Russian Math. Surveys **62**:1 1–43

Uspekhi Mat. Nauk **62**:1 3–50
DOI 10.1070/RM2007v062n01ABEH00????

On organizing principles of discrete differential geometry. Geometry of spheres

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Abstract. Discrete differential geometry aims to develop discrete equivalents of the geometric notions and methods of classical differential geometry. This survey contains a discussion of the following two fundamental *discretization principles*: the transformation group principle (smooth geometric objects and their discretizations are invariant with respect to the same transformation group) and the consistency principle (discretizations of smooth parametrized geometries can be extended to multidimensional consistent nets). The main concrete geometric problem treated here is discretization of curvature-line parametrized surfaces in Lie geometry. Systematic use of the discretization principles leads to a discretization of curvature-line parametrization which unifies circular and conical nets.

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This research was supported by DFG Research Unit 565 "Polyhedral Surfaces" and the DFG Research Centre MATHEON "Mathematics for Key Technologies", Berlin.

AMS 2000 *Mathematics Subject Classification*. Primary 53Axx, 51Bxx; Secondary 39A12, 52Cxx.

§ 1. Introduction

The new field of discrete differential geometry is presently emerging on the border between differential and discrete geometry. Whereas classical differential geometry investigates smooth geometric shapes, discrete differential geometry studies geometric shapes with finite numbers of elements and aims to develop discrete equivalents of the geometric notions and methods of classical differential geometry. The latter appears then as a limit of refinements of the discretization. Current interest in this field derives not only from its importance in pure mathematics but also from its relevance for computer graphics. An important example one should keep in mind here is that of polyhedral surfaces approximating smooth surfaces.

One may suggest many different reasonable discretizations with the same smooth limit. Which one is the best? From the theoretical point of view the best discretization is the one which preserves all the fundamental properties of the smooth theory. Often such a discretization clarifies the structures of the smooth theory and possesses important connections to other areas of mathematics (projective geometry, integrable systems, algebraic geometry, complex analysis, and so on). On the other hand, for applications the crucial point is the approximation: the best discretization should possess good convergence properties and should represent a smooth shape by a discrete shape with just a few elements. Although these theoretical and applied criteria for the best discretization are completely different, in many cases natural 'theoretical' discretizations turn out to possess remarkable approximation properties and are very useful for applications [1], [2].

This interaction of the discrete and smooth versions of the theory led to important results in the surface theory as well as in the geometry of polyhedra. The fundamental results of A. D. Alexandrov and A. V. Pogorelov on the metric geometry of polyhedra and convex surfaces are classical achievements of discrete differential geometry: for example, Alexandrov's theorem [3] states that any abstract convex polyhedral metric is uniquely realized by a convex polyhedron in Euclidean 3-space, and Pogorelov proved [4] the corresponding existence and uniqueness result for abstract convex metrics by approximating smooth surfaces by polyhedra.

Simplicial surfaces, that is, discrete surfaces made from triangles, are basic in computer graphics. This class of discrete surfaces, however, is too unstructured for analytical investigation. An important tool in the theory of smooth surfaces is the introduction of (special) parametrizations of a surface. Natural analogues of parametrized surfaces are *quadrilateral surfaces*, that is, discrete surfaces made from (not necessarily planar) quadrilaterals. The strips of quadrilaterals obtained by gluing quadrilaterals along opposite edges are analogues of coordinate lines. Probably the first non-trivial example of quadrilateral surfaces studied this way are the discrete surfaces with constant negative Gaussian curvature introduced by Sauer and Wunderlich [5], [6]. Discrete parametrized surfaces are currently becoming more important in computer graphics. They lead to meshes that better represent the shape of a surface and look regular [7]–[9], [2].

It is well known that differential equations describing interesting special classes of surfaces and parametrizations are integrable (in the sense of the theory of integrable systems), and conversely, many interesting integrable systems admit a differential-geometric interpretation. Progress in understanding the unifying fundamental structure the classical differential geometers were looking for, and simultaneously in understanding the very nature of integrability, came from efforts to discretize these theories. It turns out that many sophisticated properties of differential-geometric objects find their simple explanation within discrete differential geometry. The early period of this development is documented in the work of Sauer (see [10]). The modern period began with the work of Bobenko and Pinkall [11], [12], and of Doliwa and Santini [13], [14]. A closely related development of the spectral theory of difference operators on graphs was initiated by Novikov and collaborators [15]–[17]; see also [18] for a further development of discrete complex analysis on simplicial manifolds.

Discrete surfaces in Euclidean 3-space serve as a basic example in this survey. This case has all the essential features of the theory in all its generality, and generalizations to higher dimensions are straightforward. On the other hand, our three-dimensional geometric intuition helps to understand their properties.

Discrete differential geometry relating to integrable systems deals with multidimensional discrete nets, that is, maps from the regular cubic lattice $\mathbb{Z}^m$ into $\mathbb{R}^N$ specified by certain geometric properties (as mentioned above, we will be most interested in the case $N = 3$ in this survey). In this setting, discrete surfaces appear as two-dimensional layers of multidimensional discrete nets, and their transformations correspond to shifts in the transversal lattice directions. A characteristic feature of the theory is that all lattice directions are on equal footing with respect to the defining geometric properties. Discrete surfaces and their transformations become indistinguishable. We associate such a situation with *multidimensional consistency*, and this is one of our fundamental discretization principles. Multidimensional consistency, and therefore the existence and construction of multidimensional nets, are based on certain incidence theorems of elementary geometry.

Conceptually one can think of passing to a continuum limit by refining the mesh size in some of the lattice directions. In these directions the net converges to smooth surfaces, whereas those directions that remain discrete correspond to transformations of the surfaces (see Figure 1).

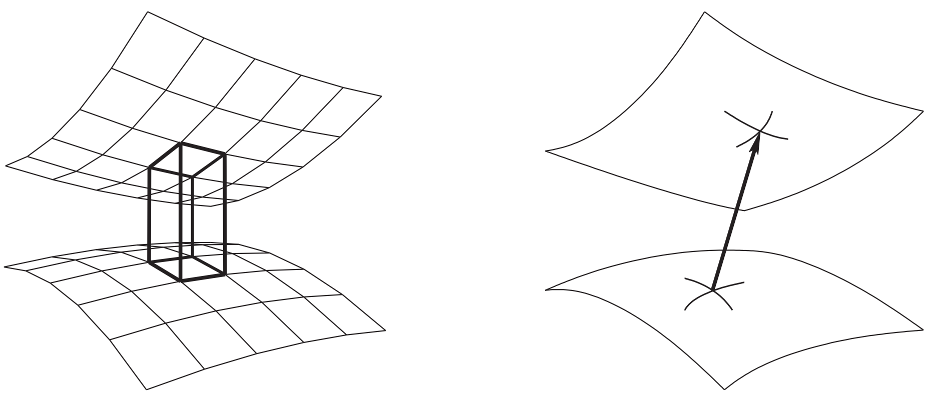

Figure 1. From the discrete master theory to the classical theory: surfaces and their transformations appear by refining two of three net directions.

The smooth theory comes as a corollary of a more fundamental discrete master theory. The true roots of the classical surface theory are found, quite unexpectedly, in various incidence theorems of elementary geometry. This phenomenon, which has been shown for many classes of surfaces and coordinate systems [19], [20], is currently becoming accepted as one of the fundamental features of classical integrable differential geometry.

We remark that finding simple discrete explanations for complicated differential geometric theories is not the only outcome of this development. Having identified the roots of integrable differential geometry in multidimensional consistency of discrete nets, we are led to a new (geometric) understanding of the integrability itself [21], [22], [20].

The simplest and at the same time the basic example of multidimensional consistent nets is multidimensional Q-nets [13], or discrete conjugate nets [10], which are characterized by planarity of all quadrilaterals. The planarity property is preserved by projective transformations, and thus Q-nets are a subject of projective geometry (like conjugate nets, which are smooth counterparts of Q-nets).

Here we come to the next basic discretization principle. According to Klein's Erlangen program, geometries are classified by their transformation groups. Classical examples are projective, affine, Euclidean, spherical, and hyperbolic geometries, and the sphere geometries of Lie, Möbius, and Laguerre. We postulate that the transformation group, as the most fundamental feature, should be preserved by a discretization. This can be seen as a sort of *discrete Erlangen program.*

Thus we come to the following fundamental principles.

Discretization Principles:

1. *Transformation group principle*: smooth geometric objects and their discretizations belong to the same geometry, that is, are invariant with respect to the same transformation group.
2. *Multidimensional consistency principle*: discretizations of surfaces, coordinate systems, and other smooth parametrized objects can be extended to multidimensional consistent nets.

Let us explain why such different imperatives as the transformation group principle and the consistency principle can be simultaneously imposed for discretization of classical geometries. The transformation groups of various geometries, including those of Lie, Möbius, and Laguerre, are subgroups of the projective transformation group. Classically, such a subgroup is described as consisting of projective transformations which preserve some distinguished quadric called the absolute. A remarkable result by Doliwa [23] is that multidimensional Q-nets can be restricted to an arbitrary quadric. This is the reason why the discretization principles work for the classical geometries.

In this survey we deal with three classical geometries described in terms of spheres: the Möbius, Laguerre, and Lie geometries. They were developed by classical scholars of geometry. The most elaborate presentation of these geometries can be found in Blaschke's book [24].

Möbius geometry is the most popular one of these three geometries. It describes properties invariant with respect to Möbius transformations, which are compositions of reflections in spheres. For $N \geqslant 3$ the Möbius transformations of $\mathbb{R}^N$ coincide with conformal transformations. Möbius geometry does not distinguish between spheres and planes (planes are regarded as spheres through the infinitely remote point ∞, which compactifies $\mathbb{R}^N$ to form the N-sphere $\mathbb{S}^N$). On the other hand, points are regarded as objects different from spheres. Surfaces are described in terms of their points. Classical examples of Möbius-geometric properties of surfaces are conformal parametrization and the Willmore functional [25]. Recent progress in this field is to a large extent due to interrelations with the theory of integrable systems [26], [27].

Laguerre geometry does not distinguish between points and spheres (points are treated as spheres of zero radius). On the other hand, planes are regarded as independent elements. Surfaces are described in terms of their tangent planes. A particular Laguerre transformation of a surface is a shift of all tangent planes in the normal direction by a constant distance. This transformation is called a normal shift.

Lie geometry is a natural unification of Möbius and Laguerre geometries: points, planes, and spheres are treated on an equal footing. The transformation group is generated by Möbius transformations and normal shift transformations. Surfaces are described in terms of their contact elements. A contact element can be understood as a surface point together with the corresponding tangent plane. The one-parameter family of spheres through a given point and with a common tangent plane at the point gives an invariant Lie-geometric description of a contact element. The point of the surface and the tangent plane at this point are just two elements of this family.

Some integrability aspects of surface theory in Lie geometry have been studied by Ferapontov [28], [29], Musso and Nicolodi [30], and Burstall and Hertrich-Jeromin [31], [32].

The main concrete geometric problem discussed in this survey is a discretization of curvature-line parametrized surfaces. Curvature lines are integral curves of the principal directions. Any surface away from its umbilic points can be parametrized by curvature lines. Curvature-line parametrization has been attracting the attention of mathematicians and physicists for two centuries. The classical results in this area can be found in books by Darboux [33], [34] and Bianchi [35]. In particular, a classical result of Dupin [36] asserts that the coordinate surfaces of triply orthogonal coordinate systems intersect along their common curvature lines. A. Ribaucour discovered a transformation of surfaces preserving the curvature-line parametrization (see [37]). A surface and its Ribaucour transform envelope a special sphere congruence. Bianchi showed [38] that Ribaucour transformations are permutable: for any two Ribaucour transforms of a surface there exists a one-parameter family of their common Ribaucour transforms (see also [39], [40]). Ganzha and Tsarev [41] established a three-dimensional non-linear superposition principle for Ribaucour transformations of triply orthogonal coordinate systems.

Curvature-line parametrizations and orthogonal systems have recently come back into the focus of interest in mathematical physics as examples of an integrable system. Zakharov [42] has constructed a variety of explicit solutions with the help of the dressing method. Algebro-geometric orthogonal coordinate systems were constructed by Krichever [43]. The recent interest in this problem is motivated, in particular, by applications to the theory of the associativity equations developed by

Dubrovin [44]. Remarkable geometric properties make curvature-line parametrizations especially useful for visualization of surfaces in computer graphics [7], [2].

The question of proper discretization of curvature-line parametrized surfaces and orthogonal systems has become a subject of intensive study in the past few years. *Circular nets*, which are Q-nets with circular quadrilaterals, were mentioned as discrete analogues of curvature-line parametrized surfaces by Nutbourne and Martin [45]. Special circular nets as discrete isothermic surfaces were investigated in [12]. Circular discretization of triply orthogonal coordinate systems was first proposed in [46]. Doliwa and Santini [13] took the next crucial step in the development of the theory. They considered discrete orthogonal systems as a reduction of discrete conjugate systems [14], generalized them to arbitrary dimension, and proved their multidimensional consistency based on the classical Miquel theorem [47].

Matthes and the authors of this survey proved [48] that circular nets approximate smooth curvature-line parametrized surfaces and orthogonal systems with all derivatives. Numerical experiments show that circular nets have the desired geometric properties already at a coarse level and not just in the refinement limit upon convergence to a smooth curvature-line parameterized surface. This is important for applications in computer graphics [2].

A convenient analytic description of circular nets was given by Konopelchenko and Schief [49]. Analytic methods of soliton theory have been applied to circular nets by Doliwa, Manakov, and Santini [50] ($\overline{\partial}$-method) and by Akhmetshin, Vol'vovskii, and Krichever [51] (algebro-geometric solutions). A Clifford algebra description of circular nets was given by Bobenko and Hertrich-Jeromin [52].

Circular nets are preserved by Möbius transformations, and thus should be treated as a discretization of curvature-line parametrizations in Möbius geometry. A recent development by Liu, Pottmann, Wallner, Yang, and Wang [2] is the introduction of *conical nets*, which should be treated as a discretization of curvature-line parametrizations in Laguerre geometry. These are special Q-nets characterized by the property that four quadrilaterals meeting at a vertex are tangent to a common cone of revolution. Equivalently, conical nets can be characterized as Q-nets with circular Gauss maps, that is, the unit normals to the quadrilaterals comprise a circular net on the unit sphere S^2. Circular Gauss maps defined at vertices of a given circular net were previously introduced by Schief [53], [54], but without any connection with conical nets. Conical nets, like circular nets, satisfy the second discretization principle (consistency). In the present survey we find a discretization of a curvature-line parametrization which unifies the theory of circular and conical nets by systematically applying the discretization principles.

It is well known that curvature lines are a subject of Lie geometry, that is, are invariant with respect to Möbius transformations and normal shifts. To see this, consider an infinitesimal neighbourhood U of a point x of an oriented smooth surface in $\mathbb{R}^3$, and the pencil of spheres $S(r)$ of signed radii r (with the sign corresponding to the orientation) tangent to the surface at x (see Figure 2). The signed radius r is assumed positive if $S(r)$ lies on the same side of the tangent plane as the normal n, and negative otherwise; $S(\infty)$ is the tangent plane. For small $r_0 > 0$ the spheres $S(r_0)$ and $S(-r_0)$ intersect U only at x. The set of the tangent spheres with this property (intersecting U only at x) has two connected components: M_+

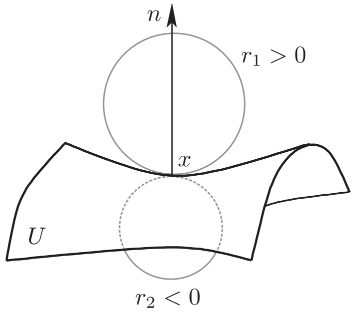

Figure 2. Principal directions in terms of touching spheres

containing $S(r_0)$ and M_- containing $S(-r_0)$ for small $r_0 > 0$. The boundary values

$$r_1 = \sup\{r : S(r) \in M_+\}, \qquad r_2 = \inf\{r : S(r) \in M_-\}$$

give the principal curvatures $k_1 = 1/r_1$ and $k_2 = 1/r_2$ of the surface at x. The directions in which $S(r_1)$ and $S(r_2)$ are tangent to U are the principal directions.

Clearly, all ingredients of this description are Möbius-invariant. Under a normal shift by the distance d the centres of the principal-curvature spheres are preserved and their radii are shifted by d. This implies that the principal directions and thus the curvature lines are also preserved under normal shifts.

The Lie-geometric nature of the curvature-line parametrization means that it has a Lie-invariant description. Such a description can be found in Blaschke's book [11]. A surface in Lie geometry, as already noted, is regarded as consisting of contact elements. Two infinitesimally close contact elements (sphere pencils) belong to the same curvature line if and only if they have a sphere in common, which is the principal-curvature sphere.

