PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2007) **463**, 3171–3193
doi:10.1098/rspa.2007.1902
Published online 25 September 2007

Isothermic surfaces in sphere geometries as Moutard nets

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We give an elaborated treatment of discrete isothermic surfaces and their analogues in different geometries (projective, Möbius, Laguerre and Lie). We find the core of the theory to be a novel characterization of discrete isothermic nets as Moutard nets. The latter are characterized by the existence of representatives in the space of homogeneous coordinates satisfying the discrete Moutard equation. Moutard nets admit also a projective geometric characterization as nets with planar faces with a five-point property: a vertex and its four diagonal neighbours span a three-dimensional space.

Restricting the projective theory to quadrics, we obtain Moutard nets in sphere geometries. In particular, Moutard nets in Möbius geometry are shown to coincide with discrete isothermic nets. The five-point property, in this particular case, states that a vertex and its four diagonal neighbours lie on a common sphere, which is a novel characterization of discrete isothermic surfaces. Discrete Laguerre isothermic surfaces are defined through the corresponding five-plane property, which requires that a plane and its four diagonal neighbours share a common touching sphere. Equivalently, Laguerre isothermic surfaces are characterized by having an isothermic Gauss map. S-isothermic surfaces as an instance of Moutard nets in Lie geometry are also discussed.

Keywords: discrete differential geometry; discrete surfaces; Möbius geometry; Moutard equation; Lie quadric

1. Introduction

This paper is a sequel to our paper 'On organizing principles of discrete differential geometry. Geometry of spheres' (Bobenko & Suris 2007), where the following discretization principles were formulated.

— *Transformation group principle.* Smooth geometric objects and their discretizations belong to the same geometry, i.e. they are invariant with respect to the same transformation group.
— *Consistency principle.* Discretizations of smooth parametrized geometries can be extended to multidimensional consistent nets.

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Received 2 November 2006
Accepted 31 August 2007

Being applied to discretization of curvature line parametrizations of general surfaces, these principles led to the definition of *principal contact element nets*. These are nets of contact elements with the property that neighbouring contact elements share a common sphere. In particular, it was shown that the points and the planes of principal contact element nets build circular and conical nets, respectively.

In the present paper, we turn to isothermic surfaces, which are a special class of surfaces, with the first fundamental form in the curvature line parametrization being conformal, possibly upon a re-parametrization of independent variables. Thus, isothermic surfaces are immersions $f:\mathbb{R}^2\to\mathbb{R}^3$, with

$$\langle\partial_1 f,\partial_2 f\rangle=0,\quad |\partial_1 f|^2=\alpha_1 s^2\quad\text{and}\quad |\partial_2 f|^2=\alpha_2 s^2, \tag{1.1}$$

where the function $s:\mathbb{R}^2\to\mathbb{R}_+$ is the metric of the surface and the functions α_i depend only on u_i (where $u=(u_1, u_2)$ are the independent variables). These conditions may be equivalently represented as

$$\partial_1\partial_2 f=(\partial_2\log s)\partial_1 f+(\partial_1\log s)\partial_2 f\quad\text{and}\quad\langle\partial_1 f,\partial_2 f\rangle=0. \tag{1.2}$$

This important class of surfaces has been studied by classics (Darboux 1914–1927). In particular, Darboux found a class of transformations of isothermic surfaces, nowadays carrying his name. Remarkable permutability properties of Darboux transformations have been established by Bianchi (1923; see also Eisenhart 1923). Another important discovery by Darboux was the following characterization of isothermic surfaces: if $\hat{f}=f+\boldsymbol{e}_0+|f|^2\boldsymbol{e}_\infty$ is the standard lift of the surface f into the Minkowski space $\mathbb{R}^{4,1}$ of pentaspherical coordinates, so that $\hat{f}$ belongs to the light cone $\mathbb{L}^{4,1}$ (the set of isotropic elements of this space), then the special lift $y=s^{-1}\hat{f}:\mathbb{R}^2\to\mathbb{L}^{4,1}$ satisfies a *Moutard equation* (Moutard 1878)

$$\partial_1\partial_2 y=q_{12}y, \tag{1.3}$$

with a scalar coefficient $q_{12}=s\partial_1\partial_2(s^{-1}):\mathbb{R}^2\to\mathbb{R}$. Conversely, any solution of a Moutard equation in the light cone $\mathbb{L}^{4,1}$ is a lift of an isothermic surface f. (For the reader's convenience, we briefly recall the Möbius-geometric formalism in §3*a*.) In the 1990s, a relation to the theory of integrable systems was discovered (Cieśliński *et al.* 1995; Bobenko & Pinkall 1996; Burstall *et al.* 1997). The theory was extended for isothermic surfaces in spaces of arbitrary dimension in Schief (2001) and Burstall (2006). A modern overview of isothermic surfaces can be found in Hertrich-Jeromin (2003).

In Bobenko & Pinkall (1996), the theory was discretized: discrete isothermic surfaces were defined as special circular nets with factorized cross-ratios of elementary quadrilaterals. This property is manifestly Möbius invariant. Moreover, it can be consistently imposed on three-dimensional nets (Bobenko & Pinkall 1999; Hertrich-Jeromin *et al.* 1999). Thus, discrete isothermic surfaces are an instance of geometry satisfying both the discretization principles.

In this paper, we give an elaborated treatment of discrete isothermic surfaces and their analogues in different geometries (projective, Möbius, Laguerre and Lie), applying the discretization principles systematically. We find the core of the theory to be a novel projective characterization of discrete isothermic nets as *discrete Moutard nets in the light cone*.

In the context of discrete differential geometry, discrete Moutard nets in Euclidean spaces were introduced in Nimmo & Schief (1997), as solutions of the *discrete Moutard equation*

$$\tau_1\tau_2 y + y = a_{12}(\tau_1 y + \tau_2 y), \qquad (1.4)$$

where τ_i is the shift in the ith coordinate direction of $\mathbb{Z}^2$, and $a_{12}:\mathbb{Z}^2\to\mathbb{R}$ is a discrete scalar coefficient. In the context of discrete integrable systems, this equation appeared earlier in Date *et al.* (1983). The multidimensional consistency of discrete Moutard nets, or, more precisely, of their close relatives satisfying the *discrete Moutard equation with minus sign*,

$$\tau_1\tau_2 y - y = a_{12}(\tau_2 y - \tau_1 y) \qquad (1.5)$$

is related to the fact that equation (1.5) expresses the permutability properties of the so-called Moutard transformation for the differential Moutard equation (Bianchi 1923; Ganzha & Tsarev 1996; Nimmo & Schief 1997). The role played by the discrete Moutard equation in the discrete differential geometry turns out to be manifold. In particular, the so-called Lelieuvre representation of discrete asymptotic nets involves discrete Moutard nets in $\mathbb{R}^3$ (Konopelchenko & Pinkall 2000; Doliwa 2001; Doliwa *et al.* 2001). A closely related notion of discrete Koenigs nets is worked out in Doliwa (2003).

Moutard nets, whose ambient space is regarded as the space of homogeneous coordinates of $\mathbb{RP}^N$, turn out to admit a projectively invariant interpretation. For multidimensional Moutard nets, $f:\mathbb{Z}^m\to\mathbb{RP}^N$, with $m\geq 3$, such an interpretation has been given previously (Doliwa 2007; planarity of tetrahedra formed by odd or even vertices of any elementary cube). For two-dimensional Moutard nets, $f:\mathbb{Z}^2\to\mathbb{RP}^N$, a projective characterization is given in §2. In the case $N\geq 4$, discrete two-dimensional Moutard nets are characterized (definition 2.1) as nets with planar faces possessing a five-point property: a vertex and its four diagonal neighbours span a three-dimensional space (thus, in comparison with a generic net having planar faces, the dimension drops by 1). We learned about this characterization of two-dimensional Moutard nets from conversations with A. Doliwa (personal communication). In the case $N=3$, the characterization is more involved (definition 2.3).

Why are both discretization principles (transformation group principle and multidimensional consistency principle) applicable simultaneously? As discussed in Bobenko & Suris (2007), the ultimate reason is the possibility of restricting the basic multidimensional systems of the projective geometric origin (Q-nets, asymptotic nets and line congruences) to quadrics. Recall that Q-nets, one of the basic objects of discrete differential geometry, have been introduced in Doliwa & Santini (1997) as maps $f:\mathbb{Z}^m\to\mathbb{RP}^N$, such that all elementary quadrilaterals $(f, \tau_i f, \tau_i\tau_j f, \tau_j f)$ are planar. In Doliwa (1999), it was shown that Q-nets can be restricted to quadrics. Restriction to quadrics is of crucial importance, since many of the classical geometries (such as Möbius, Laguerre, Lie geometries, as well as Plücker line geometry, hyperbolic geometry, etc.) can be characterized by a group of transformations preserving certain quadrics in a projective space. Since Moutard nets are shown to belong to projective geometry, they also can be restricted to quadrics. In the present paper, we investigate applications of this idea to sphere geometries.

Moutard nets in the light cone $\mathbb{L}^{N+1,1}$ of Möbius geometry are shown, in §3, to coincide with discrete isothermic nets defined via factorized real cross-ratios of elementary quadrilaterals. This result has been first found in

(Bobenko & Suris 2005). Surprisingly, this very natural generalization of Darboux's characterization of smooth isothermic surfaces had to wait for almost a decade after discrete isothermic surfaces were introduced in Bobenko & Pinkall (1996). The five-point property, in this particular case, states that a vertex and its four diagonal neighbours lie on a common sphere, which is a novel characterization of discrete isothermic surfaces.

