discrete differential geometry* [8]. In particular, it is demonstrated that the deformation parameter may be identified as the 'spectral parameter' which constitutes the key ingredient in the theory of integrable systems [1].

Here, in the main, we focus on isometric deformations of *discrete conjugate nets* which constitute discrete surfaces composed of planar quadrilaterals. We adopt the kinematic approach due to Sauer [24] and retrieve his fundamental notion of *reciprocal-parallel* discrete surfaces associated with the existence of *infinitesimal* isometric deformations. We show that, remarkably, discrete conjugate nets admit infinitesimal isometric deformations of *second-order* if and only if the reciprocal-parallel surfaces constitute *discrete Bianchi surfaces*. These constitute natural discrete analogues of an integrable class of classical surfaces which was analysed by Bianchi [4] in connection with isometric deformations of conjugate nets. The nonlinear equation underlying discrete Bianchi surfaces has been shown to be integrable [26] in a different geometric context, namely discrete isothermic surfaces [8]. By construction, discrete Bianchi surfaces constitute particular *discrete asymptotic nets*. Integrable reductions of discrete asymptotic nets including discrete Bianchi surfaces have been the subject of [14].

If a discrete conjugate net admits a *finite* isometric deformation, then any quadrilateral undergoes a rigid motion which may be decomposed into a translation and a rotation. We demonstrate that the rotational component interpreted as an $SU(2)$-valued lattice function obeys a *pair* of *linear* equations which bears the hallmarks of a 'Lax pair' [1] for a discrete integrable system in that it depends parametrically on the deformation parameter and gives rise to a *discrete zero-curvature condition*. As an illustration, it is shown that the discrete zero-curvature condition contains as a special case the 'Gauss map' of *discrete K-surfaces*. These are integrable [6] and have been proposed as natural discrete analogues of surfaces of constant negative Gaussian curvature by Sauer [23] and Wunderlich [29]. As observed by Sauer, discrete K-surfaces are reciprocal-parallel to *discrete Voss surfaces* which have indeed been shown by Sauer and Graf [25] to admit *finite* isometric deformations. Finally, it is established that the discrete zero-curvature condition may be formulated in terms of a standard integrable *discrete nonlinear σ-model* [8, 26] subject to a constraint involving the deformation parameter.

2. Infinitesimal deformations of discrete surfaces

In the following, a *discrete surface F* is defined as (the image of) a mapping

$$F : V(\mathcal{G}) \to \mathbb{R}^3,$$

where $V(\mathcal{G})$ denotes the set of vertices of a cellular decomposition $\mathcal{G}$ of the plane. The edges and (combinatorial) faces of the discrete surface are those induced naturally by the mapping F. A *dual* discrete surface F^* is a mapping which is defined on the vertices of

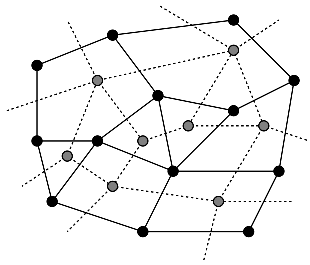

FIGURE 1. A cellular decomposition $\mathcal{G}$ (black vertices) and its dual $\mathcal{G}^*$ (grey vertices).

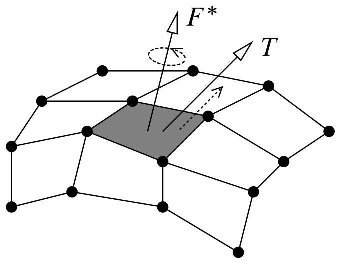

FIGURE 2. Infinitesimal rigid motion of a quadrilateral.

the dual cellular decomposition $\mathcal{G}^*$ (cf. Figure 1), that is,

$$F^* : V(\mathcal{G}^*) \to \mathbb{R}^3.$$

In the present paper, we are mainly concerned with *quadrilateral surfaces* corresponding to the choice $\mathcal{G} = \mathbb{Z}^2$ as illustrated in Figure 2. However, we begin with a generic discrete surface $F : V(\mathcal{G}) \to \mathbb{R}^3$ and consider a 'deformed surface'

$$F^\epsilon = F + \epsilon \bar{F},$$

where the constant ϵ constitutes a 'small' deformation parameter, that is, $|\epsilon| \ll 1$, and $\bar{F} : V(\mathcal{G}) \to \mathbb{R}^3$ defines the displacement of the vertices of F. It is convenient to imagine the edges of any face of the discrete surface as the boundary of a small piece of a surface. Accordingly, it is meaningful to define an *infinitesimal isometric deformation* F^ϵ of a discrete surface F as a deformation which does not change the shapes of the faces to order $O(\epsilon)$. In kinematic terms, when the discrete surface is being deformed, any face undergoes an infinitesimal rigid motion, that is, a combination of an infinitesimal translation and an infinitesimal rotation. Accordingly, if P is a point on a face f, then its displacement $P^\epsilon - P = \epsilon \bar{P}$ is given by

$$\bar{P} = T + F^* \times P, \tag{2.1}$$

where the translation T and the oriented axis of rotation F^* are *independent* of P. This is illustrated in Figure 2 for a quadrilateral surface. Since each face is associated with a vector of rotation F^*, we may regard F^* as another discrete surface which is dual to F.

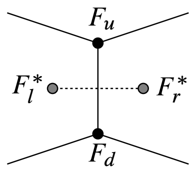

FIGURE 3. Two adjacent faces and their associated vectors of rotation F_r^* and F_l^*.

Thus, the vertices, edges and faces of F are in one-to-one correspondence with the faces, edges and vertices of F^*, respectively. Similarly, T is defined on the faces of $\mathcal{G}$, that is,

$$T : V(\mathcal{G}^*) \to \mathbb{R}^3.$$

We now focus on the relative motion of two faces f_r and f_l which are joined by an edge e linking the vertices F_u and F_d as depicted in Figure 3. Thus, if we evaluate the displacement relation (2.1) at the vertices F_u and F_d which belong to both faces f_r and f_l, then we obtain the four relations

$$\bar{F}_u = T_r + F_r^* \times F_u, \qquad \bar{F}_d = T_r + F_r^* \times F_d, \tag{2.2a}$$

$$\bar{F}_u = T_l + F_l^* \times F_u, \qquad \bar{F}_d = T_l + F_l^* \times F_d. \tag{2.2b}$$

The latter imply that the *dual* edges $[F_l^*, F_r^*]$ and $[F_d, F_u]$ are *parallel*, that is,

$$(F_r^* - F_l^*) \times (F_u - F_d) = 0.$$

This merely expresses the fact that the relative motion of the two adjacent faces f_r and f_l represents an infinitesimal rotation about the common edge $[F_d, F_u]$. Thus, the following definition is natural.

Definition 2.1. Two combinatorially dual discrete surfaces F and F^* are *reciprocal-parallel* if dual edges are parallel.

The above reasoning may now be inverted and hence we are led to a result which is due to Sauer [24] in the case of quadrilateral surfaces.

Theorem 2.2. *A discrete surface F admits an infinitesimal isometric deformation if and only if there exists a reciprocal-parallel discrete surface F^*.*

Proof. In the preceding, it has been established that an infinitesimal isometric deformation of a discrete surface F gives rise to a reciprocal-parallel surface generated by the vectors of rotation F^*. Conversely, if F^* constitutes a reciprocal-parallel discrete surface, then the relations (2.2) imply that

$$T_r - T_l = -(F_r^* - F_l^*) \times F_u, \quad T_r - T_l = -(F_r^* - F_l^*) \times F_d \tag{2.3}$$

constitute necessary conditions on the vectors of translation associated with two adjacent faces. In fact, the above pair uniquely defines a dual discrete surface $T : V(\mathcal{G}^*) \to \mathbb{R}^3$ up to a single vector T_0 defined on a face f_0. In kinematic terms, the latter corresponds to a uniform translation of the discrete surface F. In order to make good the assertion that T

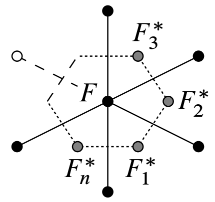

FIGURE 4. The vertices F_k^* of a face dual to a vertex F.