By a literal discretization of Blaschke's Lie-geometric description of smooth curvature-line parametrized surfaces, we define a discrete *principal contact-element net* as a map $\mathbb{Z}^2 \to \{\text{contact elements of surfaces in } \mathbb{R}^3\}$ such that any two neighbouring contact elements have a sphere in common.

In the projective model of Lie geometry spheres in $\mathbb{R}^3$ (including points and planes) are represented by elements of the so-called Lie quadric $\mathbb{L} \subset \mathbb{RP}^5$, contact elements are represented by isotropic lines, that is, lines in $\mathbb{L}$, and surfaces are represented by congruences of isotropic lines. In the curvature-line parametrization the parametric families of isotropic lines form developable surfaces in $\mathbb{L}$.

Accordingly, a discrete principal contact-element net in the projective model of Lie geometry is a discrete congruence of isotropic lines

$$\ell \colon \mathbb{Z}^2 \to \{\text{isotropic lines in } \mathbb{L}\}$$

such that any two neighbouring lines intersect. Intersection points of neighbouring lines correspond, as in the smooth case, to principal-curvature spheres. They are associated with edges of $\mathbb{Z}^2$. Four principal-curvature spheres corresponding to edges with a common vertex belong to the same contact element, that is, have a common tangent point.

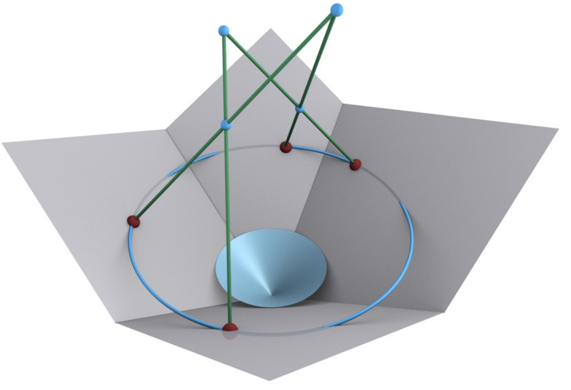

Figure 3. Geometry of principal contact-element nets. Four neighbouring contact elements are represented by points and (tangent) planes. The points lie on a circle, and the planes are tangent to a cone of revolution. Neighboring normal lines intersect at the centres of the principal-curvature spheres.

In projective geometry, discrete line congruences were introduced by Doliwa, Santini, and Mañas [55]. Discrete line congruences are closely related to Q-nets, and, like the latter, are multidimensionally consistent. It follows from our results that they can be restricted to the Lie quadric (actually, to any ruled quadric). Thus, principal contact-element nets satisfy the second discretization principle. In particular, this yields discrete Ribaucour transformations between principal contact-element nets.

The Lie-geometric notion of discrete principal contact-element nets unifies the Möbius-geometric notion (circular nets) and the Laguerre-geometric notion (conical nets). Indeed, any contact element ℓ contains a point x and a plane P. It turns out that for a discrete surface

$$\ell\colon \mathbb{Z}^2 \to \{\text{isotropic lines in } \mathbb{L}\} = \{\text{contact elements in } \mathbb{R}^3\},$$

the points form a circular net

$$x\colon \mathbb{Z}^2 \to \mathbb{R}^3,$$

whereas the planes form a conical net

$$P\colon \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}.$$

The corresponding geometry is depicted in Figure 3.

Schematically, this Lie-geometric merging of the Möbius- and Laguerre-geometric notions is presented in Figure 4.

This survey is organized as follows. In § 2 we start with a review of the basic multidimensionally consistent systems — the Q-nets and discrete line congruences. The basic notions of Lie, Möbius, and Laguerre geometries are briefly presented in § 3. In § 4 are mainly new results on discrete curvature-line parametrized surfaces: the Lie-geometric Definitions 18 and 19 and Theorem 32, which describes interrelations

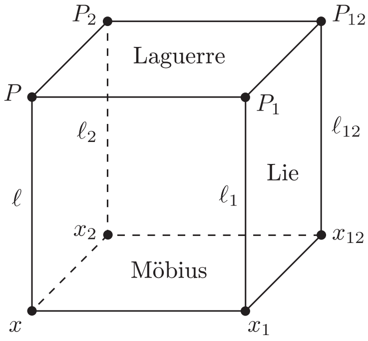

Figure 4. Geometry of principal contact-element nets. Four neighbouring contact elements ℓ, ℓ_1, ℓ_2, ℓ_{12} produce a hexahedron with vertices in the Lie quadric $\mathbb{L}$ and with planar faces. The bottom quadrilateral is the intersection of the three-dimensional space $V = \operatorname{span}(\ell, \ell_1, \ell_2, \ell_{12})$ with the 4-space in $\mathbb{RP}^5$ representing points in $\mathbb{R}^3$. The top quadrilateral is the intersection of V with the 4-space in $\mathbb{RP}^5$ representing planes in $\mathbb{R}^3$. Each side quadrilateral lies in the plane of two intersecting lines $\ell \subset \mathbb{L}$.

of discrete curvature-line nets in Lie, Möbius, and Laguerre geometries. A geometric characterization of Ribaucour transformations and discrete R-congruences of spheres as Q-nets in the Lie quadric is given in § 5.

Let us note that, in view of Lie's classical sphere-line correspondence (see [24]), the Lie-geometric theory presented in this survey can be carried over to the context of projective line geometry in three-space: the Lie quadric is replaced by the Plücker quadric, and curvature lines and R-congruences of spheres correspond to asymptotic lines and W-congruences of lines, respectively. The projective theory of discrete asymptotic nets was developed by Doliwa [56].

Our research in discrete differential Lie geometry has been stimulated by the recent introduction of conical nets by Liu, Pottmann, Wallner, Yang, and Wang [2]. The advent of a second (after circular nets) discretization of curvature-line parametrizations posed the question of the relation between the different discretizations. A connection between circular and conical nets was found independently by Pottmann [57]. We are grateful to H. Pottmann and J. Wallner for numerous communications on conical nets and for providing us with their unpublished results. We also thank U. Pinkall for useful discussions.

§ 2. Multidimensional consistency as a discretization principle

2.1. Q-nets. We use the following standard notation: for a function f on $\mathbb{Z}^m$ we write

$$\tau_i f(u) = f(u + e_i),$$

where e_i is the unit vector of the i-th coordinate direction, $1 \leqslant i \leqslant m$. We also use the shortcut notations f_i for $\tau_i f$, f_{ij} for $\tau_i \tau_j f$, and so on.

The most general of the known discrete 3D systems possessing the property of 4D consistency are nets consisting of planar quadrilaterals, or Q-nets. Two-dimensional

Q-nets were introduced by Sauer in [10], and a multidimensional generalization was given by Doliwa and Santini in [13]. Our presentation in this section follows the latter paper. The fundamental importance of multidimensional consistency of discrete systems as their integrability has been put forward by the authors [21], [22], [20].

Definition 1 (Q-net). A map $f\colon \mathbb{Z}^m \to \mathbb{RP}^N$ is called an m-dimensional Q-net (quadrilateral net, or discrete conjugate net) in $\mathbb{RP}^N$ $(N \geqslant 3)$ if all its elementary quadrilaterals (f, f_i, f_{ij}, f_j) (for any $u \in \mathbb{Z}^m$ and for all pairs $1 \leqslant i \neq j \leqslant m$) are planar.

Thus, for any elementary quadrilateral, any representatives $\widetilde{f}$, $\widetilde{f}_i$, $\widetilde{f}_j$, $\widetilde{f}_{ij}$ of its vertices in the space $\mathbb{R}^{N+1}$ of homogeneous coordinates satisfy an equation of the type

$$\widetilde{f}_{ij} = c_{ij}\widetilde{f}_j + c_{ji}\widetilde{f}_i + \rho_{ij}\widetilde{f}. \tag{1}$$

Representatives in any hyperplane of $\mathbb{R}^{N+1}$, for instance, in the affine part $\mathbb{R}^N$ of the projective space $\mathbb{RP}^N = \mathbb{P}(\mathbb{R}^{N+1})$, satisfy such an equation with $1 = c_{ij} + c_{ji} + \rho_{ij}$, that is,

$$\widetilde{f}_{ij} - \widetilde{f} = c_{ij}(\widetilde{f}_j - \widetilde{f}) + c_{ji}(\widetilde{f}_i - \widetilde{f}). \tag{2}$$

Given three points f, f_1, f_2 in $\mathbb{RP}^N$, one can take any point of the plane through these three points as the fourth vertex f_{12} of an elementary quadrilateral (f, f_1, f_{12}, f_2) of a Q-net. Correspondingly, given any two discrete curves $f\colon \mathbb{Z} \times \{0\} \to \mathbb{RP}^N$ and $f\colon \{0\} \times \mathbb{Z} \to \mathbb{RP}^N$ with common point $f(0,0)$, one can construct infinitely many Q-surfaces $f\colon \mathbb{Z}^2 \to \mathbb{RP}^N$ with these curves as coordinate curves: the construction goes inductively; in each step one has the freedom of choosing a point in the corresponding plane (two real parameters).

On the other hand, the construction of elementary hexahedra of Q-nets corresponding to elementary 3D cubes of the lattice $\mathbb{Z}^m$ is a well-posed initial-value problem with a unique solution, and therefore one says that Q-nets are described by a *discrete* 3D *system*:

Theorem 2 (elementary hexahedron of a Q-net). *Given seven points f, f_i, f_{ij} $(1 \leqslant i < j \leqslant 3)$ in $\mathbb{RP}^N$ such that each of the three quadrilaterals (f, f_i, f_{ij}, f_j) is planar (that is, f_{ij} lies in the plane Π_{ij} through f, f_i, f_j), define the three planes $\tau_k\Pi_{ij}$ to be those passing through the point triples f_k, f_{ik}, f_{jk}, respectively. Then these three planes intersect generically at one point*:

$$f_{123} = \tau_1\Pi_{23} \cap \tau_2\Pi_{13} \cap \tau_3\Pi_{12}.$$

Proof. Planarity of the quadrilaterals (f, f_i, f_{ij}, f_j) assures that all seven initial points f, f_i, f_{ij} belong to the three-dimensional space Π_{123} through the four points f, f_1, f_2, f_3. Hence, the planes $\tau_k\Pi_{ij}$ lie in this three-dimensional space, and therefore generically they intersect at exactly one point. Theorem 2 is proved.

The elementary construction step from Theorem 2 is symbolically represented in Figure 5, which is the picture we have in mind when thinking and speaking about discrete three-dimensional systems with dependent variables (fields) associated with the vertices of a regular cubic lattice.

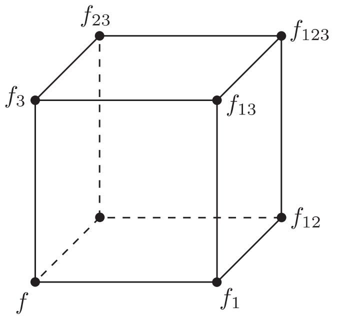

Figure 5. 3D system on an elementary cube: the field at the white vertex is determined by the seven fields at the black vertices (initial data).

As follows from Theorem 2, a three-dimensional Q-net $f\colon \mathbb{Z}^3 \to \mathbb{RP}^N$ is completely determined by its three coordinate surfaces

$$f\colon \mathbb{Z}^2 \times \{0\} \to \mathbb{RP}^N, \quad f\colon \mathbb{Z} \times \{0\} \times \mathbb{Z} \to \mathbb{RP}^N, \quad f\colon \{0\} \times \mathbb{Z}^2 \to \mathbb{RP}^N.$$

Turning to an elementary cube of dimension $m \geqslant 4$, we see that one can prescribe all the points f, f_i, f_{ij} for all $1 \leqslant i < j \leqslant m$. Indeed, the data are clearly independent, and one can construct all the other vertices of an elementary cube starting from these data, *provided one does not encounter contradictions*. To see the possible source of contradictions, let us consider in detail first the case $m = 4$. From f, f_i, f_{ij} ($1 \leqslant i < j \leqslant 4$), one determines all the f_{ijk} uniquely. Then there are, in principle, four different ways to determine f_{1234} from the four 3D cubic faces adjacent to this point (see Figure 6). The absence of contradictions means that these four values for f_{1234} automatically coincide. We call this property 4D consistency.

Definition 3 (4D consistency). A 3D system is said to be 4D consistent if it can be imposed on all three-dimensional faces of an elementary cube of $\mathbb{Z}^4$ (see Figure 6).

Remarkably, the construction of Q-nets based on the planarity of all elementary quadrilaterals enjoys this property.

Theorem 4 (Q-nets are 4D consistent). *The* 3D *system governing Q-nets is* 4D *consistent.*

Proof. In the construction above, one of the possible values of f_{1234} is

$$f_{1234} = \tau_1\tau_2\Pi_{34} \cap \tau_1\tau_3\Pi_{24} \cap \tau_1\tau_4\Pi_{23},$$

and the other three values are obtained by cyclic shifts of the indices. Thus, we have to prove that the six planes $\tau_i\tau_j\Pi_{k\ell}$ intersect in one point.

First, assume that the ambient space $\mathbb{RP}^N$ has dimension $N \geqslant 4$. Then, in general position, the space Π_{1234} through the five points f, f_i ($1 \leqslant i \leqslant 4$) is four-dimensional. It is easy to see that the plane $\tau_i\tau_j\Pi_{k\ell}$ is the intersection of two three-dimensional subspaces $\tau_i\Pi_{jk\ell}$ and $\tau_j\Pi_{ik\ell}$. Indeed, the subspace $\tau_i\Pi_{jk\ell}$

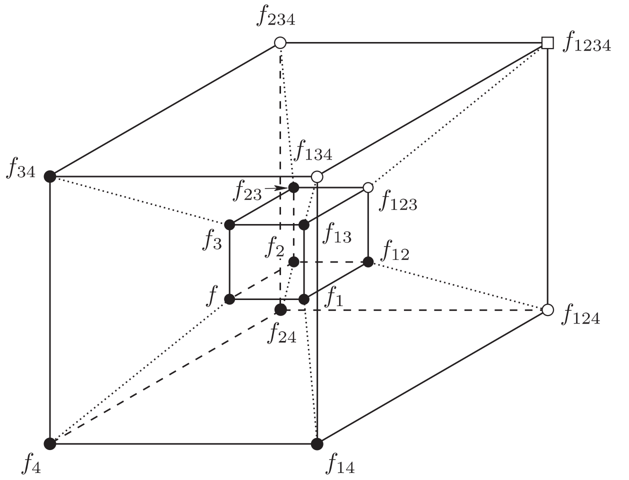

Figure 6. 4D consistency of 3D systems: the fields at the black vertices (initial data) determine, by virtue of the 3D system, the fields f_{ijk} at the white vertices. Then the 3D system gives four a priori different values for f_{1234}. The system is 4D consistent if these four values coincide for any initial data.

through the four points f_i, f_{ij}, f_{ik}, $f_{i\ell}$ contains also f_{ijk}, $f_{ij\ell}$, and $f_{ik\ell}$. Therefore, both $\tau_i\Pi_{jk\ell}$ and $\tau_j\Pi_{ik\ell}$ contain the three points f_{ij}, f_{ijk}, $f_{ij\ell}$, which determine the plane $\tau_i\tau_j\Pi_{k\ell}$. Now the intersection in question can be alternatively described as the intersection of the four three-dimensional subspaces $\tau_1\Pi_{234}$, $\tau_2\Pi_{134}$, $\tau_3\Pi_{124}$, and $\tau_4\Pi_{123}$ of one and the same four-dimensional space Π_{1234}. This intersection consists of exactly one point in the generic case.

In the case $N = 3$ we embed the ambient space into $\mathbb{RP}^4$, then slightly perturb the point f_4 by adding a small component in the fourth coordinate direction. Then we apply the above argument, and after that send the perturbation to zero. This proof works since, as one can easily see, at each step of the construction the perturbation remains regular. This implies the required statement about 4D consistency, and Theorem 4 follows.

The m-dimensional consistency of a 3D system for $m > 4$ is defined in a way analogous to that in the $m = 4$ case. Remarkably and quite generally, four-dimensional consistency already implies m-dimensional consistency for all $m > 4$.

Theorem 5 (4D consistency implies consistency in all higher dimensions). *Any* 4D *consistent discrete* 3D *system is also m-dimensionally consistent for any $m > 4$.*

Proof. The proof is by induction from $(m-1)$-dimensional to m-dimensional consistency, but for notational simplicity we present the details only for the case $m = 5$, the general case being completely similar.