In §4, we proceed with Moutard nets in the basic quadric $\mathbb{L}^{4,2}$ of Lie geometry. Such nets are special Ribaucour sphere congruences $\mathbb{Z}^2 \to \{\text{spheres in } \mathbb{R}^3\}$ with the corresponding five-sphere property, a particular case being S-isothermic surfaces (Bobenko & Pinkall 1999; Bobenko *et al.* 2006; Hoffmann, in preparation).

In Laguerre geometry, discrete surfaces are maps $\mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$. The planarity of faces in the Laguerre quadric is equivalent to the conical property, while the five-plane property requires that a plane and its four diagonal neighbours share a common touching sphere. This is a definition of discrete Laguerre isothermic surfaces (§5). Equivalently, discrete Laguerre isothermic surfaces are characterized by having an isothermic Gauss map. The latter class was independently introduced in Wallner & Pottmann (in press). Smooth Laguerre isothermic surfaces have been studied previously (Eisenhart 1923; Musso & Nicolodi 1997, 2000).

2. Discrete Moutard nets

In consideration of various nets $f: \mathbb{Z}^2 \to \mathcal{X}$, we use the following notational conventions: for some fixed $u \in \mathbb{Z}^2$, we write f for $f(u)$; further f_i for $\tau_i f(u) = f(u+e_i)$; and f_{-i} for $\tau_i^{-1} f(u) = f(u-e_i)$. Also, we freely use notations, definitions and results from Bobenko & Suris (2007).

(a) Projective Moutard nets

Definition 2.1 (Discrete Moutard net). A two-dimensional Q-net $f: \mathbb{Z}^2 \to \mathbb{RP}^N$ $(N \geq 4)$ is called a discrete Moutard net if, for every $u \in \mathbb{Z}^2$, the five points f and $f_{\pm 1, \pm 2}$ lie in a three-dimensional subspace $V = V(u) \subset \mathbb{RP}^N$, not containing some (and then any) of the four points $f_{\pm 1}, f_{\pm 2}$.

Thus, the defining condition of a discrete Moutard net deals with four elementary planar quadrilaterals adjacent to one vertex. As a consequence of this definition, all nine vertices of the four quadrilaterals of a discrete Moutard net lie in a four-dimensional subspace of $\mathbb{RP}^N$.

Theorem 2.2 (Discrete Moutard equation). *A discrete Moutard net $f: \mathbb{Z}^2 \to \mathbb{RP}^N$ possesses a lift to the space of homogeneous coordinates $y: \mathbb{Z}^2 \to \mathbb{R}^{N+1}$, satisfying the discrete Moutard equation (1.5) with some $a_{12}: \mathbb{Z}^2 \to \mathbb{R}$ (it is natural to assign the real numbers a_{12} to the elementary squares of $\mathbb{Z}^2$).*

Proof. We start with the observation that, for *any* Q-net f in $\mathbb{RP}^N$, it is always possible (and almost trivial) to find homogeneous coordinates for the four vertices of *one* elementary quadrilateral satisfying the discrete Moutard equation on that quadrilateral. Moreover, one can do this for an arbitrary choice of homogeneous coordinates for any two neighbouring vertices of the quadrilateral. Indeed, consider any homogeneous coordinates $\tilde{f}, \tilde{f}_1, \tilde{f}_2, \tilde{f}_{12} \in \mathbb{R}^{N+1}$ for the

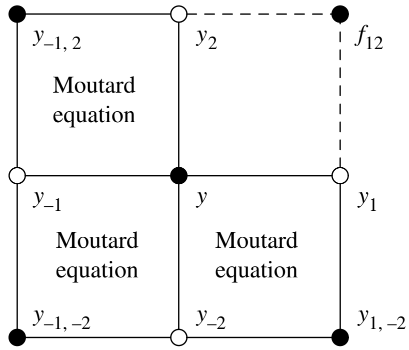

Figure 1. Constructing a Moutard representative for a projective Q-net with a three-dimensional black cross.

vertices of a planar quadrilateral, connected by a linear relation

$$\tilde{f}_{12} = \tilde{c}_{21}\tilde{f}_1 + \tilde{c}_{12}\tilde{f}_2 + \rho_{12}\tilde{f}.$$

If we keep the representatives $y = \tilde{f}$, $y_1 = \tilde{f}_1$ (say) and set $\tilde{f}_{12} = \rho_{12} y_{12}$ and $\tilde{f}_2 = a y_2$, with $a = -\tilde{c}_{21}/\tilde{c}_{12}$, then y satisfies the discrete Moutard equation (1.5) within one elementary quadrilateral.

Now, for Q-nets with a special property formulated in definition 2.1, this construction can be extended to the whole net. We start with arbitrary representatives y, y_1, and proceed clockwise around the vertex y. We then find consecutively the representatives y_{-2}, $y_{1,-2}$, which assure the Moutard equation on the quadrilateral $(y,\ y_1,\ y_{1,-2},\ y_{-2})$, the representatives y_{-1}, $y_{-1,-2}$, which assure the Moutard equation on the quadrilateral $(y, y_{-1}, y_{-1,-2}, y_{-2})$, and then the representatives y_2, $y_{-1,2}$, which assure the Moutard equation on the quadrilateral $(y,\ y_{-1},\ y_{-1,2},\ y_2)$ (figure 1).

In the remaining quadrilateral $(y,\ y_1,\ y_{12},\ y_2)$, the representatives y, y_1, y_2 are already fixed on the previous steps of the construction, so that we can dispose of the representative y_{12} of f_{12} only. Observe that the point with the representative $y_1 - y_2$ belongs to the plane $\Pi \subset \mathbb{RP}^N$ of the quadrilateral (f, f_1, f_{12}, f_2) (obviously) and to the three-dimensional space $V \subset \mathbb{RP}^N$ through the points f, $f_{1,-2}$, $f_{-1,-2}$ and $f_{-1,2}$, owing to the equation

$$\begin{aligned} y_1 - y_2 &= (y_1 - y_{-2}) + (y_{-2} - y_{-1}) + (y_{-1} - y_2) \\ &= \alpha(y_{1,-2} - y) + \beta(y_{-1,-2} - y) + \gamma(y_{-1,2} - y). \end{aligned}$$

By the hypothesis of the theorem, the point $f_{1,2}$ lies in the latter space V. Therefore, the whole line through f and $f_{1,2}$ lies in the intersection $\Pi \cap V$. Since $N \geq 4$, we conclude that, in general position, $\Pi \cap V$ *is* the line through f and f_{12}. Thus, the point with the representative $y_1 - y_2$ belongs to this line, therefore $y_1 - y_2$ is a linear combination of y and y_{12}. By a suitable choice of the representative y_{12} of f_{12}, we can make $y_1 - y_2$ proportional to $y_{12} - y$. Thus, the construction of representatives, satisfying the Moutard equation, closes up around any vertex. This allows the construction to be extended to the whole lattice $\mathbb{Z}^2$. ■

A related analytical observation is due to Doliwa *et al.* (2007): a four-point difference hyperbolic equation $a\tau_1\tau_2 y + b\tau_1 y + c\tau_2 y + dy = 0$ yields a certain five-point equation on the even and the odd sublattices of $\mathbb{Z}^2$ if and only if it is gauge equivalent to the discrete Moutard equation (1.4). However, the paper by Doliwa *et al.* (2007) does not address the geometry behind the equations (in particular, they consider only the complex fields $y\in\mathbb{C}$, and, in this context, there is no issue of dimensions of spaces spanned by solutions of equations).

Definition 2.1 is not applicable in the case when some, and then all, of the points $f_{\pm 1}$, $f_{\pm 2}$ lie in the three-dimensional space V through f, $f_{\pm 1,\pm 2}$; in particular, it cannot be used to define discrete Moutard nets in $\mathbb{RP}^3$. We show that theorem 2.2 remains valid if one defines discrete Moutard nets in $\mathbb{RP}^3$ as follows.

Definition 2.3 (Discrete Moutard net in $\mathbb{RP}^3$). A two-dimensional Q-net $f:\mathbb{Z}^2\to\mathbb{RP}^3$ is called a discrete Moutard net if, for every $u\in\mathbb{Z}^2$, the following condition is satisfied: the three planes

$$\Pi^{(\mathrm{up})}=(f,f_{12},f_{-1,2}),\quad \Pi^{(\mathrm{down})}=(f,f_{1,-2},f_{-1,-2})\quad \text{and}\quad \Pi^{(1)}=(f,f_1,f_{-1})$$

have a common line $\ell^{(1)}$.

Remark 2.4. It is not difficult to see that, in the context of definition 2.1, with $N\geq 4$, the requirement of definition 2.3 is automatically satisfied. Indeed, in this case, all nine points f, $f_{\pm 1}$, $f_{\pm 2}$ and $f_{\pm 1,\pm 2}$ lie in a four-dimensional subspace of $\mathbb{RP}^N$. In this subspace, one can consider, along with the three-dimensional subspace V, the three-dimensional subspaces $V^{(\mathrm{up})}$ containing the two quadrilaterals (f, f_1, f_{12}, f_2) and $(f, f_{-1}, f_{-1,2}, f_2)$, and $V^{(\mathrm{down})}$ containing the quadrilaterals $(f, f_1, f_{1,-2}, f_{-2})$ and $(f, f_{-1}, f_{-1,-2}, f_{-2})$. Obviously, one has

$$\Pi^{(\mathrm{up})}=V^{(\mathrm{up})}\cap V,\quad \Pi^{(\mathrm{down})}=V^{(\mathrm{down})}\cap V\quad \text{and}\quad \Pi^{(1)}=V^{(\mathrm{up})}\cap V^{(\mathrm{down})}.$$

Generically, the 3 three-dimensional subspaces V, $V^{(\mathrm{up})}$ and $V^{(\mathrm{down})}$ of a four-dimensional space intersect along a line $\ell^{(1)}$.