is well defined, it is required to verify two properties. Firstly, the two relations (2.3) are equivalent since F^* is reciprocal-parallel to F and hence $(F_r^* - F_l^*) \times (F_u - F_d) = 0$. Secondly, the 'closing condition' associated with any closed polygon composed of edges of F^* is satisfied. Indeed, it is sufficient to consider the boundary of a dual face as depicted in Figure 4. If we denote the vertices of the face which is dual to the vertex F by $F_1^*, \ldots, F_n^*$, then the n relations

$$T_{k+1} - T_k = -(F_{k+1}^* - F_k^*) \times F, \quad k = 1, \ldots, n,$$

hold (with the identification $T_{n+1} = T_1$) and the corresponding closing condition

$$\sum_{k=1}^{n} (T_{k+1} - T_k) = -\sum_{k=1}^{n} (F_{k+1}^* - F_k^*) \times F = 0$$

is satisfied. The 'displacement' $\bar{F}$ of the vertex F may now be defined by

$$\bar{F} = T_k + F_k^* \times F$$

since the latter is independent of k. In this manner, one may construct a discrete surface

$$F^\epsilon = F + \epsilon \bar{F}$$

which represents an infinitesimal isometric deformation of F. □

2.1. Quadrilateral surfaces

We are now concerned with (portions of) quadrilateral surfaces

$$F : \mathbb{Z}^2 \to \mathbb{R}^3.$$

In this case, the relation between two reciprocal-parallel surfaces F and F^* is illustrated in Figure 5. Here and throughout the remainder of the paper, we indicate unit increments of the discrete variables n_1 and n_2 which label the lattice $\mathbb{Z}^2$ by subscripts, that is, for instance,

$$F = F(n_1, n_2), \quad F_1 = F(n_1 + 1, n_2), \quad F_{12} = F(n_1 + 1, n_2 + 1),$$

so that a quadrilateral $\diamond$ is represented by $[F, F_1, F_{12}, F_2]$. Similarly, overbars on subscripts designate unit decrements. Thus, a vertex F is linked to the quadrilaterals $\diamond$, $\diamond_{\bar{1}}$, $\diamond_{\bar{1}\bar{2}}$ and $\diamond_{\bar{2}}$. Furthermore, we adopt the notation

$$\Delta_i F = F_i - F, \quad \Delta_{12} F = F_{12} - F_1 - F_2 + F$$

for the first-order and mixed second-order difference operators, respectively.

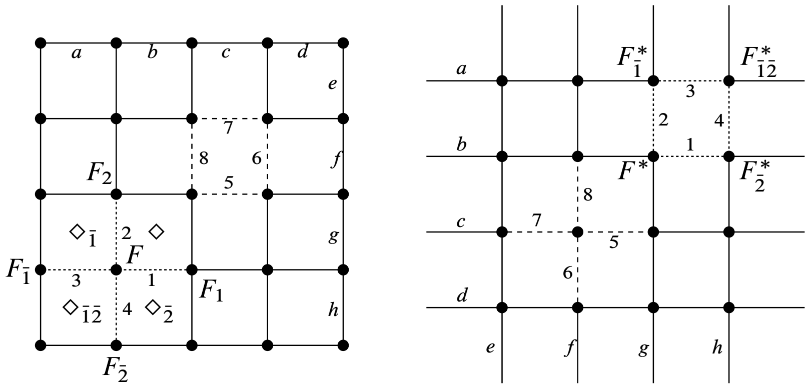

FIGURE 5. Schematic depiction of reciprocal-parallel quadrilateral surfaces F and F^*.

In the following, we are concerned with the deformation of discrete surfaces which are composed of *planar* quadrilaterals.

Definition 2.3. A quadrilateral surface is termed a *discrete conjugate net* if all quadrilaterals are planar.

Discrete conjugate nets constitute natural analogues of conjugate nets in classical differential geometry (see [8] and references therein). Their reciprocal-parallel counterparts (if they exist) represent discrete versions of classical asymptotic nets.

Definition 2.4. A quadrilateral surface is termed a *discrete asymptotic net* if all stars are planar.

The following observation [25] which provides an important connection between discrete conjugate and asymptotic nets is a direct consequence of the analysis undertaken in the previous subsection. It is illustrated in Figure 6.

Theorem 2.5. *A discrete conjugate net with nonplanar stars is infinitesimally isometrically deformable if and only if there exists a reciprocal-parallel discrete asymptotic net. The latter is uniquely determined up to a scaling and hence the deformation is unique.*

The uniqueness of the reciprocal-parallel discrete asymptotic net is due to the fact that any star of the discrete conjugate net determines the directions of the edges of the corresponding dual quadrilateral of the discrete asymptotic net. By assumption, this quadrilateral is nonplanar and hence it is known up to a scaling. However, if the length of one edge of the discrete asymptotic net is arbitrarily prescribed, the scalings of all quadrilaterals are uniquely determined. In this connection, it proves useful to adopt the following definition [19].

Definition 2.6. Two discrete conjugate nets are *Combescure transforms* of each other if corresponding edges are parallel.

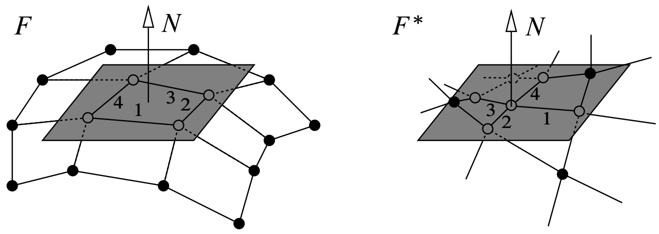

FIGURE 6. Reciprocal-parallel discrete conjugate and asymptotic nets F and F^*.

It is evident that any discrete conjugate net admits an infinite number of Combescure transforms. Their relevance in the context of discrete asymptotic nets is the content of the following theorem.

Theorem 2.7. *Any discrete asymptotic net with nonplanar quadrilaterals possesses an infinity of reciprocal-parallel discrete conjugate nets. These are related by Combescure transformations and admit an infinitesimal isometric deformation.*

Proof. Since the stars of an asymptotic net are planar, each star may be associated with a unit normal N as illustrated in Figure 6. The reciprocal-parallel conjugate nets are constructed by successively drawing planes which are orthogonal to the vectors N and have the property that the four planes associated with any four 'neighbouring' normals meet at a point. □

3. Finite deformations

The subject of *finite isometric deformations* of polyhedral surfaces is classical and is highlighted by Cauchy's non-existence theorem [10] for deformations of convex polyhedra. Here, we confine ourselves to isometric deformations of a discrete conjugate net $F : \mathbb{Z}^2 \to \mathbb{R}^3$, that is a one-parameter family of discrete surfaces

$$F^\epsilon : \mathbb{Z}^2 \to \mathbb{R}^3$$

which depends continuously on the parameter ϵ in such a manner that the planar quadrilaterals of F^ϵ are congruent to those of the undeformed discrete conjugate net $F^0 = F$. In the following, it is understood that the term deformation implies its finite character while infinitesimal deformations are explicitly referred to as such.

In the previous section, it has been concluded that for any given discrete conjugate net with nonplanar stars there exists at most a one-parameter family of infinitesimal isometric deformations. Accordingly, finite deformations of such conjugate nets are unique if they exist. In fact, this statement is still correct if one replaces the condition of nonplanar stars by the assumption that the discrete conjugate net is *non-degenerate* in the sense that opposite edges of any star are not collinear. This may be deduced by considering a non-degenerate complex of 2×2 planar quadrilaterals meeting at a vertex (cf. Figure 7). Indeed, if we vary the angle between two adjacent quadrilaterals, then the remaining two

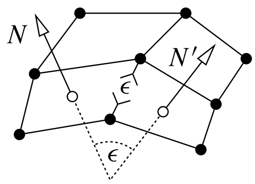

FIGURE 7. The deformability of a 2×2 complex of four planar quadrilaterals.

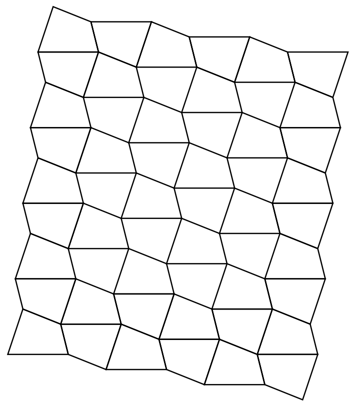

FIGURE 8. A deformable 'tessellation of the plane'.

quadrilaterals move in such a way that the complex is not torn. Thus, deformations of 2×2 complexes always exist. If we hold that angle constant, then the complex is rigid and if the complex forms part of a discrete conjugate net, then the entire discrete surface is rigid.