The initial data for a 3D system on the 5D cube $\mathscr{C}_{12345}$ with fields at the vertices consist of the fields f, f_i, f_{ij} for all $1 \leqslant i < j \leqslant 5$. From these data one first gets ten fields f_{ijk} for $1 \leqslant i < j < k \leqslant 5$, and then five fields $f_{ijk\ell}$ for $1 \leqslant i < j < k < \ell \leqslant 5$

(the fact that the latter are well defined is nothing but the assumed 4D consistency for the 4D cubes $\mathscr{C}_{ijkl}$). Now, one has ten possibly different values for f_{12345}, coming from the ten 3D cubes $\tau_i\tau_j\mathscr{C}_{k\ell m}$. To prove that these ten values coincide, consider the five 4D cubes $\tau_i\mathscr{C}_{jk\ell m}$. For instance, for the 4D cube $\tau_1\mathscr{C}_{2345}$ the assumed consistency assures that the four values for f_{12345} coming from the four 3D cubes

$$\tau_1\tau_2\mathscr{C}_{345}, \quad \tau_1\tau_3\mathscr{C}_{245}, \quad \tau_1\tau_4\mathscr{C}_{235}, \quad \tau_1\tau_5\mathscr{C}_{234}$$

are all the same. Similarly, for the 4D cube $\tau_2\mathscr{C}_{1345}$ the 4D consistency leads to the conclusion that the four values for f_{12345} coming from

$$\tau_1\tau_2\mathscr{C}_{345}, \quad \tau_2\tau_3\mathscr{C}_{145}, \quad \tau_2\tau_4\mathscr{C}_{135}, \quad \tau_2\tau_5\mathscr{C}_{134}$$

coincide. Note that the 3D cube $\tau_1\tau_2\mathscr{C}_{345}$, the intersection of $\tau_1\mathscr{C}_{2345}$ and $\tau_2\mathscr{C}_{1345}$, is present in both lists, so that we now have seven coinciding values for f_{12345}. By similar arguments for the other 4D cubes $\tau_i\mathscr{C}_{jk\ell m}$, we arrive at the desired result. Theorem 5 is proved.

Theorems 4 and 5 give us that Q-nets are m-dimensionally consistent for any $m \geqslant 4$. This fact, in turn, leads to the existence of transformations of Q-nets with remarkable permutability properties. Referring to [55] and [20] for the details, we mention here only the definition.

Definition 6 (F-transformation of Q-nets). Two m-dimensional Q-nets $f, f^+\colon \mathbb{Z}^m \to \mathbb{RP}^N$ are called F-transforms (fundamental transforms) of each other if all the quadrilaterals (f, f_i, f_i^+, f^+) (for any $u \in \mathbb{Z}^m$ and for all $1 \leqslant i \leqslant m$) are planar, that is, if the net $F\colon \mathbb{Z}^m \times \{0,1\} \to \mathbb{RP}^N$ defined by $F(u,0) = f(u)$ and $F(u,1) = f^+(u)$ is a two-layer $(m+1)$-dimensional Q-net. We will also call the two latter Q-nets F-transforms of each other.

It follows from Theorem 2 that for any Q-net f, its F-transform f^+ is uniquely defined as soon as its points along the coordinate axes are suitably prescribed.

2.2. Discrete line congruences. Another class of important geometric objects described by a 4D consistent discrete 3D system is the class of *discrete line congruences*. A corresponding theory has been developed by Doliwa, Santini, and Mañas [55], whose presentation we follow in this section.

Let $\mathscr{L}^N$ be the space of lines in $\mathbb{RP}^N$; it can be identified with the Grassmannian $\mathrm{Gr}(N+1,2)$ of two-dimensional vector subspaces of $\mathbb{R}^{N+1}$.

Definition 7 (discrete line congruence). A map $\ell\colon \mathbb{Z}^m \to \mathscr{L}^N$ is called an m-dimensional discrete line congruence in $\mathbb{RP}^N$ ($N \geqslant 3$) if any two neighbouring lines ℓ, ℓ_i (for any $u \in \mathbb{Z}^m$ and for any $1 \leqslant i \leqslant m$) intersect (are co-planar).

For example, lines $\ell = (ff^+)$ that join corresponding points of two Q-nets $f, f^+\colon \mathbb{Z}^m \to \mathbb{RP}^N$ connected by an F-transformation clearly form a discrete line congruence.

A discrete line congruence is said to be *generic* if for any $u \in \mathbb{Z}^m$ and for any $1 \leqslant i \neq j \neq k \neq i \leqslant m$ the four lines ℓ, ℓ_i, ℓ_j, ℓ_k span a four-dimensional space (that is, a space of maximal possible dimension). This implies, in particular, that for any

$u \in \mathbb{Z}^m$ and any $1 \leqslant i \neq j \leqslant m$ the three lines ℓ, ℓ_i, ℓ_j span a three-dimensional space.

The construction of line congruences is similar to that of Q-nets. Given three lines ℓ, ℓ_1, ℓ_2 of a congruence, one has a two-parameter family of lines admissible as the fourth line ℓ_{12}: it suffices to connect by a line any point of ℓ_1 with any point of ℓ_2. Thus, given any two sequences $\ell\colon \mathbb{Z} \times \{0\} \to \mathscr{L}^N$ and $\ell\colon \{0\} \times \mathbb{Z} \to \mathscr{L}^N$ of lines with a common line $\ell(0,0)$ such that any two neighbouring lines are co-planar, one can extend them to a two-dimensional line congruence $f\colon \mathbb{Z}^2 \to \mathscr{L}^N$ in an infinite number of ways: at each step of the inductive procedure one has the freedom of choosing a line from the two-parameter family.

The next theorem shows that non-degenerate line congruences are described by a *discrete* 3D *system*.

Theorem 8 (elementary hexahedron of a discrete line congruence). *For seven given lines ℓ, ℓ_i, ℓ_{ij} $(1 \leqslant i < j \leqslant 3)$ in $\mathbb{RP}^N$ such that ℓ intersects each ℓ_i, the space V_{123} spanned by ℓ, ℓ_1, ℓ_2, ℓ_3 has dimension four, and each ℓ_i intersects both ℓ_{ij} and ℓ_{ik}, there is generically a unique line ℓ_{123} that intersects all three ℓ_{ij}.*

Proof. All seven lines, and therefore also the three-dimensional spaces $\tau_i V_{jk} = \operatorname{span}(\ell_i, \ell_{ij}, \ell_{ik})$, lie in V_{123}. A line that intersects all three ℓ_{ij} must lie in the intersection of these three three-dimensional spaces. But a generic intersection of three three-dimensional spaces in V_{123} is a line:

$$\ell_{123} = \tau_1 V_{23} \cap \tau_2 V_{13} \cap \tau_3 V_{12}.$$

It is now not difficult to show that this line does indeed intersect all three ℓ_{ij}. For instance, $\tau_1 V_{23} \cap \tau_2 V_{13} = \operatorname{span}(\ell_{12}, \ell_{13}) \cap \operatorname{span}(\ell_{12}, \ell_{23})$ is a plane containing ℓ_{12}, and therefore its intersection with $\tau_3 V_{12}$ (that is, the line ℓ_{123}) intersects ℓ_{12}. Theorem 8 is proved.

A similar argument proves the following theorem.

Theorem 9 (discrete line congruences are 4D consistent). *The* 3D *system governing discrete line congruences is* 4D *consistent.*

As in the case of Q-nets, this theorem yields the existence of transformations of discrete line congruences with remarkable permutability properties.

Definition 10 (F-transformation of line congruences). Two m-dimensional line congruences $\ell, \ell^+\colon \mathbb{Z}^m \to \mathscr{L}^N$ are called F-transforms of each other if the corresponding lines ℓ and ℓ^+ intersect (for any $u \in \mathbb{Z}^m$), that is, if the map $L\colon \mathbb{Z}^m \times \{0,1\} \to \mathscr{L}^N$ defined by $L(u,0) = \ell(u)$ and $L(u,1) = \ell^+(u)$ is a two-layer $(m+1)$-dimensional line congruence.

It follows from Theorem 8 that, for a given line congruence ℓ, its F-transform ℓ^+ is uniquely defined as soon as its lines along the coordinate axes are suitably prescribed.

According to Definition 7, any two neighbouring lines $\ell = \ell(u)$ and $\ell_i = \ell(u+e_i)$ of a line congruence intersect at exactly one point $f = \ell \cap \ell_i \in \mathbb{RP}^N$, which is thus combinatorially associated with the edge $(u, u+e_i)$ of the lattice $\mathbb{Z}^m$: $f = f(u, u+e_i)$. It is, however, sometimes more convenient to use the notation $f(u, u+e_i) =$

$f^{(i)}(u)$ for this point, thus associating it with the vertex u of the lattice (and, of course, with the coordinate direction i). See Figure 7.

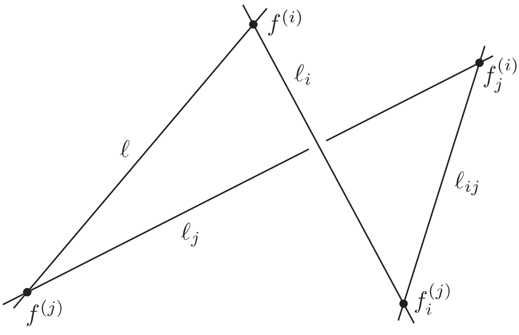

Figure 7. Four lines of a discrete line congruence

Definition 11 (focal net). For a discrete line congruence $\ell\colon \mathbb{Z}^m \to \mathscr{L}^N$ the map $f^{(i)}\colon \mathbb{Z}^m \to \mathbb{RP}^N$ defined by $f^{(i)}(u) = \ell(u) \cap \ell(u+e_i)$ is called its i-th focal net.

Theorem 12. *For a non-degenerate discrete line congruence $\ell\colon \mathbb{Z}^m \to \mathscr{L}^N$, all its focal nets $f^{(k)}\colon \mathbb{Z}^m \to \mathbb{RP}^N$, $1 \leqslant k \leqslant m$, are Q-nets.*

Proof. The proof consists of two steps.

1. First, one shows that for the k-th focal net $f^{(k)}$ all its elementary quadrilaterals $(f^{(k)}, f_i^{(k)}, f_{ik}^{(k)}, f_k^{(k)})$ are planar. This is true for any line congruence. Indeed, both points $f^{(k)}$ and $f_k^{(k)}$ lie on the line ℓ_k, while both points $f_i^{(k)}$ and $f_{ik}^{(k)}$ lie on the line ℓ_{ik}. Therefore, all four points lie in the plane spanned by these two lines ℓ_k and ℓ_{ik}, which intersect by the definition of a line congruence.

2. Second, one shows that for the k-th focal net $f^{(k)}$, all elementary quadrilaterals $(f^{(k)}, f_i^{(k)}, f_{ij}^{(k)}, f_j^{(k)})$ with both $i \neq j$ different from k are planar. Here one uses essentially the assumption that the line congruence ℓ is generic. All four points in question lie in each of the three-dimensional spaces

$$V_{ij} = \mathrm{span}(\ell, \ell_i, \ell_j, \ell_{ij}) \quad \text{and} \quad \tau_k V_{ij} = \mathrm{span}(\ell_k, \ell_{ik}, \ell_{jk}, \ell_{ijk})$$

(Figure 8). Both 3-spaces lie in the four-dimensional space $V_{ijk} = \mathrm{span}(\ell, \ell_i, \ell_j, \ell_k)$, so that generically their intersection is a plane. Theorem 12 is proved.

Corollary 13 (focal net of an F-transformation of a line congruence). *For any two generic line congruences $\ell, \ell^+\colon \mathbb{Z}^m \to \mathscr{L}^N$ connected by an F-transformation, the intersection points $f = \ell \cap \ell^+$ form a Q-net $f\colon \mathbb{Z}^m \to \mathbb{RP}^N$.*

2.3. Q-nets in quadrics. We consider an important admissible reduction of Q-nets: they can be restricted to an arbitrary quadric in $\mathbb{RP}^N$. In smooth differential geometry, that is, for conjugate nets, this is due to Darboux [33]. In discrete differential geometry it was shown by Doliwa [23].

The deep reason for this result is the following fundamental fact, well known in classical projective geometry (see, for example, [58]):

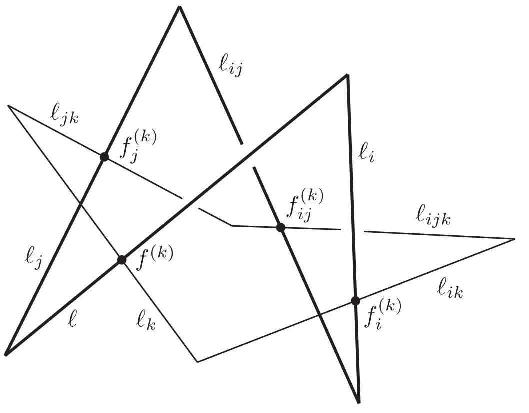

Figure 8. Elementary (ij) quadrilateral of the k-th focal net

Theorem 14 (associated point). *For any seven points of $\mathbb{CP}^3$ in general position, there exists an eighth point (called the associated point) which belongs to any quadric through the original seven points.*

Proof. The proof is based on the following computations. The equation $\mathscr{Q} = 0$ of a quadric in $\mathbb{CP}^3$ has ten coefficients (homogeneous polynomial in four variables). Therefore, there is a unique quadric $\mathscr{Q} = 0$ through nine points in general position. Similarly, a pencil (one-parameter linear family) of quadrics $\mathscr{Q} + \lambda\mathscr{Q}' = 0$ can be drawn through eight points in general position, and a two-parameter linear family of quadrics $\mathscr{Q} + \lambda\mathscr{Q}' + \mu\mathscr{Q}'' = 0$ can be drawn through seven points in general position. Generically, the solution of a system of three quadratic equations

$$\mathscr{Q} = 0, \qquad \mathscr{Q}' = 0, \qquad \mathscr{Q}'' = 0$$

for the intersection of three quadrics in $\mathbb{CP}^3$ consists of eight points. It can be shown that the three quadrics spanning the indicated two-parameter family can be considered generic enough for such a conclusion. Clearly, the resulting eight points lie on every quadric of the two-parameter family, and Theorem 14 follows.

Theorem 15 (elementary hexahedron of a Q-net in a quadric). *If seven points f, f_i, f_{ij} $(1 \leqslant i < j \leqslant 3)$ of an elementary hexahedron of a Q-net $f\colon \mathbb{Z}^m \to \mathbb{RP}^N$ belong to a quadric $\mathscr{Q} \subset \mathbb{RP}^N$, then so does the eighth point f_{123}.*

Proof. The original seven points can be assumed to lie in a three-dimensional space, and they are known to belong to three (degenerate) quadrics: the pairs of planes $\Pi_{jk} \cup \tau_i\Pi_{jk}$ for $(jk) = (12), (23), (31)$. Clearly, the eighth intersection point of these quadrics is $f_{123} = \tau_1\Pi_{23} \cap \tau_2\Pi_{31} \cap \tau_3\Pi_{12}$, and this has to be the associated point. According to Theorem 14, it belongs to any quadric through the original seven points, in particular, to $\mathscr{Q}$, and Theorem 15 follows.

§ 3. Geometries of spheres

3.1. Lie geometry. A classical source on *Lie geometry* is Blaschke's book [24]; see also a modern account by Cecil [59].

The following geometric objects in the Euclidean space $\mathbb{R}^N$ are elements of Lie geometry.

- *Oriented hyperspheres.* The hypersphere in $\mathbb{R}^N$ with centre $c \in \mathbb{R}^N$ and radius $r > 0$ is described by the equation $S = \{x \in \mathbb{R}^N : |x - c|^2 = r^2\}$. It divides $\mathbb{R}^N$ into two parts, the interior and the exterior. If one designates one of the two parts of $\mathbb{R}^N$ as 'positive', one comes to the notion of an oriented hypersphere. Thus, there are two oriented hyperspheres $S^\pm$ for any S. One can take the orientation of a hypersphere into account by assigning a signed radius $\pm r$ to it. For instance, one can assign positive radii $r > 0$ to hyperspheres with the inward field of unit normals and negative radii $r < 0$ to hyperspheres with the outward field of unit normals.
- *Oriented hyperplanes.* A hyperplane in $\mathbb{R}^N$ is given by an equation $P = \{x \in \mathbb{R}^N : \langle v, x\rangle = d\}$, with a unit normal $v \in \mathbb{S}^{N-1}$ and a number $d \in \mathbb{R}$. Clearly, the pairs (v, d) and $(-v, -d)$ represent one and the same hyperplane. It divides $\mathbb{R}^N$ into two half-spaces. Designating one of the two as positive, we arrive at the notion of an oriented hyperplane. Thus, there are two oriented hyperplanes $P^\pm$ for any P. One can take the orientation of a hyperplane into account by assigning to it the pair (v, d) with unit normal v pointing into the positive half-space.
- *Points.* Points $x \in \mathbb{R}^N$ are regarded as hyperspheres of zero radius.
- *Infinity.* One compactifies the space $\mathbb{R}^N$ by adding a point ∞ at infinity, with the understanding that a basis of open neighbourhoods of ∞ is given, for example, by the exteriors of the hyperspheres $|x|^2 = r^2$. Topologically, the so-defined compactification is equivalent to a sphere $\mathbb{S}^N$.
- *Contact elements.* A contact element of a hypersurface is a pair consisting of a point $x \in \mathbb{R}^N$ and an (oriented) hyperplane P through x; alternatively, one can use a normal vector v to P at x. In the framework of Lie geometry a contact element can be identified with the set (pencil) of all hyperspheres S through x which are in oriented contact with P (and with one another), thus sharing the normal vector v at x (see Figure 9).