Remark 2.5. There is an asymmetry between the coordinate directions 1 and 2 in definition 2.3. However, this asymmetry is apparent: the condition in definition 2.3 is equivalent to the requirement that the three planes

$$\Pi^{(\mathrm{left})}=(f,f_{-1,2},f_{-1,-2}),\quad \Pi^{(\mathrm{right})}=(f,f_{1,2},f_{1,-2}),\quad \Pi^{(2)}=(f,f_2,f_{-2})$$

have a common line $\ell^{(2)}$. One way to see this is to consider a central projection of the whole picture from the point f to some plane not containing f. In this projection, the planarity of elementary quadrilaterals (f, f_i, f_{ij}, f_j) turns into collinearity of the triples of points f_i, f_j and f_{ij}. The traces of the planes $\Pi^{(\mathrm{up})}$, $\Pi^{(\mathrm{down})}$ and $\Pi^{(1)}$ on the projection plane are the lines $(f_{12}, f_{-1,2})$, $(f_{1,-2}, f_{-1,-2})$ and (f_1, f_{-1}), respectively, and the requirement of definition 2.3 turns into the requirement for these three lines to meet in a point. Similarly, the traces of the planes $\Pi^{(\mathrm{left})}$, $\Pi^{(\mathrm{right})}$ and $\Pi^{(2)}$ on the projection plane are the lines $(f_{-1,2}, f_{-1,-2})$, $(f_{1,2}, f_{1,-2})$ and (f_2, f_{-2}), respectively. The requirement for the latter three lines to meet in a point is equivalent to the previous one—this is the statement of the famous Desargues theorem (figure 2).

Another way to demonstrate the actual symmetry between the coordinate directions 1 and 2 in definition 2.3 is to show that theorem 2.2 still holds in $\mathbb{RP}^3$. Indeed, the discrete Moutard equation (1.5) is manifestly symmetric with respect to the flip $1\leftrightarrow 2$.

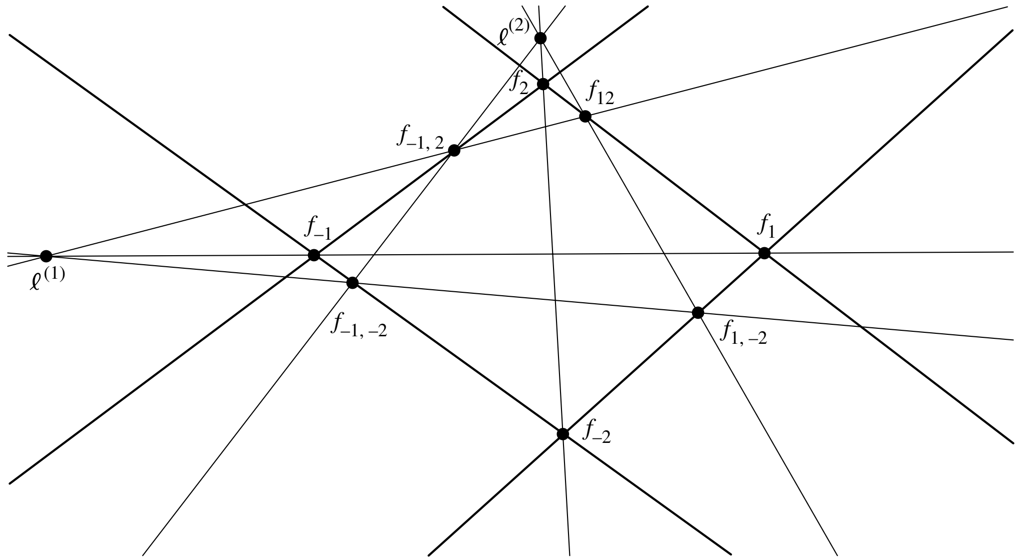

Figure 2. Desargues theorem.

Proof of theorem 2.2 for $N=3$. We start the proof exactly as in the general case $N\geq 4$. The only thing to be changed is the demonstration of the fact that the point with the representative y_2-y_1 lies on the line through f and f_{12}. To do this in the present situation, we first observe that, due to

$$y_1-y_{-1}=(y_1-y_{-2})+(y_{-2}-y_{-1})=\alpha(y_{1,-2}-y)+\beta(y_{-1,-2}-y),$$

the point with the representative y_1-y_{-1} lies in the plane $\Pi^{(\mathrm{down})}$. Obviously, it lies also in $\Pi^{(1)}$; therefore, it lies on the line $\ell^{(1)}$. As a consequence of the property of definition 2.3, it belongs also to the plane $\Pi^{(\mathrm{up})}$. Now, from

$$y_2-y_1=(y_2-y_{-1})-(y_1-y_{-1})=\gamma(y_{-1,2}-y)-(y_1-y_{-1}),$$

we find that the point with the representative y_2-y_1 belongs to $\Pi^{(\mathrm{up})}$, as well. Since the point with the representative y_2-y_1 also belongs (obviously) to the plane of the quadrilateral (f,f_1,f_{12},f_2), we conclude that it lies in the intersection of the latter plane, with $\Pi^{(\mathrm{up})}=(f,f_{12},f_{-1,2})$, which is, in the generic case, the line through f and f_{12}. ■

(*b*) *T-nets*

Definitions 2.1 and 2.3 are essentially dealing with two-dimensional Q-nets. However, the characterization of discrete Moutard nets given in theorem 2.2 opens a way to define multidimensional Moutard nets, and, in particular, to define transformations of Moutard nets with remarkable permutability properties. Namely, it turns out that equation (1.5) can be posed on multidimensional lattices.

Definition 2.6 (T-net). A map $y:\mathbb{Z}^m\to\mathbb{R}^N$ is called an m-dimensional T-net if, for every $u\in\mathbb{Z}^m$ and for every pair of indices $i\neq j$, there holds the discrete Moutard equation

$$\tau_i\tau_j y-y=a_{ij}(\tau_j y-\tau_i y), \tag{2.1}$$

with some $a_{ij}:\mathbb{Z}^m\to\mathbb{R}$, in other words, if all elementary quadrilaterals $(y, y_i, y_{ij,}, y_j)$ are planar and have parallel diagonals.

Of course, coefficients a_{ij} have to be skew-symmetric, $a_{ij}=-a_{ji}$. We show that three-dimensional T-nets are described by a well-defined three-dimensional system (cf. Nimmo & Schief 1997).

Theorem 2.7 (Elementary hexahedron of a T-net). *Given seven points y, y_i and y_{ij} $(1\leq i\neq j\leq 3)$ in $\mathbb{R}^N$, such that equation (2.1) is satisfied on the three quadrilaterals (y, y_i, y_{ij}, y_j) adjacent to the vertex y, there exists a unique point y_{123}, such that equation (2.1) is satisfied on the three quadrilaterals $(y_i, y_{ij}, y_{123}, y_{ik})$ adjacent to the vertex y_{123}.*

Proof. Three equations (2.1) for the faces of an elementary cube of $\mathbb{Z}^3$ adjacent to y_{123} give

$$\tau_i y_{jk} = \big(1+(\tau_i a_{jk})(a_{ij}+a_{ki})\big)y_i-(\tau_i a_{jk})a_{ij}y_j-(\tau_i a_{jk})a_{ki}y_k.$$

They lead to consistent results for y_{123} for arbitrary initial data if and only if the following conditions are satisfied:

$$1+(\tau_1 a_{23})(a_{12}+a_{31}) = -(\tau_2 a_{31})a_{12} = -(\tau_3 a_{12})a_{31},$$
$$1+(\tau_2 a_{31})(a_{23}+a_{12}) = -(\tau_3 a_{12})a_{23} = -(\tau_1 a_{23})a_{12} \quad \text{and}$$
$$1+(\tau_3 a_{12})(a_{23}+a_{31}) = -(\tau_1 a_{23})a_{31} = -(\tau_2 a_{31})a_{23}.$$

These conditions constitute a system of six (linear) equations for three unknown variables $\tau_i a_{jk}$ in terms of the known ones a_{jk}. A direct computation shows that this system is not overdetermined but admits a unique solution

$$\frac{\tau_1 a_{23}}{a_{23}} = \frac{\tau_2 a_{31}}{a_{31}} = \frac{\tau_3 a_{12}}{a_{12}} = -\frac{1}{a_{12}a_{23}+a_{23}a_{31}+a_{31}a_{12}}. \tag{2.2}$$

With $\tau_i a_{jk}$ so defined, equations (2.1) are fulfilled on all three quadrilaterals adjacent to y_{123}. ■

Equations (2.2) represent a well-defined birational map $\{a_{jk}\}\mapsto\{\tau_i a_{jk}\}$, which can be considered as the fundamental three-dimensional system related to T-nets. It is sometimes called the 'star-triangle map'.