In conclusion, if a discrete conjugate net is isometrically deformable, then any Combescure transform is likewise isometrically deformable. In fact, the deformability of a discrete conjugate net is a property of its *normals only* and may therefore be dealt with in the realm of spherical geometry. Accordingly, we may state the following:

Theorem 3.1. *A non-degenerate discrete conjugate net admits at most a one-parameter family of isometric deformations. It is isometrically deformable if and only if all its Combescure transforms are isometrically deformable.*

3.1. General considerations

The complete classification of deformable discrete conjugate nets has not been achieved yet. Particular classes of such discrete surfaces were constructed by Sauer and Graf [25] and Kokotsakis [17] in the 1930s. An interesting example is displayed in Figure 8 and represents the 'tessellation of the plane' by means of copies of an arbitrary quadrilateral which are glued at corresponding edges. Some of the known deformable discrete conjugate nets do not possess smooth counterparts. This is a first indication that the class of

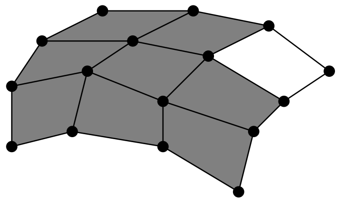

FIGURE 9. A complex of eight planar quadrilaterals is always deformable while a 3×3 complex is generically rigid.

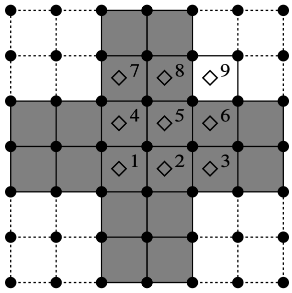

FIGURE 10. Illustration of the fact that a discrete conjugate net is deformable if and only if its 3×3 complexes are deformable.

deformable discrete conjugate nets is richer than in the smooth setting. In this connection, it is interesting to note that the conditions for the existence of isometric deformations which preserve conjugate nets on smooth surfaces were given by Bianchi [4] in as early as 1892 and related to an integrable class of surfaces which have been termed *Bianchi surfaces*. This will be discussed in more detail in Section 4.4.

The problem of isometric deformations becomes nontrivial as soon as the discrete conjugate net consists of at least 3×3 quadrilaterals. Indeed, if we consider a 3×3 complex and remove a corner quadrilateral, then deformation of the 'opposite' 2×2 complex determines the new position of the remaining four adjacent quadrilaterals (cf. Figure 9). However, in general, it will be impossible to re-insert the ninth quadrilateral and form a 3×3 complex. Thus, a 3×3 complex is generically rigid but admits a unique one-parameter family of isometric deformations if certain constraints on the normals to the quadrilaterals are satisfied. In fact, it is sufficient to focus on 3×3 complexes in the following sense:

Theorem 3.2. *A non-degenerate discrete conjugate net is isometrically deformable if and only if its 3×3 complexes are isometrically deformable.*

Proof. Let F be a non-degenerate discrete conjugate net which is such that all its 3×3 complexes are isometrically deformable. We remove all quadrilaterals up to two neighbouring 'horizontal' and two neighbouring 'vertical' strips which intersect at four quadrilaterals $\diamond^1, \diamond^2, \diamond^4, \diamond^5$ as indicated in Figure 10. This 'cross' of quadrilaterals admits a

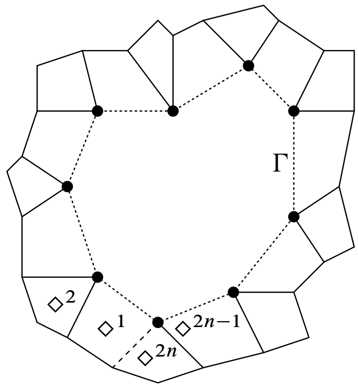

FIGURE 11. A polyhedral surface of Kokotsakis type.

one-parameter family of deformations. If it is deformed, then the assumption of the deformability of the 3×3 complexes implies that the quadrilateral $\diamond^9$ may be inserted into the existing part of the deformed surface F^ϵ to form a 3×3 complex $[\diamond^{1\epsilon},\dots,\diamond^{9\epsilon}]$. The complete deformed surface F^ϵ is constructed by repeating this procedure iteratively. □

Kokotsakis [17] investigated infinitesimal and finite deformations of a special class of (open) polyhedral surfaces which contains, for instance, the above-mentioned 3×3 complexes and closed octahedra. His investigations naturally led to Cauchy's theorem in the case of convex octahedra and the infinitesimal deformability of the octahedra of Bricard type [9]. Specifically, his polyhedral surfaces consist of a closed strip of $2n$ planar quadrilaterals $\diamond^1,\dots,\diamond^{2n}$ which are alternately attached to the n edges and n vertices of a rigid and closed (not necessarily planar) polygon Γ as indicated in Figure 11. In particular, if Γ constitutes a planar quadrilateral, then it may be regarded as the central quadrilateral of a 3×3 complex. In order to investigate the (infinitesimal) deformability of this polyhedral surface, one now cuts the surface at the edge between the quadrilaterals $\diamond^1$ and $\diamond^{2n}$ and (infinitesimally) rotates the quadrilateral $\diamond^1$ about the corresponding edge of Γ. The remaining quadrilaterals then move accordingly and, in general, there now exists a gap between the quadrilaterals $\diamond^1$ and $\diamond^{2n}$. If the motion is infinitesimal, then the condition of a closed deformed strip imposes one constraint on the polyhedral surface. However, if the motion is finite and one demands that the strip is closed for *all* possible positions of the quadrilateral $\diamond^1$, then this constraint constitutes a one-parameter family of constraints the solution of which is, in general, unknown. In the case of a 3×3 complex, it may be shown that the determination of solutions of these constraints amounts to finding common zeros of certain polynomials. Once again, in general, the solution to this problem is unknown. Nevertheless, as noted by Kokotsakis, direct inspection of these constraints gives rise to a class of isometrically deformable 3×3 complexes which was first recorded by Sauer and Graf [25]. Its extension to discrete surfaces is discussed below.

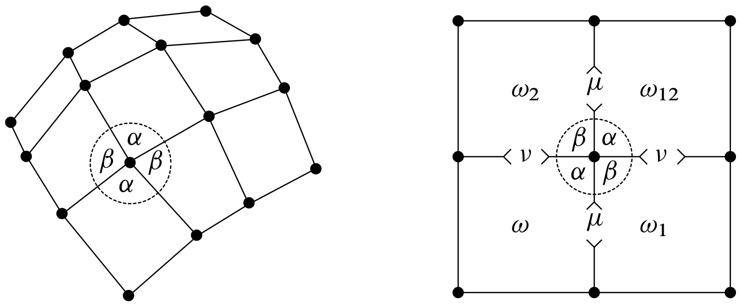

FIGURE 12. Definition and properties of discrete Voss surfaces.

3.2. The deformability of discrete Voss surfaces

As stated in the preceding, it is not known under which circumstances a discrete conjugate net is isometrically deformable. However, there exists an important class of discrete conjugate nets which admit isometric deformations. It consists of natural discrete analogues of classical Voss surfaces [27]. The latter terminology is due to the fact that, in the formal continuum limit, the coordinate polygons of any discrete Voss surface become two families of *geodesic* conjugate lines. The definition of discrete Voss surfaces is depicted in Figure 12.

Definition 3.3. A non-degenerate discrete conjugate net which is such that opposite angles made by the edges of any star are equal is termed a *discrete Voss surface*.

The existence of discrete Voss surfaces is readily shown by considering well-posed Cauchy problems. For instance, it is not difficult to verify that any arbitrarily prescribed spatial stairway gives rise to a unique discrete Voss surface. Moreover, it may be shown that the angle made by two adjacent quadrilaterals (dihedral angle) is constant along the coordinate polygon containing the common edge. This property may be taken as an alternative definition of discrete Voss surfaces and is illustrated in Figure 12. The connection between the 'interior' angles α, β and the dihedral angles μ, ν is given by [25]

$$\tan\frac{\mu}{2}\tan\frac{\nu}{2} = \frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta}. \tag{3.1}$$

Thus, the one-parameter family of deformations of 2×2 complexes of Voss type may be described algebraically by the transformation

$$\tan\frac{\mu}{2} \to \lambda\tan\frac{\mu}{2}, \quad \tan\frac{\nu}{2} \to \frac{1}{\lambda}\tan\frac{\nu}{2}.$$

The deformability of discrete Voss surfaces is now an immediate consequence of the above invariance.

Theorem 3.4. *Discrete Voss surfaces admit a one-parameter family of isometric deformations.*

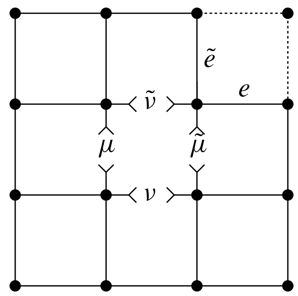

FIGURE 13. Illustration of the deformability of discrete Voss surfaces.

