NONLINEAR HYPERBOLIC EQUATIONS IN SURFACE THEORY: INTEGRABLE DISCRETIZATIONS AND APPROXIMATION RESULTS

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Abstract. A lattice-discretization of the Goursat problem for a class of nonlinear hyperbolic systems is proposed. Local C^∞-convergence of the discrete solutions is proven, and the approximation error for the smooth limit is estimated. The results hold in arbitrary dimensions, and for an arbitrary number of dependent variables. The abstract approximation theory is matched by a guiding example, which is the sine-Gordon-equation. As the main application, a geometric Goursat problem for surfaces of constant negative Gaussian curvature (K-surfaces) is formulated, and approximation by discrete K-surfaces is proven. The result extends to the simultaneous approximation of Bäcklund transformations. This puts on a firm basis on the generally accepted belief that the theory of integrable surfaces and their transformations may be obtained as the continuum limit of a unifying multi–dimensional discrete theory.

1. Introduction

The development of the classical differential geometry led to the introduction and investigation of various classes of surfaces which are of interest both for the internal differential–geometric reasons and for application in other sciences. Well-known examples are minimal surfaces, constant curvature surfaces, isothermic surfaces, ... The rich theory of such surface classes is, to a large extent, a classical heritage. The theory of *discrete differential geometry*, on the other hand, is more recent and is nowadays a flourishing area which parallels to a large extent its classical (continuous) counterpart. Many important classes of surfaces have been discretized up to now, see a review in [BP2]. Their properties are well understood. Today, classes of discrete surfaces are widely employed for visualization needs and for numerical approximation.

The available rigorous convergence results, however, apply mostly to problems described by elliptic partial differential equations, like the Plateau problem in the theory of minimal surfaces (see, e.g., [PP, Hin, DH1, DH2]). In this article, surfaces of constant negative Gaussian curvature (K-surfaces) are studied, which are analytically described by the sine-Gordon-equation, which is a *hyperbolic* PDE. Analogously, discrete K-surfaces are described

by a hyperbolic difference equation. A discrete approximation theory for equations of this type is developed, which applies to much more general situations than the sine-Gordon-equation. As the main geometric payoff, a rigorous proof of the convergence of discrete K-surfaces to smooth K-surfaces is provided. Fig. 1 illustrates the approximation of a continuous Amsler K-surface by a discrete one. Recall that the defining property of discrete K-surfaces $F : (\epsilon\mathbb{Z})^2 \to \mathbb{R}^3$ is that the five points $F(x, y)$ and $F(x \pm \epsilon, y \pm \epsilon)$ are coplanar. More picture of discrete K-surfaces as well as a visualization of the convergence of the Amsler family in form of a movie can be found on the home page of A. Bobenko
`http://www-sfb288.math.tu-berlin.de/~bobenko`

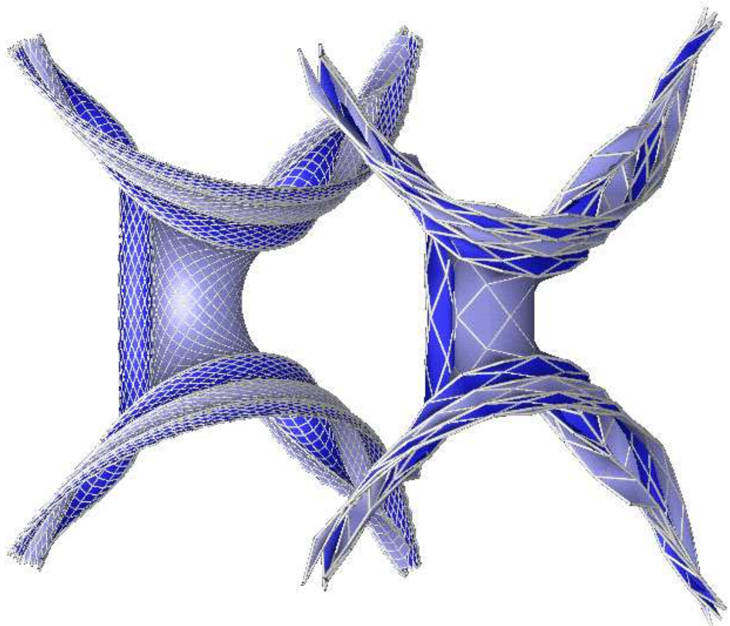


FIGURE 1. A countinous and a discrete Amsler surfaces

The characteristic property of various special classes of surfaces studied by the classical differential geometry turns out to be their integrability. One of the manifestations of integrability is the existence of a rich transformations theory, unified under the name "Bäcklund–Darboux" transformations. Classically, the theory of surfaces and that of their transformations were dealt with separately to a large extent. Recently, it became clear that both theories can be unified in the framework of the discrete differential geometry (cf. [Sau, BP2]). In this framework, multidimensional lattices with certain geometrical properties become the basic mathematical structures. Passing to the continuum limit in some of the coordinate directions (mesh size $\epsilon \to 0$), the respective smooth surface is obtained. The directions, where the mesh

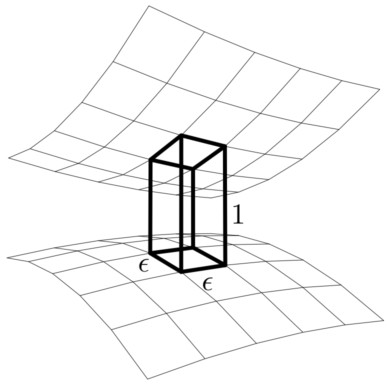


FIGURE 2. Surfaces and their transformations as a limit of multidimensional lattices

size remains constant correspond to the transformations of smooth surfaces (see Fig. 2).

The developed framework allows to give a precise formulation for this conjecture and also provides a scheme for the proof in specific situations. Essentially, if the lattice is described by consistent hyperbolic difference equations, the abovementioned limiting procedure can be carried out and yields smooth surfaces and their transformations. In the case of interest here, the result is that a smooth K-surface and a Bäcklund transformation are simultaneously approximated by discrete K-surfaces and respective discrete transformations. Other examples where the theory has been put into pratice can be found in [BMS], where convergence of discrete orthogonal coordinate systems and conjugate nets is proven.

The structure of the paper is the following. In Sect. 2 we formulate the continuous and discrete setup of the two–dimensional hyperbolic systems and the corresponding Goursat problems. The C^1–convergence result is proven which holds for all difference schemes with a local approximation property. The C^r–approximation under the appropriate conditions is established in Sect. 3. The theory is extended to the case of three independent variables in Sect. 4. At this point the notion of three–dimensional compatibility starts to play the key role; it turns out to be intimately related to the integrability. Therefore, the convergence result holds only for difference schemes with these properties. The theory is illustrated in Sect. 5, where we apply the convergence results to an integrable discretization of the sine–Gordon equation, and thus prove the convergence of discrete K–surfaces and their Bäcklund transformations to the continuous counterparts. Finally, in Sect. 6 the theory is extended to the case of an arbitrary number of independent variables.

2. Two-dimensional theory

In this section we prove an approximation theorem for a certain class of hyperbolic differential and difference equations in two dimensions. More general d–dimensional systems are considered in sections 4 and 6. The following notations for domains are used: let $\mathbf{r} = (r_1, \ldots, r_d)$ consist of positive numbers $r_i > 0$, then

$$\mathcal{B}(\mathbf{r}) = [0, r_1] \times \ldots \times [0, r_d] \subset \mathbb{R}^d. \tag{1}$$

As domains for discrete equations, we use parts of rectangular lattices inside $\mathcal{B}(\mathbf{r})$, with possibly different grid sizes along different coordinate axes $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$:

$$\mathcal{B}^{\boldsymbol{\epsilon}}(\mathbf{r}) = [0, r_1]^{\epsilon_1} \times \ldots \times [0, r_d]^{\epsilon_d} \subset \prod_{i=1}^{d} (\epsilon_i \mathbb{Z}). \tag{2}$$

where $[0, r]^\epsilon = [0, r] \cap (\epsilon\mathbb{Z})$. The dependent variables of the differential and difference equations under consideration belong to a vector space $\mathcal{X}$ with norm $|\cdot|$.

In the two–dimensional situation the notations are simplified as follows: $\mathcal{B}(r) = [0, r] \times [0, r] \subset \mathbb{R}^2$ and $\mathcal{B}^\epsilon(r) = [0, r]^\epsilon \times [0, r]^\epsilon \subset (\epsilon\mathbb{Z})^2$ denote continuous and discrete domains, respectively. Each $\mathcal{B}^\epsilon(r)$ contains $O(\epsilon^{-2})$ grid points. It is convenient to assume that ϵ attain only values of the form 2^{-k} with a positive integer k. Then $\epsilon_1 < \epsilon_2$ implies that ϵ_2 is an integer multiple of ϵ_1, and hence $\mathcal{B}^{\epsilon_2}(r) \subset \mathcal{B}^{\epsilon_1}(r)$. The limiting domains

$$\mathcal{B}^0(r) = \bigcup_{\epsilon = 2^{-k}} \mathcal{B}^\epsilon(r), \tag{3}$$

lie dense in $\mathcal{B}(r)$. Each point $x \in \mathcal{B}^0(r)$ belongs to $\mathcal{B}^\epsilon(r)$ with $\epsilon = 2^{-k}$ for all k large enough. Hence, one can speak about pointwise convergence of functions $\{a^\epsilon : \mathcal{B}^\epsilon(r) \to \mathcal{X}\}_{\epsilon = 2^{-k}}$ as $\epsilon \to 0$: the limiting function a^0 is naturally defined on $\mathcal{B}^0(r)$. If a^0 is Lipschitz on $\mathcal{B}^0(r)$, it extendeds to a Lipschitz function $a : \mathcal{B}(r) \to \mathcal{X}$.

Introduce the difference quotient operators δ_x^ϵ and δ_y^ϵ, acting on functions $a^\epsilon : \mathcal{B}^\epsilon(r) \to \mathcal{X}$,

$$\begin{aligned} \delta_x^\epsilon a^\epsilon &= \tfrac{1}{\epsilon}(a^\epsilon(x+\epsilon, y) - a^\epsilon(x, y)) \\ \delta_y^\epsilon a^\epsilon &= \tfrac{1}{\epsilon}(a^\epsilon(x, y+\epsilon) - a^\epsilon(x, y)). \end{aligned}$$

Definition 2.1. A *continuous 2D hyperbolic system* is a system of partial differential equations for functions $a, b : \mathcal{B}(r) \to \mathcal{X}$ of the form

$$\partial_x a = f(a, b), \qquad \partial_y b = g(a, b), \tag{4}$$

with smooth functions $f, g : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. A *Goursat problem* consists of prescribing the initial values

$$a(x, 0) = a_0(x), \qquad b(0, y) = b_0(y) \tag{5}$$

for $x \in [0, r]$ and $y \in [0, r]$, respectively. The functions $a_0, b_0 : [0, r] \to \mathcal{X}$ are supposed to belong to some C^k.