Theorem 2.7 means that the defining condition of T-nets (parallel diagonals of elementary planar quadrilaterals) yields a discrete three-dimensional system with fields on vertices taking values in an affine space $\mathbb{R}^N$. This system can be considered as an admissible reduction of the three-dimensional system describing Q-nets in $\mathbb{R}^N$. Indeed, if one has an elementary hexahedron of an affine Q-net $y:\mathbb{Z}^3\to\mathbb{R}^N$, such that its elementary quadrilaterals (y, y_i, y_{ij}, y_j) have parallel diagonals, then the elementary quadrilaterals $(y_i, y_{ij,}, y_{123}, y_{ik})$ have this property as well. To see this, observe that the point y_{123} from theorem 2.7 satisfies the planarity condition, and therefore it has to coincide with the unique point defined by planarity of the quadrilaterals $(y_i, y_{ij}, y_{123}, y_{ik})$.

The four-dimensional consistency of T-nets is a consequence of the analogous property of Q-nets, since T-constraint propagates in the construction of a Q-net from its coordinate surfaces. On the level of formulae, we have for T-nets, with

$m \geq 4$, the system (2.1), while the map $\{a_{jk}\} \mapsto \{\tau_i a_{jk}\}$ is given by

$$\frac{\tau_i a_{jk}}{a_{jk}} = -\frac{1}{a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij}}. \tag{2.3}$$

All indices i, j, k vary now between 1 and m, and, for any triple of pairwise different indices (i, j, k), equations involving these indices solely form a closed subset.

The multidimensional consistency of T-nets yields, in a usual fashion, Darboux transformations with permutability properties (which in the present context should be called discrete Moutard transformations). We refer to Bobenko & Suris (2005) for the background on the relation of multidimensional consistency to Darboux transformations, and give here only the formulae for the discrete Moutard transformation of equation (2.1) into

$$y_{ij}^{+} - y^{+} = a_{ij}^{+}(y_j^{+} - y_i^{+}). \tag{2.4}$$

These formulae can be written as

$$y_i^{+} - y = b_i(y^{+} - y_i), \tag{2.5}$$

where the quantities b_i and the transformed coefficients a_{ij}^{+} are defined by equations

$$\frac{\tau_i b_j}{b_j} = \frac{a_{ij}^{+}}{a_{ij}} = \frac{1}{(b_i - b_j)a_{ij} + b_i b_j}. \tag{2.6}$$

It is not difficult to recognize in equations (2.4) and (2.5) the same Moutard equation (2.1) on the $(m+1)$-dimensional lattice, with the superscript '+' used to denote the shift τ_{m+1}. Similarly, equations (2.6) are nothing but the star-triangle formulae (2.3), with $b_i = a_{i,m+1}$.

(*c*) *Discrete Moutard nets in quadrics*

We have seen that discrete Moutard nets (or, more precisely, their T-net representatives) constitute an admissible reduction of Q-nets. The restriction to a quadric constitutes another admissible reduction (Doliwa 1999). Imposing two admissible reductions simultaneously, one comes to T-nets in quadrics. Let $\mathbb{R}^N$ be equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ (which does not need to be positive definite), and let

$$\mathcal{Q} = \{y \in \mathbb{R}^N : \langle y, y \rangle = \kappa_0\}, \tag{2.7}$$

be a quadric in $\mathbb{R}^N$. We study T-nets $y : \mathbb{Z}^m \to \mathcal{Q}$. This leads to a *discrete two-dimensional system*, since constructing elementary quadrilaterals of T-nets in $\mathcal{Q}$ corresponding to elementary squares of the lattice $\mathbb{Z}^m$ admits a well-posed initial-value problem: given three points y, y_1, $y_2 \in \mathcal{Q}$, one finds a unique fourth point $y_{12} \in \mathcal{Q}$, $y_{12} \neq y$, satisfying the discrete Moutard equation

$$y_{12} - y = a_{12}(y_2 - y_1).$$

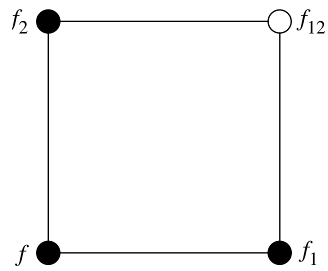

Figure 3. Two-dimensional system on an elementary quadrilateral.

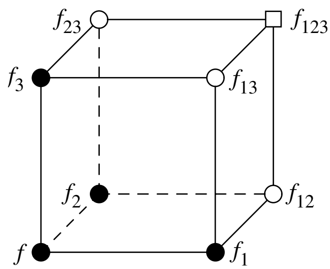

Figure 4. Three-dimensional consistency of two-dimensional systems.

Indeed, the condition

$$\langle y_{12}, y_{12}\rangle = \langle y + a_{12}(y_2 - y_1),\ y + a_{12}(y_2 - y_1)\rangle = \kappa_0$$

leads to a quadratic equation for a_{12}, which has one trivial solution $a_{12}=0 \Leftrightarrow y_{12}=y$ and one non-trivial

$$a_{12} = \frac{\langle y, y_1 - y_2\rangle}{\kappa_0 - \langle y_1, y_2\rangle}.$$

This elementary construction step, i.e. finding the fourth vertex of an elementary quadrilateral out of the known three vertices, is symbolically represented in figure 3.

Turning to an elementary cube of dimension $m \geq 3$, we see that one can prescribe all points y and y_i for all $1 \leq i \leq m$. Indeed, these data are independent and one can construct all other vertices of an elementary cube from these data, *provided one does not encounter contradictions.* To see the possible source of contradictions, consider in detail the case of $m=3$. From y and y_i $(1 \leq i \leq 3)$, one determines all y_{ij} by

$$y_{ij} - y = a_{ij}(y_j - y_i) \quad \text{and} \quad a_{ij} = \frac{\langle y, y_i - y_j\rangle}{\kappa_0 - \langle y_i, y_j\rangle}. \tag{2.8}$$

After that one has, in principle, three different ways to determine y_{123}, from three squares adjacent to this point (figure 4). These three values for y_{123} have to coincide, independently of initial conditions.

Definition 2.8 (Three-dimensional consistency). A two-dimensional system is called three-dimensional consistent if it can be imposed on all two-dimensional faces of an elementary cube of $\mathbb{Z}^3$.

There holds a quite general theorem analogous to theorem 2.7 of Bobenko & Suris (2007).

Theorem 2.9 (Three-dimensional consistency yields consistency in all higher dimensions). *Any three-dimensional consistent discrete two-dimensional system is also m-dimensionally consistent for all $m>3$.*

Proof. Goes by induction in m and is analogous to the proof of theorem 2.7 from Bobenko & Suris (2007). ■

Theorem 2.10 (T-nets in quadrics are three-dimensional consistent). *The two-dimensional system (2.8) governing T-nets in $\mathcal{Q}$ is three-dimensional consistent.*

Proof. This can be checked by a tiresome computation, which can, however, be avoided by the following conceptual argument. T-nets in $\mathcal{Q}$ are a result of imposing two admissible reductions on Q-nets in $\mathbb{R}^N$, namely the T-reduction and the restriction to a quadric $\mathcal{Q}$. This reduces the effective dimension of the system by 1 (which allows determination of the fourth vertex of an elementary quadrilateral from the three known ones), and transfers the original three-dimensional equation into the three-dimensional consistency of the reduced two-dimensional equation. Indeed, after finding y_{12}, y_{23} and y_{13}, one can construct y_{123} according to the planarity condition (as intersection of three planes). Then, both the T- and the $\mathcal{Q}$-condition are fulfilled for all three quadrilaterals adjacent to y_{123}, according to theorem 2.7 and the result of Doliwa (1999). Therefore, these quadrilaterals satisfy our two-dimensional system. ■

We also mention an important property of T-nets in quadrics used in the sequel: the functions

$$\alpha_i = \langle y, y_i \rangle, \tag{2.9}$$

defined on edges of $\mathbb{Z}^m$ parallel to the ith coordinate axes, satisfy

$$\tau_i \alpha_j = \alpha_j, \quad i \neq j, \tag{2.10}$$

i.e. any two opposite edges of any elementary square carry the same value of the corresponding α_i. Indeed, equations

$$\langle y_{ij}, y_j \rangle = \langle y_i, y \rangle \quad \text{and} \quad \langle y_{ij}, y_i \rangle = \langle y_j, y \rangle$$

follow from (2.8) by a direct computation.

A last but not least remark: quadrics $\mathcal{Q}$, which are given by equation (2.7) with $\kappa_0 = 0$, can be projectivized, so that it is admissible to interpret $y \in \mathbb{R}^N$ as homogeneous coordinates in $\mathbb{RP}^{N-1}$. In this case, restrictions of Q-nets and Moutard nets in $\mathbb{RP}^{N-1}$ to $\mathbb{P}(\mathcal{Q})$ are well defined.

3. Isothermic surfaces in Möbius geometry

(*a*) *Projective model of Möbius geometry*

Recall (e.g. Hertrich-Jeromin (2003) or Bobenko & Suris 2005) that the basic space of the projective model of Möbius geometry in $\mathbb{R}^N$ is the projectivization $\mathbb{P}(\mathbb{R}^{N+1,1})$ of the Minkowski space $\mathbb{R}^{N+1,1}$. The latter is the space spanned by $N+2$ linearly independent vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{N+2}$ and equipped with the Minkowski scalar product

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \begin{cases} 1, & i = j \in \{1, \ldots, N+1\}, \\ -1, & i = j = N+2, \\ 0, & i \neq j. \end{cases}$$

It is convenient to introduce two isotropic vectors $\boldsymbol{e}_0 = (1/2)(\boldsymbol{e}_{N+2} - \boldsymbol{e}_{N+1})$ and $\boldsymbol{e}_\infty = (1/2)(\boldsymbol{e}_{N+2} + \boldsymbol{e}_{N+1})$, satisfying $\langle \boldsymbol{e}_0, \boldsymbol{e}_\infty \rangle = -(1/2)$.