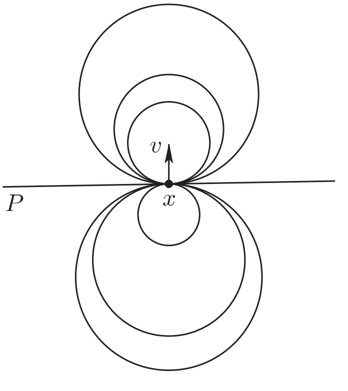

Figure 9. Contact element

All these elements are modelled in Lie geometry as points and lines, respectively, in the $(N+2)$-dimensional projective space $\mathbb{P}(\mathbb{R}^{N+1,2})$ with the space $\mathbb{R}^{N+1,2}$ of homogeneous coordinates. The latter is the space spanned by $N+3$ linearly independent vectors $\mathbf{e}_1,\dots,\mathbf{e}_{N+3}$ and equipped with the pseudo-Euclidean scalar product

$$\langle \mathbf{e}_i,\mathbf{e}_j\rangle = \begin{cases} 1, & i=j\in\{1,\dots,N+1\},\\ -1, & i=j\in\{N+2,N+3\},\\ 0, & i\neq j.\end{cases}$$

It is convenient to introduce the two isotropic vectors

$$\mathbf{e}_0=\frac{1}{2}(\mathbf{e}_{N+2}-\mathbf{e}_{N+1}),\qquad \mathbf{e}_\infty=\frac{1}{2}(\mathbf{e}_{N+2}+\mathbf{e}_{N+1}), \tag{3}$$

for which

$$\langle \mathbf{e}_0,\mathbf{e}_0\rangle=\langle \mathbf{e}_\infty,\mathbf{e}_\infty\rangle=0,\qquad \langle \mathbf{e}_0,\mathbf{e}_\infty\rangle=-\frac{1}{2}\,.$$

The models of the above elements in the space $\mathbb{R}^{N+1,2}$ of homogeneous coordinates are as follows.

- *Oriented hypersphere with centre $c\in\mathbb{R}^N$ and signed radius $r\in\mathbb{R}$ (with sign corresponding to the orientation)*:

$$\widehat{s}=c+\mathbf{e}_0+(|c|^2-r^2)\mathbf{e}_\infty+r\mathbf{e}_{N+3}. \tag{4}$$

- *Oriented hyperplane $\langle v,x\rangle=d$ with $v\in\mathbb{S}^{N-1}$ and $d\in\mathbb{R}$*:

$$\widehat{p}=v+0\cdot\mathbf{e}_0+2d\mathbf{e}_\infty+\mathbf{e}_{N+3}. \tag{5}$$

- *Point $x\in\mathbb{R}^N$*:

$$\widehat{x}=x+\mathbf{e}_0+|x|^2\mathbf{e}_\infty+0\cdot\mathbf{e}_{N+3}. \tag{6}$$

- *Infinity ∞*:

$$\widehat{\infty}=\mathbf{e}_\infty. \tag{7}$$

- *Contact element (x,P)*:

$$\mathrm{span}(\widehat{x},\widehat{p}). \tag{8}$$

In the projective space $\mathbb{P}(\mathbb{R}^{N+1,2})$ the first four types of elements are represented by points which are equivalence classes of (4)–(7) with respect to the relation $\xi\sim\eta\iff\xi=\lambda\eta$ with $\lambda\in\mathbb{R}^*$ for $\xi,\eta\in\mathbb{R}^{N+1,2}$. A contact element is represented by the line in $\mathbb{P}(\mathbb{R}^{N+1,2})$ through the points with the representatives $\widehat{x}$ and $\widehat{p}$. We mention several fundamentally important features of this model.

(i) All the above elements belong to the *Lie quadric* $\mathbb{P}(\mathbb{L}^{N+1,2})$, where

$$\mathbb{L}^{N+1,2}=\big\{\xi\in\mathbb{R}^{N+1,2}:\langle\xi,\xi\rangle=0\big\}. \tag{9}$$

Moreover, the points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ are in a one-to-one correspondence with the oriented hyperspheres in $\mathbb{R}^N$, including the degenerate cases: proper hyperspheres correspond to points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with $\mathbf{e}_0$- and $\mathbf{e}_{N+3}$-components both non-vanishing, hyperplanes correspond to points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with vanishing $\mathbf{e}_0$-component, points correspond to points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with vanishing $\mathbf{e}_{N+3}$-component, and infinity corresponds to the only point of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with both $\mathbf{e}_0$- and $\mathbf{e}_{N+3}$-components vanishing.

(ii) Two oriented hyperspheres S_1, S_2 are in oriented contact (that is, are tangent to each other, with the unit normals at the tangency pointing in the same direction) if and only if

$$|c_1 - c_2|^2 = (r_1 - r_2)^2, \tag{10}$$

and this is equivalent to $\langle \widehat{s}_1, \widehat{s}_2 \rangle = 0$.

(iii) An oriented hypersphere $S = \{x \in \mathbb{R}^N : |x - c|^2 = r^2\}$ is in oriented contact with an oriented hyperplane $P = \{x \in \mathbb{R}^N : \langle v, x \rangle = d\}$ if and only if

$$\langle c, v \rangle - r - d = 0. \tag{11}$$

Indeed, $\langle x_0 - c, x - c \rangle = r^2$ is the equation of the hyperplane P tangent to S at $x_0 \in S$. Denoting by $v = (c - x_0)/r$ the unit normal vector of P (recall that positive radii are assigned to spheres with inward unit normals), we can write the above equation as $\langle v, x \rangle = d$ with $d = \langle c, (c - x_0)/r \rangle - r = \langle c, v \rangle - r$, which proves (11). The last equation is equivalent to $\langle \widehat{s}, \widehat{p} \rangle = 0$.

(iv) A point x can be regarded as a hypersphere of radius $r = 0$ (in this case the two oriented hyperspheres coincide). The incidence relation $x \in S$ with a hypersphere S (respectively, $x \in P$ with a hyperplane P) can be interpreted as a particular case of oriented contact of a sphere of radius $r = 0$ with S (respectively, with P), and it holds if and only if $\langle \widehat{x}, \widehat{s} \rangle = 0$ (respectively, $\langle \widehat{x}, \widehat{p} \rangle = 0$).

(v) Any hyperplane P satisfies $\langle \widehat{\infty}, \widehat{p} \rangle = 0$. One can interpret hyperplanes as hyperspheres (of infinite radius) passing through ∞. More precisely, a hyperplane $\langle v, x \rangle = d$ can be interpreted as a limit, as $r \to \infty$, of the hyperspheres of radii r with centres at $c = rv + u$, with $\langle v, u \rangle = d$. Indeed, the representatives (4) of such spheres are

$$\begin{aligned}
\widehat{s} &= (rv + u) + \mathbf{e}_0 + (2dr + \langle u, u \rangle)\mathbf{e}_\infty + r\mathbf{e}_{N+3} \\
&\sim (v + O(1/r)) + (1/r)\mathbf{e}_0 + (2d + O(1/r))\mathbf{e}_\infty + \mathbf{e}_{N+3} \\
&= \widehat{p} + O(1/r).
\end{aligned}$$

Moreover, for similar reasons, the infinity ∞ can be regarded as a limiting position of any sequence of points x with $|x| \to \infty$.

(vi) Any two hyperspheres S_1, S_2 in oriented contact determine a contact element (their point of contact and their common tangent hyperplane). For their representatives $\widehat{s}_1$, $\widehat{s}_2$ in $\mathbb{R}^{N+1,2}$, the line in $\mathbb{P}(\mathbb{R}^{N+1,2})$ through the corresponding points in $\mathbb{P}(\mathbb{L}^{N+1,2})$ is *isotropic*, that is, lies entirely on the Lie quadric $\mathbb{P}(\mathbb{L}^{N+1,2})$. This follows from

$$\langle \alpha_1 \widehat{s}_1 + \alpha_2 \widehat{s}_2, \alpha_1 \widehat{s}_1 + \alpha_2 \widehat{s}_2 \rangle = 2\alpha_1\alpha_2 \langle \widehat{s}_1, \widehat{s}_2 \rangle = 0.$$

Such a line contains exactly one point whose representative $\widehat{x}$ has vanishing $\mathbf{e}_{N+3}$-component (and corresponds to x, the common point of contact of all the hyperspheres), and if $x \neq \infty$, then exactly one point whose representative $\widehat{p}$ has vanishing $\mathbf{e}_0$-component (and corresponds to P, the common tangent hyperplane of all the hyperspheres). In the case when an isotropic line contains $\widehat{\infty}$, all its points represent parallel hyperplanes, which constitute a contact element through ∞.

Thus, if one regards hyperplanes as hyperspheres of infinite radii, and points as hyperspheres of vanishing radii, then one can conclude that:

(i) oriented hyperspheres are in a one-to-one correspondence with points of the Lie quadric $\mathbb{P}(\mathbb{L}^{N+1,2})$ in the projective space $\mathbb{P}(\mathbb{R}^{N+1,2})$;

(ii) oriented contact of two oriented hyperspheres corresponds to orthogonality of (any) representatives of the corresponding points in $\mathbb{P}(\mathbb{R}^{N+1,2})$;

(iii) contact elements of hypersurfaces are in a one-to-one correspondence with isotropic lines in $\mathbb{P}(\mathbb{R}^{N+1,2})$, and $\mathscr{L}_0^{N+1,2}$ will denote the set of such lines.

According to Klein's Erlangen Program, Lie geometry is the study of properties of transformations which map oriented hyperspheres (including points and hyperplanes) to oriented hyperspheres and, moreover, preserve oriented contact of hypersphere pairs. In the projective model described above, Lie geometry is the study of projective transformations of $\mathbb{P}(\mathbb{R}^{N+1,2})$ which leave $\mathbb{P}(\mathbb{L}^{N+1,2})$ invariant and, moreover, preserve orthogonality of points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ (which is understood as orthogonality of their lifts to $\mathbb{L}^{N+1,2} \subset \mathbb{R}^{N+1,2}$; clearly, this relation does not depend on the choice of lifts). Such transformations are called *Lie sphere transformations.*

Theorem 16 (fundamental theorem of Lie geometry).

a) *The group of Lie sphere transformations is isomorphic to $O(N+1,2)/\{\pm I\}$.*

b) *Any diffeomorphism of $\mathbb{P}(\mathbb{L}^{N+1,2})$ preserving the class of isotropic lines is the restriction to $\mathbb{P}(\mathbb{L}^{N+1,2})$ of a Lie sphere transformation.*

Since (non-)vanishing of the $\mathbf{e}_0$- or the $\mathbf{e}_{N+3}$-component of a point in $\mathbb{P}(\mathbb{L}^{N+1,2})$ is not invariant under a general Lie sphere transformation, there is no distinction among oriented hyperspheres, oriented hyperplanes, and points in Lie geometry.

3.2. Möbius geometry. Blaschke's book [24] serves also as a classical source on *Möbius geometry*, of which a modern account can be found in [39].

Möbius geometry is a subgeometry of Lie geometry, with points distinguished among all hyperspheres as those of radius zero. Thus, Möbius geometry studies properties of hyperspheres invariant under the subgroup of Lie sphere transformations preserving the set of points. In the projective model, points of $\mathbb{R}^N$ are distinguished as points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with zero $\mathbf{e}_{N+3}$-component. (Of course, here one could replace $\mathbf{e}_{N+3}$ by any time-like vector.) Thus, Möbius geometry studies the subgroup of Lie sphere transformations preserving the subset of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with zero $\mathbf{e}_{N+3}$-component. The following geometric objects in $\mathbb{R}^N$ are elements of Möbius geometry:

- (*non-oriented*) *hyperspheres* $S = \{x \in \mathbb{R}^N : |x-c|^2 = r^2\}$ with centres $c \in \mathbb{R}^N$ and radii $r > 0$;
- (*non-oriented*) *hyperplanes* $P = \{x \in \mathbb{R}^N : \langle v, x\rangle = d\}$ with unit normals $v \in \mathbb{S}^{N-1}$ and $d \in \mathbb{R}$;
- *points* $x \in \mathbb{R}^N$;
- *infinity* ∞ which compactifies $\mathbb{R}^N$ into $\mathbb{S}^N$.

In modelling these elements one can use the Lie-geometric description and just omit the $\mathbf{e}_{N+3}$-component. The resulting objects are points of the $(N+1)$-dimensional projective space $\mathbb{P}(\mathbb{R}^{N+1,1})$ with the space $\mathbb{R}^{N+1,1}$ of homogeneous coordinates. The latter is the space spanned by $N+2$ linearly independent vectors

$\mathbf{e}_1, \dots, \mathbf{e}_{N+2}$ and equipped with the Minkowski scalar product

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1, & i = j \in \{1, \dots, N+1\}, \\ -1, & i = j = N+2, \\ 0, & i \neq j. \end{cases}$$

We continue to use the notation in (3) in the context of the Möbius geometry. The above elements are modelled in the space $\mathbb{R}^{N+1,1}$ of homogeneous coordinates as follows.

- *Hypersphere with centre $c \in \mathbb{R}^N$ and radius $r > 0$*:

$$\widehat{s} = c + \mathbf{e}_0 + (|c|^2 - r^2)\mathbf{e}_\infty. \tag{12}$$

- *Hyperplane $\langle v, x \rangle = d$ with $v \in \mathbb{S}^{N-1}$ and $d \in \mathbb{R}$*:

$$\widehat{p} = v + 0 \cdot \mathbf{e}_0 + 2d\mathbf{e}_\infty. \tag{13}$$

- *Point $x \in \mathbb{R}^N$*:

$$\widehat{x} = x + \mathbf{e}_0 + |x|^2\mathbf{e}_\infty. \tag{14}$$

- *Infinity ∞*:

$$\widehat{\infty} = \mathbf{e}_\infty. \tag{15}$$

In the projective space $\mathbb{P}(\mathbb{R}^{N+1,1})$ these elements are represented by points which are equivalence classes of (12)–(15) with respect to the usual equivalence relation $\xi \sim \eta \iff \xi = \lambda\eta$ with $\lambda \in \mathbb{R}^*$ for $\xi, \eta \in \mathbb{R}^{N+1,1}$. The fundamental features of these identifications are:

(i) The infinity $\widehat{\infty}$ can be regarded as a limit of any sequence of $\widehat{x}$ for $x \in \mathbb{R}^N$ with $|x| \to \infty$. Elements $x \in \mathbb{R}^N \cup \{\infty\}$ are in a one-to-one correspondence with points of the projectivized *light cone* $\mathbb{P}(\mathbb{L}^{N+1,1})$, where

$$\mathbb{L}^{N+1,1} = \big\{\xi \in \mathbb{R}^{N+1,1} : \langle \xi, \xi \rangle = 0\big\}. \tag{16}$$

Points $x \in \mathbb{R}^N$ correspond to points of $\mathbb{P}(\mathbb{L}^{N+1,1})$ with a non-vanishing $\mathbf{e}_0$-component, while ∞ corresponds to the only point of $\mathbb{P}(\mathbb{L}^{N+1,1})$ with vanishing $\mathbf{e}_0$-component.

(ii) Hyperspheres $\widehat{s}$ and hyperplanes $\widehat{p}$ belong to $\mathbb{P}(\mathbb{R}^{N+1,1}_{\text{out}})$, where

$$\mathbb{R}^{N+1,1}_{\text{out}} = \big\{\xi \in \mathbb{R}^{N+1,1} : \langle \xi, \xi \rangle > 0\big\} \tag{17}$$

is the set of space-like vectors of the Minkowski space $\mathbb{R}^{N+1,1}$. Hyperplanes can be interpreted as hyperspheres (of infinite radius) through ∞.