Definition 2.2. A (one-parameter family of) *discrete 2D hyperbolic systems* consists of two partial difference equations for $a^\epsilon, b^\epsilon : \mathcal{B}^\epsilon(r) \to \mathcal{X}$ of the form

$$\delta_x^\epsilon a = f^\epsilon(a, b), \qquad \delta_y^\epsilon b = g^\epsilon(a, b), \tag{6}$$

with smooth functions $f^\epsilon, g^\epsilon : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. A *Goursat problem* for this system consists of prescribing the initial values

$$a^\epsilon(x, 0) = a_0^\epsilon(x), \qquad b^\epsilon(0, y) = b_0^\epsilon(y) \tag{7}$$

for $x \in [0, r]^\epsilon$ and $y \in [0, r]^\epsilon$, respectively.

Remark 1. The notations suggest that the variables (a^ϵ, b^ϵ) are attached to the points of the two-dimensional lattice $\mathcal{B}^\epsilon(r)$. But they are naturally associated to the *edges* of this lattice: $a^\epsilon(x, y)$ to the horizontal edge connecting the vertices (x, y) and $(x + \epsilon, y)$, and $b^\epsilon(x, y)$ to the vertical edge connecting the vertices (x, y) and $(x, y + \epsilon)$. See Fig. 3. The equations (6) give the fields on the right and on the top edges of an elementary square, provided the fields sitting on the left and on the bottom ones are known.

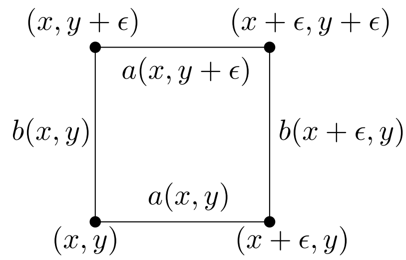


FIGURE 3. An elementary quadrilateral

The following result is almost obvious:

Proposition 2.1. *The Goursat problem for a discrete 2D hyperbolic system (6) has a unique solution (a^ϵ, b^ϵ) on $\mathcal{B}^\epsilon(r)$.*

Proof. The discrete solution is calculated by induction from the data on the coordinate axes. □

Example 1. The constructions are now illustrated for the Sine-Gordon equation,

$$\partial_x \partial_y \phi = \sin \phi. \tag{8}$$

A Goursat problem for (8) is posed as follows

$$\phi(x, 0) = \phi_1(x), \quad \phi(0, y) = \phi_2(y). \tag{9}$$

The canonical way to bring (8) into the form (4) is to introduce two new dependent variables

$$a = \partial_x \phi, \qquad b = \phi, \tag{10}$$

which have to satisfy the following equations and initial conditions

$$\partial_y a = \sin b, \qquad \partial_x b = a, \tag{11}$$

$$a(x,0) = \partial_x \phi_1(x), \quad b(0,y) = \phi_2(y). \tag{12}$$

A naive discretization is obtained by replacing partial derivatives by their difference quotients,

$$\delta_x^\epsilon \delta_y^\epsilon \phi = \sin \phi. \tag{13}$$

Introduce two new dependent variables

$$a^\epsilon = \delta_x^\epsilon \phi, \qquad b^\epsilon = \phi, \tag{14}$$

then they have to satisfy a discrete 2D hyperbolic system,

$$\delta_y^\epsilon a^\epsilon = \sin b^\epsilon, \qquad \delta_x^\epsilon b^\epsilon = a^\epsilon \tag{15}$$

There are various choices for the discrete initial data. The canonical one is to take simply

$$a^\epsilon(x,0) = \partial_x \phi_1(x), \quad b^\epsilon(0,y) = \phi_2(y) \tag{16}$$

at grid points $(x,0),(0,y) \in \mathcal{B}^\epsilon(r)$. The main Theorem 2.2 below implies that the solutions of (15) and (16) converge as $\epsilon \to 0$ to the solutions of (11) and (12), and hence to the solution of (8) and (9) on a suitable domain $\mathcal{B}(\bar{r})$ with $0 < \bar{r} \le r$.

However, the discretization (13) is non-geometric: recall that the sine-Gordon-equation describes smooth K-surfaces, whereas (13) does not posses an immediate geometric interpretation. There exists an alternative discretization of (8), which is due to Hirota [Hir] and has become famous,

$$\begin{aligned} &\sin \frac{1}{4}\big(\phi^\epsilon(x+\epsilon,y+\epsilon) - \phi^\epsilon(x+\epsilon,y) - \phi^\epsilon(x,y+\epsilon) + \phi^\epsilon(x,y)\big) \\ &= \frac{\epsilon^2}{4} \sin \frac{1}{4}\big(\phi^\epsilon(x+\epsilon,y+\epsilon) + \phi^\epsilon(x+\epsilon,y) + \phi^\epsilon(x,y+\epsilon) + \phi^\epsilon(x,y)\big). \end{aligned} \tag{17}$$

Its solutions *do* correspond to discrete K-surfaces, see [BP1] for the interpretation of the angle ϕ. Introducing a^ϵ as in (14) and

$$b^\epsilon(x,y) = \phi^\epsilon(x,y) + \frac{\epsilon}{2}\,\delta_y^\epsilon \phi^\epsilon(x,y) = \tfrac{1}{2}(\phi^\epsilon(x,y+\epsilon) + \phi^\epsilon(x,y)), \tag{18}$$

then (17) becomes equivalent to

$$\begin{aligned} b^\epsilon(x+\epsilon,y) - b^\epsilon(x,y) &= \frac{\epsilon}{2}(a^\epsilon(x,y+\epsilon) + a^\epsilon(x,y)), \\ e^{i\epsilon a^\epsilon(x,y+\epsilon)/2} - e^{i\epsilon a^\epsilon(x,y)/2} &= \frac{\epsilon^2}{4}\Big(e^{ib^\epsilon(x+\epsilon,y)} - e^{-ib^\epsilon(x,y)}\Big). \end{aligned}$$

Solving for $a^\epsilon(x, y+\epsilon)$ and $b^\epsilon(x+\epsilon, y)$, one ends up with

$$(19)\quad \delta_y^\epsilon a^\epsilon = \frac{2}{i\epsilon^2}\log\frac{1-(\epsilon^2/4)\exp(-ib^\epsilon - i\epsilon a^\epsilon/2)}{1-(\epsilon^2/4)\exp(ib^\epsilon + i\epsilon a^\epsilon/2)}, \qquad \delta_x^\epsilon b^\epsilon = a^\epsilon + \frac{\epsilon}{2}\,\delta_y^\epsilon a^\epsilon.$$

Both discrete 2D hyperbolic systems (15), (19) approximate the continuous one (11) in the sense of the next definition.

Definition 2.3. A discrete 2D hyperbolic system (6) $\mathcal{O}(\epsilon)$–*approximates* the continuous one (4), if the functions f^ϵ, g^ϵ satisfy

$$(20)\qquad f^\epsilon(a,b) = f(a,b) + \mathcal{O}(\epsilon), \qquad g^\epsilon(a,b) = g(a,b) + \mathcal{O}(\epsilon),$$

uniformly on compact subsets of $\mathcal{X}\times\mathcal{X}$. Moreover, $\mathcal{O}(\epsilon)$–*approximation in* C^k means that (20) also holds for all k–th partial derivatives.

Remark 2. In the context of difference equations, approximation properties as (20) are often referred to as *consistency* of the discrete approximation. This notion is not to be confused with consistency in our sense, which means multi–dimensional compatibility of the equations.

The main result of this section is

Theorem 2.2. *Let a family of discrete 2D hyperbolic systems (6) $\mathcal{O}(\epsilon)$–approximate the continuous 2D hyperbolic system (4) in C^1. Let also the discrete initial data (7) approximate the continuous ones (5) as*

$$(21)\qquad a_0^\epsilon(x) = a_0(x) + \mathcal{O}(\epsilon), \qquad b_0^\epsilon(y) = b_0(y) + \mathcal{O}(\epsilon)$$

uniformly for $x \in [0,r]^\epsilon$ and $y \in [0,r]^\epsilon$, respectively. Then the sequence of solutions (a^ϵ, b^ϵ) converges pointwise uniformly to a pair of Lipschitz–continuous functions (a,b),

$$(22)\qquad a^\epsilon(x,y) = a(x,y) + \mathcal{O}(\epsilon), \qquad b^\epsilon(x,y) = b(x,y) + \mathcal{O}(\epsilon)$$

for $(x,y) \in \mathcal{B}(\bar r)$, with a suitable $\bar r \in (0,r]$. The functions a, b solve the continuous Goursat problem for (4) on $\mathcal{B}(\bar r)$.

In general one cannot expect $\bar r = r$ because the solutions of the limiting equations may develop blow-ups that are absent in the discretization. Consequently, the essential prerequisite for the proof of Theorem 2.2 are ϵ-independent *á priori* bounds on a^ϵ and b^ϵ.

Lemma 2.3 (Uniform bound)**.** *Let the norms of initial data a_0^ϵ, b_0^ϵ be bounded by ϵ-independent constants. Then there exists $\bar r \in (0,r]$ such that the norms of the solutions (a^ϵ, b^ϵ) are bounded on the respective $\mathcal{B}^\epsilon(\bar r)$ independently of ϵ.*

Remark 3. If f and g posses a global Lipschitz constant, then $\bar r = r$ in Lemma 2.3 and also in Theorem 2.2.

