GEOMETRY OF DISCRETE INTEGRABILITY. THE CONSISTENCY APPROACH

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1 ORIGIN AND MOTIVATION: DIFFERENTIAL GEOMETRY

Long before the theory of solitons, geometers used integrable equations to describe various special curves, surfaces etc. At that time no relation to mathematical physics was known, and quite different geometries appeared in this context (we will call them integrable) were unified by their common geometric features:

- Integrable surfaces, curves etc. have *nice* geometric properties,
- Integrable geometries come with their *interesting* transformations (Bäcklund–Darboux transformations) acting within the class,
- These transformations are *permutable* (Bianchi permutability).

Since "nice" and "interesting" can hardly be treated as mathematically formulated features, let us address to the permutability property. We explain it for the classical example of surfaces with constant negative Gaussian curvature (K-surface) with their Bäcklund transformations.

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a K-surface and $F_{1,0}$ and $F_{0,1}$ its two Bäcklund transformed. The classical Bianchi permutability theorem claims that there exists a unique K-surface $F_{1,1}$ which is the Bäcklund transformed of $F_{1,0}$ and $F_{0,1}$. Proceeding further this way for a given point $F_{0,0}$ on the original K-surface one obtains a $\mathbb{Z}^2$ lattice $F_{k,\ell}$ of permutable Bäcklund transformations. From the geometric properties of the Bäcklund transformations it is easy to see [1] that $F_{k,\ell}$ defined this way is a discrete K-surface.

The discrete K-surfaces have the same properties and transformations as their smooth counterparts [2]. There exist deep reasons for that. The classical

* Partially supported by the SFB 288 "Differential geometry and quantum physics" and by the DFG research center "Mathematics for key technologies" (FZT 86) in Berlin

L. Faddeev et al. (eds.),
Bilinear Integrable Systems: From Classical to Quantum, Continuous to Discrete, 43–53.

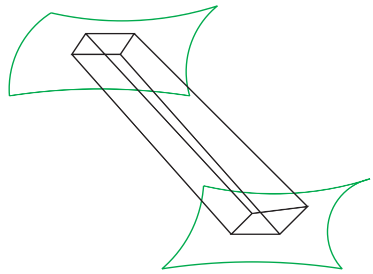

Figure 1. Surfaces and their transformations as a limit of multidimensional lattices

differential geometry of integrable surfaces may be obtained from a unifying multidimensional discrete theory by a refinement of the coordinate mesh-size in some of the directions.

Indeed, by refining of the coordinate mesh-size,

$$F : (\epsilon\mathbb{Z})^2 \to \mathbb{R}^3 \quad \longrightarrow \quad F : \mathbb{R}^2 \to \mathbb{R}^3,$$
$$\text{discrete surface } \epsilon \to 0 \text{ smooth surface}$$

in the limit one obtains classical smooth K-surfaces from discrete K-surfaces. Starting with an n-dimensional net of permutable Bäcklund transformations

$$F : (\epsilon_1\mathbb{Z}) \times \cdots \times (\epsilon_n\mathbb{Z}) \to \mathbb{R}^3$$

in the limit $\epsilon_1 \to 0, \epsilon_2 \to 0, \epsilon_3 = \cdots = \epsilon_n = 1$ one arrives to a smooth K-surface with its $n-2$-dimensional discrete family of permutable Bäcklund transformations:

$$F : \mathbb{R}^2 \times \mathbb{Z}^{n-2} \to \mathbb{R}^3.$$

This simple idea is quite fruitful. In the discrete case all directions of the multidimensional lattices appear in quite symmetric way. It leads to:

- A *unification* of surfaces and their transformations. Discrete surfaces and their transformations are indistinguishable.
- A fundamental *consistency principle*. Due to the symmetry of the discrete setup the same equations hold on all elementary faces of the lattice. This leads us beyond the pure differential geometry to a new understanding of the integrability, classification of integrable equations and derivation of the zero curvature (Lax) representation from the first principles.
- Interesting *generalizations* to: $n > 2$-dimensional systems, quantum systems, discrete systems with the fields on various lattice elements (vertices, edges, faces etc.).