(iii) Two hyperspheres S_1, S_2 with centres c_1, c_2 and radii r_1, r_2 intersect orthogonally if and only if

$$|c_1 - c_2|^2 = r_1^2 + r_2^2, \tag{18}$$

which is equivalent to $\langle \widehat{s}_1, \widehat{s}_2 \rangle = 0$. Similarly, a hypersphere S intersects orthogonally with a hyperplane P if and only if its centre lies in P:

$$\langle c, v \rangle - d = 0, \tag{19}$$

which is equivalent to $\langle \widehat{s}, \widehat{p} \rangle = 0$.

(iv) A point x can be regarded as a limiting case of a hypersphere with radius $r = 0$. The incidence relation of a point $x \in S$ with a hypersphere S (respectively, $x \in P$ with a hyperplane P) can be interpreted as a particular case of an orthogonal intersection of a sphere of radius $r = 0$ with S (respectively, P), and it takes place if and only if $\langle \widehat{x}, \widehat{s} \rangle = 0$ (respectively, $\langle \widehat{x}, \widehat{p} \rangle = 0$).

We remark that a hypersphere S can also be interpreted as the set of points $x \in S$. Correspondingly, it admits, along with the representation $\widehat{s}$, the dual representation as a transversal intersection of $\mathbb{P}(\mathbb{L}^{N+1,1})$ with the projective N-space $\mathbb{P}(\widehat{s}^\perp)$ that is polar to the point $\widehat{s}$ with respect to $\mathbb{P}(\mathbb{L}^{N+1,1})$; here, of course, $\widehat{s}^\perp = \{\widehat{x} \in \mathbb{R}^{N+1,1} : \langle \widehat{s}, \widehat{x} \rangle = 0\}$. This can be generalized to model lower-dimensional spheres.

- *Spheres.* A k-sphere is a generic intersection of $N-k$ hyperspheres S_i ($i = 1, \dots, N-k$). The intersection of $N-k$ hyperspheres represented by $\widehat{s}_i \in \mathbb{R}^{N+1,1}_{\rm out}$ ($i = 1, \dots, N-k$) is generic if the $(N-k)$-dimensional linear subspace of $\mathbb{R}^{N+1,1}$ spanned by the $\widehat{s}_i$ is space-like:

$$\Sigma = \mathrm{span}(\widehat{s}_1, \dots, \widehat{s}_{N-k}) \subset \mathbb{R}^{N+1,1}_{\rm out}.$$

As a set of points, this k-sphere is represented as $\mathbb{P}(\mathbb{L}^{N+1,1} \cap \Sigma^\perp)$, where

$$\Sigma^\perp = \bigcap_{i=1}^{N-k} \widehat{s}_i^\perp = \big\{\widehat{x} \in \mathbb{R}^{N+1,1} : \langle \widehat{s}_1, \widehat{x} \rangle = \dots = \langle \widehat{s}_{N-k}, \widehat{x} \rangle = 0\big\}$$

is a $(k+2)$-dimensional linear subspace of $\mathbb{R}^{N+1,1}$ of signature $(k+1, 1)$. Through any $k+2$ points $x_1, \dots, x_{k+2} \in \mathbb{R}^N$ in general position one can draw a unique k-sphere. It corresponds to the $(k+2)$-dimensional linear subspace

$$\Sigma^\perp = \mathrm{span}(\widehat{x}_1, \dots, \widehat{x}_{k+2})$$

of signature $(k+1, 1)$, with $k+2$ linearly independent isotropic vectors $\widehat{x}_1, \dots, \widehat{x}_{k+2} \in \mathbb{L}^{N+1,1}$. In the polar formulation this k-sphere corresponds to the $(N-k)$-dimensional space-like linear subspace

$$\Sigma = \bigcap_{i=1}^{k+2} \widehat{x}_i^\perp = \big\{\widehat{s} \in \mathbb{R}^{N+1,1} : \langle \widehat{s}, \widehat{x}_1 \rangle = \dots = \langle \widehat{s}, \widehat{x}_{k+2} \rangle = 0\big\}.$$

Möbius geometry is the study of properties of (non-)oriented hyperspheres invariant with respect to projective transformations of $\mathbb{P}(\mathbb{R}^{N+1,1})$ which map points to points, that is, which leave $\mathbb{P}(\mathbb{L}^{N+1,1})$ invariant. Such transformations are called *Möbius transformations.*

Theorem 17 (fundamental theorem of Möbius geometry).

a) *The group of Möbius transformations is isomorphic to* $O(N+1,1)/\{\pm I\} \simeq O^+(N+1,1)$, *the group of Lorentz transformations of* $\mathbb{R}^{N+1,1}$ *preserving the time-like direction.*

b) *For* $N \geqslant 3$ *every conformal diffeomorphism of* $\mathbb{S}^N \simeq \mathbb{R}^N \cup \{\infty\}$ *is induced by the restriction to* $\mathbb{P}(\mathbb{L}^{N+1,1})$ *of a Möbius transformation.*

The group $O^+(N+1,1)$ is generated by reflections

$$A_{\widehat{s}}\colon \mathbb{R}^{N+1,1} \to \mathbb{R}^{N+1,1}, \qquad A_{\widehat{s}}(\widehat{x}) = \widehat{x} - \frac{2\langle \widehat{s}, \widehat{x}\rangle}{\langle \widehat{s}, \widehat{s}\rangle}\,\widehat{s}. \tag{20}$$

If $\widehat{s}$ is the hypersphere (12), then the transformation induced on $\mathbb{R}^N$ by $A_{\widehat{s}}$ is obtained from (20) by a computation with the representatives (14) for points and is given by

$$x \mapsto c + \frac{r^2}{|x-c|^2}\,(x-c) \tag{21}$$

(inversion in the hypersphere $S = \{x \in \mathbb{R}^N : |x-c|^2 = r^2\}$); similarly, if $\widehat{s} = \widehat{p}$ is the hyperplane (13), then the transformation induced on $\mathbb{R}^N$ by $A_{\widehat{p}}$ is easily computed to be

$$x \mapsto x - \frac{2(\langle v, x\rangle - d)}{\langle v, v\rangle}\, v \tag{22}$$

(reflection in the hyperplane $P = \{x \in \mathbb{R}^N : \langle v, x\rangle = d\}$).

Since (non-)vanishing of the $\mathbf{e}_\infty$-component of a point in $\mathbb{P}(\mathbb{R}^{N+1,1})$ is not invariant under a general Möbius transformation, there is no distinction in Möbius geometry between hyperspheres and hyperplanes.

3.3. Laguerre geometry. Blaschke's book [24] serves as the indispensable classical source also in the case of *Laguerre geometry.* One can find a modern account, for example, in [60], [59], [61].

Laguerre geometry is a subgeometry of Lie geometry, with hyperplanes distinguished among all hyperspheres as those passing through ∞. Thus, Laguerre geometry studies properties of hyperspheres invariant under the subgroup of Lie sphere transformations which preserve the set of hyperplanes. The following objects in $\mathbb{R}^N$ are elements of Laguerre geometry.

- (*Oriented*) *hyperspheres* $S = \{x \in \mathbb{R}^N : |x-c|^2 = r^2\}$ with centres $c \in \mathbb{R}^N$ and signed radii $r \in \mathbb{R}$, can be put into correspondence with $(N+1)$-tuples (c, r).
- *Points* $x \in \mathbb{R}^N$ are regarded as hyperspheres of radius zero, and are put into correspondence with $(N+1)$-tuples $(x, 0)$.
- (*Oriented*) *hyperplanes* $P = \{x \in \mathbb{R}^N : \langle v, x\rangle = d\}$ with unit normals $v \in \mathbb{S}^{N-1}$ and $d \in \mathbb{R}$ can be put into correspondence with $(N+1)$-tuples (v, d).

In the projective model of Lie geometry, hyperplanes are distinguished as elements of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with vanishing $\mathbf{e}_0$-component. (Of course, one could replace $\mathbf{e}_0$ here by any isotropic vector.) Thus, Laguerre geometry studies the subgroup of Lie sphere transformations preserving the subset of $\mathbb{P}(\mathbb{L}^{N+1,2})$ with vanishing $\mathbf{e}_0$-component.

There seems to exist no model of Laguerre geometry with hyperspheres and hyperplanes modelled as points of one and the same space. Depending on which of the two types of elements is modelled by points, one comes to the *Blaschke cylinder model* or the *cyclographic model* of Laguerre geometry. We will use the first of these models, which has the advantage of a simpler description of the distinguished objects of Laguerre geometry which are hyperplanes. The main advantage of the second model is a simpler description of the group of Laguerre transformations.

The setting in both models consists of *two* $(N+1)$-dimensional projective spaces whose spaces $\mathbb{R}^{N,1,1}$ and $(\mathbb{R}^{N,1,1})^*$ of homogeneous coordinates are dual to each other and arise from $\mathbb{R}^{N+1,2}$ by 'forgetting' the $\mathbf{e}_0$- and $\mathbf{e}_\infty$-components, respectively. Thus, $\mathbb{R}^{N,1,1}$ is spanned by $N+2$ linearly independent vectors $\mathbf{e}_1,\dots,\mathbf{e}_N$, $\mathbf{e}_{N+3}$, $\mathbf{e}_\infty$, and is equipped with a degenerate bilinear form of signature $(N,1,1)$ in which the indicated vectors are pairwise orthogonal, the first N being space-like, that is, $\langle\mathbf{e}_i,\mathbf{e}_i\rangle=1$ for $1\leqslant i\leqslant N$, and the last two being time-like and isotropic, respectively, that is, $\langle\mathbf{e}_{N+3},\mathbf{e}_{N+3}\rangle=-1$ and $\langle\mathbf{e}_\infty,\mathbf{e}_\infty\rangle=0$. Similarly, $(\mathbb{R}^{N,1,1})^*$ is assumed to have an orthogonal basis consisting of $\mathbf{e}_1,\dots,\mathbf{e}_N$, $\mathbf{e}_{N+3}$, $\mathbf{e}_0$, again with an isotropic last vector: $\langle\mathbf{e}_0,\mathbf{e}_0\rangle=0$. Note that one and the same symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ is used to denote two degenerate bilinear forms in our two spaces. We will overload this symbol even more and use it also for the (non-degenerate) pairing between these two spaces, which is established by setting $\langle\mathbf{e}_0,\mathbf{e}_\infty\rangle=-\frac{1}{2}$, in addition to the above relations. (We note that a degenerate bilinear form cannot be used to identify a vector space with its dual.)

In both the above models:

- a *hyperplane* $P=(v,d)$ *is modelled as a point in the space* $\mathbb{P}(\mathbb{R}^{N,1,1})$, *with representative*
$$\widehat{p}=v+2d\mathbf{e}_\infty+\mathbf{e}_{N+3};\tag{23}$$
- a *hypersphere* $S=(c,r)$ *is modelled as a point in the space* $\mathbb{P}((\mathbb{R}^{N,1,1})^*)$, *with representative*
$$\widehat{s}=c+\mathbf{e}_0+r\mathbf{e}_{N+3}.\tag{24}$$

Each of the models appears if we regard one of the spaces as a preferred (fundamental) space and interpret the points of the second space as hyperplanes in the preferred space. In the *Blaschke cylinder model* the preferred space is the space $\mathbb{P}(\mathbb{R}^{N,1,1})$ whose points model hyperplanes $P\subset\mathbb{R}^N$. A hypersphere $S\subset\mathbb{R}^N$ is then modelled as a hyperplane $\{\xi\in\mathbb{P}(\mathbb{R}^{N,1,1}):\langle\widehat{s},\xi\rangle=0\}$ in the space $\mathbb{P}(\mathbb{R}^{N,1,1})$. The basic features of this model are as follows.

(i) Oriented hyperplanes $P\subset\mathbb{R}^N$ are in a one-to-one correspondence with points $\widehat{p}$ of the quadric $\mathbb{P}(\mathbb{L}^{N,1,1})$, where
$$\mathbb{L}^{N,1,1}=\bigl\{\xi\in\mathbb{R}^{N,1,1}:\langle\xi,\xi\rangle=0\bigr\}.\tag{25}$$

(ii) Two oriented hyperplanes $P_1,P_2\subset\mathbb{R}^N$ are in oriented contact (parallel) if and only if their representatives $\widehat{p}_1$, $\widehat{p}_2$ differ by a vector parallel to $\mathbf{e}_\infty$.

(iii) An oriented hypersphere $S\subset\mathbb{R}^N$ is in oriented contact with an oriented hyperplane $P\subset\mathbb{R}^N$ if and only if $\widehat{p}\in\widehat{s}$, that is, $\langle\widehat{p},\widehat{s}\rangle=0$. Thus, a hypersphere S is interpreted as the set of all its tangent hyperplanes.

The quadric $\mathbb{P}(\mathbb{L}^{N,1,1})$ is diffeomorphic to the *Blaschke cylinder*
$$\mathscr{Z}=\bigl\{(v,d)\in\mathbb{R}^{N+1}:|v|=1\bigr\}=\mathbb{S}^{N-1}\times\mathbb{R}\subset\mathbb{R}^{N+1};\tag{26}$$
two points of it represent parallel hyperplanes if they lie on one straight-line generator of $\mathscr{Z}$ parallel to its axis. In the ambient space $\mathbb{R}^{N+1}$ of the Blaschke cylinder, oriented hyperspheres $S\subset\mathbb{R}^N$ are in a one-to-one correspondence with hyperplanes non-parallel to the axis of $\mathscr{Z}$:
$$S\sim\bigl\{(v,d)\in\mathbb{R}^{N+1}:\langle c,v\rangle-d-r=0\bigr\}.\tag{27}$$

The intersection of such a hyperplane with $\mathscr{X}$ consists of points in $\mathscr{X}$ which represent tangent hyperplanes to $S \subset \mathbb{R}^N$, as follows from equation (11).

In this paper we will not use the cyclographic model of Laguerre geometry; there is a short description of it in Appendix 1.

§ 4. Discrete curvature-line parametrization in the Lie, Möbius, and Laguerre geometries

From now on, we confine ourselves to the geometry of surfaces in three-dimensional Euclidean space $\mathbb{R}^3$. Accordingly, one should set $N = 3$ in all previous considerations.

It is natural to regard the following objects as discrete surfaces in the various geometries discussed above.

- In *Lie geometry* a surface is viewed as built of its contact elements. These contact elements are interpreted as points of the surface and tangent planes (or, equivalently, normals) at these points. This can be discretized in a natural way: a discrete surface is a map

  $$(x, P)\colon \mathbb{Z}^2 \to \{\text{contact elements of surfaces in } \mathbb{R}^3\},$$

  or, in the projective model of Lie geometry, a map

  $$\ell\colon \mathbb{Z}^2 \to \mathscr{L}_0^{4,2}, \tag{28}$$

  where, we recall, $\mathscr{L}_0^{4,2}$ denotes the set of isotropic lines in $\mathbb{P}(\mathbb{R}^{4,2})$.
- In *Möbius geometry* a surface is viewed simply as built of its points. A discrete surface is a map

  $$x\colon \mathbb{Z}^2 \to \mathbb{R}^3,$$

  or, in the projective model, a map

  $$\widehat{x}\colon \mathbb{Z}^2 \to \mathbb{P}(\mathbb{L}^{4,1}). \tag{29}$$

- In *Laguerre geometry* a surface is viewed as the envelope of the system of its tangent planes. A discrete surface is a map

  $$P\colon \mathbb{Z}^2 \to \{\text{oriented planes in } \mathbb{R}^3\},$$

  or, in the projective model, a map

  $$\widehat{p}\colon \mathbb{Z}^2 \to \mathbb{P}(\mathbb{L}^{3,1,1}). \tag{30}$$

  It should be mentioned that a substantial part of the description of a surface in Laguerre geometry is its *Gauss map*

  $$v\colon \mathbb{Z}^2 \to \mathbb{S}^2, \tag{31}$$

  consisting of the unit normals v to the tangent planes $P = (v, d)$.

Thus, the description of a discrete surface in Lie geometry contains more information than the description of a discrete surface in Möbius or Laguerre geometry. Actually, the former description merges the two latter ones.

4.1. Lie geometry. The definition below is a discretization of the Lie-geometric description of curvature-line parametrized surfaces, as found, for example, in [24].

Definition 18 (principal contact-element nets, Euclidean model). A map

$$(x, P)\colon \mathbb{Z}^2 \to \{\text{contact elements of surfaces in } \mathbb{R}^3\}$$

is called a principal contact-element net if any two neighbouring contact elements (x, P), (x_i, P_i) have a sphere $S^{(i)}$ in common, that is, a sphere touching both planes P, P_i at the corresponding points x, x_i.

Hence, the normals to the neighbouring planes P, P_i at the corresponding points x, x_i intersect at a point $c^{(i)}$ (the centre of the sphere $S^{(i)}$), and the distances from $c^{(i)}$ to x and to x_i are equal (see Figure 10). These spheres $S^{(i)}$, which are associated with edges of $\mathbb{Z}^2$ parallel to the i-th coordinate axis, will be called *principal-curvature spheres* of the discrete surface.