Proof of Lemma 2.3. Let $M_0 > 0$ such that $|a_0^\epsilon|, |b_0^\epsilon| \le M_0$, and choose $M_1 > M_0$ arbitrary. Define

$$\bar r = (M_1 - M_0)/\sup_\epsilon \sup_{|a|,|b|<M_1} \{|f^\epsilon(a,b)| + |g^\epsilon(a,b)|\}.$$

It is easily shown that $|a^\epsilon|, |b^\epsilon| < M_1$ on $\mathcal{B}^\epsilon(\bar{r})$: rewrite the difference equations (6) as

$$a^\epsilon(x,y) = a^\epsilon(x,y-\epsilon) + \epsilon f^\epsilon(a^\epsilon(x,y-\epsilon), b^\epsilon(x,y-\epsilon)), \tag{23}$$

$$b^\epsilon(x,y) = b^\epsilon(x-\epsilon,y) + \epsilon g^\epsilon(a^\epsilon(x-\epsilon,y), b^\epsilon(x-\epsilon,y)), \tag{24}$$

and then conclude by induction that

$$|a^\epsilon(x,y)| \le M_0 + (M_1 - M_0)\,\frac{y}{\bar{r}} < M_1,$$

$$|b^\epsilon(x,y)| \le M_0 + (M_1 - M_0)\,\frac{x}{\bar{r}} < M_1,$$

for $(x,y) \in \mathcal{B}^\epsilon(\bar{r})$. □

With the bounds on the absolute value at hand, estimates on the difference quotients can be derived, using

Lemma 2.4 (Discrete Gronwall estimate). *Assume a nonnegative function $\Delta : \mathbb{N}_0 \to \mathbb{R}$ satisfies the implicit estimate*

$$\Delta(n+1) \le (1+\epsilon\mathcal{K})\Delta(n) + \kappa \tag{25}$$

with nonnegative constants $\mathcal{K}$ and κ for all $n = 0,1,2,\dots,N$, then this explicit estimate follows:

$$\Delta(n) \le (\Delta(0) + n\kappa)\exp(\mathcal{K}n\epsilon). \tag{26}$$

Proof. Iterate (25) to confirm (26) by induction on $n > 0$, observing that $\exp(\epsilon\mathcal{K}) \le (1+\epsilon\mathcal{K})$ for $\mathcal{K} \ge 0$. □

Lemma 2.5 (Lipschitz bound). *Assume the continuous Goursat data a_0, b_0 are C^1 functions, and the discrete Goursat data $a_0^\epsilon, b_0^\epsilon$ satisfy*

$$|a_0^\epsilon(x) - a_0(x)| \le M\epsilon, \quad |b_0^\epsilon(y) - b_0(y)| \le M\epsilon \tag{27}$$

with an ϵ-independent constant M. Then the difference quotients

$$\delta_x a^\epsilon,\ \delta_y a^\epsilon,\ \delta_x b^\epsilon,\ \delta_y b^\epsilon$$

are bounded independently of ϵ on the respective $\mathcal{B}^\epsilon(\bar{r})$, where $\bar{r} \in (0,r]$ chosen according to Lemma 2.3.

Proof. In virtue of the equations (6) and Lemma 2.3 it is clear that the difference quotients $\delta_y^\epsilon a^\epsilon$ and $\delta_x^\epsilon b^\epsilon$ are uniformly bounded.
Let M_1 be an absolute bound on the solutions (a^ϵ, b^ϵ) of the discrete Goursat problems, and

$$M_2 = \sup_\epsilon \sup_{|a|,|b|\le M_1} \Big\{ |f^\epsilon(a,b)|, |g^\epsilon(a,b)|, |\partial_a f^\epsilon(a,b)|, \dots, |\partial_b g^\epsilon(a,b)| \Big\}, \tag{28}$$

which is finite since $f^\epsilon \to f$ and $g^\epsilon \to g$ locally uniformly in C^1.
Without loss of generality, $M > M_1$ and $M > M_2$. By the mean value theorem,

$$|\delta_x^\epsilon a_0^\epsilon(x)| \le |\delta_x^\epsilon a_0(x)| + \epsilon^{-1}|a_0^\epsilon(x+\epsilon) - a_0(x+\epsilon)| + \epsilon^{-1}|a_0(x) - a_0^\epsilon(x)| \le 3M. \tag{29}$$

Proceeding from y to $y+\epsilon$,

$$\begin{aligned}|\delta_x^\epsilon a^\epsilon(x,y+\epsilon)| &\leq |\delta_x^\epsilon a^\epsilon(x,y)| + \epsilon|\delta_x^\epsilon f^\epsilon(a^\epsilon(x,y),b^\epsilon(x,y))| \\ &\leq |\delta_x^\epsilon a^\epsilon(x,y)| + \epsilon M(|\delta_x^\epsilon a^\epsilon(x,y)| + |\delta_x^\epsilon b^\epsilon(x,y)|) \\ &\leq (1+\epsilon M)|\delta_x^\epsilon a^\epsilon(x,y)| + \epsilon M^2.\end{aligned}$$

Now Lemma 2.4 yields the desired estimate:

$$|\delta_x^\epsilon a^\epsilon(x,y)| \leq 4M\exp(M\bar{r}).$$

The same reasoning applies to $\delta_y^\epsilon b^\epsilon$. □

Proof of Theorem 2.2. Consider the family $\{(\tilde{a}^\epsilon,\tilde{b}^\epsilon)\}_{\epsilon=2^{-k}}$ of functions $\tilde{a}^\epsilon,\tilde{b}^\epsilon : \mathcal{B}(\bar{r}) \to \mathcal{X}$ obtained from a^ϵ and b^ϵ by linear interpolation. By Lemma 2.5, there is a Lipschitz constant $L>0$, so that

$$|\tilde{a}^\epsilon(x',y') - \tilde{a}^\epsilon(x,y)| + |\tilde{b}^\epsilon(x',y') - \tilde{b}^\epsilon(x,y)| \quad \leq \quad L(|x'-x| + |y'-y|).$$

In combination with Lemma 2.3, it follows that the family is *equicontinuous*, i.e., it satisfies the hypothesis of the Arzelá-Ascoli theorem. Consequently, there exist continuous functions $a,b : \mathcal{B}(\bar{r}) \to \mathcal{X}$ such that $\tilde{a}^{\epsilon'} \to a$ and $\tilde{b}^{\epsilon'} \to b$ uniformly for an infinite subsequence $\epsilon' = 2^{-k'}$. Moreover, a and b are Lipschitz continuous, and L is a Lipschitz constant.

To show that (a,b) solve the differential equations (4), observe that relation (23) and Lipschitz-continuity of $\tilde{a}^\epsilon$ imply

$$\tilde{a}^{\epsilon'}(x,y) \quad = \quad \tilde{a}_0^{\epsilon'}(x) + \epsilon \sum_{k=0}^{[y/\epsilon']-1} f^{\epsilon'}[\tilde{a}^{\epsilon'},\tilde{b}^{\epsilon'}](x,k\epsilon') + \mathcal{O}(\epsilon') \tag{30}$$

for $(x,y)\in\mathcal{B}(\bar{r})$. As the convergence of $\tilde{a}^{\epsilon'}$ and $\tilde{b}^{\epsilon'}$ is uniform, and $f^\epsilon \to f$ in C^1, one may pass to the limit $\epsilon' \to 0$ on both sides of (30),

$$a(x,y) = a_0(x) + \int_0^y f[a,b](x,\eta)d\eta. \tag{31}$$

It follows that a is everywhere differentiable with respect to y, and $\partial_y a = f(a,b)$. The function b is treated in the same manner.

Eventually, the convergence (22) can be proven. For arbitrary $\epsilon = 2^{-k}$ define the approximation error

$$\begin{aligned}\Delta^\epsilon(n) \quad = \quad \max\{&|a^\epsilon(x,y) - a(x,y)| + |b^\epsilon(x,y) - b(x,y)|, \\ &(x,y)\in\mathcal{B}^\epsilon(\bar{r}),\ x+y = n\epsilon\}.\end{aligned} \tag{32}$$

Combining formula (23) with the integral representation (31) yields

$$\begin{aligned}\Delta^\epsilon(n+1) \quad &\leq \quad \Delta^\epsilon(n) + \epsilon \max_{x+y=n\epsilon}\big(|\delta_x^\epsilon(a^\epsilon - a)|(x,y) + |\delta_y^\epsilon(b^\epsilon - b)|(x,y)\big) \\ &\qquad + |a_0^\epsilon - a_0|(n\epsilon+\epsilon) + |b_0^\epsilon - b^0|(n\epsilon+\epsilon) \\ &\leq \quad \Delta^\epsilon(n) + \epsilon \max_{x+y=n\epsilon}\big(|f^\epsilon[a^\epsilon,b^\epsilon] - f[a,b]|(x,y) \\ &\qquad + |g^\epsilon[a^\epsilon,b^\epsilon] - g[a,b]|(x,y)\big) + \mathcal{O}(\epsilon) \\ &\leq \quad (1+\mathcal{O}(\epsilon))\Delta^\epsilon(n) + \mathcal{O}(\epsilon).\end{aligned}$$

By the Gronwall estimate in Lemma 2.4, $\Delta^\epsilon(n) = \mathcal{O}(\epsilon)$ for $n\epsilon \le \bar{r}$. This implies the estimate (22). □

Corollary 2.6. *The two dimensional hyperbolic Goursat problem (4), (5) possesses a unique classical solution.*

Proof. Existence of a classical solution is already part of the conclusions of Theorem 2.2. Uniqueness follows from the proof above: the estimates for Δ^ϵ introduced in (32) are independent of the specific solution (a,b) to (4), (5). In fact, only the integral representation (31) has been used. Hence, every solution to the continuous Goursat problem appears as uniform limit of the discrete solutions (a^ϵ, b^ϵ) as $\epsilon \to 0$. On the other hand, the discrete solutions are unique, and so is their limit. □

This chapter is concluded with a Theorem about a stronger kind of convergence. The following definition introduces a strengthend version of $\mathcal{O}(\epsilon)$–approximation considered in Definition 2.3.

Definition 2.4. A 2D hyperbolic system (6) $\mathcal{O}(\epsilon^2)$*–approximates* the continous one (4) if

$$\begin{aligned} f^\epsilon &= f + \tfrac{\epsilon}{2}\left(D_a f \cdot f + D_b f \cdot g\right) + \mathcal{O}(\epsilon^2) \\ g^\epsilon &= g + \frac{\epsilon}{2}\left(D_a g \cdot f + D_b g \cdot g\right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

uniformly on compact subsets of $\mathcal{X} \times \mathcal{X}$.