Proof. It is sufficient to consider a discrete Voss surface composed of nine quadrilaterals as displayed in Figure 13. If the top-right quadrilateral is removed and the discrete surface is isometrically deformed, then the dihedral angles change according to

$$\tan\frac{\mu}{2} \to \lambda \tan\frac{\mu}{2}, \qquad \tan\frac{\nu}{2} \to \frac{1}{\lambda}\tan\frac{\nu}{2}$$
$$\tan\frac{\tilde{\mu}}{2} \to \lambda \tan\frac{\tilde{\mu}}{2}, \qquad \tan\frac{\tilde{\nu}}{2} \to \frac{1}{\lambda}\tan\frac{\tilde{\nu}}{2}$$

so that the quantity

$$\tan\frac{\tilde{\mu}}{2}\tan\frac{\tilde{\nu}}{2}$$

is preserved. Relation (3.1) now implies that the angle made by the edges e and $\tilde{e}$ is unchanged. Thus, the missing quadrilateral may be inserted back into the discrete surface and the 3×3 complex is indeed deformable. □

Since discrete Voss surfaces are isometrically deformable, any discrete Voss surface F admits a discrete asymptotic reciprocal-parallel net F^* which is unique up to an overall scaling. Moreover, the defining property of discrete Voss surfaces implies that opposite angles in the quadrilaterals of F^* are equal. This is equivalent to stating that the lengths of opposite edges of the quadrilaterals are equal. The latter is the defining property of *discrete (generalized) Chebyshev nets*. Thus, based on the definition below, we have come to the following conclusion [24]:

Definition 3.5. A discrete surface is called a *discrete K-surface* or *discrete pseudospherical* if the coordinate polygons form both a discrete asymptotic net and a discrete (generalized) Chebyshev net.

Corollary 3.6. *The reciprocal-parallel counterparts of discrete Voss surfaces are discrete K-surfaces.*

Discrete K-surfaces have been proposed independently by Sauer [23] and Wunderlich [29] as natural discrete versions of classical pseudospherical surfaces, that is, surfaces of constant negative Gaussian curvature. The significance of discrete K-surfaces in *integrable* discrete differential geometry has been revealed in [6]. At the level of discrete

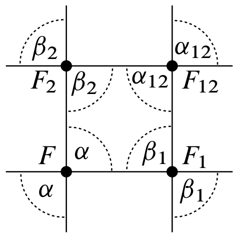

FIGURE 14. Definition of the lattice functions α and β.

Voss surfaces, the connection with the theory of integrable systems [1] is readily established. Thus, relation (3.1) may be exploited to characterize Voss surfaces in the following manner. If we arbitrarily prescribe two families of dihedral angles which are taken to be constant along the respective coordinate polygons, then the angles of a discrete Voss surface are constrained by the relation (3.1) which holds on each star and by the condition that the four angles in any quadrilateral must add up to 2π. If we regard the angles α and β as introduced in Figure 14 as functions defined on the vertices of the discrete Voss surfaces, then the latter condition may be formulated as

$$\alpha + \beta_1 + \alpha_{12} + \beta_2 = 2\pi.$$

It may be satisfied identically by introducing a lattice function ω which is defined on the quadrilaterals as illustrated in Figure 12. Thus, on each star of Voss type with angles α, β and dihedral angles μ, ν, the angles α and β are parametrized by

$$\alpha = \frac{\omega_1 + \omega_2}{2}, \quad \beta = \pi - \frac{\omega_{12} + \omega}{2}$$

and the relation (3.1) becomes

$$\sin\left(\frac{\omega_{12} - \omega_2 - \omega_1 + \omega}{4}\right) = \tan\frac{\mu}{2}\tan\frac{\nu}{2}\sin\left(\frac{\omega_{12} + \omega_2 + \omega_1 + \omega}{4}\right).$$

For given dihedral angles and modulo the Combescure transformation, any solution of the above lattice equation gives rise to a unique discrete Voss surface. Remarkably, this lattice equation is but another avatar of Hirota's integrable discrete version of the classical sine-Gordon equation [16]. Moreover, as demonstrated in Section 5.2, the deformation parameter λ turns out to play the role of a 'spectral parameter' in the corresponding Lax pair [6].

4. Infinitesimal deformations of second order

It has been demonstrated that the existence of an infinitesimal isometric deformation of a discrete surface F corresponds to the existence of a reciprocal-parallel discrete surface F^*. Accordingly, if there exists an infinitesimal isometric deformation of F of *second order*, then the reciprocal-parallel surface is infinitesimally deformable and the deformation is such that the stars are unchanged except for the length of their edges. In particular, the

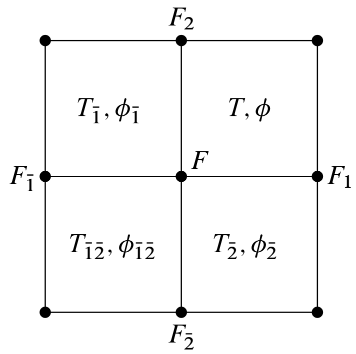

FIGURE 15. The translation T and rotation ϕ associated with finite deformations.

angles between the edges of the coordinate polygons are preserved and hence the deformation is infinitesimally *conformal*. If the original discrete surface constitutes a discrete conjugate net F, then the second-order isometric deformation corresponds to an infinitesimal conformal deformation of its discrete asymptotic reciprocal-parallel counterpart F^*.

4.1. Finite deformations

In order to analyse second-order isometric deformations, it is convenient to make some general statements about finite isometric deformations of arbitrary quadrilateral surfaces. Thus, if a discrete surface F is isometrically deformable, then its (nonplanar) quadrilaterals are subjected to rigid motions. Since any rigid motion may be decomposed into a translation and a rotation, the motion of any point P on the quadrilateral $\diamond = [F, F_1, F_{12}, F_2]$ may be described by

$$P^\epsilon = T(\epsilon) + \phi^{-1}(\epsilon) P \phi(\epsilon),$$

where we have made the standard identification of the ambient space $\mathbb{R}^3$ with the Lie algebra $su(2)$, that is,

$$P_{su(2)} = \langle P_{\mathbb{R}^3}, e \rangle, \quad e = (e_1, e_2, e_3),$$
$$e_1 = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad e_3 = \frac{1}{\mathrm{i}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $T \in su(2)$ constitutes the translation and $\phi \in SU(2)$ encodes the rotation. Here, T and ϕ are independent of P and $T(0) = 0$, $\phi(0) = \mathbb{1}$ so that $P^0 = P$ as required. As in the case of infinitesimal deformations, each quadrilateral is associated with a translation T and a rotation ϕ. Hence, T and ϕ are defined on the dual lattice (cf. Figure 15).

If we now consider the motion of the vertices F and F_2 which are common to the quadrilaterals $\diamond$ and $\diamond_{\bar{1}}$, then

$$F^\epsilon = T + \phi^{-1} F \phi, \qquad F_2^\epsilon = T + \phi^{-1} F_2 \phi,$$
$$F^\epsilon = T_{\bar{1}} + \phi_{\bar{1}}^{-1} F \phi_{\bar{1}}, \qquad F_2^\epsilon = T_{\bar{1}} + \phi_{\bar{1}}^{-1} F_2 \phi_{\bar{1}},$$

and elimination of $F^\epsilon, F_2^\epsilon, T, T_{\bar 1}$ produces

$$\phi^{-1}(F_2 - F)\phi = \phi_{\bar 1}^{-1}(F_2 - F)\phi_{\bar 1}. \tag{4.1}$$

Similarly, consideration of the quadrilaterals $\diamond$ and $\diamond_{\bar 2}$ leads to

$$\phi^{-1}(F_1 - F)\phi = \phi_{\bar 2}^{-1}(F_1 - F)\phi_{\bar 2}. \tag{4.2}$$

Thus, if the discrete surface F is isometrically deformable, then the latter two edge relations hold. Conversely, if for a given discrete surface F there exists a one-parameter family of matrices $\phi(\epsilon) \in SU(2)$ with $\phi(0) = \mathbb{1}$ which satisfies the edge relations (4.1), (4.2), then the compatible relations

$$T - T_{\bar 1} = \phi_{\bar 1}^{-1} F \phi_{\bar 1} - \phi^{-1} F \phi, \tag{4.3a}$$

$$T - T_{\bar 2} = \phi_{\bar 2}^{-1} F \phi_{\bar 2} - \phi^{-1} F \phi, \tag{4.3b}$$

and $T(0) = 0$ uniquely[1] define $T \in su(2)$ and the discrete surface F admits a one-parameter family of isometric deformations given by

$$F^\epsilon = T + \phi^{-1} F \phi.$$

It is noted that the existence of T is guaranteed since, as in the case of infinitesimal deformations, the associated closing condition is satisfied modulo the edge relations (4.1) and (4.2).