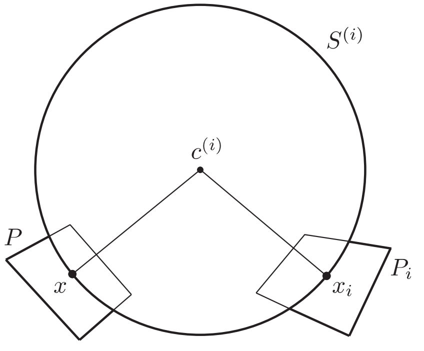

Figure 10. Principal-curvature sphere

A direct translation of Definition 18 into the language of the projective model is as follows.

Definition 19 (principal contact-element nets, projective model). A map $\ell\colon \mathbb{Z}^2 \to \mathscr{L}_0^{4,2}$ is called a principal contact-element net if it is a discrete congruence of isotropic lines in $\mathbb{P}(\mathbb{R}^{4,2})$, that is, any two neighbouring lines intersect:

$$\ell(u) \cap \ell(u + e_i) = \widehat{s}^{(i)}(u) \in \mathbb{P}(\mathbb{L}^{4,2}) \qquad \forall\, u \in \mathbb{Z}^2, \quad \forall\, i = 1, 2. \tag{32}$$

In the projective model the representatives of the principal-curvature spheres $S^{(i)}$ of the i-th coordinate direction form the corresponding focal net of the line congruence ℓ:

$$\widehat{s}^{(i)}\colon \mathbb{Z}^2 \to \mathbb{P}(\mathbb{L}^{4,2}), \qquad i = 1, 2 \tag{33}$$

(cf. Definition 11). According to Theorem 12, both focal nets are Q-nets in $\mathbb{P}(\mathbb{R}^{4,2})$. This motivates the following definition.

Definition 20 (discrete R-congruence of spheres). A map

$$S\colon \mathbb{Z}^m \to \{\text{oriented spheres in } \mathbb{R}^3\}$$

is called a discrete R-congruence (Ribaucour congruence) of spheres if the corresponding map

$$\widehat{s}\colon \mathbb{Z}^m \to \mathbb{P}(\mathbb{L}^{4,2})$$

is a Q-net in $\mathbb{P}(\mathbb{R}^{4,2})$.

A geometric characterization of discrete R-congruences will be given in § 5.

Corollary 21 (principal-curvature spheres form an R-congruence). *For a discrete contact-element net, the principal-curvature spheres of the i-th coordinate direction* ($i = 1, 2$) *form a two-dimensional discrete R-congruence.*

Turning to transformations of principal contact-element nets, we introduce the following definition.

Definition 22 (Ribaucour transforms, Euclidean model). Two principal contact-element nets

$$(x, P), (x^+, P^+)\colon \mathbb{Z}^2 \to \{\text{contact elements of surfaces in } \mathbb{R}^3\}$$

are called Ribaucour transforms of each other if any two corresponding contact elements (x, P) and (x^+, P^+) have a sphere S in common, that is, a sphere which touches both planes P, P^+ at the corresponding points x, x^+.

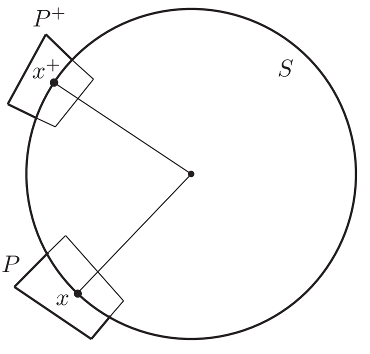

Figure 11. Ribaucour transformation

Again, a direct translation of Definition 22 into the language of the projective model gives the following.

Definition 23 (Ribaucour transforms, projective model). Two principal contact-element nets $\ell, \ell^+\colon \mathbb{Z}^2 \to \mathscr{L}_0^{4,2}$ are called Ribaucour transforms of each other if these discrete congruences of isotropic lines are in the relation of F-transformation, that is, if any pair of the corresponding lines intersect:

$$\ell(u) \cap \ell^+(u) = \widehat{s}(u) \in \mathbb{P}(\mathbb{L}^{4,2}) \qquad \forall\, u \in \mathbb{Z}^2. \tag{34}$$

The spheres S of a Ribaucour transformation are associated with the vertices u of the lattice $\mathbb{Z}^2$, or, better, with the 'vertical' edges connecting the vertices $(u, 0)$ and $(u, 1)$ of the lattice $\mathbb{Z}^2 \times \{0, 1\}$. In the projective model their representatives

$$\widehat{s}\colon \mathbb{Z}^2 \to \mathbb{P}(\mathbb{L}^{4,2}) \tag{35}$$

form the focal net of the three-dimensional line congruence for the third coordinate direction. From Theorem 12 we get the following result.

Corollary 24 (spheres of a Ribaucour transformation form an R-congruence). *The spheres of a generic Ribaucour transformation form a discrete R-congruence.*

Now, we turn to the study of the geometry of an elementary quadrilateral of contact elements of a principal contact-element net, consisting of $\ell \sim (x, P)$, $\ell_1 \sim (x_1, P_1)$, $\ell_2 \sim (x_2, P_2)$, and $\ell_{12} \sim (x_{12}, P_{12})$.

We leave aside the degenerate *umbilic* situation, when the four lines have a common point and span a four-dimensional space. Geometrically, this means that one is dealing with four contact elements of a sphere $S \subset \mathbb{R}^3$. In this situation, one cannot draw any further conclusion about the four points x, x_1, x_2, x_{12} on the sphere S: they can be arbitrary.

In the non-umbilic situation the space spanned by the four lines ℓ, ℓ_1, ℓ_2, ℓ_{12} is three-dimensional. The four elements $\widehat{x}, \widehat{x}_1, \widehat{x}_2, \widehat{x}_{12} \in \mathbb{P}(\mathbb{L}^{4,2})$, corresponding to the points $x, x_1, x_2, x_{12} \in \mathbb{R}^3$, are obtained as the intersection of the four isotropic lines ℓ, ℓ_1, ℓ_2, ℓ_{12} with the projective hyperplane $\mathbb{P}(\mathbf{e}_6^\perp)$ in $\mathbb{P}(\mathbb{R}^{4,2})$. Therefore, the four elements $\widehat{x}$, $\widehat{x}_1$, $\widehat{x}_2$, $\widehat{x}_{12}$ lie in a plane. A suitable framework for the study of this configuration is the projective model of Möbius geometry. Namely, omitting the inessential (vanishing) $\mathbf{e}_6$-component, we arrive at a planar quadrilateral in the Möbius sphere $\mathbb{P}(\mathbb{L}^{4,1})$. We devote § 4.2 to the study of such objects.

Analogously, the four elements $\widehat{p}, \widehat{p}_1, \widehat{p}_2, \widehat{p}_{12} \in \mathbb{P}(\mathbb{L}^{4,2})$ corresponding to the planes $P, P_1, P_2, P_{12} \in \mathbb{R}^3$ are obtained as the intersection of the four isotropic lines ℓ, ℓ_1, ℓ_2, ℓ_{12} with the projective hyperplane $\mathbb{P}(\mathbf{e}_\infty^\perp)$ in $\mathbb{P}(\mathbb{R}^{4,2})$. Therefore, the four elements $\widehat{p}$, $\widehat{p}_1$, $\widehat{p}_2$, $\widehat{p}_{12}$ also lie in a plane. A suitable framework for the study of such a configuration is the projective model of Laguerre geometry; this will be realized in § 4.3.

4.2. Möbius geometry: circular nets. Circular nets were introduced and studied in the context of integrable systems in [46], [14], and [49].

Caution. In this subsection the notation $\widehat{x}$ refers to the Möbius-geometric representatives in $\mathbb{L}^{4,1}$, and not to the Lie-geometric ones in $\mathbb{L}^{4,2}$. The former are obtained from the latter by omitting the (vanishing) $\mathbf{e}_6$-component.

It is assumed that the principal contact-element nets under consideration are generic, that is, do not contain umbilic quadruples. The main result of this subsection is the following.

Theorem 25 (points of principal contact-element nets form circular nets). *For a principal contact-element net*

$$(x, P) \colon \mathbb{Z}^2 \to \{\textit{contact elements of surfaces in } \mathbb{R}^3\}$$

its points $x \colon \mathbb{Z}^2 \to \mathbb{R}^3$ *form a circular net.*

This statement refers to a new notion which can be defined in two different ways.

Definition 26 (circular net, Euclidean model). A net $x \colon \mathbb{Z}^m \to \mathbb{R}^3$ is said to be circular if the vertices of any elementary quadrilateral (x, x_i, x_{ij}, x_j) (for any $u \in \mathbb{Z}^m$ and for all pairs $1 \leqslant i \neq j \leqslant m$) lie on a circle (in particular, are co-planar).

Definition 27 (circular net, projective model). A net $x\colon \mathbb{Z}^m \to \mathbb{R}^3$ is said to be circular if the corresponding $\widehat{x}\colon \mathbb{Z}^m \to \mathbb{P}(\mathbb{L}^{4,1})$ is a Q-net in $\mathbb{P}(\mathbb{R}^{4,1})$.

This time the translation between the Euclidean model and the projective model is not straightforward and actually constitutes the matter of Theorem 25: indeed, this theorem has already been demonstrated (or rather, is obvious) in terms of Definition 27, and it remains to establish the equivalence of Definitions 26 and 27.

Conceptual proof. The linear subspace of $\mathbb{R}^{4,1}$ spanned by the isotropic vectors $\widehat{x}$, $\widehat{x}_i$, $\widehat{x}_j$, $\widehat{x}_{ij}$ is three-dimensional. Its orthogonal complement is thus two-dimensional and lies in $\mathbb{R}^{4,1}_{\mathrm{out}}$. Therefore, it represents a circle (an intersection of two spheres).

Computational proof. For arbitrary representatives $\widetilde{x} \in \mathbb{L}^{4,1}$ of $\widehat{x}$ the requirement of Definition 27 is equivalent to an equation of type (1). Since the representatives $\widehat{x} = x + \mathbf{e}_0 + |x|^2\mathbf{e}_\infty$ fixed in (14) lie in an affine hyperplane of $\mathbb{R}^{4,1}$ (their $\mathbf{e}_0$-component is equal to 1), one has an equation of type (2) for them. Clearly, this holds if and only if x is a Q-net in $\mathbb{R}^3$ and $|x|^2$ satisfies the same equation (2) as x. We show that the latter condition is equivalent to circularity. On a single planar elementary quadrilateral (x, x_i, x_{ij}, x_j) the function $|x|^2$ satisfies equation (2) simultaneously with $|x - c|^2 = |x|^2 - 2\langle x, c\rangle + |c|^2$ for any $c \in \mathbb{R}^3$. Choose c to be the centre of the circle through the three points x, x_i, x_j, so that $|x - c|^2 = |x_i - c|^2 = |x_j - c|^2$. Then equation (2) for $|x - c|^2$ turns into $|x_{ij} - c|^2 = |x - c|^2$, which means that x_{ij} lies on the same circle, and Theorem 25 follows.

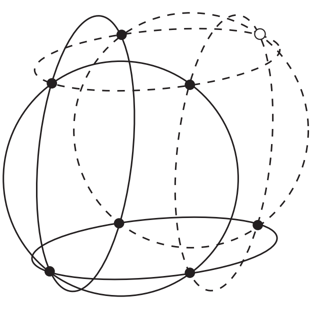

Figure 12. An elementary hexahedron of a circular net

Two-dimensional circular nets ($m = 2$) are discrete analogues of curvature-line parametrized surfaces, while the case $m = 3$ discretizes orthogonal coordinate systems in $\mathbb{R}^3$. A construction of an elementary hexahedron of a circular net is based on the following geometric theorem.

Theorem 28 (elementary hexahedron of a circular net). *Given seven points x, x_i, and x_{ij} $(1 \leqslant i < j \leqslant 3)$ in $\mathbb{R}^3$ such that each of the three quadruples (x, x_i, x_j, x_{ij}) lies on a circle C_{ij}, define three new circles $\tau_i C_{jk}$ as those passing through the triples (x_i, x_{ij}, x_{ik}), respectively. Then these new circles intersect at one point*

$$x_{123} = \tau_1 C_{23} \cap \tau_2 C_{31} \cap \tau_3 C_{12}$$

(*see Figure* 12).

Proof. This is a particular case of Theorem 15, applied to the quadric $\mathbb{P}(\mathbb{L}^{4,1})$.

Theorem 28 can be proven also by elementary geometric considerations. If one notes that under the conditions of the theorem the seven points x, x_i, x_{ij} lie on a two-dimensional sphere, and performs the stereographic projection of this sphere with pole at x, then one arrives at a planar picture which is nothing but the classical Miquel theorem.

4.3. Laguerre geometry: conical nets. Conical meshes have been introduced recently in [2].

Caution. In this subsection the notation $\widehat{p}$ refers to the Laguerre-geometric representatives in $\mathbb{L}^{3,1,1}$, and not to the Lie-geometric ones in $\mathbb{L}^{4,2}$. The former are obtained from the latter by omitting the (vanishing) $\mathbf{e}_0$-component.

As in the previous subsection, we assume that the principal contact-element nets under consideration do not contain umbilic quadruples. The main result of this subsection is the following.

Theorem 29 (tangent planes of principal contact-element nets form conical nets). *For a principal contact-element net*

$$(x, P)\colon \mathbb{Z}^2 \to \{\textit{contact elements of surfaces in } \mathbb{R}^3\}$$

its tangent planes $P\colon \mathbb{Z}^2 \to \{\textit{oriented planes in } \mathbb{R}^3\}$ *form a conical net.*

This statement refers to a notion which can be defined in two different ways.

Definition 30 (conical net, Euclidean model). A net

$$P\colon \mathbb{Z}^m \to \{\text{oriented planes in } \mathbb{R}^3\}$$

is called conical if for any $u \in \mathbb{Z}^m$ and for all pairs $1 \leqslant i \neq j \leqslant m$ the four planes P, P_i, P_{ij}, P_j touch a cone of revolution (in particular, intersect at the tip of the cone).

Definition 31 (conical net, projective model). A net

$$P\colon \mathbb{Z}^m \to \{\text{oriented planes in } \mathbb{R}^3\}$$

is called conical if the corresponding $\widehat{p}\colon \mathbb{Z}^m \to \mathbb{P}(\mathbb{L}^{3,1,1})$ is a Q-net in $\mathbb{P}(\mathbb{R}^{3,1,1})$.

Theorem 29 is obvious in terms of Definition 31, so the real content of this theorem is the translation between the Euclidean model and the projective model, that is, the equivalence of Definitions 30 and 31.

Proof. Representatives $\widehat{p}$ in (23) form a Q-net if and only if they satisfy equation (2), that is, $v\colon \mathbb{Z}^m \to \mathbb{S}^2$ and $d\colon \mathbb{Z}^m \to \mathbb{R}$ satisfy this equation. Equation (2) for v implies that $v\colon \mathbb{Z}^m \to \mathbb{S}^2$ is actually a Q-net in $\mathbb{S}^2$, so that any quadrilateral (v, v_i, v_{ij}, v_j) in $\mathbb{S}^2$ is planar and therefore circular. Equation (2) for (v, d) ensures that the (unique) intersection point of the three planes P, P_i, P_j lies on P_{ij} as well, so that all four planes intersect in one point. Thus, we have arrived at a characterization of conical nets in the sense of Definition 31 as those nets of planes for which every quadruple (P, P_i, P_{ij}, P_j) of planes is concurrent and every quadrilateral (v, v_i, v_{ij}, v_j) of unit normal vectors is planar. It is clear that this description is equivalent to that of Definition 30. The direction of the axis of the tangent cone coincides with the spherical centre of the quadrilateral (v, v_i, v_{ij}, v_j) in $\mathbb{S}^2$. Theorem 29 is proved.

Thus, conical nets are Q-nets with circular Gauss maps. It is worthwhile to mention that, in order to prescribe a conical net, it is enough to prescribe a circular Gauss map $v\colon \mathbb{Z}^m \to \mathbb{S}^2$ and additionally the numbers d (that is, the planes $P = (v, d)$) along the coordinate axes of $\mathbb{Z}^m$. Indeed, these data allow one to reconstruct the conical net uniquely. This is done via a recursive procedure, whose elementary step consists in finding the fourth plane P_{ij} from the three planes P, P_i, P_j and the normal direction v_{ij} of the fourth plane. But this is easy: P_{ij} is the plane normal to v_{ij} through the unique intersection point of the three planes P, P_i, P_j.