Theorem 2.7. *Let a family of discrete 2D hyperbolic systems (6)* $\mathcal{O}(\epsilon)$*–approximate the continuous hyperbolic system (4) in* C^1*. Assume further, that the discrete family is also* $\mathcal{O}(\epsilon^2)$*–approximative. Let the discrete initial data (7) converge to the continuous initial data (5) as*

$$a_0^\epsilon(x) = a_0(x + \tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^2), \quad b_0^\epsilon(y) = b_0(y + \tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^2). \tag{33}$$

Then the conclusions of Theorem 2.2 hold and in addition, the discrete solutions (a^ϵ, b^ϵ) *converge to the continuous ones* (a,b) *as*

$$a^\epsilon(x,y) = a(x + \tfrac{\epsilon}{2}, y) + \mathcal{O}(\epsilon^2), \quad b^\epsilon(x,y) = b(x, y + \tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^2) \tag{34}$$

uniformly on $\mathcal{B}(\bar{r})$*.*

Remark 4. The estimate (34) underlines the statement that it is more natural to think of $a^\epsilon(x,y)$ and $b^\epsilon(x,y)$ as associated to (the midpoints of) the edges from (x,y) to $(x+\epsilon, y)$ and from (x,y) to $(x, y+\epsilon)$, respectively.

Proof. Since the initial data a_0, b_0 are Lipschitz-continuous, it is clear that the hypothesis (33) above implies (21), and therefore all of the conclusions of Theorem 2.2 as well.

To obtain (34), one modifies the proof of estimate (22). In obvious analogy to Δ^ϵ defined in (32), let

$$\begin{aligned} \Delta^\epsilon(n) &= \sup\{|a^\epsilon(x,y) - a(x + \tfrac{\epsilon}{2}, y)| + |b^\epsilon(x,y) - b(x, y + \tfrac{\epsilon}{2})|, \\ &\qquad (x,y) \in \mathcal{B}^\epsilon(\bar{r}),\, x + y = n\epsilon\}. \end{aligned}$$

As before, one obtains

$$\Delta^\epsilon(n+1) \le \Delta^\epsilon(n) + \epsilon \max_{x+y=n\epsilon} \big(|\delta_x(a^\epsilon - a)|(x,y) + |\delta_y(b^\epsilon - b)|(x,y) \big)$$
$$+ |a_0^\epsilon((n+1)\epsilon) - a_0((n+\tfrac{3}{2})\epsilon)| + |b_0^\epsilon((n+1)\epsilon) - b_0((n+\tfrac{3}{2})\epsilon)|.$$

The difference quotient $\delta^\epsilon a(x+\frac{\epsilon}{2},y)$ is now analyzed up to $\mathcal{O}(\epsilon^3)$. For shortness, let $\bar{a} = a(x+\frac{\epsilon}{2},y)$ and $\bar{b} = b(x,y+\frac{\epsilon}{2})$.

$$\begin{aligned} a(x+\tfrac{\epsilon}{2},y+\epsilon) - a(x+\tfrac{\epsilon}{2},y) &= \int_0^\epsilon f[a,b](x+\tfrac{\epsilon}{2},y+\eta)\,d\eta \\ &= \epsilon f[a,b](x+\tfrac{\epsilon}{2},y+\tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^3) \\ &= \epsilon\big(f(\bar{a},\bar{b}) + D_a f(\bar{a},\bar{b}) \cdot (a(x+\tfrac{\epsilon}{2},y+\tfrac{\epsilon}{2}) - \bar{a}) + \\ &\qquad D_b f(\bar{a},\bar{b}) \cdot (b(x+\tfrac{\epsilon}{2},y+\tfrac{\epsilon}{2}) - \bar{b}) + \mathcal{O}(\epsilon^2)\big) + \mathcal{O}(\epsilon^3) \\ &= \epsilon\big(f(\bar{a},\bar{b}) + \tfrac{\epsilon}{2} D_a f(\bar{a},\bar{b}) \cdot f(\bar{a},\bar{b}) + \tfrac{\epsilon}{2} D_b f(\bar{a},\bar{b}) \cdot g(\bar{a},\bar{b})\big) + \mathcal{O}(\epsilon^3). \end{aligned}$$

As the discrete equations are $\mathcal{O}(\epsilon^2)$-approximative,

$$|\delta_x a^\epsilon(x,y) - \delta_x a(x+\tfrac{\epsilon}{2},y)| \quad \le \quad C\epsilon\Delta^\epsilon(n) + \mathcal{O}(\epsilon^3),$$

where the constant C depends only on the functions f, g and their first derivatives. The same reasoning applies to b. In summary,

$$\Delta^\epsilon(n) \quad \le \quad (1+\mathcal{O}(\epsilon))\Delta^\epsilon(n) + \mathcal{O}(\epsilon^3),$$

and therefore $\Delta^\epsilon(n) = \mathcal{O}(\epsilon^2)$ by Lemma 2.4. □

Remark 5. Convergence of order $\mathcal{O}(\epsilon^{2+\delta})$ with $\delta > 0$ *cannot* be achieved simply by imposing stronger conditions on the convergence $f^\epsilon \to f$, $g^\epsilon \to g$.

3. Additional Smoothness

This section is devoted to the proof of

Theorem 3.1. *In addition to the hypothesis of Theorem 2.2, assume that the nonlinearities f^ϵ, g^ϵ are $\mathcal{O}(\epsilon)$-approximative in C^{S+1}, $S > 1$. Assume further that the continuous initial data a_0, b_0 are actually $C^{S,1}$-functions*[1] *and that the discrete data approximates them as*

(35) $$(\delta_x^\epsilon)^\ell a_0^\epsilon(x) = \partial_x^\ell a_0(x) + \mathcal{O}(\epsilon),\ (\delta_y^\epsilon)^\ell b_0^\epsilon(y) = \partial_y^\ell b_0(y) + \mathcal{O}(\epsilon)$$

for all $\ell \le S$. Then the continuous solutions a, b belong to $C^{S,1}(\mathcal{B}^\epsilon(\bar{r}))$, with the same $\bar{r} > 0$ as in Theorem 2.2. Moreover, the limits are uniform in C^S,

(36) $$(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a^\epsilon = \partial_x^m\partial_y^n a + \mathcal{O}(\epsilon),\ (\delta_x^\epsilon)^m(\delta_y^\epsilon)^n b^\epsilon = \partial_x^m\partial_y^n b + \mathcal{O}(\epsilon),$$

for all m, n with $m+n \le S$.

Assumption (35) above is quite natural. In fact, if a_0 belongs to $C^{S,1}$, and a_0^ϵ is its restriction to the ϵ-lattice, then (35) is fulfilled.

---

[1] their S-th derivative is Lipschitz-continuous

Remark 6. If the hypothesis of Theorem 3.1 is met for all positive integers S, then one may then loosely speak of C^∞-approximation of (a, b) by (a^ϵ, b^ϵ).

First, *á priori* estimates for higher-order difference quotients of a^ϵ, b^ϵ are derived. As discrete analogue of the C^s-norm, define for $u : \mathcal{B}^\epsilon(r) \to \mathcal{X}$

$$\|u\|_s = \max_{k+\ell\le s} \sup_{\mathcal{B}^\epsilon(r-s\epsilon)} |(\delta_x^\epsilon)^k(\delta_y^\epsilon)^\ell u|. \tag{37}$$

Recall that a^ϵ and b^ϵ are bounded on $\mathcal{B}^\epsilon(\bar{r})$, $|a^\epsilon|, |b^\epsilon| \le M_1$ independent of $\epsilon > 0$. Introduce for a smooth function $h : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$

$$\|h\|_s = \max_{m\le s} \sup_{|a|,|b|<M_1} |D^m h(a, b)|,$$

which is the C^s-norm of f on the ball of radius M_1. The following is essential to estimate the norm of compositions with smooth functions.

Lemma 3.2. *Let $h : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a smooth function, and $a, b : \mathcal{B}^\epsilon(r) \to \mathcal{X}$ be bounded by M_1. Let further (m, n) be a pair of nonnegative integers. Then, there is a constant C_{mn} such that*

$$|(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n h[a, b](x, y)| \le$$

$$\|h\|_{m+n+1} \Big(|(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a(x, y)| + |(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n b(x, y)| + Q_{mn}\Big), \tag{38}$$

where Q_{mn} is a continuous, non-decreasing function of $\|a\|_{m+n-1}$ and $\|b\|_{m+n-1}$.

The proof of this Lemma is technical. It is a consequence of a much more general formula that replaces the chain rule for difference quotients. A detailed proof can be found in [Mat].

Lemma 3.3. *Under the conditions of Theorem 3.1,*

$$\sup_\epsilon \|a^\epsilon\|_{S+1} < \infty, \quad \sup_\epsilon \|b^\epsilon\|_{S+1} < \infty. \tag{39}$$

Proof. The proof goes by induction over the total degree $s = m + n = K \le S + 1$ of the difference quotient. So assume that $\sup_\epsilon \|a^\epsilon\|_{s-1}, \sup_\epsilon \|b^\epsilon\|_{s-1} < \infty$ is already proved.
Then, for $n > 0$,

$$|(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a^\epsilon(x, y)| = |(\delta_x^\epsilon)^m(\delta_y^\epsilon)^{n-1} f^\epsilon[a^\epsilon, b^\epsilon](x, y)| \le Q_{mn} < \infty$$

by Lemma 3.2, and similarly for $(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n b^\epsilon$ with $m > 0$. Otherwise, observe that for once,

$$|(\delta_x^\epsilon)^m a^\epsilon(x, 0)| \le M < \infty$$

by (35) for all $m = 1, \dots, S + 1$, and further

$$(\delta_x^\epsilon)^m a^\epsilon(x, y) = (\delta_x^\epsilon)^m a^\epsilon(x, y - \epsilon) + \epsilon(\delta_x^\epsilon)^m f^\epsilon[a^\epsilon, b^\epsilon](x, y - \epsilon),$$

so by Lemma 3.2,

$$\begin{aligned}|(\delta_x^\epsilon)^m a^\epsilon(x, y)| \le{}& (1 + \epsilon\|f^\epsilon\|_{m+1})|(\delta_x^\epsilon)^m a^\epsilon(x, y - \epsilon)| \\ &+\epsilon\|f^\epsilon\|_{m+1}\big(|(\delta_x^\epsilon)^m b^\epsilon(x, y)| + Q_{mn}\big).\end{aligned}$$

Knowing that the norm of $(\delta_y^\epsilon)^m$ is bounded independently of ϵ, the Gronwall Lemma 2.4 yields an ϵ-independent bound on $(\delta_x^\epsilon)^m a^\epsilon$. The same reasoning applies to $(\delta_y^\epsilon)^n b^\epsilon$. □

Proof of the smoothness of a and b. Recall that the continuous function a was obtained as the uniform limit of a suitable subsequence $a^{\epsilon'}$ as $\epsilon' \to 0$. Under the hypothesis of Theorem 3.1, this subsequence can be chosen so that also $(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a^\epsilon \to a^{(mn)}$ converge uniformly on $\mathcal{B}^\epsilon(\bar{r})$ for $m+n \leq S$, where $a^{(mn)}$ are Lipschitz functions. This follows directly from the Arzelá-Ascoli theorem: by estimate (39), each family $\{(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a^\epsilon\}$ posseses an ϵ-independent Lipschitz constant, as long as $m+n \leq S$. It is then easily seen that $a^{(mn)} = \partial_x a^{(m-1\,n)} = \partial_y a^{(m\,n-1)}$. In conclusion, a is S times differentiable with $\partial_x^m \partial_y^n a = a^{(mn)}$, which are Lipschitz functions. The same argument is true with b in place of a. □

To show also the convergence in (36), another technical lemma is needed.