4.2. Second-order deformations

Instead of demanding that the edge relations (4.1) and (4.2) be satisfied for all values of ϵ, we may now require that only the first n nontrivial orders in ϵ vanish. Here, we are interested in the case $n = 2$ so that it is sufficient to deal with Taylor series about $\epsilon = 0$ which are truncated at the second level. Thus, we consider deformations of the form

$$P^\epsilon = P + \epsilon(T^* + [P, F^*]) + \frac{\epsilon^2}{2}(S^* + [P, G^*] + [[P, F^*], F^*]),$$

where P is any point on the quadrilateral $\diamond$ and

$$T^* = T_\epsilon, \quad S^* = T_{\epsilon\epsilon}, \quad F^* = \phi_\epsilon \phi^{-1}, \quad G^* = (\phi_\epsilon \phi^{-1})_\epsilon,$$

evaluated at $\epsilon = 0$. Once again, T^*, S^*, F^* and G^* are $su(2)$-valued objects defined on the quadrilaterals of F. Deformations of this type do not change the quadrilaterals to order $O(\epsilon^2)$ since it is readily seen that

$$\left\langle \hat P^\epsilon - P^\epsilon, \tilde P^\epsilon - P^\epsilon \right\rangle = \left\langle \hat P - P, \tilde P - P \right\rangle + O(\epsilon^3)$$

for any three points $P, \hat P, \tilde P$ on $\diamond$. Hence, if such deformations exist, then we refer to them as (infinitesimal) isometric deformations of second order.

[1] Up to $T(\epsilon)$ at one vertex, corresponding to a trivial translation of F^0.

Differentiation of the edge relation (4.1) and evaluation at $\epsilon = 0$ now produce

$$\begin{aligned}&\epsilon[F_2 - F, F^*] + \frac{\epsilon^2}{2}([F_2 - F, G^*] + [[F_2 - F, F^*], F^*])\\&= \epsilon[F_2 - F, F^*_{\bar{1}}] + \frac{\epsilon^2}{2}([F_2 - F, G^*_{\bar{1}}] + [[F_2 - F, F^*_{\bar{1}}], F^*_{\bar{1}}]) + O(\epsilon^3),\end{aligned}$$

and an analogous expansion obtains in the case of the second edge relation (4.2). The requirement that the terms are linear in ϵ vanish therefore imposes the conditions

$$[F_2 - F, F^* - F^*_{\bar{1}}] = 0, \quad [F_1 - F, F^* - F^*_{\bar{2}}] = 0. \tag{4.4}$$

These encapsulate nothing but the fact that an isometric deformation of first order exists if and only if there exists a reciprocal-parallel discrete surface F^*. The terms quadratic in ϵ may then be simplified and application of the Jacobi identity results in

$$[F_2 - F, G^* - G^*_{\bar{1}} + [F^*_{\bar{1}}, F^*]] = 0, \tag{4.5a}$$

$$[F_1 - F, G^* - G^*_{\bar{2}} + [F^*_{\bar{2}}, F^*]] = 0. \tag{4.5b}$$

Since the closing condition for the translation T is satisfied for finite isometric deformations and its nature is such that it holds separately for any order in ϵ, the defining relations for the coefficients T^* and S^* which are obtained from the Taylor expansion of (4.3) are compatible. We therefore conclude that a discrete surface F admits an isometric deformation of second order if and only if there exists a reciprocal-parallel surface F^* and another dual surface G^* which obey the pair (4.5).

4.3. Second-order deformations of discrete conjugate nets

In the case of second-order deformations of discrete conjugate nets, progress may be made by exploiting a discrete version of the classical Lelieuvre formulae which encapsulate the Gauss map for surfaces parametrized in terms of asymptotic coordinates [15]. To this end, it is recalled that if a discrete conjugate net F admits an infinitesimal isometric deformation, then the associated reciprocal-parallel net F^* is discrete asymptotic. Thus, if N denotes the unit normals to the planar stars of the discrete asymptotic net as depicted in Figure 6, then there exist lattice functions ρ and σ such that the connection between the discrete asymptotic net and its unit normals is represented by

$$F^*_1 - F^* = \rho N_1 \times N, \quad F^*_2 - F^* = \sigma N \times N_2.$$

The compatibility ('closing') condition $F^*_{12} = F^*_{21}$ is given by

$$\rho_2 N_{12} \times N_2 - \rho N_1 \times N = \sigma_1 N_1 \times N_{12} - \sigma N \times N_2$$

so that multiplication by N_1 and N_2, respectively, yields

$$\rho\rho_2 = \sigma\sigma_1.$$

The functions ρ and σ may therefore be parametrized according to

$$\rho = \tau\tau_1, \quad \sigma = \tau\tau_2,$$

where τ constitutes an arbitrary lattice function. Introduction of the scaled normal

$$\mathcal{V} = \tau N$$

then reduces the compatibility condition to

$$(\mathcal{V}_{12} + \mathcal{V}) \times (\mathcal{V}_1 + \mathcal{V}_2) = 0.$$

Thus, the following statement may be made:

Theorem 4.1. *If F^* constitutes a discrete asymptotic net, then there exists a scaled normal $\mathcal{V}$ such that the* discrete Lelieuvre formulae [7, 18]

$$F_1^* - F^* = \mathcal{V}_1 \times \mathcal{V}, \quad F_2^* - F^* = \mathcal{V} \times \mathcal{V}_2 \tag{4.6}$$

hold. These imply that $\mathcal{V}$ satisfies the discrete Moutard equation

$$\mathcal{V}_{12} + \mathcal{V} = H(\mathcal{V}_1 + \mathcal{V}_2) \tag{4.7}$$

for some scalar lattice function H. Conversely, for any given solution $(\mathcal{V}, H)$ of the discrete Moutard equation (4.7), the discrete Lelieuvre formulae (4.6) are compatible and uniquely define a discrete asymptotic net F^* with $\mathcal{V}$ being its scaled normal.

Since we are concerned with second-order isometric deformations of discrete conjugate nets, the condition (4.4), which enshrines the existence of an asymptotic reciprocal-parallel net F^*, may be replaced by the discrete Moutard equation (4.7) by virtue of Theorem 2.7. Moreover, the identity

$$AB = \langle A \times B, e\rangle - \langle A, B\rangle\, \mathbb{1}$$

which holds for any $A, B \in \mathbb{R}^3 \cong su(2)$ shows that the remaining constraints (4.5) are satisfied if and only if there exist functions a and b such that

$$G_1^* - G^* = 2a(F_1^* - F^*) + 2F_1^* \times F^*,$$
$$G_2^* - G^* = 2b(F_2^* - F^*) + 2F_2^* \times F^*.$$

Elimination of G^* leads to the compatibility condition

$$a_2(F_{12}^* - F_2^*) - a(F_1^* - F^*) \\ - b_1(F_{12}^* - F_1^*) + b(F_2^* - F^*) = (F_{12}^* - F^*) \times (F_1^* - F_2^*)$$

which may be expressed entirely in terms of $\mathcal{V}$ by virtue of the discrete Lelieuvre formulae. In fact, on use of the discrete Moutard equation, multiplication by $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}$ results in

$$a_2 - b = \langle \mathcal{V}_{12} + \mathcal{V}, \mathcal{V}_1\rangle, \quad a - b_1 = \langle \mathcal{V}_{12} + \mathcal{V}, \mathcal{V}_2\rangle, \quad a_2 - b_1 = \langle \mathcal{V}_1 + \mathcal{V}_2, \mathcal{V}\rangle,$$

respectively. These relations imply that $\Delta_2(a + \langle \mathcal{V}_1, \mathcal{V}\rangle) = 0$ and $\Delta_1(b - \langle \mathcal{V}_2, \mathcal{V}\rangle) = 0$ so that summation yields

$$a = -\langle \mathcal{V}_1, \mathcal{V}\rangle + f(n_1), \quad b = \langle \mathcal{V}_2, \mathcal{V}\rangle - g(n_2)$$

and hence

$$\langle \mathcal{V}_{12} + \mathcal{V}, \mathcal{V}_1 + \mathcal{V}_2\rangle = f(n_1) + g(n_2).$$

Since the latter may be regarded as a definition of the function H in the discrete Moutard equation, we have established the following theorem:

Theorem 4.2. *A discrete conjugate net F admits an infinitesimal isometric deformation of second order if and only if there exists a (discrete asymptotic) reciprocal-parallel net F^* whose scaled normal $\mathcal{V}$ obeys the vector equation*

$$\mathcal{V}_{12} + \mathcal{V} = \frac{f(n_1) + g(n_2)}{|\mathcal{V}_1 + \mathcal{V}_2|^2}(\mathcal{V}_1 + \mathcal{V}_2)$$

for some functions f and g or, equivalently,

$$\mathcal{V}_{12} + \mathcal{V} = H(\mathcal{V}_1 + \mathcal{V}_2), \quad \Delta_{12} \langle \mathcal{V}_{12} + \mathcal{V}, \mathcal{V}_1 + \mathcal{V}_2 \rangle = 0. \tag{4.8}$$

In particular, if the scaled normal $\mathcal{V}$ of a discrete asymptotic net F^* satisfies the constraint (4.8)$_2$*, then the associated reciprocal-parallel discrete conjugate nets F admit an infinitesimal isometric deformation of second order.*

4.4. Discussion

In 1892, Bianchi [4] observed that a conjugate net (x, y) on a surface F is preserved by a finite isometric deformation if and only if there exists a correspondence between the conjugate lines on F and the asymptotic lines on another surface F^* such that the Gauss maps N and N^* coincide and the Gaussian curvature of F^* is constrained by

$$\left(\frac{1}{\sqrt{-K^*}}\right)_{xy} = 0.$$

Based on a parameter-dependent linear representation (cf. Section 5), Bianchi [3] derived a Bäcklund transformation for the surfaces F^* which have come to be known as *Bianchi surfaces* [11, 20]. An analytic description of Bianchi surfaces is readily obtained on use of the Lelieuvre formulae [15]

$$F^*_x = \mathcal{V}_x \times \mathcal{V}, \quad F^*_y = \mathcal{V} \times \mathcal{V}_y,$$

where the scaled normal $\mathcal{V}$ obeys the Moutard equation

$$\mathcal{V}_{xy} = h\mathcal{V}. \tag{4.9}$$

The Lelieuvre formulae immediately provide an expression for the Gaussian curvature of F^*, namely $K^* = -1/|\mathcal{V}|^4$. Accordingly, the constraint which defines Bianchi surfaces may be formulated as

$$(|\mathcal{V}|^2)_{xy} = 0. \tag{4.10}$$

It is evident that the pair (4.9), (4.10) which entirely encodes Bianchi surfaces constitutes the natural continuum limit of (4.8). However, the latter has been derived in connection with *second-order* deformations while the classical differential-geometric derivation is based on *finite* deformations. This discrepancy may be partly resolved by referring to the fact that, in the classical setting, second-order isometric deformations of conjugate nets may be shown to be finite. Thus, Bianchi surfaces are, in fact, retrieved by imposing the *a priori* weaker condition of the existence of second-order deformations. Nevertheless, it is emphasized that, in general, this statement *does not* apply in the discrete context, that is, second-order isometric deformations of discrete conjugate nets are not necessarily finite.

As in the smooth setting, the discrete Bianchi system (4.8) has been shown to be integrable [26]. Second-order isometric deformations of discrete conjugate nets may therefore be considered integrable and, remarkably, any discrete conjugate net which admits a *finite* isometric deformation corresponds to a particular solution of the integrable Bianchi system (4.8).

At present, it is not known how to formulate the existence of finite isometric deformations in terms of constraints on the discrete Bianchi system. However, if we assume that the modulus of the scaled normal $\mathcal{V}$ is constant, that is,

$$|\mathcal{V}| = 1$$

without loss of generality, then the discrete Moutard equation $(4.8)_1$ reduces to

$$\mathcal{V}_{12} + \mathcal{V} = \frac{\langle \mathcal{V}, \mathcal{V}_1 + \mathcal{V}_2 \rangle}{1 + \langle \mathcal{V}_1, \mathcal{V}_2 \rangle}(\mathcal{V}_1 + \mathcal{V}_2),$$

and its algebraic consequences

$$\Delta_1 \langle \mathcal{V}_2, \mathcal{V} \rangle = 0, \quad \Delta_2 \langle \mathcal{V}_1, \mathcal{V} \rangle = 0$$

reveal that the constraint $(4.8)_2$ is identically satisfied. The discrete Lelieuvre formulae (4.6) then imply that F^* constitutes a discrete (generalized) Chebyshev net, that is,

$$\Delta_1 |F_2^* - F^*| = 0, \quad \Delta_2 |F_1^* - F^*| = 0,$$

so that discrete K-surfaces together with their reciprocal-parallel isometrically deformable discrete Voss surfaces are retrieved.

5. Integrability of finite deformations

In the preceding, it has been demonstrated that the discrete surfaces which are reciprocal-parallel to isometrically deformable conjugate nets are necessarily of discrete Bianchi type. In particular, discrete Voss surfaces correspond to integrable discrete K-surfaces. Here, we investigate in more detail the connection with the theory of integrable systems and demonstrate that the appearance of *integrable* discrete differential geometry in the context of isometric deformations is, in fact, no coincidence. Thus, it has been shown that a discrete quadrilateral surface F is isometrically deformable if and only if there exists an $SU(2)$–valued function $\phi(\epsilon)$ with $\phi(0) = \mathbb{1}$ which obeys the relations (4.1) and (4.2), that is,

$$[F_2 - F, \phi\phi_{\bar{1}}^{-1}] = 0, \quad [F_1 - F, \phi\phi_{\bar{2}}^{-1}] = 0. \tag{5.1}$$

Now, since $\phi(0) = \mathbb{1}$, the quantity

$$F^* = \phi_\epsilon(0)$$

is $su(2)$-valued and differentiation of (5.1) with respect to the deformation parameter ϵ reproduces the condition for reciprocal-parallelism

$$[F_2 - F, F^* - F_{\bar{1}}^*] = 0, \quad [F_1 - F, F^* - F_{\bar{2}}^*] = 0.$$

The pair (5.1) then implies that

$$[F_1^* - F^*, \phi_1\phi^{-1}] = 0, \quad [F_2^* - F^*, \phi_2\phi^{-1}] = 0,$$

which, in turn, shows that there exist real lattice functions $a(\epsilon), b(\epsilon)$ and $c(\epsilon), d(\epsilon)$ such that

$$\phi_1 = \mathcal{L}(\epsilon)\phi, \qquad \mathcal{L}(\epsilon) = a(\epsilon)(F_1^* - F^*) + b(\epsilon)\mathbb{1}, \tag{5.2a}$$

$$\phi_2 = \mathcal{M}(\epsilon)\phi, \qquad \mathcal{M}(\epsilon) = c(\epsilon)(F_2^* - F^*) + d(\epsilon)\mathbb{1}. \tag{5.2b}$$

Finally, the compatibility condition $\phi_{12} = \phi_{21}$ produces the *discrete zero-curvature condition*

$$\mathcal{L}_2(\epsilon)\mathcal{M}(\epsilon) = \mathcal{M}_1(\epsilon)\mathcal{L}(\epsilon). \tag{5.3}$$

The latter encodes a system of lattice equations if the $SU(2)$-valued functions $\mathcal{L}(\epsilon)$ and $\mathcal{M}(\epsilon)$ are assumed to admit the power series expansions

$$\mathcal{L}(\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \mathcal{L}^{(k)}, \quad \mathcal{M}(\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \mathcal{M}^{(k)}.$$

The above analysis shows that any isometrically deformable quadrilateral surface F gives rise to a system of *nonlinear* lattice equations encoded in the discrete zero-curvature condition (5.3) which constitutes the compatibility condition associated with the ϵ-dependent *linear* system (5.2). A parameter-dependent linear representation of the form $(5.2\text{a})_1$, $(5.2\text{b})_1$ constitutes the key ingredient in the mathematical treatment of a discrete integrable system represented by the corresponding discrete zero-curvature condition (5.3) [1]. In fact, all isometrically deformable quadrilateral surfaces are encapsulated in the following theorem.

Theorem 5.1. *Let $\mathcal{L}(\epsilon)$ and $\mathcal{M}(\epsilon)$ be two parameter-dependent $SU(2)$-valued lattice functions of the form*

$$\mathcal{L}(\epsilon) = a(\epsilon)A + b(\epsilon)\mathbb{1}, \quad \mathcal{M}(\epsilon) = c(\epsilon)B + d(\epsilon)\mathbb{1},$$

which obey the discrete zero-curvature condition

$$\mathcal{L}_2(\epsilon)\mathcal{M}(\epsilon) = \mathcal{M}_1(\epsilon)\mathcal{L}(\epsilon) \tag{5.4}$$

for some lattice functions $A, B \in su(2)$ and $a(\epsilon), b(\epsilon), c(\epsilon), d(\epsilon) \in \mathbb{R}$ with

$$a(0) = c(0) = 0, \quad b(0) = d(0) = 1 \tag{5.5}$$

and $a_\epsilon(0) \neq 0$, $c_\epsilon(0) \neq 0$. Then, there exists a unique $\phi(\epsilon) \in SU(2)$ (up to an irrelevant gauge matrix depending on ϵ) with $\phi(0) = \mathbb{1}$ which satisfies the linear system

$$\phi_1 = \mathcal{L}(\epsilon)\phi, \quad \phi_2 = \mathcal{M}(\epsilon)\phi. \tag{5.6}$$

If the quadrilateral surface

$$F^* = \phi_\epsilon(0)$$

admits a reciprocal-parallel discrete surface F, then F is isometrically deformable.