4.4. Synthesis. In view of Theorems 25 and 29 it is natural to ask whether for a given circular net $x\colon \mathbb{Z}^2 \to \mathbb{R}^3$ or a conical net

$$P\colon \mathbb{Z}^2 \to \{\text{oriented planes in } \mathbb{R}^3\}$$

there exists a principal contact-element net

$$(x, P)\colon \mathbb{Z}^2 \to \{\text{contact elements of surfaces in } \mathbb{R}^3\}$$

with the prescribed half of the data (x or P). The positive answer to this question is a corollary of the following general theorem.

Theorem 32 (extending R-congruences of spheres to principal contact-element nets). *For any discrete R-congruence of spheres*

$$S\colon \mathbb{Z}^2 \to \{\textit{oriented spheres in } \mathbb{R}^3\},$$

there exists a two-parameter family of principal contact-element nets

$$(x, P)\colon \mathbb{Z}^2 \to \{\textit{contact elements of surfaces in } \mathbb{R}^3\}$$

such that S belongs to the contact element (x, P), that is, P is the tangent plane to S at the point $x \in S$, for all $u \in \mathbb{Z}^2$. Such a principal contact-element net is uniquely determined by prescribing a contact element $(x, P)(0, 0)$ containing the sphere $S(0, 0)$.

Proof. The input data is a Q-net $\widehat{s}\colon \mathbb{Z}^2 \to \mathbb{P}(\mathbb{L}^{4,2})$ in the Lie quadric, and we are looking for a congruence of isotropic lines $\ell\colon \mathbb{Z}^2 \to \mathscr{L}_0^{4,2}$ such that $\widehat{s}(u) \in \ell(u)$ for all $u \in \mathbb{Z}^2$. The construction starts with an arbitrary isotropic line $\ell(0, 0)$ through $\widehat{s}(0, 0)$, and hinges on the following lemma.

Lemma 33. *For an isotropic line $\ell \in \mathscr{L}_0^{4,2}$ and a point $\widehat{s}_1 \in \mathbb{P}(\mathbb{L}^{4,2})$ not lying on ℓ, there is a unique isotropic line ℓ_1 through $\widehat{s}_1$ intersecting ℓ.*

Proof. Let $\widehat{s}$ and $\widehat{\sigma}$ be two arbitrary points on ℓ (in homogeneous coordinates), so that the line ℓ is given by the linear combinations $\alpha\widehat{s} + \beta\widehat{\sigma}$. The relation $\langle \alpha\widehat{s} + \beta\widehat{\sigma}, \widehat{s}_1 \rangle = 0$ implies that

$$\alpha : \beta = -\langle \widehat{\sigma}, \widehat{s}_1 \rangle : \langle \widehat{s}, \widehat{s}_1 \rangle.$$

Thus, there exists a unique point $\widehat{s}^{(1)} \in \ell$ such that $\langle \widehat{s}^{(1)}, \widehat{s}_1 \rangle = 0$. Now ℓ_1 is the line through $\widehat{s}_1$ and $\widehat{s}^{(1)}$. Lemma 33 is proved.

Let us continue the proof of Theorem 32. With the help of Lemma 33, one can construct the isotropic lines of the congruence along the coordinate axes:

$$\ell\colon \mathbb{Z}\times\{0\}\to\mathscr{L}_0^{4,2} \quad \text{and} \quad \ell\colon \{0\}\times\mathbb{Z}\to\mathscr{L}_0^{4,2}.$$

Next, one has to extend the congruence ℓ from the coordinate axes to the whole of $\mathbb{Z}^2$. An elementary step of this extension consists in finding, for three given isotropic lines ℓ, ℓ_1, ℓ_2 (such that ℓ intersects both ℓ_1 and ℓ_2), a fourth line ℓ_{12} intersecting ℓ_1 and ℓ_2 and going through a given point $\widehat{s}_{12}$. One can use Lemma 33 here, but then one has to demonstrate that this construction is consistent, that is, that the lines ℓ_{12} obtained from the requirements of intersecting with ℓ_1 and with ℓ_2 coincide. We show this with the following argument. The space $V=\mathrm{span}(\ell,\ell_1,\ell_2)$ is three-dimensional. The points $\widehat{s}$, $\widehat{s}_1$, $\widehat{s}_2$ lie in V. By the hypothesis of the theorem, the quadrilateral $(\widehat{s},\widehat{s}_1,\widehat{s}_{12},\widehat{s}_2)$ is planar, and therefore $\widehat{s}_{12}$ also lies in V. Draw two planes in V: $\Pi_1=\mathrm{span}(\ell_1,\widehat{s}_{12})$ and $\Pi_2=\mathrm{span}(\ell_2,\widehat{s}_{12})$. Their intersection is a line ℓ_{12} through $\widehat{s}_{12}$. It remains to prove that this line is isotropic. To this end, note that ℓ_{12} can be alternatively described as the line through the two points $\widehat{s}_1^{(2)}=\ell_1\cap\ell_{12}$ and $\widehat{s}_2^{(1)}=\ell_2\cap\ell_{12}$. Both these points lie in $\mathbb{P}(\mathbb{L}^{4,2})$, since they belong to the isotropic lines ℓ_1 and ℓ_2, respectively. But it is easy to see that a line in $\mathbb{P}(\mathbb{R}^{4,2})$ through two points in $\mathbb{P}(\mathbb{L}^{4,2})$ either is isotropic or contains no other points in $\mathbb{P}(\mathbb{L}^{4,2})$, depending on whether or not these two points are polar to each other (with respect to $\mathbb{P}(\mathbb{L}^{4,2})$). In our case the line ℓ_{12} contains, by construction, one other point $\widehat{s}_{12}$ in $\mathbb{P}(\mathbb{L}^{4,2})$, and hence it has to be isotropic. Theorem 32 is proved.

Since the representatives $\widehat{x}$ in $\mathbb{P}(\mathbb{L}^{4,2})$ of a circular net $x\colon\mathbb{Z}^2\to\mathbb{R}^3$ form a Q-net in $\mathbb{P}(\mathbb{R}^{4,2})$, and the same holds for the representatives $\widehat{p}$ in $\mathbb{P}(\mathbb{L}^{4,2})$ of a conical net $P\colon\mathbb{Z}^2\to\{\text{oriented planes in }\mathbb{R}^3\}$, we come to the following conclusion (obtained independently by Pottmann [57]).

Corollary 34 (extending circular and conical nets to principal contact-element nets).

i) *For any circular net* $x\colon\mathbb{Z}^2\to\mathbb{R}^3$ *there is a two-parameter family of conical nets* $P\colon\mathbb{Z}^2\to\{\textit{planes in }\mathbb{R}^3\}$ *such that* $x\in P$ *for all* $u\in\mathbb{Z}^2$*, and the contact-element net*

$$(x,P)\colon\mathbb{Z}^2\to\{\textit{contact elements of surfaces in }\mathbb{R}^3\}$$

is principal. Such a conical net is uniquely determined by prescribing a plane $P(0,0)$ *through the point* $x(0,0)$.

ii) *For any conical net* $P\colon\mathbb{Z}^2\to\{\textit{oriented planes in }\mathbb{R}^3\}$ *there is a two-parameter family of circular nets* $x\colon\mathbb{Z}^2\to\mathbb{R}^3$ *such that* $x\in P$ *for all* $u\in\mathbb{Z}^2$*, and the contact-element net*

$$(x,P)\colon\mathbb{Z}^2\to\{\textit{contact elements of surfaces in }\mathbb{R}^3\}$$

is principal. Such a circular net is uniquely determined by prescribing a point $x(0,0)$ *in the plane* $P(0,0)$.

These relations can be summarized as in Figure 13. We note that the axes of conical nets corresponding to a given circular net coincide with the Gauss map at its vertices, which was considered by Schief [54].

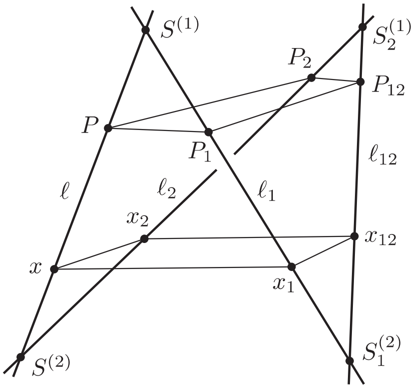

Figure 13. Elementary quadrilateral of a curvature-line parametrized discrete surface with vertices x and tangent planes P in the projective model. The vertices x form a circular net (Möbius geometry), and lie in the planes P forming a conical net (Laguerre geometry). The contact elements (x, P) are represented by isotropic lines ℓ (Lie geometry). The principal-curvature spheres $S^{(i)}$ pass through pairs of neighbouring points x, x_i and are tangent to the corresponding pairs of planes P, P_i.

Remark. In the situation of Corollary 34, that is, when the R-congruence S consists of points x (and is therefore a circular net) or of planes P (and is therefore a conical net), the elementary construction step of Lemma 33 allows for a very simple description from the Euclidean perspective in $\mathbb{R}^3$. This has been given by Pottmann [57].

i) *Given a contact element (x, P) and a point x_1, find a plane P_1 through x_1 such that there exists a sphere $S^{(1)}$ tangent to both planes P and P_1 at the points x and x_1, respectively.* Solution: P_1 is obtained from P by reflection in the bisecting orthogonal plane $\mathscr{P}$ of the edge $[x, x_1]$. The centre $c^{(1)}$ of $S^{(1)}$ is found as the intersection of the normal to P at x with the plane $\mathscr{P}$.

ii) *Given a contact element (x, P) and a plane P_1, find a point x_1 in P_1 such that there exists a sphere $S^{(1)}$ tangent to both planes P and P_1 at the points x and x_1, respectively.* Solution: the point x_1 is obtained from x by reflection in the bisecting plane $\overline{\mathscr{P}}$ of the dihedral angle formed by P and P_1. The centre $c^{(1)}$ of $S^{(1)}$ is found as the intersection of the normal to P at x with the plane $\overline{\mathscr{P}}$.

§ 5. R-congruences of spheres

In § 4 (Corollaries 21 and 24) we saw that the principal-curvature spheres of a principal contact-element net and the spheres of a Ribaucour transformation form discrete R-congruences, introduced in Definition 20. In this section we study the geometry of discrete R-congruences of spheres. Definition 20 can be re-formulated as follows: a map

$$S\colon \mathbb{Z}^m \to \{\text{oriented spheres in } \mathbb{R}^3\}$$

or the corresponding map

$$\widehat{s}\colon \mathbb{Z}^m \to \mathbb{L}^{4,2} \subset \mathbb{R}^{4,2}$$

into the space of homogeneous coordinates is called a discrete R-congruence of spheres if for any $u \in \mathbb{Z}^m$ and for any pair $1 \leqslant i \neq j \leqslant m$ the linear subspace

$$\Sigma = \operatorname{span}(\widehat{s}, \widehat{s}_i, \widehat{s}_j, \widehat{s}_{ij})$$

is three-dimensional. Thus, to any elementary square of $\mathbb{Z}^m$ there corresponds a three-dimensional linear subspace $\Sigma \subset \mathbb{R}^{4,2}$.

The R-congruence of principal-curvature spheres $S^{(i)}$ of the i-th coordinate direction is degenerate in the sense that the subspaces of its elementary quadrilaterals

$$\Sigma = \operatorname{span}(\widehat{s}^{(i)}, \widehat{s}^{(i)}_i, \widehat{s}^{(i)}_{ij}, \widehat{s}^{(i)}_j)$$

contain two-dimensional isotropic subspaces (corresponding to ℓ_i and ℓ_{ij}). The R-congruence of spheres of a generic Ribaucour transformation is, on the contrary, non-degenerate: its subspaces Σ do not contain two-dimensional isotropic subspaces, and its elementary quadrilaterals are included in planar families of spheres, which we introduce in the following definition.

Definition 35 (planar family of spheres). A planar family of spheres is a set of spheres whose representatives $\widehat{s} \in \mathbb{P}(\mathbb{L}^{4,2})$ are contained in a projective plane $\mathbb{P}(\Sigma)$, where Σ is a three-dimensional linear subspace of $\mathbb{R}^{4,2}$ such that the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle$ to Σ is non-degenerate.

Thus, a planar family of spheres is an intersection $\mathbb{P}(\Sigma \cap \mathbb{L}^{4,2})$. Clearly, there are two possibilities:

(a) $\langle\,\cdot\,,\,\cdot\,\rangle|_\Sigma$ has signature $(2,1)$, so that the signature of $\langle\,\cdot\,,\,\cdot\,\rangle|_{\Sigma^\perp}$ is also $(2,1)$;

(b) $\langle\,\cdot\,,\,\cdot\,\rangle|_\Sigma$ has signature $(1,2)$, so that the signature of $\langle\,\cdot\,,\,\cdot\,\rangle|_{\Sigma^\perp}$ is $(3,0)$.

It is easy to see that a planar family is one-parametric, parametrized by a circle $\mathbb{S}^1$. Indeed, if e_1, e_2, e_3 is an orthogonal basis of Σ such that, say, $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = -\langle e_3, e_3\rangle = 1$, then the spheres of the planar family come from the linear combinations $\widehat{s} = \alpha_1 e_1 + \alpha_2 e_2 + e_3$ with

$$\langle \alpha_1 e_1 + \alpha_2 e_2 + e_3, \alpha_1 e_1 + \alpha_2 e_2 + e_3\rangle = 0 \iff \alpha_1^2 + \alpha_2^2 = 1.$$

In the second of the cases mentioned above, the space $\Sigma^\perp$ has only a trivial intersection with $\mathbb{L}^{4,2}$, so that the spheres of the planar family $\mathbb{P}(\mathbb{L}^{4,2} \cap \Sigma)$ have no common tangent spheres. This case has no counterpart in smooth differential geometry. From the point of view of discrete differential geometry the first case is more significant.

Definition 36 (cyclidic family of spheres). A planar family of spheres is called cyclidic if the signature of $\langle\,\cdot\,,\,\cdot\,\rangle|_\Sigma$ is $(2,1)$, so that the signature of $\langle\,\cdot\,,\,\cdot\,\rangle|_{\Sigma^\perp}$ is also $(2,1)$.

Thus, for a cyclidic family $\mathbb{P}(\mathbb{L}^{4,2} \cap \Sigma)$ there is a *dual* cyclidic family $\mathbb{P}(\mathbb{L}^{4,2} \cap \Sigma^\perp)$ such that any sphere of the first family is in oriented contact with any sphere of the second family. The family $\mathbb{P}(\mathbb{L}^{4,2} \cap \Sigma)$, as a one-parameter family of spheres, envelopes a canal surface in $\mathbb{R}^3$, and this surface is an envelope of the dual family

$\mathbb{P}(\mathbb{L}^{4,2} \cap \Sigma^\perp)$. Such surfaces are called *Dupin cyclides*. Thus, to any elementary quadrilateral of a discrete R-congruence whose spheres $(\widehat{s}, \widehat{s}_i, \widehat{s}_{ij}, \widehat{s}_j)$ span a subspace of signature $(2, 1)$ there corresponds a Dupin cyclide.

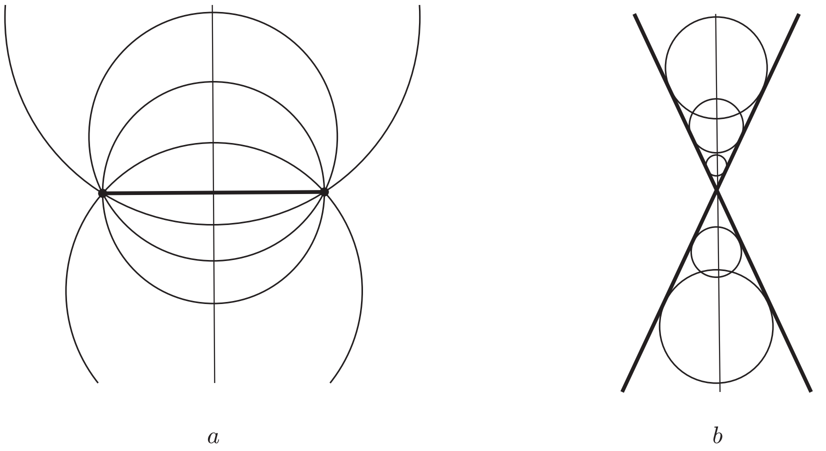

Figure 14. a: A cyclidic family of spheres through a circle; b: a cyclidic family of spheres tangent to a cone

Example. a) The points of a circle form a planar cyclidic family of spheres (of radius zero). The dual family consists of all (oriented) spheres containing this circle, with centres lying on the line through the centre of the circle and orthogonal to its plane (see Figure 14a). The corresponding Dupin cyclide is the circle itself. It can be shown that any Dupin cyclide is an image of this case under a Lie sphere transformation. For a circular net, regarded as a discrete R-congruence, each elementary quadrilateral carries such a structure.

b) The planes tangent to a cone of revolution also form a planar cyclidic family of spheres. The dual family consists of all (oriented) spheres tangent to the cone, with centres lying on the axis of the cone (see Figure 14b). The corresponding Dupin cyclide is the cone itself. For a conical net, regarded as a discrete R-congruence, each elementary quadrilateral carries such a structure.