As it was mentioned above, all this suggests that it might be possible to develop the classical differential geometry, including both the theory of surfaces and of their transformations, as a mesh refining limit of the discrete constructions. On the other hand, the good quantitative properties of approximations delivered by the discrete differential geometry suggest that they might be put at the basis of the practical numerical algorithms for computations in the differential geometry. However until recently there were no rigorous mathematical statements supporting this observation.

The first step in closing this gap was made in the paper [3] where the convergence of the corresponding integrable geometric numerical scheme has been proven for nonlinear hyperbolic systems (including the K-surfaces and the sine–Gordon equation).

Thus, summarizing we arrive at the following philosophy of discrete differential geometry: surfaces and their transformations can be obtained as a special limit of a discrete master-theory. The latter treats the corresponding discrete surfaces and their transformation in absolutely symmetric way. This is possible because these are merged into multidimensional nets such that their all sublattices have the same geometric properties. The possibility of this multidimensional extension results to *consistency* of the corresponding difference equations characterizing the geometry. The latter is the main topic of this paper.

2 EQUATIONS ON QUAD-GRAPHS. INTEGRABILITY AS CONSISTENCY

Traditionally discrete integrable systems were considered with fields defined on the $\mathbb{Z}^2$ lattice. One can define integrable systems on arbitrary graphs as flat connections with the values in loop groups. However, one should not go that far with the generalization. As we have shown in [4], there is a special class of graphs, called *quad-graphs*, supporting the most fundamental properties of the integrability theory. This notion turns out to be a proper generalization of the $\mathbb{Z}^2$ lattice as far as the integrability theory is concerned.

Definition 1 *A cellular decomposition $\mathcal{G}$ of an oriented surface is called a* quad-graph*, if all its faces are quadrilateral.*

Note that if one considers an arbitrary cellular decomposition C jointly with its dual C^* one obtains a quad-graph $\mathcal{D}$ connecting by the edges the neighboring vertices of C and C^*. Let us stress that the edges of the quad-graph $\mathcal{D}$ differ from the edges of C and C^*.

For the integrable systems on quad-graphs we consider in this section the fields $z : V(\mathcal{D}) \mapsto \hat{\mathbb{C}}$ are attached to the vertices of the graph. They are subject to an equation $Q(z_1, z_2, z_3, z_4) = 0$, relating four fields sitting on the four vertices

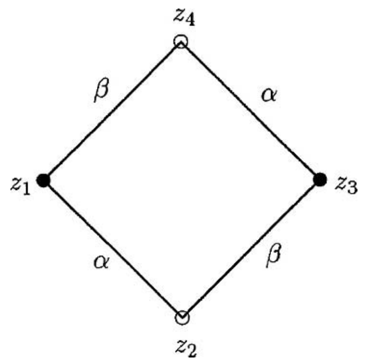

Figure 2. A face of the labelled quad-graph

of an arbitrary face from $F(\mathcal{D})$. The Hirota equation

$$\frac{z_4}{z_2} = \frac{\alpha z_3 - \beta z_1}{\beta z_3 - \alpha z_1} \tag{1}$$

is such an example. We observe that the equation carries parameters α, β which can be naturally associated to the edges, and the opposite edges of an elementary quadrilateral carry equal parameters (see Figure 2). At this point we specify the setup further. The example illustrated in Figure 2 can be naturally generalized. An integrable system on a quad-graph

$$Q(z_1, z_2, z_3, z_4; \alpha, \beta) = 0 \tag{2}$$

is parametrized by a function on the edges of the quad-graph which takes equal values on the opposite edges of any elementary quadrilateral. We call such a function a *labelling* of the quad-graph.