Lemma 3.4. *Let $h^\epsilon, h : \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ be smooth functions, and $u^\epsilon, u : \mathcal{B}^\epsilon(r) \to \mathcal{X}$ be bounded by M_1. Given a pair (m,n) of nonnegative integers, let $s = m+n$. Then, there is a constant C_s such that*

$$\begin{aligned} \left|(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n(h^\epsilon(u^\epsilon) - h(u))(x,y)\right| \ &\leq C_s Q_s \|h^\epsilon - h\|_s \\ +\|h\|_{s+1}\left(\left|(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n(u^\epsilon - u)(x,y)\right| \right. &+ \left. C_s\|u^\epsilon - u\|_{s-1} Q_s\right), \end{aligned} \tag{40}$$

holds for all $(x,y) \in \mathcal{B}^\epsilon(r)$, and Q_s is a continuous, non-decreasing function of $\|u^\epsilon\|_s$ and $\|u\|_s$.

As for Lemma 3.2, the proof follows from the chain rule for difference operators on lattices.

Proof of estimate (36). Again, it is natural to make an induction on the total degree $s = m+n \leq S$. Convergence in C^0, i.e. the case $s = 0$, is already settled by Theorem 2.2. Assume that (36) holds for $s-1$. Let $m+n = s$ below. To estimate $A_{mn}^\epsilon(x,y) := (\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a^\epsilon(x,y) - \partial_x^m\partial_y^n a(x,y)$, three cases have to be considered.

(1) $y = 0$ and $n = 0$. Then $A_{m0}^\epsilon(x,0) = \mathcal{O}(\epsilon)$ by the hypothesis (35).

(2) $n \geq 1$. As a, b are $C^{S,1}$-smooth and f is C^∞,

$$\partial_x^m\partial_y^{n-1} f[a,b](x,y) \ = \ (\delta_x^\epsilon)^m(\delta_y^\epsilon)^{n-1} f[a,b](x,y) + \mathcal{O}(\epsilon)$$

Uniformly on $B(\bar{r})$. One obtains

$$A_{mn}^\epsilon(x,y) \ = \ (\delta_x^\epsilon)^m(\delta_y^\epsilon)^{n-1}(f^\epsilon[a^\epsilon,b^\epsilon] - f[a,b])(x,y) + \mathcal{O}(\epsilon). \tag{41}$$

Recall that f^ϵ $\mathcal{O}(\epsilon)$-approximates f in C^{S+1}, and that by induction hypothesis, $\|a^\epsilon - a\|_{s-1}$, $\|b^\epsilon - b\|_{s-1} = \mathcal{O}(\epsilon)$. Apply Lemma 3.4 to (41) to find $A_{mn}^\epsilon(x,y) = \mathcal{O}(\epsilon)$.

(3) $y > 0$ but $n = 0$. Again, smoothness of $f(a)$ is used to derive

$$\begin{aligned} A_{m0}^\epsilon(x,y) \ &= \ A_{m0}^\epsilon(x,y-\epsilon) \\ &\quad +\epsilon(\delta_x^\epsilon)^m(f^\epsilon[a^\epsilon,b^\epsilon] - f[a,b])(x,y-\epsilon) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Now estimate the second term in the sum by Lemma 3.4. After trivial manipulations,

$$\begin{aligned}|A^\epsilon_{m0}(x,y)| &\leq (1+\epsilon\|f\|_{s-1})|A^\epsilon_{m0}(x,y-\epsilon)| \\ &\quad +\epsilon\|f\|_{s-1}|(\delta^\epsilon_x)^m(b^\epsilon-b)|(x,y-\epsilon)+\mathcal{O}(\epsilon^2).\end{aligned}$$

But $(\delta^\epsilon_x)^m(b^\epsilon - b)$ can be estimated along the same lines as $(\delta^\epsilon_y)^n a^\epsilon - \partial^n_y a = \mathcal{O}(\epsilon)$ in item (2). Hence,

$$|A^\epsilon_{m0}(x,y)| \leq (1+\epsilon\|f\|_{s-1})|A^\epsilon_{m0}(x,y-\epsilon)| + \mathcal{O}(\epsilon^2),$$

and an application of the Gronwall Lemma 2.4 gives $A^\epsilon_{m0}(x,y) = \mathcal{O}(\epsilon)$ for all $(x,y) \in \mathcal{B}^\epsilon(\bar r)$.

This proves the estimates for a^ϵ, and the same reasoning applies to b^ϵ. □

4. Three–dimensional theory: approximating Bäcklund transformations

The Sine-Gordon equation (8) posseses Bäcklund transformations. From a given solution ϕ, new solutions can be constructed by solving ordinary differential equations only. The famous formula for a family of elementary Bäcklund transformations $\phi \to \widetilde{\phi}$ for (8) reads:

$$\partial_x\widetilde{\phi} + \partial_x\phi = 2\alpha\sin\frac{\widetilde{\phi}-\phi}{2}, \qquad \partial_y\widetilde{\phi} - \partial_y\phi = \frac{2}{\alpha}\sin\frac{\widetilde{\phi}+\phi}{2}. \tag{42}$$

A direct calculation shows that this system is compatible, $\partial_y(\partial_x\widetilde{\phi}) = \partial_x(\partial_y\widetilde{\phi})$, provided ϕ is a solution of the Sine-Gordon equation, and then $\widetilde{\phi}$ is also a solution. An equivalent way to express this state of affairs is to introduce, along with the variables a, b from (10), also the auxilary function $\theta = (\widetilde{\phi} - \phi)/2$, which satisfies the following system of ordinary differential equations:

$$\partial_x\theta = -a + \alpha\sin\theta, \qquad \partial_y\theta = \frac{1}{\alpha}\sin(b+\theta). \tag{43}$$

Compatibility $\partial_y(\partial_x\theta) = \partial_x(\partial_y\theta)$ holds provided (a,b) solves the system (11), and the initial value

$$\theta(0,0) = \theta_0 \tag{44}$$

determines a unique solution. Then the formulas

$$\widetilde{a} = a + 2\partial_x\theta = -a + 2\alpha\sin\theta, \qquad \widetilde{b} = b + 2\theta \tag{45}$$

deliver a new solution $(\widetilde{a}, \widetilde{b})$ of the 2D hyperbolic system (11) equivalent to the Sine-Gordon equation. Clearly, Bäcklund transformations can be iterated in a straightforward manner.

Definition 4.1. A *continuous 2D hyperbolic system with a Bäcklund tranformation* is a compatible system of partial differential and difference equations

$$\partial_y a = f(a,b), \qquad \partial_x b = g(a,b), \tag{46}$$

$$\partial_x \theta = u(a,\theta), \qquad \partial_y \theta = v(b,\theta), \tag{47}$$

$$\delta_z a = \xi(a,\theta), \qquad \delta_z b = \eta(b,\theta) \tag{48}$$

for functions $a, b, \theta : \mathcal{B}(r,R) \to \mathcal{X}$, where

$$\mathcal{B}(r,R) = \{(x,y,z) \mid (x,y) \in \mathcal{B}(r),\ z = 0,1,\ldots,R\}. \tag{49}$$

Here $f, g, u, v, \xi, \eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are asuumed to be smooth functions. A *Goursat problem* is posed by the requirement

$$a(x,0,0) = a_0(x),\ \ b(0,y,0) = b_0(y),\ \ \theta(0,0,z) = \theta_0(z) \tag{50}$$

for $x \in [0,r]$, $y \in [0,r]$, and $z \in \{0,1,\ldots,R\}$, respectively, with given smooth functions $a_0(x)$, $b_0(y)$ and a sequence $\theta_0(0),\ldots,\theta_0(R-1)$.