Proof. The zero-curvature condition (5.4) guarantees that the linear pair (5.6) is compatible. Its $SU(2)$-valued solution is uniquely determined by the value of $\phi(\epsilon)$ at one vertex which we may choose to be $\mathbb{1}$ without loss of generality. The constraints (5.5) imply that

$\mathcal{L}(0) = \mathcal{M}(0) = \mathbb{1}$ so that $\phi(0) = \mathbb{1}$ everywhere and hence $F^* = \phi_\epsilon(0) \in su(2)$. Differentiation of (5.6) and evaluation at $\epsilon = 0$ produce

$$F_1^* - F^* = a_\epsilon(0)A + b_\epsilon(0), \quad F_2^* - F^* = c_\epsilon(0)B + d_\epsilon(0),$$

wherein $b_\epsilon(0) = d_\epsilon(0) = 0$ since $F^* \in su(2)$. Accordingly, the pair (5.6) gives rise to the commutator relations

$$[F_1^* - F^*, \phi_1\phi^{-1}] = 0, \quad [F_2^* - F^*, \phi_2\phi^{-1}] = 0,$$

which are equivalent to the conditions (5.1) for the existence of an isometric deformation of a reciprocal-parallel surface F. □

5.1. Finite deformations of discrete conjugate nets

In general, there is no guarantee that the quadrilateral surface F^* in Theorem 5.1 admits a reciprocal-parallel counterpart F. This property must therefore be regarded as a constraint on the $SU(2)$-valued lattice functions $\mathcal{L}(\epsilon)$ and $\mathcal{M}(\epsilon)$. In the case of isometric deformations of discrete conjugate nets, this constraint may be implemented explicitly since the quantities A and B may be expressed in terms of the scaled normal $\mathcal{V}$ by virtue of the discrete Lelieuvre formulae (4.6) and the representation (5.2) of $\mathcal{L}(\epsilon)$ and $\mathcal{M}(\epsilon)$. Accordingly, Theorem 5.1 may be simplified and brought into the following form:

Theorem 5.2. *Let $\mathcal{A}(\epsilon)$ and $\mathcal{B}(\epsilon)$ be two parameter-dependent matrix-valued lattice functions of the form*

$$\mathcal{A}(\epsilon) = \alpha(\epsilon)S_1S + \beta(\epsilon)\mathbb{1}, \quad \mathcal{B}(\epsilon) = \gamma(\epsilon)S_2S + \delta(\epsilon)\mathbb{1},$$

which obey the discrete zero-curvature condition

$$\mathcal{A}_2(\epsilon)\mathcal{B}(\epsilon) = \mathcal{B}_1(\epsilon)\mathcal{A}(\epsilon) \tag{5.7}$$

for some lattice functions $S \in su(2)$ and $\alpha(\epsilon), \beta(\epsilon), \gamma(\epsilon), \delta(\epsilon) \in \mathbb{R}$ with

$$\alpha(0) = \gamma(0) = 0$$

and the inequalities

$$\beta(0) > 0, \quad \delta(0) > 0, \quad \alpha_\epsilon(0) \neq 0, \quad \gamma_\epsilon(0) \neq 0, \quad \det\mathcal{A}(\epsilon) > 0, \quad \det\mathcal{B}(\epsilon) > 0.$$

Then, the determinants of $\mathcal{A}(\epsilon)$ and $\mathcal{B}(\epsilon)$ may be parametrized according to

$$\det\mathcal{A}(\epsilon) = \frac{\tau_1^2(\epsilon)}{\tau^2(\epsilon)}, \quad \det\mathcal{B}(\epsilon) = \frac{\tau_2^2(\epsilon)}{\tau^2(\epsilon)}, \quad \tau(\epsilon) > 0,$$

and the linear system

$$\psi_1 = \mathcal{A}(\epsilon)\psi, \quad \psi_2 = \mathcal{B}(\epsilon)\psi \tag{5.8}$$

admits a unique solution (up to an irrelevant gauge matrix depending on ϵ) with

$$\phi(\epsilon) := \frac{\psi(\epsilon)}{\tau(\epsilon)} \in SU(2), \quad \phi(0) = \mathbb{1}.$$

The quadrilateral surface

$$F^* = \phi_\epsilon(0)$$

is discrete asymptotic and its reciprocal-parallel counterparts F constitute isometrically deformable discrete conjugate nets with S being a common normal.

The above theorem states that all isometrically deformable discrete conjugate nets are enshrined in the discrete zero-curvature condition (5.7). Before we analyse this condition in more detail, we present the proof and an illustration of Theorem 5.2.

Proof. It is readily verified that

$$\mathcal{A}^{\dagger}\mathcal{A} = \mathbb{1}\det\mathcal{A}, \quad \mathcal{B}^{\dagger}\mathcal{B} = \mathbb{1}\det\mathcal{B},$$

where

$$\det\mathcal{A} = \alpha^2|S_1|^2|S|^2 - 2\alpha\beta\,\langle S_1, S\rangle + \beta^2 \tag{5.9a}$$

$$\det\mathcal{B} = \beta^2|S_2|^2|S|^2 - 2\gamma\delta\,\langle S_2, S\rangle + \delta^2. \tag{5.9b}$$

The discrete zero-curvature condition implies that

$$\det\mathcal{A}_2\det\mathcal{B} = \det\mathcal{B}_1\det\mathcal{A},$$

whence there exists a positive lattice function $\tau(\epsilon)$ such that

$$\det\mathcal{A}(\epsilon) = \frac{\tau_1^2(\epsilon)}{\tau^2(\epsilon)}, \quad \det\mathcal{B}(\epsilon) = \frac{\tau_2^2(\epsilon)}{\tau^2(\epsilon)}.$$

Thus, if we introduce the quantities

$$\phi(\epsilon) = \frac{\psi(\epsilon)}{\tau(\epsilon)}, \quad \mathcal{L}(\epsilon) = \frac{\tau(\epsilon)}{\tau_1(\epsilon)}\mathcal{A}(\epsilon), \quad \mathcal{M}(\epsilon) = \frac{\tau(\epsilon)}{\tau_2(\epsilon)}\mathcal{B}(\epsilon),$$

then $\mathcal{L}(\epsilon), \mathcal{M}(\epsilon) \in SU(2)$ and the linear system (5.8) becomes

$$\phi_1 = \mathcal{L}(\epsilon)\phi, \quad \phi_2 = \mathcal{M}(\epsilon)\phi, \tag{5.10}$$

so that Theorem 5.1 applies. Indeed, evaluation of the identities (5.9) at $\epsilon = 0$ reveals that

$$\beta(0) = \frac{\tau_1(0)}{\tau(0)}, \quad \delta(0) = \frac{\tau_2(0)}{\tau(0)},$$

and hence $\mathcal{L}(0) = \mathcal{M}(0) = \mathbb{1}$. It therefore remains to show that the quadrilateral surface $F^* = \phi_\epsilon(0)$ is discrete asymptotic. To this end, it is observed that differentiation of (5.10) yields

$$\phi_{1\epsilon}(0) = \frac{\alpha_\epsilon(0)}{\beta(0)}S_1 S + \left(\beta(\epsilon)\frac{\tau(\epsilon)}{\tau_1(\epsilon)}\right)_\epsilon\Bigg|_{\epsilon=0}\mathbb{1} + \phi_\epsilon(0)$$

and an analogous expression for $\phi_{2\epsilon}(0)$. Since $F^* = \phi_\epsilon(0) \in su(2)$, we conclude that

$$F_1^* - F^* = \frac{\alpha_\epsilon(0)}{\beta(0)}S_1 \times S, \quad F_2^* - F^* = \frac{\gamma_\epsilon(0)}{\delta(0)}S_2 \times S,$$

so that S is indeed orthogonal to the stars of the quadrilateral surface F^*. This concludes the proof since, according to Theorem 2.7, any (non-degenerate) discrete asymptotic net F^* admits an infinite number of reciprocal-parallel discrete conjugate nets F. □