An elementary quadrilateral of a quad-graph can be viewed from various directions. This implies that the system (2) is well defined on a general quad-graph only if it possesses the rhombic symmetry, i.e., each of the equations

$$Q(z_1, z_4, z_3, z_2; \beta, \alpha) = 0, \quad Q(z_3, z_2, z_1, z_4; \beta, \alpha) = 0$$

is equivalent to (2).

2.1 3D-Consistency

Now we introduce a crucial property of discrete integrable systems which later on will be taken as a characteristic one.

Let us extend a quad–graph $\mathcal{D}$ into the third dimension. We take the second copy $\mathcal{D}'$ of $\mathcal{D}$ and add edges connecting the corresponding vertices of $\mathcal{D}$ and $\mathcal{D}'$. Elementary building blocks of so obtained “three-dimensional quad-graph” **D**

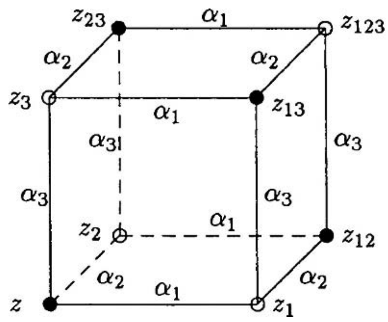

Figure 3. Elementary cube

are "cubes" as shown in Figure 3. The labelling of $\mathcal{D}$ can be extended to **D** so that the opposite edges of all elementary faces (including the "vertical" ones) carry equal parameters (see Figure 3).

Now, the fundamental property of discrete integrable system mentioned above is the *three-dimensional consistency*.

Definition 2 *Consider an elementary cube, as on Figure 3. Suppose that the values of the field z are given at the vertex z and at its three neighbors z_1, z_2, and z_3. Then the Eq. (2) uniquely determines the values z_{12}, z_{23}, and z_{13}. After that the same Eq. (2) delivers three a priori different values for the value of the field z_{123} at the eighth vertex of the cube, coming from the faces* $[z_1, z_{12}, z_{123}, z_{13}]$, $[z_2, z_{12}, z_{123}, z_{23}]$, *and* $[z_3, z_{13}, z_{123}, z_{23}]$, *respectively. The Eq.(2) is called* 3D-consistent *if these three values for z_{123} coincide for any choice of the initial data z, z_1, z_2, z_3.*

Proposition 3 *The Hirota equation*

$$\frac{z_{12}}{z} = \frac{\alpha_2 z_1 - \alpha_1 z_2}{\alpha_1 z_1 - \alpha_2 z_2}$$

is 3D-consistent.

This can be checked by a straightforward computation. For the field at the eighth vertex of the cube one obtains

$$z_{123} = \frac{(l_{21} - l_{12})z_1 z_2 + (l_{32} - l_{23})z_2 z_3 + (l_{13} - l_{31})z_1 z_3}{(l_{23} - l_{32})z_1 + (l_{31} - l_{13})z_2 + (l_{12} - l_{21})z_3}, \tag{3}$$

where $l_{ij} = \frac{\alpha_i}{\alpha_j}$.

In [4, 5] we suggested to treat the consistency property (in the sense of Definition 2) as the characteristic one for discrete integrable systems. Thus we come to the central.

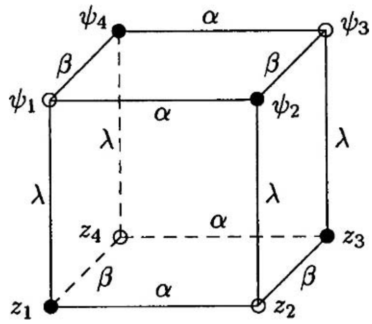

Figure 4. Zero curvature representation from the consistency

Definition 4 *A discrete equation is called* integrable *if it is* consistent.

Note that this definition of the integrability is *conceptually transparent* and *algorithmic*: for any equation it can be easily checked whether it is integrable or not.