The *compatibility conditions* mentioned in this definition, are to assure the existence of solutions of the above Goursat problem. They follow from

$$\partial_y(\partial_x\theta) = \partial_x(\partial_y\theta), \quad \partial_y(\delta_z a) = \delta_z(\partial_y a), \quad \partial_x(\delta_z b) = \delta_z(\partial_x b).$$

In length, these conditions for (46)-(48) to be Bäcklund transformations of the 2D hyperbolic system read:

$$\begin{aligned}
& D_a u(a,\theta)\cdot f(a,b) + D_\theta u(a,\theta)\cdot v(b,\theta) = \\
& \quad = \ D_b v(b,\theta)\cdot g(a,b) + D_\theta v(b,\theta)\cdot u(a,\theta), \\
& D_a \xi(a,\theta)\cdot f(a,b) + D_\theta \xi(a,\theta)\cdot v(b,\theta) = \\
& \quad = \ f\Big(a+\xi(a,\theta), b+\eta(b,\theta)\Big) - f(a,b), \\
& D_b \eta(b,\theta)\cdot g(a,b) + D_\theta \eta(b,\theta)\cdot u(a,\theta) = \\
& \quad = \ g\Big(a+\xi(a,\theta), b+\eta(b,\theta)\Big) - g(a,b).
\end{aligned} \tag{51}$$

The existence of Bäcklund transformations may be regarded as one of the possible definitions of the *integrability* of a given 2D continuous hyperbolic system. For a given 2D continuous hyperbolic system with Bäcklund transformations, not every discretization possesses the analogous property. For instance, the naive discretization (13) of the Sine-Gordon equation does not admit Bäcklund transformations, while the integrable discretization (17) does. The difference analogs of the formulas (42) read:

$$\sin\frac{1}{4}\big(\widetilde{\phi}(x+\epsilon,y)-\widetilde{\phi}(x,y)+\phi(x+\epsilon,y)-\phi(x,y)\big)=$$

$$=\frac{\epsilon\alpha}{2}\sin\frac{1}{4}\big(\widetilde{\phi}(x+\epsilon,y)+\widetilde{\phi}(x,y)-\phi(x+\epsilon,y)-\phi(x,y)\big). \tag{52}$$

$$\sin\frac{1}{4}\big(\widetilde{\phi}(x,y+\epsilon)-\widetilde{\phi}(x,y)-\phi(x,y+\epsilon)+\phi(x,y)\big)=$$

$$=\frac{\epsilon}{2\alpha}\sin\frac{1}{4}\big(\widetilde{\phi}(x,y+\epsilon)+\widetilde{\phi}(x,y)+\phi(x,y+\epsilon)+\phi(x,y)\big). \tag{53}$$

Obviously, in the limit $\epsilon\to 0$ these equations approximate (42). A very remarkable feature is that these equations closely resemble the original difference equation (17), if one considers the tilde as the shift in the third z–direction. Upon introducing the quantity $\theta=(\widetilde{\phi}-\phi)/2$, one rewrites (52), (53) in the form of the system of first order equations approximating (43), (45):

$$\delta_x^\epsilon\theta = -a+\frac{1}{i\epsilon}\log\frac{1-(\epsilon\alpha/2)\exp(-i\theta+i\epsilon a/2)}{1-(\epsilon\alpha/2)\exp(i\theta-i\epsilon a/2)}, \tag{54}$$

$$\delta_y^\epsilon\theta = \frac{1}{i\epsilon}\log\frac{1-(\epsilon/2\alpha)\exp(-ib-i\theta)}{1-(\epsilon/2\alpha)\exp(ib+i\theta)}, \tag{55}$$

and

$$\widetilde{a}=a+2\delta_x^\epsilon\theta,\qquad \widetilde{b}=b+2\theta+\epsilon\delta_y^\epsilon\theta. \tag{56}$$

This suggests the following definition.

Definition 4.2. A *discrete 3D hyperbolic system* is a collection of compatible partial difference equations of the form

$$\delta_y^\epsilon a=f^\epsilon(a,b),\qquad \delta_x^\epsilon b=g^\epsilon(a,b), \tag{57}$$

$$\delta_x^\epsilon\theta=u^\epsilon(a,\theta),\qquad \delta_y^\epsilon\theta=v^\epsilon(b,\theta), \tag{58}$$

$$\delta_z a=\xi^\epsilon(a,\theta),\qquad \delta_z b=\eta^\epsilon(b,\theta), \tag{59}$$

for functions $a,b,\theta:\mathcal{B}^\epsilon(r,R)\to\mathcal{X}$, where

$$\mathcal{B}^\epsilon(r,R)=\{(x,y,z)\mid (x,y)\in\mathcal{B}^\epsilon(r),\ z=0,1,\dots,R\}. \tag{60}$$

Here the functions $f^\epsilon,g^\epsilon,u^\epsilon,v^\epsilon,\xi^\epsilon,\eta^\epsilon:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ are smooth enough. A *Goursat problem* consists of prescribing the initial data

$$a(x,0,0)=a_0^\epsilon(x),\ \ b(0,y,0)=b_0^\epsilon(y),\ \ \theta(0,0,z)=\theta_0^\epsilon(z) \tag{61}$$

for $x\in[0,r]^\epsilon$, $y\in[0,r]^\epsilon$, and $z\in\{0,1,\dots,R\}$, respectively.

Compatibility conditions are necessary for solutions of (57)-(59) to exist. These conditions express the following identities that have to be fulfilled for the solutions:

$$\delta_y^\epsilon(\delta_x^\epsilon\theta)=\delta_x^\epsilon(\delta_y^\epsilon\theta),\quad \delta_y^\epsilon(\delta_z a)=\delta_z(\delta_y^\epsilon a),\quad \delta_x^\epsilon(\delta_z b)=\delta_z(\delta_x^\epsilon b).$$

In length, these formulas read:

$$
\begin{aligned}
& u^\epsilon\Big(a+\epsilon f^\epsilon(a,b),\theta+\epsilon v^\epsilon(b,\theta)\Big)-u^\epsilon(a,\theta)= \\
& \quad = \; v^\epsilon\Big(b+\epsilon g^\epsilon(a,b),\theta+\epsilon u^\epsilon(a,\theta)\Big)-v^\epsilon(b,\theta), \\
& \xi^\epsilon\Big(a+\epsilon f^\epsilon(a,b),\theta+\epsilon v^\epsilon(b,\theta)\Big)-\xi^\epsilon(a,\theta)= \\
& \quad = \; \epsilon f^\epsilon\Big(a+\xi^\epsilon(a,\theta),b+\eta^\epsilon(b,\theta)\Big)-\epsilon f^\epsilon(a,b), \\
& \eta^\epsilon\Big(b+\epsilon g^\epsilon(a,b),\theta+\epsilon u^\epsilon(a,\theta)\Big)-\eta^\epsilon(b,\theta)= \\
& \quad = \; \epsilon g^\epsilon\Big(a+\xi^\epsilon(a,\theta),b+\eta^\epsilon(b,\theta)\Big)-\epsilon g^\epsilon(a,b).
\end{aligned}
\tag{62}
$$

This has to be satisfied identically in $a, b, \theta \in \mathcal{X}$.
As demonstrated in [BS], the compatibility of a discrete 3D hyperbolic system is closely related to its integrability in the sense of the soliton theory. Moreover, such a key attribute of integrability as a *discrete zero curvature representation with a spectral parameter* can be derived from the fact of compatibility.

Proposition 4.1. *The Goursat problem for a discrete 3D hyperbolic system (57)–(59) satisfying the compatibility conditions (62) has a unique solution* $(a^\epsilon, b^\epsilon, \theta^\epsilon)$ *on* $\mathcal{B}^\epsilon(r,R)$.

Proof. Like in the proof of Proposition 2.1, it is enough to demonstrate that the solution can be propagated along an elementary "cube" of the three–dimensional lattice, then the solution is constructed by induction from the Goursat data. Again, it is convenient to assume that the variables $a(x,y,z)$, $b(x,y,z)$, $\theta(x,y,z)$ are attached not to the points $(x,y,z) \in \mathcal{B}^\epsilon(r,R)$, but to the edges $[(x,y,z),(x+\epsilon,y,z)]$, $[(x,y,z),(x,y+\epsilon,z)]$, $[(x,y,z),(x,y,z+1)]$, respectively. Denote (in this proof only) shifts of the edge variables in the directions of x, y, z axes by the subscripts $1, 2, 3$, respectively. (See Fig. 4.) Then the values (a_2, b_1) are determined by equations (57) the values (θ_1, θ_2) by (58), and the values (a_3, b_3) by (59). a_{23} is calculated either from a_3, b_3 by (57), or from a_2, θ_2 by (59); compatibility guarantees that the same results are obtained. The same is true for b_{13} and θ_{12}). □

Theorem 4.2. *Let the family of discrete 3D hyperbolic systems (57)-(59) satisfying the compatibility conditions (62) approximate the continuous 2D hyperbolic system with a Bäcklund transformation (46)-(48). Assume* $\mathcal{O}(\epsilon)$*-approximation in* C^1 *of the nonlinearities and uniform approximation of the initial data as usual,*

$$a_0^\epsilon(x) = a_0(x) + \mathcal{O}(\epsilon), \quad b_0^\epsilon(y) = b_0(y) + \mathcal{O}(\epsilon), \quad \theta_0^\epsilon(z) = \theta_0(z) + \mathcal{O}(\epsilon)$$

for $x \in [0,r]^\epsilon$, $y \in [0,r]^\epsilon$, $z \in \{0,1,\ldots,R\}$.
Then, for some $\bar{r} \in (0,r]$, *the sequence of solutions* $(a^\epsilon, b^\epsilon, \theta^\epsilon)$ *has a uniform limit of Lipschitz–continuous functions* (a, b, θ) *on* $\mathcal{B}(\bar{r}, R)$ *in the sense that*

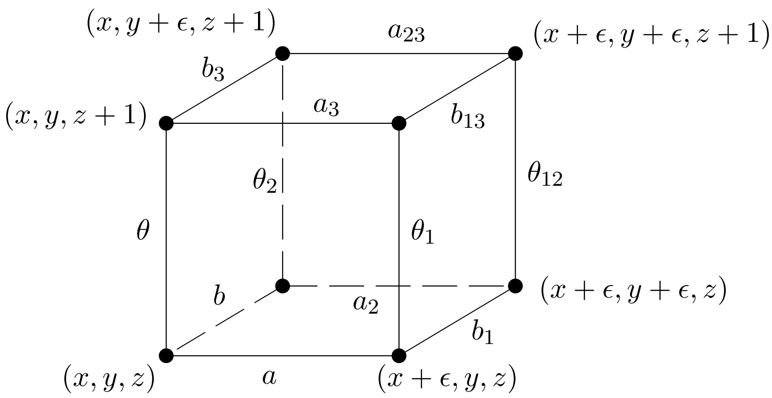


FIGURE 4. Three-dimensional consistency

the relations

$$\begin{aligned} a^\epsilon(x, y, z) &= a(x, y, z) + \mathcal{O}(\epsilon), \\ b^\epsilon(x, y, z) &= b(x, y, z) + \mathcal{O}(\epsilon), \\ \theta^\epsilon(x, y, z) &= \theta(x, y, z) + \mathcal{O}(\epsilon) \end{aligned}$$

hold uniformly on $\mathcal{B}^\epsilon(\bar{r}, R)$. Furthermore, (a, b, θ) solve the Goursat problem for the continuous 2D system with a sequence of Bäcklund transformations.

The proof parallels the proof of Theorem 2.2, and starts with *á priori* estimates for a^ϵ, b^ϵ, θ^ϵ, and their first order difference quotients.