5.2. Finite deformations of discrete Voss surfaces

As an illustration of Theorem 5.2, we make the choice

$$\alpha = \epsilon, \quad \beta = 1, \quad \gamma = -\epsilon, \quad \delta = 1, \quad S^2 = -\mathbb{1} \Leftrightarrow |S| = 1.$$

In this case, the discrete zero-curvature condition associated with the linear system

$$\psi_1 = (\epsilon S_1 S + \mathbb{1})\psi, \quad \psi_2 = (-\epsilon S_2 S + \mathbb{1})\psi$$

reduces to

$$\epsilon[S_{12}(S_1 + S_2) - (S_1 + S_2)S] = 0$$

or, equivalently,

$$(S_{12} + S) \times (S_1 + S_2) = 0, \quad \langle S_{12} - S, S_1 + S_2 \rangle = 0. \tag{5.11}$$

The former relation is equivalent to the discrete Moutard equation

$$S_{12} + S = H(S_1 + S_2)$$

in which the function H is determined by the consistency condition $|S_{12}| = 1$, leading to

$$S_{12} + S = \frac{\langle S, S_1 + S_2 \rangle}{1 + \langle S_1, S_2 \rangle}(S_1 + S_2). \tag{5.12}$$

The algebraic consequences

$$\Delta_1 \langle S_2, S \rangle = 0, \quad \Delta_2 \langle S_1, S \rangle = 0$$

then show that the additional requirement $(5.11)_2$ is identically satisfied. Thus, S represents the Gauss map of discrete K-surfaces as discussed in Section 4.4 and the isometrically deformable reciprocal-parallel discrete surfaces are therefore of discrete Voss type.

The above example highlights the geometric interpretation of the 'eigenfunction' $\phi(\epsilon)$ in the sense of soliton theory as the rotational component of the rigid motion undergone by the quadrilaterals during an isometric deformation. Here, the deformation parameter ϵ plays the role of the 'spectral parameter' [1].

5.3. A discrete nonlinear σ-model

Here, we embark on an analysis of the discrete zero-curvature condition (5.7) given by

$$(\alpha_2 S_{12} S_2 + \beta_2 \mathbb{1})(\gamma S_2 S + \delta \mathbb{1}) = (\gamma_1 S_{12} S_1 + \delta_1 \mathbb{1})(\alpha S_1 S + \beta \mathbb{1}).$$

Therein, we may assume without loss of generality that the normal S obeys the discrete Moutard equation

$$S_{12} + S = H(S_1 + S_2) \tag{5.13}$$

by virtue of Theorems 4.1 and 5.2. Decomposition of the discrete zero-curvature condition into its trace and trace-free parts yields

$$\begin{aligned} &\alpha_2 \gamma |S_2|^2 \langle S_{12}, S \rangle - \alpha_2 \delta \langle S_{12}, S_2 \rangle - \beta_2 \gamma \langle S_2, S \rangle + \beta_2 \delta \\ &= \gamma_1 \alpha |S_1|^2 \langle S_{12}, S \rangle - \gamma_1 \beta \langle S_{12}, S_1 \rangle - \delta_1 \alpha \langle S_1, S \rangle + \delta_1 \beta \end{aligned}$$

and

$$\begin{aligned}&\alpha_2\gamma|S_2|^2 S_{12}\times S-\alpha_2\delta S_{12}\times S_2-\beta_2\gamma S_2\times S\\&=\gamma_1\alpha|S_1|^2 S_{12}\times S-\gamma_1\beta S_{12}\times S_1-\delta_1\alpha S_1\times S.\end{aligned}$$

The inner product of the latter with S and S_1+S_2, respectively, gives rise to

$$\alpha_2\delta=-\gamma_1\beta,\quad \beta_2\gamma=-\delta_1\alpha, \tag{5.14}$$

so that the above relations reduce to

$$H(\alpha_2\gamma|S_2|^2-\gamma_1\alpha|S_1|^2)=\alpha_2\delta+\beta_2\gamma, \tag{5.15a}$$

$$\frac{1}{H}(\alpha_2\delta|S_{12}|^2+\beta_2\gamma|S^2|)=\beta_2\delta-\delta_1\beta. \tag{5.15b}$$

Now, on the one hand, it is readily verified that the linear system (5.8) is invariant under

$$\psi\to\mu\psi,\quad (\alpha,\beta)\to\frac{\mu_1}{\mu}(\alpha,\beta),\quad (\gamma,\delta)\to\frac{\mu_2}{\mu}(\gamma,\delta).$$

On the other hand, the quadratic relations (5.14) guarantee that there exist ‘potentials’ μ and ν such that

$$\mu_1=\alpha\nu,\quad \mu_2=-\gamma\nu,\quad \nu_1=\beta\mu,\quad \nu_2=\delta\mu.$$

Without loss of generality, we may therefore introduce the parametrization

$$\alpha=\frac{\mu}{\nu},\quad \beta=\frac{\nu_1}{\mu_1},\quad \gamma=-\frac{\mu}{\nu},\quad \delta=\frac{\nu_2}{\mu_2}.$$

Finally, if we set

$$\sigma=\frac{\nu}{\mu},$$

then the relations (5.15) adopt the form

$$\sigma_{12}-\sigma=H\left(\frac{|S_2|^2}{\sigma_2}-\frac{|S_1|^2}{\sigma_1}\right),\quad \sigma_2-\sigma_1=\frac{1}{H}\left(\frac{|S_{12}|^2}{\sigma_{12}}-\frac{|S|^2}{\sigma}\right). \tag{5.16}$$

Thus, it is required to determine all one-parameter (ϵ) families of solutions of the coupled system (5.13), (5.16) which are such that S is *independent* of ϵ. It is observed that the pair (5.16) may be brought into the form

$$\Delta_2\left(2\langle S_1,S\rangle-\sigma_1\sigma-\frac{|S_1|^2}{\sigma_1}\frac{|S|^2}{\sigma}\right)=0,\quad \Delta_1\left(2\langle S_2,S\rangle+\sigma_2\sigma+\frac{|S_2|^2}{\sigma_2}\frac{|S|^2}{\sigma}\right)=0,$$

which provides two first integrals.

In order to proceed, it turns out convenient to define the quantity

$$\tau=\frac{|S|^2}{\sigma}.$$

The system (5.13), (5.16) is then equivalent to the *linear* system

$$\begin{aligned}S_{12}+S&=H(S_1+S_2),\\ \sigma_{12}-\sigma&=H(\tau_2-\tau_1),\\ \tau_{12}-\tau&=H(\sigma_2-\sigma_1),\end{aligned}$$

subject to the *nonlinear* constraint

$$\sigma\tau = |S|^2.$$

It is noted in passing that the choice

$$\sigma = \frac{1}{f(\epsilon)}, \quad \tau = f(\epsilon)$$

reduces this system to the Gauss map (5.12) of discrete K-surfaces corresponding to isometrically deformable discrete Voss surfaces. In general, the change of variables

$$A = (-1)^{n_1}\frac{\sigma - \tau}{2}, \quad B = (-1)^{n_2}\frac{\sigma + \tau}{2}, \quad \mathsf{V} = \begin{pmatrix} S \\ A \\ B \end{pmatrix}$$

gives rise to the vector equation

$$\mathsf{V}_{12} + \mathsf{V} = H(\mathsf{V}_1 + \mathsf{V}_2), \quad \langle \mathsf{V}, \mathsf{V}\rangle = 0,$$

where the inner product is defined by

$$\langle \mathsf{V}, \tilde{\mathsf{V}}\rangle = \langle S, \tilde{S}\rangle + A\tilde{A} - B\tilde{B}.$$

Since the function H is determined by the requirement that V constitutes a null-vector, discrete conjugate nets which are isometrically deformable are encapsulated in the following theorem:

Theorem 5.3. *Isometrically deformable discrete conjugate nets are encoded in one-parameter (ϵ) families of solutions of the vector equation*

$$\mathsf{V}_{12} + \mathsf{V} = \frac{\langle \mathsf{V}, \mathsf{V}_1 + \mathsf{V}_2\rangle}{\langle \mathsf{V}_1, \mathsf{V}_2\rangle}(\mathsf{V}_1 + \mathsf{V}_2), \quad \langle \mathsf{V}, \mathsf{V}\rangle = 0,$$

which are such that the first three components of V *are* independent *of* ϵ. *Here, the inner product is taken with respect to the metric* $\mathrm{diag}(1, 1, 1, 1, -1)$,

Remarkably, the above vector equation constitutes the standard integrable discretization of a particular *nonlinear σ-model* (see, e.g., [8, 26]). The complete characterization of its families of solutions which admit the required dependence on a parameter ϵ and are compatible with the technical assumptions made in Theorem 5.2 is the subject of ongoing research.

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