2.2 Zero Curvature Representation from the 3D-Consistency

Our Definition 2 of discrete integrable systems is more fundamental then the traditional one as systems having a zero curvature representation in a loop group. Here we demonstrate how the corresponding flat connection in a loop group can be derived from the equation. Independently this was found in [6].

We get rid of our symmetric notations, consider the system

$$Q(z_1, z_2, z_3, z_4; \alpha, \beta) = 0 \tag{4}$$

on the base face of the cube and choose the vertical direction to carry an additional (spectral) parameter λ (see Figure 4).

Assume the left-hand-side of (4) is affine in each z_k. This gives z_4 as a fractional–linear (Möbius) transformation z_2 with the coefficients depending on z_1, z_3 and α, β. One can of course freely interchange $z_1, \dots, z_4$ in this statement. Consider now the equations on the vertical faces of the cube in Figure 4. One gets ψ_2 as a Möbius transformation of ψ_1

$$\psi_2 = L(z_2, z_1; \alpha, \lambda)[\psi_1],$$

with the coefficients depending of the fields z_2, z_1, on the parameter α in the system (4) and on the additional parameter λ which is to be treated as the spectral parameter. The mapping $L(z_2, z_1; \alpha, \lambda)$ is associated to the oriented edge (z_1, z_2). Going from ψ_1 to ψ_3 in two different ways and using the arbitrariness of ψ_1 we get

$$L(z_3, z_2; \beta, \lambda)L(z_2, z_1; \alpha, \lambda) = L(z_3, z_4; \alpha, \lambda)L(z_4, z_1; \beta, \lambda). \quad (5)$$

Using the matrix representation of Möbius transformations

$$\frac{az+b}{cz+d} = L[z], \quad \text{where} \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and normalizing the matrices (for example by the condition det $L = 1$) we arrive at the zero curvature representation (5).

Let us apply this derivation method to the Hirota equation. Equation (1) can be written as $Q = 0$ with the affine

$$Q(z_1, z_2, z_3, z_4; \alpha, \beta) = \alpha(z_2 z_3 + z_1 z_4) - \beta(z_3 z_4 + z_1 z_2).$$

Performing the computations as above in this case we derive the well known zero curvature representation (5) with the matrices

$$L(z_2, z_1, \alpha, \lambda) = \begin{pmatrix} \alpha & -\lambda z_2 \\ \dfrac{\lambda}{z_1} & -\alpha \dfrac{z_2}{z_1} \end{pmatrix} \quad (6)$$

for the Hirota equation.

3 CLASSIFICATION

Here we classify all integrable (in the sense of Definition 2) one-field equations on quad-graphs satisfying some natural symmetry conditions.

We consider equations

$$Q(x, u, v, y; \alpha, \beta) = 0, \quad (7)$$

on quad-graphs. Equations are associated to elementary quadrilaterals, the fields $x, u, v, y \in \mathbb{C}$ are assigned to the four vertices of the quadrilateral, and the parameters $\alpha, \beta \in \mathbb{C}$ are assigned to its edges. We now list more precisely the assumptions under which we classify the equations.

1. ***Consistency***. Equation (7) is integrable (in the sense it is 3D-consistent).
2. ***Linearity***. The function $Q(x, u, v, y; \alpha, \beta)$ is linear in each argument (affine linear):

$$Q(x, u, v, y; \alpha, \beta) = a_1 xuvy + \cdots + a_{16}, \quad (8)$$

where coefficients a_i depend on α, β. This is equivalent to the condition that Eq. (7) can be uniquely solved for any one of its arguments $x, u, v, y \in \hat{\mathbb{C}}$.

3. ***Symmetry***. The Eq. (7) is invariant under the group D_4 of the square symmetries, that is function Q satisfies the symmetry properties

$$Q(x,u,v,y;\alpha,\beta)=\varepsilon Q(x,v,u,y;\beta,\alpha)=\sigma Q(u,x,y,v;\alpha,\beta) \tag{9}$$

with $\varepsilon,\sigma=\pm 1$.