A point $f \in \mathbb{R}^N$ is modelled in the space $\mathbb{P}(\mathbb{R}^{N+1,1})$ by the element with homogeneous coordinates $\hat{f} = f + \boldsymbol{e}_0 + |f|^2 \boldsymbol{e}_\infty$, while a hypersphere $S \subset \mathbb{R}^N$ with centre $c \in \mathbb{R}^N$ and radius $r > 0$ is modelled by the element with homogeneous coordinates $\hat{s} = c + \boldsymbol{e}_0 + (|c|^2 - r^2)\boldsymbol{e}_\infty$. Thus, points $f \in \mathbb{R}^N \cup \{\infty\}$ are in a one-to-one correspondence with points of the projectivized light cone $\mathbb{P}(\mathbb{L}^{N+1,1})$, while hyperspheres (including planes) are in a one-to-one correspondence with $\mathbb{P}(\mathbb{R}^{N+1,1}_{\text{out}})$, where

$$\mathbb{L}^{N+1,1} = \{\xi \in \mathbb{R}^{N+1,1} : \langle \xi, \xi \rangle = 0\} \quad \text{and} \quad \mathbb{R}^{N+1,1}_{\text{out}} = \{\xi \in \mathbb{R}^{N+1,1} : \langle \xi, \xi \rangle > 0\}.$$

The incidence relation $f \in S$ is represented in the projective model by $\langle \hat{f}, \hat{s} \rangle = 0$.

A k-sphere is a (generic) intersection of $N - k$ hyperspheres $S_1, \ldots, S_{N-k}$. As a set of points, a k-sphere is modelled as a projectivization of the orthogonal complement of the linear subspace spanned by $\hat{s}_1, \ldots, \hat{s}_{N-k}$. The latter space is a $(k+2)$-dimensional linear subspace of $\mathbb{R}^{N+1,1}$ of signature $(k+1, 1)$. Through any $k+2$ points $f_1, \ldots, f_{k+2} \in \mathbb{R}^N$, in general position, one can draw a unique k-sphere. It corresponds to the $(k+2)$-dimensional linear subspace spanned by the vectors $\hat{f}_1, \ldots, \hat{f}_{k+2}$.

(b) *Discrete isothermic surfaces as Moutard nets*

Definition 3.1 (Discrete isothermic surface). A two-dimensional circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is called a discrete isothermic surface if the corresponding net $\hat{f} = f + \boldsymbol{e}_0 + |f|^2 \boldsymbol{e}_\infty : \mathbb{Z}^2 \to \mathbb{L}^{N+1,1}$ is a lift of a discrete Moutard net in $\mathbb{P}(\mathbb{L}^{N+1,1})$.

From definitions 2.1 and 2.3, there follows a geometric characterization of discrete isothermic nets.

Theorem 3.2 (Central spheres for discrete isothermic nets).

(i) A circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ not lying on a two-dimensional sphere is a discrete isothermic net if and only if, for every $u \in \mathbb{Z}^2$, the five points f and $f_{\pm 1, \pm 2}$ lie on a two-dimensional sphere not containing some (and then any) of the four points $f_{\pm 1}, f_{\pm 2}$.

(ii) A circular net $f : \mathbb{Z}^2 \to \mathbb{S}^2 \subset \mathbb{R}^N$ is a discrete isothermic net if and only if, for every $u \in \mathbb{Z}^2$, the three circles through f,

$$C^{(\text{up})} = \text{circle}(f, f_{12}, f_{-1,2}), \quad C^{(\text{down})} = \text{circle}(f, f_{1,-2}, f_{-1,-2}) \quad \textit{and}$$
$$C^{(1)} = \text{circle}(f, f_1, f_{-1})$$

have one additional point in common, which is also equivalent for the three circles through f,

$$C^{(\text{left})} = \text{circle}(f, f_{-1,2}, f_{-1,-2}), \quad C^{(\text{right})} = \text{circle}(f, f_{1,2}, f_{1,-2}) \quad \textit{and}$$
$$C^{(2)} = \text{circle}(f, f_2, f_{-2})$$

to have one additional point in common.

Figure 5. Four circles of a generic discrete isothermic surface, with a central sphere.

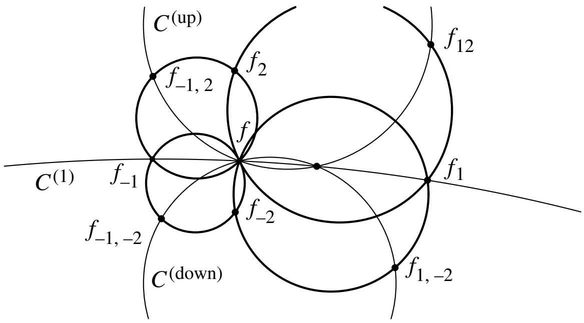

Figure 6. Four circles of a planar (or spherical) discrete isothermic net.

Cases (i) and (ii) of theorem 3.2 are illustrated in figures 5 and 6, respectively.

Another characterization of discrete isothermic surfaces can be given in terms of the cross-ratios. Recall (Bobenko & Pinkall 1996; Cieśliński 1997; Hertrich-Jeromin 2003) that, for any four concircular points $f, f_1, f_2, f_{12} \in \mathbb{R}^N$, their (real-valued) cross-ratio can be defined as

$$q(f, f_1, f_{12}, f_2) = (f_1 - f)(f_{12} - f_1)^{-1}(f_{12} - f_2)(f_2 - f)^{-1}. \tag{3.1}$$

Here multiplication is interpreted as the Clifford multiplication in the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^N)$. Recall that for $x, y \in \mathbb{R}^N$, the Clifford product satisfies $xy + yx = -2\langle x, y\rangle$, and that the inverse element of $x \in \mathbb{R}^N$ in the Clifford algebra is given by $x^{-1} = -x/|x|^2$. Alternatively, one can identify the plane of the quadrilateral (f, f_1, f_{12}, f_2) with the complex plane $\mathbb{C}$, and then interpret multiplication in equation (3.1) as the complex multiplication. An important property of the cross-ratio is its invariance under Möbius transformations.

Theorem 3.3 (Four cross-ratios of a discrete isothermic net). *A circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is a discrete isothermic net if and only if the cross-ratios $q = q(f, f_1, f_{12}, f_2)$ of its elementary quadrilaterals satisfy the following condition:*

$$q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2}. \tag{3.2}$$

Here, like in §2, the negative indices $-i$ are used to denote the backward shifts τ_i^{-1}, so that, for example, $q_{-1} = q(f_{-1}, f, f_2, f_{-1,2})$.

Proof. Perform a Möbius transformation sending f to ∞. Under such a transformation, the four adjacent circles through f turn into four straight lines $f_{\pm 1}f_{\pm 2}$, containing the corresponding points $f_{\pm 1,\pm 2}$. Formula (3.2) turns into the following relation for the quotients of (directed) lengths:

$$\frac{l(f_2,f_{12})}{l(f_{12},f_1)}\cdot\frac{l(f_1,f_{1,-2})}{l(f_{1,-2},f_{-2})}\cdot\frac{l(f_{-2},f_{-1,-2})}{l(f_{-1,-2},f_{-1})}\cdot\frac{l(f_{-1},f_{-1,2})}{l(f_{-1,2},f_2)}=1. \tag{3.3}$$

If the affine space through the points $f_{\pm 1}$, $f_{\pm 2}$ is three-dimensional, then equation (3.3) is equivalent to the fact that the four points $f_{\pm 1,\pm 2}$ lie in a plane, which is a sphere through $f=\infty$. This is the $n=4$ particular case of lemma 3.4.

If, on the contrary, the four points $f_{\pm 1}$, $f_{\pm 2}$ are coplanar, then we are in the situation as in figure 2, described by the Desargues theorem. Here, we apply the Menelaus theorem (case $n=3$ of lemma 3.4) twice to the triangle $\triangle(f_{-1},f_2,f_1)$ intersected by the line $(f_{-1,2},f_{12})$, and to the triangle $\triangle(f_{-1},f_{-2},f_1)$ intersected by the line $(f_{-1,-2}f_{1,-2})$: both lines meet the line $(f_{-1}f_1)$ at the same point $\ell^{(1)}$ if and only if

$$\frac{l(f_2,f_{12})}{l(f_{12},f_1)}\cdot\frac{l(f_{-1},f_{-1,2})}{l(f_{-1,2},f_2)}=-\frac{l(f_{-1},\ell^{(1)})}{l(\ell^{(1)},f_1)}=\frac{l(f_{-2},f_{1,-2})}{l(f_{1,-2},f_1)}\cdot\frac{l(f_{-1},f_{-1,-2})}{l(f_{-1,-2},f_{-2})}.$$

This yields (3.3). ∎

Lemma 3.4 (Generalized Menelaus theorem). *Let $P_1, \ldots, P_n$ be n points in general position in $\mathbb{R}^{n-1}$, so that the affine space through the points P_i is $(n-1)$-dimensional. Let $P_{i,i+1}$ be some points on the lines (P_iP_{i+1}) (indices are read modulo n). The n points $P_{i,i+1}$ lie in an $(n-2)$-dimensional affine subspace if and only if the following relation for the quotients of the directed lengths holds*:

$$\prod_{i=1}^{n}\frac{l(P_i,P_{i,i+1})}{l(P_{i,i+1},P_{i+1})}=(-1)^n.$$

This statement is due to Boldescu (1970) and Budinský & Nádeník (1972).