Theorem 37 (common tangent spheres of two neighbouring quadrilaterals of an R-congruence). *For a discrete R-congruence of spheres and for two neighbouring quadrilaterals of it carrying cyclidic families there are generically exactly two spheres tangent to all six spheres of the congruence.*

Proof. Suppose that these quadrilaterals belong to the planar families generated by subspaces Σ_1 and Σ_2 of signature $(2, 1)$. The quadrilaterals share two spheres $\widehat{s}_1$ and $\widehat{s}_2$, which span a linear space of signature $(1, 1)$. Each of the planar families Σ_1 and Σ_2 adds one space-like vector, hence the linear space $\Sigma_1 \cup \Sigma_2$ spanned by all six spheres is four-dimensional and has signature $(3, 1)$. Therefore, its orthogonal complement $(\Sigma_1 \cup \Sigma_2)^\perp$ is two-dimensional and has signature $(1, 1)$. The intersection of $\mathbb{L}^{4,2}$ with a two-dimensional linear subspace of signature $(1, 1)$ contains, upon projectivization, exactly two spheres: indeed, if e_1, e_2 form an orthogonal basis of

$(\Sigma_1 \cup \Sigma_2)^\perp$ with $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1$, then the spheres in this space correspond to the combinations $\alpha_1 e_1 + \alpha_2 e_2$ with

$$\langle \alpha_1 e_1 + \alpha_2 e_2, \alpha_1 e_1 + \alpha_2 e_2\rangle = 0 \iff \alpha_1^2 = \alpha_2^2 \iff \alpha_1 : \alpha_2 = \pm 1.$$

Theorem 37 is proved.

In particular:

a) For any two neighbouring quadrilaterals of a circular net there is a single non-oriented sphere (hence two oriented spheres) containing both circles. Its centre is the intersection point of the lines passing through the centres of the circles and orthogonal to their respective planes (see Figure 15a).

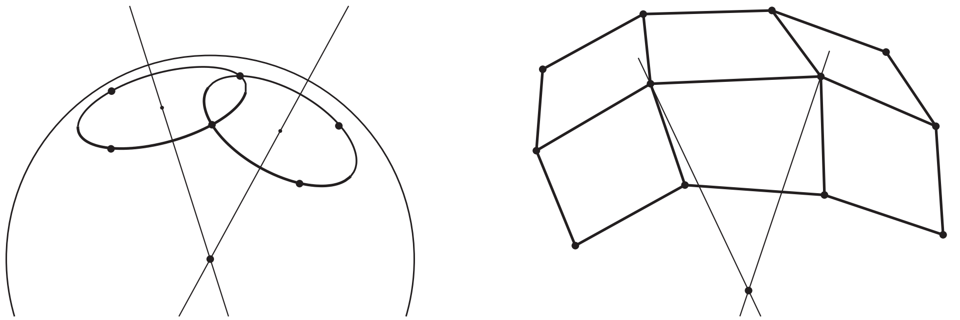

Figure 15. a: The normals of two neighbouring quadrilaterals of a circular net intersect: both lie in the bisecting orthogonal plane of the common edge. b: The axes of the cones of two neighbouring quadrilaterals of a conical net intersect: the two common planes of the quadrilaterals are tangent to both cones, therefore the axes of both cones lie in the bisecting plane of the dihedral angle of these two planes.

b) For any two neighbouring quadrilaterals of a conical net, there is a unique oriented sphere touching both cones (the second such sphere is the point at infinity). The centre of this sphere is the intersection point of the axes of the cones (see Figure 15b).

The next theorem is proved in exactly the same way as Theorem 37.

Theorem 38 (common tangent spheres of an elementary hexahedron of an R-congruence). *For an elementary hexahedron of a discrete R-congruence of spheres, with all faces carrying cyclidic families, there are generically exactly two spheres tangent to all the eight spheres at its vertices.*

It should be mentioned that these spheres associated with elementary hexahedra do not form a discrete R-congruence, contrary to what was asserted by Doliwa as a main result of [62].

We now turn to a geometric characterization of discrete R-congruences. From equation (4) it follows immediately that a map

$$S\colon \mathbb{Z}^m \to \{\text{oriented spheres in } \mathbb{R}^3\}$$

is a discrete R-congruence if and only if the centres $c\colon \mathbb{Z}^m \to \mathbb{R}^3$ of the spheres form a Q-net in $\mathbb{R}^3$, and the two real-valued functions

$$|c|^2 - r^2\colon \mathbb{Z}^m \to \mathbb{R} \quad \text{and} \quad r\colon \mathbb{Z}^m \to \mathbb{R}$$

satisfy the same equation of type (2) as the centres c. By omitting the latter requirement for the signed radii r, one arrives at a less restrictive definition than that of R-congruence. Actually, this definition belongs to Möbius geometry and uses the notation in § 3.2 (with $N = 3$).

Definition 39 (Q-congruence of spheres). A map

$$S\colon \mathbb{Z}^m \to \{\text{non-oriented spheres in } \mathbb{R}^3\} \tag{36}$$

is called a Q-congruence of spheres if the corresponding map

$$\widehat{s}\colon \mathbb{Z}^m \to \mathbb{P}(\mathbb{R}^{4,1}_{\text{out}}), \qquad \widehat{s} = c + \mathbf{e}_0 + \big(|c|^2 - r^2\big)\mathbf{e}_\infty, \tag{37}$$

is a Q-net in $\mathbb{P}(\mathbb{R}^{4,1})$.

Thus, a map (36) is a Q-congruence if and only if the centres $c\colon \mathbb{Z}^m \to \mathbb{R}^3$ of the spheres S form a Q-net in $\mathbb{R}^3$, and the function $|c|^2 - r^2$ satisfies the same equation (2) as the centres c.

Theorem 40 (characterization of R-congruences among Q-congruences). *Four (oriented) spheres (S, S_i, S_{ij}, S_j) in $\mathbb{R}^3$ comprise an elementary quadrilateral of an R-congruence if and only if they comprise (as non-oriented spheres) an elementary quadrilateral of a Q-congruence and satisfy the additional condition*

(R) *there exists a non-point sphere in oriented contact with all four oriented spheres S, S_i, S_j, S_{ij}.*

Under this condition, any sphere in oriented contact with the three spheres S, S_i, S_j is also in oriented contact with the fourth sphere S_{ij}.

Proof. Let S_0 be a sphere with centre c_0 and (finite) oriented radius $r_0 \neq 0$ in oriented contact with the three spheres S, S_i, S_j. This means that the following conditions are satisfied:

$$\langle c, c_0 \rangle - \frac{1}{2}(|c|^2 - r^2) - \frac{1}{2}(|c_0|^2 - r_0^2) - r r_0 = 0 \tag{38}$$

(tangency of S, S_0; see (10)) and the two analogous equations with (c, r) replaced by (c_i, r_i) and (c_j, r_j). Using the fact that c and $|c|^2 - r^2$ satisfy one and the same equation of type (2), we now conclude that equation (38) is fulfilled for (c_{ij}, r_{ij}) if and only if r satisfies the same equation (2) as c and $|c|^2 - r^2$. This proves the theorem in the case when the common tangent sphere S_0 for the three spheres S, S_i, S_j has finite radius. The case when S_0 has infinite radius (is actually a plane) is dealt with similarly, with the help of the equation

$$\langle c, v_0 \rangle - r - d_0 = 0, \tag{39}$$

which plays the role of (38). Theorem 40 is proved.

Remark. We have already seen that, generically, if three oriented spheres S, S_i, S_j have a common sphere in oriented contact, then they have a one-parameter (cyclidic) family of common touching spheres, represented by a three-dimensional linear subspace Σ of $\mathbb{R}^{4,2}$. It is easy to see that if the projection of Σ on $\mathbf{e}_\infty^\perp$ is non-vanishing, then the family of spheres represented by $\Sigma^\perp$ contains exactly two planes. (For a conical cyclidic family Σ all elements have vanishing $\mathbf{e}_0$-component and represent planes, while the family $\Sigma^\perp$ contains no planes.) Therefore, in all cases but the conical, the condition (R) can be replaced by the following requirement:

(R_0) *The four oriented spheres S, S_i, S_j, S_{ij} have a common tangent plane* (*actually*, *two common tangent planes*).

It remains to give a geometric characterization of Q-congruences. This is done in the following theorem.

Theorem 41 (three types of Q-congruences). *Four* (*non-oriented*) *spheres* (S, S_i, S_{ij}, S_j) *in* $\mathbb{R}^3$ *comprise an elementary quadrilateral of a Q-congruence if and only if they satisfy one of the following three conditions*:

(i) *they have a common orthogonal circle*;

(ii) *they intersect in a pair of points* (*a* 0*-sphere*);

(iii) *they intersect at exactly one point.*

Case (iii) *can be regarded as a degenerate case of both* (i) *and* (ii).

Caution. The notation in the proof below refers to Möbius-geometric objects, which are different from the Lie-geometric objects denoted by the same symbols.

Conceptual proof. The linear subspace Σ of $\mathbb{R}^{4,1}$ spanned by the points $\widehat{s}$, $\widehat{s}_i$, $\widehat{s}_j$, $\widehat{s}_{ij}$ is three-dimensional, so that its orthogonal complement $\Sigma^\perp$ is two-dimensional. If $\Sigma^\perp$ lies in $\mathbb{R}^{4,1}_{\mathrm{out}}$, that is, if the restriction of the Minkowski scalar product to $\Sigma^\perp$ is positive-definite (of signature $(2,0)$), then $\Sigma^\perp$ represents a 1-sphere (a circle) orthogonal to our four spheres, and we have the case (i). If, on the contrary, the restriction of the scalar product to $\Sigma^\perp$ has signature $(1,1)$, so that Σ lies in $\mathbb{R}^{4,1}_{\mathrm{out}}$, then Σ represents a 0-sphere which is the intersection of our four spheres, and we have the case (ii). Finally, if the restriction of the scalar product to $\Sigma^\perp$ is degenerate, then $\Sigma\cap\Sigma^\perp$ is an isotropic one-dimensional linear subspace, which represents the common point of our four spheres, and we have the case (iii).

Computational proof. The quadrilateral in $\mathbb{R}^3$ with vertices at the sphere centres c, c_i, c_j, c_{ij} is planar; denote its plane by Π. In the same way as in the proof of Theorem 25 we show that there is a point $C\in\Pi$ such that

$$|c-C|^2-r^2=|c_i-C|^2-r_i^2=|c_j-C|^2-r_j^2=|c_{ij}-C|^2-r_{ij}^2. \qquad (40)$$

Indeed, the first two of these equations define C uniquely as the intersection of two lines ℓ_i and ℓ_j in Π, where

$$\ell_i=\{x\in\Pi:\langle 2x-c_i-c,\,c_i-c\rangle=r^2-r_i^2\},$$

and then the last equation in (40) is automatically satisfied. If the common value of all four expressions in (40) is positive (say, equal to R^2), then these four spheres are orthogonal to the circle with centre C and radius R in the plane Π, so that we have the case (i) (see Figure 16). If the common value of (40) is negative (say, equal to $-R^2$), then the pair of points on the line through C orthogonal to Π and at distance R from C belong to all four spheres, so that we have the case (ii). Finally, if the common value of (40) is equal to 0, then C is the intersection point of all four spheres, and we have the case (iii). Theorem 41 is proved.

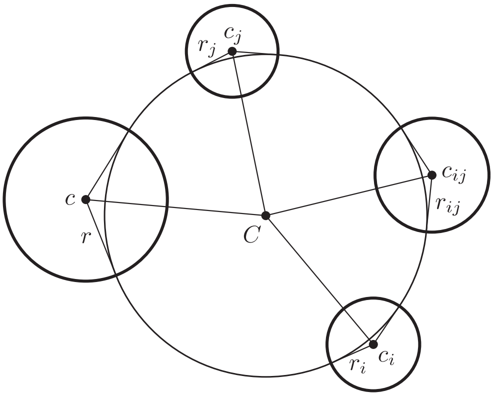

Figure 16. Elementary quadrilateral of a Q-congruence of spheres, the orthogonal circle case

Clearly, case (i) of Q-congruences reduces to circular nets if the radii of all the spheres become infinitely small (see Figure 16). Q-congruences with intersections of type (ii) are natural discrete analogues of sphere congruences parametrized along principal directions [37].

Some remarks about Q-congruences of spheres are in order here. They are multi-dimensionally consistent, with the following reservation: for any seven points $\widehat{s}$, $\widehat{s}_i$, $\widehat{s}_{ij}$ in $\mathbb{P}(\mathbb{R}^{4,1}_{\text{out}})$, the Q-property (planarity condition) uniquely determines the eighth point $\widehat{s}_{123}$ in $\mathbb{P}(\mathbb{R}^{4,1})$, which, however, might be outside of $\mathbb{P}(\mathbb{R}^{4,1}_{\text{out}})$, and therefore might not represent a real sphere. Thus, the corresponding discrete 3D system is well defined only on an open subset of the space of initial data. As long as it is defined, it can be used to produce transformations of Q-congruences, with the usual permutability properties.

We note a difference between Q-congruences and R-congruences: given three spheres S, S_i, S_j of an elementary quadrilateral, one has a two-parameter family for the fourth sphere S_{ij} in the case of a Q-congruence, and only a one-parameter family in the case of an R-congruence. This is a consequence of the fact that $\mathbb{R}^{4,1}_{\text{out}}$ is an open set in $\mathbb{R}^{4,1}$, while $\mathbb{L}^{4,2}$ is a hypersurface in $\mathbb{R}^{4,2}$.

Appendix 1. Cyclographic model of Laguerre geometry

In the cyclographic model of Laguerre geometry the preferred space is the space $(\mathbb{R}^{N,1,1})^*$ of hyperspheres, and therefore hyperspheres $S \subset \mathbb{R}^N$ are modelled as points $\widehat{s} \in \mathbb{P}\big((\mathbb{R}^{N,1,1})^*\big)$, while hyperplanes $P \subset \mathbb{R}^N$ are modelled as hyperplanes

$\{\xi : \langle \widehat{p}, \xi \rangle = 0\} \subset \mathbb{P}\big((\mathbb{R}^{N,1,1})^*\big)$. Thus, a hyperplane P is interpreted as the set of hyperspheres S which are in oriented contact with P.

Basic features of this model.

(i) The set of oriented hyperspheres $S \subset \mathbb{R}^N$ is in a one-to-one correspondence with the set of points

$$\sigma = (c, r) \tag{41}$$

of the Minkowski space $\mathbb{R}^{N,1}$ spanned by the vectors $\mathbf{e}_1, \dots, \mathbf{e}_N, \mathbf{e}_{N+3}$. This space can be interpreted as the affine part of $\mathbb{P}\big((\mathbb{R}^{N,1,1})^*\big)$.

(ii) Oriented hyperplanes $P \subset \mathbb{R}^N$ can be modelled as hyperplanes in $\mathbb{R}^{N,1}$:

$$\pi = \big\{(c, r) \in \mathbb{R}^{N,1} : \langle (v, 1), (c, r) \rangle = \langle v, c \rangle - r = d\big\}. \tag{42}$$

Thus, oriented hyperplanes $P \in \mathbb{R}^N$ are in a one-to-one correspondence with hyperplanes $\pi \subset \mathbb{R}^{N,1}$ which make an angle $\pi/4$ with the subspace $\mathbb{R}^N = \{(x, 0)\} \subset \mathbb{R}^{N,1}$.

(iii) An oriented hypersphere $S \subset \mathbb{R}^N$ is in oriented contact with an oriented hyperplane $P \subset \mathbb{R}^N$ if and only if $\sigma \in \pi$.

(iv) Two oriented hyperspheres $S_1, S_2 \subset \mathbb{R}^N$ are in oriented contact if and only if their representatives σ_1, σ_2 in the Minkowski space $\mathbb{R}^{N,1}$ differ by an isotropic vector: $|\sigma_1 - \sigma_2| = 0$.

In the cyclographic model the group of Laguerre transformations admits a beautiful description.

Theorem 42 (fundamental theorem of Laguerre geometry). *The group of Laguerre transformations is isomorphic to the subgroup of affine transformations of $\mathbb{R}^{N,1}$ of the form $y \mapsto \lambda A y + b$ with $A \in O(N,1)$, $\lambda > 0$, and $b \in \mathbb{R}^{N,1}$.*

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Received 20/NOV/06
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