4. ***Tetrahedron property***. The function $z_{123}=f(z,z_1,z_2,z_3;\alpha_1,\alpha_2,\alpha_3)$, existing due to the three-dimensional consistency, actually does not depend on the variable z, that is, $f_x=0$. This property holds (3) for the Hirota equation as well as for all other known integrable examples.

The proof of the classification theorem is rather involved and is given in [5].

Theorem 5 *Up to common Möbius transformations of the variables z and point transformations of the parameters α, the three-dimensionally consistent quad-graph equations (7) with the properties (2–4) (linearity, symmetry, tetrahedron property) are exhausted by the following three lists Q, H, A ($x=z$, $u=z_1$, $v=z_2$, $y=z_{12}$, $\alpha=\alpha_1$, $\beta=\alpha_2$).*

List Q:

(Q1) $\alpha(x-v)(u-y)-\beta(x-u)(v-y)+\delta^2\alpha\beta(\alpha-\beta)=0,$

(Q2) $\alpha(x-v)(u-y)-\beta(x-u)(v-y)+\alpha\beta(\alpha-\beta)(x+u+v+y)$
$-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0,$

(Q3) $(\beta^2-\alpha^2)(xy+uv)+\beta(\alpha^2-1)(xu+vy)-\alpha(\beta^2-1)(xv+uy)$
$-\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta)=0,$

(Q4) $a_0xuvy+a_1(xuv+uvy+vyx+yxu)+a_2(xy+uv)+\bar{a}_2(xu+vy)$
$+\bar{a}_2(xv+uy)+a_3(x+u+v+y)+a_4=0,$

where the coefficients a_i are expressed through (α,a) and (β,b) with $a^2=r(\alpha)$, $b^2=r(\beta)$, $r(x)=4x^3-g_2x-g_3$, by the following formulae:

$$a_0=a+b,\qquad a_1=-\beta a-\alpha b,\qquad \alpha_2=\beta^2a+\alpha^2b,$$

$$\bar{a}_2=\frac{ab(a+b)}{2(\alpha-\beta)}+\beta^2a-\left(2\alpha^2-\frac{g_2}{4}\right)b,$$

$$\bar{a}_2=\frac{ab(a+b)}{2(\beta-\alpha)}+\alpha^2b-\left(2\beta^2-\frac{g_2}{4}\right)a,$$

$$a_3=\frac{g_3}{2}a_0-\frac{g_2}{4}a_1,\qquad a_4=\frac{g_2^2}{16}a_0-g_3a_1.$$

List H:

(H1) $(x-y)(u-v)+\beta-\alpha=0,$

(H2) $(x-y)(u-v)+(\beta-\alpha)(x+u+v+y)+\beta^2-\alpha^2=0,$

(H3) $\alpha(xu+vy)-\beta(xv+uy)+\delta(\alpha^2-\beta^2)=0.$

List A:

(A1) $\alpha(x+v)(u+y)-\beta(x+u)(v+y)-\delta^2\alpha\beta(\alpha-\beta)=0,$
(A2) $(\beta^2-\alpha^2)(xuvy+1)+\beta(\alpha^2-1)(xv+uy)-\alpha(\beta^2-1)(xu+vy)=0.$

Remarks

1. The list A can be dropped down by allowing an extended group of Möbius transformations, which act on the variables *x, y* differently than on *u, v*. So, really independent equations are given by the lists Q and H.
2. In both lists Q, H the last equations are the most general ones. This means that Eqs. (Q1)–(Q3) and (H1), (H2) may be obtained from (Q4) and (H3), respectively, by certain degenerations and/or limit procedures. This resembles the situation with the list of six Painlevé equations and the coalescences connecting them.
3. Note that the list contains the fundamental equations only. A discrete equation which is derived as a corollary of an equation with the consistency property usually loose this property.