The claim of theorem 3.3 can be reformulated as follows.

Corollary 3.5 (Factorized cross-ratios for a discrete isothermic net). *A circular net $f:\mathbb{Z}^2\to\mathbb{R}^N$ is a discrete isothermic net if and only if the cross-ratios $q=q(f,f_1,f_{12},f_2)$ of its elementary quadrilaterals satisfy the following condition*:

$$q(f,f_1,f_{12},f_2)=\frac{\alpha_1}{\alpha_2}, \tag{3.4}$$

with some edge functions α_i satisfying the labelling property (2.10).

Clearly, functions α_i are defined up to a common constant factor. Actually, it was this characterization of discrete isothermic nets which was used as a definition in the pioneering paper (Bobenko & Pinkall 1996).

Actually, edge functions α_i in corollary 3.5 admit a nice geometric expression. According to theorem 2.2, a discrete isothermic net $f:\mathbb{Z}^2\to\mathbb{R}^N$ can be characterized by the existence of representatives $y:\mathbb{Z}^2\to\mathbb{L}^{N+1,1}$ in the light cone satisfying the discrete Moutard equation (1.5). We use the following notation for these T-net representatives:

$$y = s^{-1}\hat{f} = s^{-1}(f + \boldsymbol{e}_0 + |f|^2\boldsymbol{e}_\infty). \tag{3.5}$$

Thus, s^{-1} denotes the $\boldsymbol{e}_0$-component of the T-net representative y of an isothermic net f. Now, equation (2.9) defines functions on edges,

$$\alpha_i = -2\langle y, y_i\rangle = \frac{|f_i - f|^2}{ss_i}, \tag{3.6}$$

possessing property (2.10): any two opposite edges of any elementary square carry the same value of the corresponding α_i. The function $s : \mathbb{Z}^2 \to \mathbb{R}$ can be called the discrete metric of the isothermic net f, since it is analogous to the metric s of a smooth isothermic surface: formula $|\delta_i f|^2 = \alpha_i ss_i$ is analogous to equation (1.1), while formula (3.5) literally coincides with its smooth counterpart.

Theorem 3.6 (Cross-ratios through discrete metric). *Edge functions α_i participating in the factorization (3.4) of the cross-ratios of elementary quadrilaterals of a discrete isothermic net can be defined by equation (3.6).*

Proof. Comparing the $\boldsymbol{e}_0$-components in the Moutard equation $y_{12} - y = a_{12}(y_2 - y_1)$, we find $a_{12} = (s_{12}^{-1} - s^{-1})/(s_2^{-1} - s_1^{-1})$. Therefore, we can rewrite the Moutard equation as

$$\left(\frac{1}{s_2} - \frac{1}{s_1}\right)\left(\frac{\hat{f}_{12}}{s_{12}} - \frac{\hat{f}}{s}\right) = \left(\frac{1}{s_{12}} - \frac{1}{s}\right)\left(\frac{\hat{f}_2}{s_2} - \frac{\hat{f}_1}{s_1}\right),$$

which is equivalent to

$$\frac{\hat{f}_1 - \hat{f}}{ss_1} + \frac{\hat{f}_{12} - \hat{f}_1}{s_1 s_{12}} = \frac{\hat{f}_2 - \hat{f}}{ss_2} + \frac{\hat{f}_{12} - \hat{f}_2}{s_2 s_{12}}. \tag{3.7}$$

The $\mathbb{R}^N$-part of the latter equation, i.e.

$$\frac{f_1 - f}{ss_1} + \frac{f_{12} - f_1}{s_1 s_{12}} = \frac{f_2 - f}{ss_2} + \frac{f_{12} - f_2}{s_2 s_{12}}, \tag{3.8}$$

can be rewritten with the help of equation (3.6) as

$$\alpha_1 \frac{f_1 - f}{|f_1 - f|^2} + \alpha_2 \frac{f_{12} - f_1}{|f_{12} - f_1|^2} = \alpha_2 \frac{f_2 - f}{|f_2 - f|^2} + \alpha_1 \frac{f_{12} - f_2}{|f_{12} - f_2|^2}. \tag{3.9}$$

In terms of the inversion in the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^N)$, this can be presented as

$$\alpha_1(f_1 - f)^{-1} + \alpha_2(f_{12} - f_1)^{-1} = \alpha_2(f_2 - f)^{-1} + \alpha_1(f_{12} - f_2)^{-1}. \tag{3.10}$$

Equation (3.10) is, in the generic case $f_{12} + f \neq f_1 + f_2$, equivalent to equation (3.4). It is not quite straightforward to show this equivalence in the case of non-commutative variables $f \in \mathcal{C}\ell(\mathbb{R}^N)$. But one can identify the plane of the quadrilateral (f, f_1, f_{12}, f_2) with $\mathbb{C}$, and then equation (3.9) is the complex conjugate of equation (3.10) where, now, all variables are commutative (complex numbers), and, in this case, the equivalence to equation (3.4) is immediate after clearing the denominators. ■

T-nets in the light cone $\mathbb{L}^{N+1,1}$ are three-dimensional consistent. This also yields the three-dimensional consistency of the cross-ratio equation (3.4) with a prescribed labelling α_i of the edges, i.e. of the two-dimensional equation

$$q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j}. \tag{3.11}$$

Both constructions provide us with a well-defined notion of multidimensional discrete isothermic nets, and therefore with Darboux transformations of discrete isothermic nets with the usual permutability properties.

We conclude this section with duality of discrete isothermic nets.

Theorem 3.7 (Dual discrete isothermic net). *Let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a discrete isothermic net, with the T-net representatives in the light cone*

$$y = s^{-1}\hat{f} = s^{-1}(f + \boldsymbol{e}_0 + |f|^2\boldsymbol{e}_\infty) : \mathbb{Z}^m \to \mathbb{L}^{N+1,1}.$$

Then, the $\mathbb{R}^N$-valued discrete one-form δf^* defined by

$$\delta_i f^* = \frac{\delta_i f}{s s_i} = \alpha_i \frac{\delta_i f}{|\delta_i f|^2}, \quad i = 1, \ldots, m, \tag{3.12}$$

is closed. Its integration defines (up to translation) a net $f^* : \mathbb{Z}^m \to \mathbb{R}^N$, called dual to the net f. The net f^* is a discrete isothermic net, with

$$q(f^*, f_i^*, f_{ij}^*, f_j^*) = \frac{\alpha_i}{\alpha_j}. \tag{3.13}$$

We also define the function $s^* : \mathbb{Z}^m \to \mathbb{R}$ as *$s^* = s^{-1}$. Then, the net*

$$y^* = (s^*)^{-1}\hat{f}^* = (s^*)^{-1}(f^* + \boldsymbol{e}_0 + |f^*|^2\boldsymbol{e}_\infty) : \mathbb{Z}^m \to \mathbb{L}^{N+1,1}$$

is a T-net in the light cone.

Proof. Clearly, for any pair of indices i, j, the function $\hat{f}$ satisfies an equation analogous to equation (3.7), which expresses the closeness of the $\mathbb{R}^{N+1,1}$-valued one-form defined by $\delta_i \hat{f}/(s s_i)$. Unfortunately, the net obtained by integration of this one-form does not lie, in general, in the light cone $\mathbb{L}^{N+1,1}$ and cannot be taken as the dual net $\hat{f}^*$. We use the following trick for the construction of the dual net $\hat{f}^*$ in the light cone. The $\mathbb{R}^N$-part of equation (3.7), i.e. equation (3.8), expresses the closeness of the $\mathbb{R}^N$-valued one-form $\delta_i f^* = \delta_i f/(s s_i)$, the integration of which gives a dual net f^* in $\mathbb{R}^N$. Equation (3.13) follows immediately from equation (3.12) and implies that f^* is a discrete isothermic net. In particular, it is a circular net, so that $\hat{f}^* = f^* + \boldsymbol{e}_0 + |f^*|^2\boldsymbol{e}_\infty$ is a conjugate net in the light cone. It remains to show that the so-defined $\hat{f}^*$ is a Moutard net, with the T-net representatives $y^* = (s^*)^{-1}\hat{f}^*$. This claim is equivalent to the closeness of the discrete $\mathbb{R}^{N+1,1}$-valued one-form $\delta_i \hat{f}^*/(s^* s_i^*)$. Since $\hat{f}^*$ is a conjugate net in the light cone, it is enough to prove the closeness of the $\mathbb{R}^N$-valued one-form $\delta_i f^*/(s^* s_i^*)$. But for $s^* = s^{-1}$, we have

$$\frac{\delta_i f^*}{s^* s_i^*} = (s s_i)\delta_i f^* = (s s_i)\frac{\delta_i f}{s s_i} = \delta_i f,$$

which is automatically closed. ∎

4. Lie geometry: S-isothermic nets

(*a*) *Projective model of Lie geometry*

Lie geometry (Blaschke 1929; Cecil 1992) is the geometry of oriented spheres in $\mathbb{R}^N$ and their properties invariant with respect to Lie sphere transformations, which preserve the oriented contact of spheres. The basic space of the projective model of Lie geometry is $\mathbb{P}(\mathbb{R}^{N+1,2})$, where the space of homogeneous coordinates $\mathbb{R}^{N+1,2}$ is spanned by $N+3$ linearly independent vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{N+3}$ and is equipped with the pseudo-Euclidean scalar product