Lemma 4.3 (Uniform estimate)**.** *Assume the norms of the initial data a_0^ϵ, b_0^ϵ, θ_0^ϵ is ϵ-independently bounded. Then there exists $\bar{r} \in (0, r]$ such that the solutions $(a^\epsilon, b^\epsilon, \theta^\epsilon)$ are bounded on $\mathcal{B}^\epsilon(\bar{r}, R)$ independently of ϵ.*

Proof of Lemma 4.3. Let $|a_0^\epsilon|, |b_0^\epsilon|, |\theta_0^\epsilon| \leq M_0$ with $M_0 > 0$. Define

$$\mathcal{F}(M) = \sup_\epsilon \sup_{|a|,|b|,|\theta|<M} \{|f^\epsilon(a, b)|, \ldots, |\eta^\epsilon(b, \theta)|\}. \tag{63}$$

Choose $M_1 > M_0$ arbitrary, and define inductively $M_{j+1} = M_j + \mathcal{F}(M_j)$ for $j = 1, \ldots, R$. Finally, let

$$\bar{r} = \min_{j=1,\ldots,R+1} \frac{M_j - M_0}{2\mathcal{F}(M_j)}, \tag{64}$$

so that for all $j = 1, 2, \ldots, R+1$ one has

$$M_0 + 2\bar{r}\mathcal{F}(M_j) \leq M_j. \tag{65}$$

The following estimate is shown by induction on $z = 0, 1, \ldots, R$:

$$|a^\epsilon(x, y, z)|,\ |b^\epsilon(x, y, z)|,\ |\theta^\epsilon(x, y, z)| \leq M_{z+1}. \tag{66}$$

Let $z = 0$. As in the proof of Lemma 2.3,

$$|a^\epsilon(x,y,0)| \le M_0 + y\mathcal{F}(M_1), \quad |b^\epsilon(x,y,0)| \le M_0 + x\mathcal{F}(M_1), \tag{67}$$

follows from equations (57), and from equations (58), one concludes

$$|\theta^\epsilon(x,y,0)| \le M_0 + (x+y)\mathcal{F}(M_1), \tag{68}$$

for all $(x,y) \in \mathcal{B}^\epsilon(\bar r)$. Assuming (66) for a given $z \ge 0$, equations (59) and equations (58) immediately imply

$$\begin{aligned}
&|a^\epsilon(x,y,z+1)|,\, |b^\epsilon(x,y,z+1)| \le M_{z+1} + \mathcal{F}(M_{z+1}) \le M_{z+2},\\
&|\theta^\epsilon(x,y,z+1)| \le M_0 + (x+y)\mathcal{F}(M_{z+2}) \le M_{z+2},
\end{aligned}$$

respectively. This proves (66) for $z+1$, and thus the Lemma. 2.3. □

Lemma 4.4 (Lipschitz bound)**.** *Assume the initial data of the continuous Goursat problem are C^1 functions, and are approximated by the initial data of the discrete Goursat problem,*

$$a_0^\epsilon = a_0 + \mathcal{O}(\epsilon), \quad b_0^\epsilon = b_0 + \mathcal{O}(\epsilon). \tag{69}$$

Let $\bar r \in (0,r]$ be chosen according to Lemma 4.3. Then the difference quotients $\delta_x^\epsilon a^\epsilon$, $\delta_y^\epsilon a^\epsilon$, $\delta_x^\epsilon b^\epsilon$, $\delta_y^\epsilon b^\epsilon$, $\delta_x^\epsilon \theta^\epsilon$, and $\delta_y^\epsilon \theta^\epsilon$ are ϵ-independently bounded on $\mathcal{B}^\epsilon(\bar r, R)$.

Proof of Lemma 4.4. The reasoning from the previous proof is continued. From the equations in (57)-(59) it is immediately seen that

$$|\delta_y^\epsilon a^\epsilon|,\, |\delta_x^\epsilon b^\epsilon|,\, |\delta_x^\epsilon \theta^\epsilon|,\, |\delta_y^\epsilon \theta^\epsilon| \le \mathcal{F}(M_{z+1}).$$

Therefore, only $\delta_x^\epsilon a^\epsilon$ and $\delta_y^\epsilon b^\epsilon$ need to be estimated. By Lemma 2.5,

$$|\delta_x^\epsilon a^\epsilon(x,y,0)| \le A_0 < \infty.$$

Proceeding inductively from $z-1$ to z,

$$\begin{aligned}
|\delta_x^\epsilon a^\epsilon(x,y,z)| &\le |\delta_x^\epsilon a^\epsilon(x,y,z-1)| + |\delta_x^\epsilon \xi^\epsilon[a^\epsilon,\theta^\epsilon](x,y,z-1)|\\
&\le |\delta_x^\epsilon a^\epsilon(x,y,z-1)| + C(|\delta_x^\epsilon a^\epsilon(x,y,z-1)| + |\delta_x^\epsilon \theta^\epsilon(x,y,z-1)|)\\
&\le (1+C)|\delta_x^\epsilon a^\epsilon(x,y,z-1)| + C\mathcal{F}(M_z) =: A_z < \infty,
\end{aligned}$$

where $C > 0$ is an ϵ-independent Lipschitz constant for the nonlinearities in equations (57)-(59). □

Proof of Theorem 4.2. Now proceed as in the proof of Theorem 2.2. A Lipschitz-continuous function $(a,b,\theta) : \mathcal{B}(\bar r) \to \mathcal{X}^3$ is obtained as uniform limit of a suitable subsequence $(a^{\epsilon'}, b^{\epsilon'}, \theta^{\epsilon'})$, using linear interpolation and the Arzelá-Ascoli theorem. To obtain the $\mathcal{O}(\epsilon)$-estimate, simply choose Δ^ϵ instead as in (32) as follows:

$$\begin{aligned}
\Delta^\epsilon(n) \;=\; &\max\{|a^\epsilon - a|(x,y,z) + |b^\epsilon - b|(x,y,z) + |\theta^\epsilon - \theta|(x,y,z),\\
&(x,y,z) \in \mathcal{B}^\epsilon(\bar r),\ x + y + \epsilon z \le \epsilon n\}
\end{aligned}$$

Applying the Gronwall Lemma 2.4 yields $\Delta^\epsilon(x,y,z) = \mathcal{O}(\epsilon)$. □

5. Approximation theorems for K-Surfaces

In the present section we apply the theory developed so far to prove that the known construction of discrete surfaces of constant negative Gauss curvature $K = -1$ (*K–surfaces*, for short) may be actually used not only to modelling the geometric properties of their continuous counterparts, but also to a quantitative approximation. First we briefly recall the correspondent geometric notions.

Smooth K-surfaces. Let F be a K-surface parametrized by its asymtotic lines:

$$F : \mathcal{B}(r) \to \mathbb{R}^3. \tag{70}$$

This means that the vectors $\partial_x F$, $\partial_y F$, $\partial_x^2 F$, $\partial_y^2 F$ are orthogonal to the normal vector $N : \mathcal{B}(r) \to S^2$. Reparametrizing the asymptotic lines, if necessary, we assume that $|\partial_x F| = 1$ and $|\partial_y F| = 1$. The angle $\phi = \phi(x, y)$ between the vectors $\partial_x F$, and $\partial_y F$ satisfies the sine–Gordon equation (8). Moreover, a K-surface is determined by a solution to (8) essentially uniquely. The correspondent construction is as follows. Consider the matrices U, V defined by the formulas

$$U(a;\lambda) = \frac{i}{2}\begin{pmatrix} a & -\lambda \\ -\lambda & -a \end{pmatrix}, \tag{71}$$

$$V(b;\lambda) = \frac{i}{2}\begin{pmatrix} 0 & \lambda^{-1}\exp(ib) \\ \lambda^{-1}\exp(-ib) & 0 \end{pmatrix}, \tag{72}$$

taking values in the twisted loop algebra

$$g[\lambda] = \{\xi : \mathbb{R}_* \to \mathrm{su}(2) : \ \xi(-\lambda) = \sigma_3 \xi(\lambda) \sigma_3\}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose now that a and b are some real–valued functions on $\mathcal{B}(r)$. Then the zero curvature condition

$$\partial_y U - \partial_x V + [U, V] = 0 \tag{73}$$

is satisfied identically in λ, if and only if (a, b) satisfy the system (11), or, in other words, if $a = \partial_x \phi$ and $b = \phi$, where ϕ is a solution of (8). Given a solution ϕ, that is, a pair of matrices (71), (72) satisfying (73), the following system of linear differential equations is uniquely solvable:

$$\partial_x \Phi = U\Phi, \quad \partial_y \Phi = V\Phi, \quad \Phi(0, 0, \lambda) = \mathbf{1}. \tag{74}$$

Here $\Phi : \mathcal{B}(r) \to G[\lambda]$ takes values in the twisted loop group

$$G[\lambda] = \{\Xi : \mathbb{R}_* \to \mathrm{SU}(2) : \ \Xi(-\lambda) = \sigma_3 \Xi(\lambda) \sigma_3\}.$$

The solution $\Phi(x, y; \lambda)$ yields the immersion $F(x, y)$ by the *Sym formula*:

$$F(x, y) = \left(2\lambda \Phi(x, y; \lambda)^{-1} \partial_\lambda \Phi(x, y; \lambda)\right)\Big|_{\lambda=1}, \tag{75}$$

using the canonical identification of su(2) with $\mathbb{R}^3$. Moreover, the right–hand side of (75) at the values of λ different from $\lambda = 1$ delivers a whole family of immersions $F_\lambda : \mathcal{B}(r) \to \mathbb{R}^3$, all of which turn out to be asymptotic lines parametrized K–surfaces. These surfaces F_λ constitute the so–called *associated family* of F.