4 GENERALIZATIONS: MULTIDIMENSIONAL AND NON-COMMUTATIVE (QUANTUM) CASES

4.1 Yang–Baxter Maps

It should be mentioned, however, that to assign fields to the vertices is not the only possibility. Another large class of two-dimensional systems on quad–graphs build those with the fields assigned to the *edges*.

In this situation each elementary quadrilateral carries a map $R:\mathcal{X}^2\mapsto\mathcal{X}^2$, where $\mathcal{X}$ is the space where the fields take values. The question on the three–dimensional consistency of such maps is also legitimate and, moreover, began to be studied recently. The corresponding property can be encoded in the formula

$$R_{23}\circ R_{13}\circ R_{12}=R_{12}\circ R_{13}\circ R_{23}, \tag{10}$$

where each $R_{ij}:\mathcal{X}^3\mapsto\mathcal{X}^3$ acts as the map R on the factors *i,j* of the cartesian product $\mathcal{X}^3$ and acts identically on the third factor. The maps with this property were introduced by Drinfeld [7] under the name of "set-theoretical solutions of the Yang-Baxter equations", an alternative name is "Yang-Baxter maps" used by Veselov in his recent study [8].

The problem of classification of Yang–Baxter maps, like the one achieved in the previous section, is under current investigation.

4.2 Four-Dimensional Consistency of Three-Dimensional Systems

The consistency principle can be obviously generalized to an arbitrary dimension. We say that

> a d–dimensional discrete equation possesses the consistency property, if it may be imposed in a consistent way on all d–dimensional sublattices of a $(d+1)$–dimensional lattice

In the three–dimensional context there are also *a priori* many kinds of systems, according to where the fields are defined: on the vertices, on the edges, or on the elementary squares of the cubic lattice. Consider three–dimensional systems with the fields sitting on the vertices. In this case each elementary cube carries just one equation

$$Q(z, z_1, z_2, z_3, z_{12}, z_{23}, z_{13}, z_{123}) = 0, \tag{11}$$

relating the fields in all its vertices. The four–dimensional consistency of such equations is defined in the same way as in Section 2.1 for the case of one dimension lower.

It is tempting to accept the four–dimensional consistency of equations of the type (11) as the constructive definition of their integrability. It is important to solve the correspondent classification problem.

We present here just one example of the equation appeared first in [9].

Proposition 6 *Equation*

$$\frac{(z_1 - z_3)(z_2 - z_{123})}{(z_3 - z_2)(z_{123} - z_1)} = \frac{(z - z_{13})(z_{12} - z_{23})}{(z_{13} - z_{12})(z_{23} - z)}. \tag{12}$$

is four–dimensionally consistent.

4.3 Noncommutative (Quantum) Cases

As we have shown in [10] the consistency approach works also in the noncommutative case, where the participating fields live in an arbitrary associative (not necessary commutative) algebra $\mathcal{A}$ (over the field $\mathcal{K}$).

In particular the noncommutative Hirota equation

$$yx^{-1} = \frac{1 - (\beta/\alpha)uv^{-1}}{(\beta/\alpha) - uv^{-1}}. \tag{13}$$

belongs to this class. Now $x, u, v, y \in \mathcal{A}$ are the fields assigned to the four vertices of the quadrilateral, and $\alpha, \beta \in \mathcal{K}$ are the parameters assigned to its edges. Note that Eq. (13) preserves the Weil commutation relations. This yields the quantum Hirota equation studied in [11].

Proposition 7 *The noncommutative Hirota equation is 3D-consistent.*

Similar to the commutative case the Lax representation can be derived from the equation and the consistency property. It turns out that finding the zero curvature representation does not hinge on the particular algebra $\mathcal{A}$ or on prescribing some particular commutation rules for fields in the neighboring vertices. The fact that some commutation relations are preserved by the evolution, is thus conceptually separated from the integrability.

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