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \begin{cases} 1, & i = j \in \{1, \dots, N+1\}, \\ -1, & i = j \in \{N+2, N+3\}, \\ 0, & i \neq j. \end{cases}$$

We use the same notations $\boldsymbol{e}_0 = 1/2(\boldsymbol{e}_{N+2} - \boldsymbol{e}_{N+1})$ and $\boldsymbol{e}_\infty = 1/2(\boldsymbol{e}_{N+2} + \boldsymbol{e}_{N+1})$ as in the Möbius case. An oriented hypersphere $S \subset \mathbb{R}^N$ with centre $c \in \mathbb{R}^N$ and signed radius $r \in \mathbb{R}$ is modelled by the element of $\mathbb{P}(\mathbb{R}^{N+1,2})$ with the homogeneous coordinates $\hat{s} = c + \boldsymbol{e}_0 + (|c|^2 - r^2)\boldsymbol{e}_\infty + r\boldsymbol{e}_{N+3}$. An oriented hyperplane $P = \{x \in \mathbb{R}^N : \langle v, x \rangle = d\}$ with $v \in \mathbb{S}^{N-1}$ and $d \in \mathbb{R}$ is modelled by the element of $\mathbb{P}(\mathbb{R}^{N+1,2})$ with the homogeneous coordinates $\hat{p} = v + 0 \cdot \boldsymbol{e}_0 + 2d\boldsymbol{e}_\infty + \boldsymbol{e}_{N+3}$. A point $x \in \mathbb{R}^N$ is modelled by the element of $\mathbb{P}(\mathbb{R}^{N+1,2})$ with the homogeneous coordinates $\hat{x} = x + \boldsymbol{e}_0 + |x|^2 \boldsymbol{e}_\infty + 0 \cdot \boldsymbol{e}_{N+3}$. Hyperplanes are interpreted as hyperspheres of an infinite radius, while points are interpreted as hyperspheres of radius 0. All the listed elements of $\mathbb{P}(\mathbb{R}^{N+1,2})$ belong to the Lie quadric $\mathbb{P}(\mathbb{L}^{N+1,2})$, where

$$\mathbb{L}^{N+1,2} = \{\xi \in \mathbb{R}^{N+1,2} : \langle \xi, \xi \rangle = 0\}.$$

Moreover, points of $\mathbb{P}(\mathbb{L}^{N+1,2})$ are in a one-to-one correspondence with oriented hyperspheres in $\mathbb{R}^N$, including degenerate cases of hyperplanes and points. Two oriented hyperspheres S_1, S_2 are in an oriented contact (i.e. are tangent to each other with the unit normals at tangency pointing in the same direction) if and only if $\langle \hat{s}_1, \hat{s}_2 \rangle = 0$. This also holds if one or both hyperspheres turn out to be a hyperplane or a point.

(*b*) *S-isothermic surfaces as Moutard nets in the Lie quadric*

From now on we restrict our considerations to the case of surfaces and sphere congruences in $\mathbb{R}^3$, i.e. $N=3$ and $m=2$.

Two-dimensional nets in the Lie quadric $\mathbb{L}^{4,2}$ are discrete congruences of spheres. An interesting class of such congruences is constituted by discrete Moutard nets in $\mathbb{P}(\mathbb{L}^{4,2})$. We leave a general study of this class for a future research, and describe here, as an example, a particularly interesting subclass, for which the T-net representatives in $\mathbb{L}^{4,2}$ have a fixed $\boldsymbol{e}_6$-component,

$$y = \frac{\kappa}{r}\left(c + \boldsymbol{e}_0 + (|c|^2 - r^2)\boldsymbol{e}_\infty + r\boldsymbol{e}_6\right).$$

Omitting the constant and therefore non-interesting $\boldsymbol{e}_6$-component, we come to a T-net in a hyperboloid of the Lorentz space of the Möbius geometry,

$$\mathbb{L}^{4,1}_\kappa = \{\xi \in \mathbb{R}^{4,1} : \langle \xi, \xi \rangle = \kappa^2\}.$$

Definition 4.1 (S-isothermic net). A map

$$S:\mathbb{Z}^2\to\{\text{oriented spheres in } \mathbb{R}^3\}$$

is called an S-isothermic net if the corresponding map

$$\hat{s}:\mathbb{Z}^2\to\mathbb{L}_\kappa^{4,1},\qquad \hat{s}=\frac{\kappa}{r}\left(c+\boldsymbol{e}_0+(|c|^2-r^2)\boldsymbol{e}_\infty\right) \tag{4.1}$$

is a T-net.

Thus, S-isothermic nets are governed by equation

$$\hat{s}_{12}-\hat{s}=a_{12}(\hat{s}_2-\hat{s}_1)\quad\text{and}\quad a_{12}=\frac{\langle\hat{s},\hat{s}_1-\hat{s}_2\rangle}{\kappa^2-\langle\hat{s}_1,\hat{s}_2\rangle}=\frac{\alpha_1-\alpha_2}{\kappa^2-\langle\hat{s}_1,\hat{s}_2\rangle}, \tag{4.2}$$

with the quantities $\alpha_i=\langle\hat{s},\hat{s}_i\rangle$ depending only on u_i. If (signed) radii of all hyperspheres become uniformly small, $r(u)\sim\kappa s(u)$, $\kappa\to 0$, then in the limit we recover discrete isothermic nets.

Consistency of T-nets in $\mathbb{L}_\kappa^{4,1}$ (which is a particular case of theorem 2.10) yields, in particular, Darboux transformations for S-isothermic nets (Hoffmann in preparation). A Darboux transform $\hat{s}^+:\mathbb{Z}^m\to\mathbb{L}_\kappa^{4,1}$ of a given S-isothermic net $\hat{s}:\mathbb{Z}^m\to\mathbb{L}_\kappa^{4,1}$ is uniquely specified by a choice of one of its spheres $\hat{s}^+(0)$.

We turn now to geometric properties of S-isothermic nets. First of all, S-isothermic nets form a subclass of discrete R-congruences of spheres (see Bobenko & Suris (2007) for a geometric characterization of discrete R-congruences). Furthermore, consider the quantities $\langle\hat{s},\hat{s}_i\rangle$ which have the meaning of cosines of the intersection angles of the neighbouring spheres (respectively, of their so-called inversive distances if they do not intersect). Then these quantities $\langle\hat{s},\hat{s}_i\rangle$ have the labelling property, i.e. depend only on u_i.

There holds the following generalization of theorem 3.7.

Theorem 4.2 (Dual S-isothermic net). *Let*

$$S:\mathbb{Z}^m\to\{\textit{oriented spheres in } \mathbb{R}^3\}$$

be an S-isothermic net. Denote the Euclidean centres and (signed) radii of S by $c:\mathbb{Z}^m\to\mathbb{R}^3$ and $r:\mathbb{Z}^m\to\mathbb{R}$, respectively. Then, the $\mathbb{R}^3$-valued discrete one-form δc^* defined by

$$\delta_i c^*=\frac{\delta_i c}{r r_i},\quad 1\le i\le m \tag{4.3}$$

is closed, so that its integration defines (up to a translation) a function $c^*:\mathbb{Z}^m\to\mathbb{R}^3$. We also define $r^*:\mathbb{Z}^m\to\mathbb{R}$ as $r^*=r^{-1}$. Then, the spheres S^* with the centres c^* and radii r^* form an S-isothermic net, called dual to S.

Proof. Consider equation

$$\hat{s}_{ij}-\hat{s}=a_{ij}(\hat{s}_j-\hat{s}_i), \tag{4.4}$$

in terms of $\hat{s}$ from (4.1). Its $\boldsymbol{e}_0$-part yields $a_{ij}=(r_{ij}^{-1}-r^{-1})/(r_j^{-1}-r_i^{-1})$. This allows us to rewrite equation (4.4) as

$$(r_j^{-1}-r_i^{-1})(\hat{s}_{ij}-\hat{s})=(r_{ij}^{-1}-r^{-1})(\hat{s}_j-\hat{s}_i). \tag{4.5}$$

A direct computation shows that the $\mathbb{R}^3$-part of this equation can be rewritten as

$$\frac{c_i - c}{rr_i} + \frac{c_{ij} - c_i}{r_i r_{ij}} = \frac{c_j - c}{rr_j} + \frac{c_{ij} - c_j}{r_j r_{ij}}, \tag{4.6}$$

which is equivalent to the closeness of the form δc^* defined by (4.3). In the same way, the $\boldsymbol{e}_\infty$-part of equation (4.5) is equivalent to the closeness of the discrete form δw defined by

$$\delta_i w = \frac{\delta_i(|c|^2 - r^2)}{rr_i}, \quad 1 \le i \le m.$$

For similar reasons, the second claim of the theorem is equivalent to the closeness of the form

$$\delta_i w^* = \frac{\delta_i\big(|c^*|^2 - (r^*)^2\big)}{r^* r_i^*}, \quad 1 \le i \le m,$$

where, recall, $r^* = 1/r$. With the help of $c_i^* - c^* = (c_i - c)/rr_i$, one easily checks that the forms δw and δw^* can be written as

$$\delta_i w = \langle c_i^* - c^*, c_i + c\rangle - \frac{r_i}{r} + \frac{r}{r_i} \quad \text{and} \quad \delta_i w^* = \langle c_i - c, c_i^* + c^*\rangle - \frac{r}{r_i} + \frac{r_i}{r}.$$

The sum of these one-forms is closed,

$$\delta_i(w + w^*) = 2\langle c_i^*, c_i\rangle - 2\langle c^*, c\rangle,$$

therefore they are closed simultaneously. ■

An interesting particular case of S-isothermic surfaces is characterized by touching of any pair of neighbouring spheres. In this case, the limit of small spheres is not relevant, therefore it is convenient to restrict the considerations to a fixed value of $\kappa = 1$. Clearly, in this case, both scalar products $\alpha_i = \langle \hat{s}, \tau_i \hat{s}\rangle$, $i = 1$, 2, can, in principle, take values ± 1. However, it is easily seen from (4.2) that, in the case $\alpha_1 = \alpha_2$, one gets only trivial nets. Thus, we assume that

$$\langle \hat{s}, \hat{s}_1\rangle = \langle \hat{s}_2, \hat{s}_{12}\rangle = -1 \quad \text{and} \quad \langle \hat{s}, \hat{s}_2\rangle = \langle \hat{s}_1, \hat{s}_{12}\rangle = 1. \tag{4.7}$$

Interestingly, these touching conditions are enough to enforce the Moutard shape of a linear dependence of the spheres.

Theorem 4.3. *S-isothermic surfaces with touching spheres can be characterized by any of the following equivalent descriptions:*

— *Q-congruence of spheres (Q-net in the Lorentz space of Möbius geometry) with touching spheres,*
— *R-congruence (Q-net in the Lie quadric) with touching spheres, and*
— *T-net in the Lie quadric with touching spheres,*

which are listed in the order of a priori increasing restrictions.

Proof. Let $\hat{s}$, $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_{12}$ be four linearly dependent oriented spheres in $\mathbb{L}_1^{4,1}$, pairwise touching so that equation (4.7) is fulfilled. (We remark that, in the present situation, the geometric meaning of linear dependence is the existence of a common orthogonal circle through the touching points.) We make a general position assumption that the spheres $\hat{s}$ and $\hat{s}_{12}$ do not touch, and likewise that the spheres $\hat{s}_1$ and $\hat{s}_2$ do not touch. The linear dependence condition is written as

$$\hat{s}_{12} = \lambda\hat{s} + \mu\hat{s}_1 + \nu\hat{s}_2. \tag{4.8}$$

Scalar product of this with $\hat{s}_1$, $\hat{s}_2$ leads to

$$1 + \lambda = \mu + \nu\langle\hat{s}_1, \hat{s}_2\rangle = -\mu\langle\hat{s}_1, \hat{s}_2\rangle - \nu \Rightarrow \mu = -\nu = \frac{\lambda + 1}{1 - \langle\hat{s}_1, \hat{s}_2\rangle}.$$

Similarly, a scalar product of equation (4.8) with $\hat{s}$, $\hat{s}_{12}$ leads to

$$\mu - \nu = \lambda - \langle\hat{s}, \hat{s}_{12}\rangle = 1 - \lambda\langle\hat{s}, \hat{s}_{12}\rangle \quad \Rightarrow \quad \lambda = 1.$$

Thus, the linear dependence has to be of the Moutard form

$$\hat{s}_{12} - \hat{s} = a_{12}(\hat{s}_2 - \hat{s}_1) \quad \text{and} \quad a_{12} = -\frac{2}{1 - \langle\hat{s}_1, \hat{s}_2\rangle}. \tag{4.9}$$

■

5. Isothermic surfaces in Laguerre geometry

In Laguerre geometry, surfaces are viewed as envelopes of their tangent planes, so that discrete surfaces are maps $P : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$.

Definition 5.1 (Discrete L-isothermic surface). A two-dimensional conical net $P : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$ is called a discrete L-isothermic surface if the corresponding net $\hat{p} : \mathbb{Z}^2 \to \mathbb{L}^{4,2}$ is a lift of a discrete Moutard net in $\mathbb{P}(\mathbb{L}^{4,2})$.

Recall that, for an (oriented) plane $P = \{x \in \mathbb{R}^3 : \langle v, x\rangle = d\}$ with the unit normal vector $v \in \mathbb{S}^2$ and $d \in \mathbb{R}$, its representative $\hat{p}$ in the Lie quadric $\mathbb{L}^{4,2}$ is given by

$$\hat{p} = v + 0 \cdot \boldsymbol{e}_0 + 2d\boldsymbol{e}_\infty + 1 \cdot \boldsymbol{e}_6.$$

Recall also that the vectors $v : \mathbb{Z}^2 \to \mathbb{S}^2$ comprise the *Gauss map* for a given net $P : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$, and that the net P is conical if and only if its Gauss map v is circular (Bobenko & Suris 2007).

From definitions 2.1 and 2.3, there follows a geometric characterization of discrete L-isothermic nets.

Theorem 5.2 (Central spheres for discrete L-isothermic nets).

(i) A conical net $P : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$ not tangent to a two-dimensional sphere is a discrete L-isothermic net if and only if, for every $u \in \mathbb{Z}^2$, the five planes P and $P_{\pm 1, \pm 2}$ are tangent to a two-dimensional sphere not touching some (and then any) of the four planes $P_{\pm 1}$, $P_{\pm 2}$.

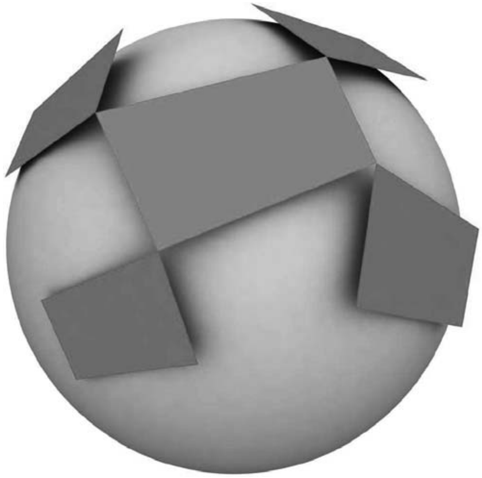

Figure 7. Five diagonally neighbouring planes of a generic discrete L-isothermic surface, with a central sphere.

(ii) A conical net $P:\mathbb{Z}^2\to\{\textit{tangent planes of } \mathbb{S}^2\subset\mathbb{R}^3\}$ *is a discrete L-isothermic net if and only if, for every* $u\in\mathbb{Z}^2$*, the three cones through P*

$$C^{(\mathrm{up})}=\mathrm{cone}(P,P_{12},P_{-1,2}),\quad C^{(\mathrm{down})}=\mathrm{cone}(P,P_{1,-2},P_{-1,-2})\quad \textit{and}$$

$$C^{(1)}=\mathrm{cone}(P,P_1,P_{-1})$$

have one additional plane in common, which is also equivalent for the three cones through P,

$$C^{(\mathrm{left})}=\mathrm{cone}(P,P_{-1,2},P_{-1,-2}),\quad C^{(\mathrm{right})}=\mathrm{cone}(P,P_{1,2},P_{1,-2})\quad \textit{and}$$

$$C^{(2)}=\mathrm{cone}(P,P_2,P_{-2}),$$

to have one additional plane in common.

The (generic) case (i) of theorem 5.2 is illustrated in figure 7.

Theorem 5.3 (Gauss map of an L-isothermic net is an isothermic net in the sphere). *A Gauss map of an L-isothermic net is a discrete isothermic net in* $\mathbb{S}^2$*. Conversely, if, for every* $u\in\mathbb{Z}^2$*, the four planes* P, P_1, P_2, P_{12} *of a net* $P:\mathbb{Z}^2\to\{\textit{planes in } \mathbb{R}^3\}$ *meet at a point and the Gauss map of the net P is isothermic, then P is an L-isothermic conical net.*

Proof. First, let P be an L-isothermic net. Then, for some $c:\mathbb{Z}^2\to\mathbb{R}$, the net $c\hat{p}$ is a T-net in the Lie quadric. As a consequence, $c(v+1\cdot \boldsymbol{e}_0+1\cdot \boldsymbol{e}_\infty)$ is a T-net in the light cone $\mathbb{L}^{4,1}$ of the Minkowski space $\mathbb{R}^{4,1}$ of the Möbius geometry for $N=3$. Therefore, the net $v:\mathbb{Z}^2\to\mathbb{S}^2\subset\mathbb{R}^3$ is isothermic.

Conversely, let the net $v:\mathbb{Z}^2\to\mathbb{S}^2\subset\mathbb{R}^3$ be isothermic. This is equivalent to the existence of the function $c:\mathbb{Z}^2\to\mathbb{R}$, such that $c(v,1)$ is a T-net. If now $\langle v,x\rangle=d$ is the equation of the plane P, then the existence of the common intersection point of the planes P, P_1, P_2, P_{12} yields that the function cd satisfies the same Moutard equation as the function cv. Therefore, $c(v,d,1)$ is a T-net, so that $\hat{p}$ is a discrete Moutard net. ∎

To conclude, we mention that, in the continuous limit, the results of this section yield the following apparently new characterization of smooth L-isothermic surfaces.

Theorem 5.4 (L-isothermic surfaces as Moutard nets in the Laguerre quadric). *A surface enveloping a two-parameter family of planes* $P : \mathbb{R}^2 \to \{\text{planes in } \mathbb{R}^3\}$, *with* $P = \{x \in \mathbb{R}^3 : \langle v, x\rangle = d\}$, $v \in \mathbb{S}^2$, $d \in \mathbb{R}$, *is L-isothermic if and only if there exists a function* $\rho : \mathbb{R}^2 \to \mathbb{R}$, *such that* $\rho^{-1}(v, d)$ *satisfies a Moutard equation (1.3).*

This research was supported by the DFG Research Unit 'Polyhedral Surfaces'.

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