Discrete K-surfaces. . Let F^ϵ be a discrete surface parametrized by asymptotic lines, i.e. an immersion

$$F^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3 \tag{76}$$

such that for each $(x,y) \in \mathcal{B}^\epsilon(r)$ the five points $F^\epsilon(x,y)$, $F^\epsilon(x \pm \epsilon, y)$, and $F^\epsilon(x, y \pm \epsilon)$ lie in a single plane $\mathcal{P}(x,y)$. It is required that all edges of the discrete surface F^ϵ have the same length $\epsilon\ell$, that is $|\delta_x^\epsilon F^\epsilon| = |\delta_y^\epsilon F^\epsilon| = \ell$, and it turns out to be convenient to assume that $\ell = (1+\epsilon^2/4)^{-1}$. The same relation we presented between K-surfaces and solutions to the (classical) sine–Gordon equation (8) can be found between discrete K-surfaces and solutions to the sine–Gordon equation in Hirota's discretization (17): define matrices $\mathcal{U}^\epsilon$, $\mathcal{V}^\epsilon$ by the formulas

$$\mathcal{U}^\epsilon(a;\lambda) = (1+\epsilon^2\lambda^2/4)^{-1/2} \begin{pmatrix} \exp(i\epsilon a/2) & -i\epsilon\lambda/2 \\ -i\epsilon\lambda/2 & \exp(-i\epsilon a/2) \end{pmatrix}, \tag{77}$$

$$\mathcal{V}^\epsilon(b;\lambda) = (1+\epsilon^2\lambda^{-2}/4)^{-1/2} \begin{pmatrix} 1 & (i\epsilon\lambda^{-1}/2)\exp(ib) \\ (i\epsilon\lambda^{-1}/2)\exp(-ib) & 1 \end{pmatrix} \tag{78}$$

Let a^ϵ, b^ϵ be real–valued functions on $\mathcal{B}^\epsilon(r)$, and consider the discrete zero curvature condition

$$\mathcal{U}^\epsilon(x, y+\epsilon;\lambda) \cdot \mathcal{V}^\epsilon(x,y;\lambda) = \mathcal{V}^\epsilon(x+\epsilon, y;\lambda) \cdot \mathcal{U}^\epsilon(x,y;\lambda), \tag{79}$$

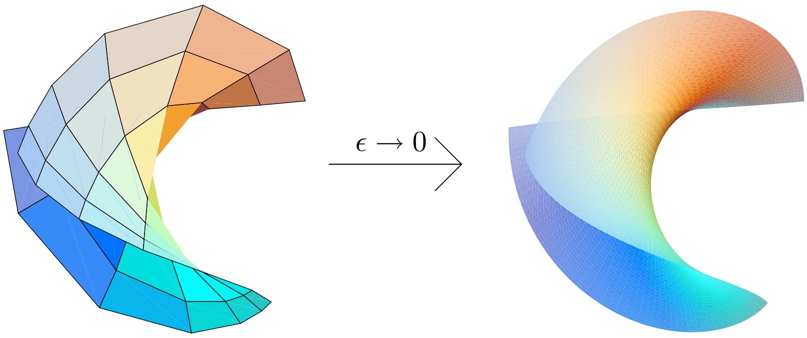
where $\mathcal{U}^\epsilon$ and $\mathcal{V}^\epsilon$ depend on $(x,y) \in \mathcal{B}^\epsilon(r)$ through the dependence of a^ϵ and b^ϵ on (x,y). A direct calculation shows that (79) is equivalent to the system (19), or, in other words, to the Hirota equation (17) for the function ϕ^ϵ defined by (18). The function ϕ^ϵ has a clear geometric meaning, see [BP1]. The formula (79) is the compatibility condition of the following system of linear difference equations:

$$\begin{aligned} \Psi^\epsilon(x+\epsilon, y;\lambda) &= \mathcal{U}^\epsilon(x,y;\lambda)\Psi^\epsilon(x,y;\lambda), \\ \Psi^\epsilon(x, y+\epsilon;\lambda) &= \mathcal{V}^\epsilon(x,y;\lambda)\Psi^\epsilon(x,y;\lambda), \\ \Psi^\epsilon(0,0;\lambda) &= \mathbf{1}. \end{aligned} \tag{80}$$

So, any solution of (17) uniquely defines a matrix $\Psi^\epsilon : \mathcal{B}^\epsilon(r) \to G[\lambda]$ satisfying (80). This can be used to finally construct the immersion by an analog of the Sym formula:

$$F^\epsilon(x,y) = \left(2\lambda\Psi^\epsilon(x,y;\lambda)^{-1}\partial_\lambda\Psi^\epsilon(x,y;\lambda)\right)\Big|_{\lambda=1}. \tag{81}$$

Again, the right–hand side of (81) at the values of λ different from $\lambda = 1$ delivers an associated family F_λ^ϵ of discrete asymptotic lines parametrized K–surfaces.



Bäcklund transformations for continuous and discrete K–surfaces. Below, only the algebraic approach is presented. For the geometrical interpretation of smooth and discrete Bäcklund transformations, see, e.g. [BP1]. Introduce the matrix

$$\mathcal{W}(\theta;\lambda) = \begin{pmatrix} \alpha\exp(i\theta) & -i\lambda \\ -i\lambda & \alpha\exp(-i\theta) \end{pmatrix}. \tag{82}$$

It is easy to see that the matrix differential equations

$$\partial_x \mathcal{W} = \widetilde{U}\mathcal{W} - \mathcal{W}U, \qquad \partial_y \mathcal{W} = \widetilde{V}\mathcal{W} - \mathcal{W}V \tag{83}$$

are equivalent to the formulas (43), (45). On the other hand, these matrix differential equations constitute a sufficient condition for the solvability of the system consisting of (74) and

$$\widetilde{\Phi} = \mathcal{W}\Phi. \tag{84}$$

So, frames Φ can be in a consistent way extended into the third direction z (shift in which is encoded by the tilde), which results also in the transformation of the K–surfaces $F \to \widetilde{F}$, and moreover of the whole associated family, via (75).
Similarly, the matrix equations

$$\mathcal{W}(x+\epsilon, y;\lambda)\mathcal{U}^\epsilon(x,y;\lambda) = \widetilde{\mathcal{U}}^\epsilon(x,y;\lambda)\mathcal{W}(x,y;\lambda), \tag{85}$$

$$\mathcal{W}(x, y+\epsilon;\lambda)\mathcal{V}^\epsilon(x,y;\lambda) = \widetilde{\mathcal{V}}^\epsilon(x,y;\lambda)\mathcal{W}(x,y;\lambda) \tag{86}$$

are equivalent to the formulas (54), (55), (56), and, on the other hand, assure the solvability of the system consisting of (80) and

$$\widetilde{\Psi}^\epsilon = \mathcal{W}\Psi^\epsilon. \tag{87}$$

Therefore, also the frames Ψ^ϵ of the discrete surfaces can be extended in the third direction z. This leads to the transformation of discrete K–surfaces and their associated families, according to (81).

Theorem 5.1. *Let two smooth functions $\phi_1, \phi_2 : [0,r] \to \mathbb{R}$ with $\phi_1(0) = \phi_2(0)$ be given. Then, for a suitable positive $\bar{r} \le r$:*

- *There exists a smooth asymptotic line parametrized K–surface* $F : \mathcal{B}(\bar{r}) \to \mathbb{R}^3$*, unique up to Euclidean motions, such that the angle* $\phi : \mathcal{B}(\bar{r}) \to \mathbb{R}$ *between the asymptotic lines satisfies*

$$\phi(x,0) = \phi_1(x), \quad \phi(0,y) = \phi_2(y), \quad 0 \le x, y \le r.$$

- *For each* $\epsilon > 0$ *there exists a unique asymptotic line parametrized discrete K–surface* $F^\epsilon : \mathcal{B}^\epsilon(\bar{r}) \to \mathbb{R}^3$ *such that its characteristic angle* $\phi^\epsilon : \mathcal{B}^\epsilon(\bar{r}) \to \mathbb{R}$ *satisfies*

$$\phi^\epsilon(x+\epsilon,0) - \phi^\epsilon(x,0) = \epsilon\partial_x\phi_1(x),\ \phi^\epsilon(0,y+\epsilon) + \phi^\epsilon(0,y) = 2\phi_2(y). \tag{88}$$

- *The discrete surfaces converge uniformly to the smooth one,*

$$\sup_{\mathcal{B}^\epsilon(r)} |F^\epsilon - F| \ \le \ C\epsilon, \tag{89}$$

where C *does not depend on* ϵ*. Moreover, the convergence is in* C^∞*: for each pair* (m,n) *of nonnegative integers,*

$$\sup_{\mathcal{B}^\epsilon(r-(m+n)\epsilon)} |(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n F^\epsilon - \partial_x^m\partial_y^n F| \ \le \ C_{mn}\epsilon. \tag{90}$$

- *The estimates (89), (90) hold for the associated families* F_λ*,* F_λ^ϵ *in place of* F*,* F^ϵ*, uniformly in* $\lambda \in [\Lambda^{-1},\Lambda]$ *for a suitable* $\Lambda > 1$*.*
- *Given a parameter* θ *for a Bäcklund transformation, then the discrete Bäcklund transformations* $\tilde{F}^\epsilon$ *of the* F^ϵ *(with respect to* θ*) converge to a smooth K-surface* $\tilde{F}$*. The latter is the unique Bäcklund transformation of* F *(with respect to* θ*). The estimates (89), (90) hold with* $\tilde{F}$*,* $\tilde{F}^\epsilon$ *in place of* F*,* F^ϵ*. The result carries over to the associated families.*

An important application of Theorem 5.1 is the derivation of the *classical Bianchi permutability Theorem* from the discrete permutability Theorem for K-surfaces. The result is formulated in the next section.

Theorem 5.2. *Under the same hypothesis as in Theorem 5.1, let* F^ϵ *be the unique discrete K-surface constructed from data*

$$\begin{aligned}\phi^\epsilon(x+\epsilon,0) - \phi^\epsilon(x,0) &= \phi_1(x+\epsilon) - \phi_1(x),\\ \phi^\epsilon(0,y+\epsilon) - \phi^\epsilon(0,y) &= \phi_2(y+\epsilon) + \phi_2(y).\end{aligned}$$

Then the uniform convergence (89) is improved as follows:

$$\sup_{\mathcal{B}^\epsilon(\bar{r})} |F^\epsilon(x,y) - F(x,y)| \le C\epsilon^2.$$

Proof of Theorems 5.1 and 5.2. Theorems 2.2 and 3.1 yield the existence and the uniqueness of solutions (a^ϵ, b^ϵ) to the difference equations, the existence and uniqueness of the solutions (a,b) to the differential equations on $\mathcal{B}(\bar{r})$, and the $\mathcal{O}(\epsilon)$-approximation of the latter by the former in C^∞. Moreover, under the hypothesis of Theorem 5.2, one has additionally $\mathcal{O}(\epsilon^2)$-approximation in C^0. This follows from Theorem 2.7... **Square-approximation left to prove.**

It remains to prove that similar approximation holds also for the immersions F^ϵ, F. Te strategy is to prove approximation of the frame Φ by Ψ^ϵ, uniformly in λ, and then use the Sym formula. Recall that the frames are the solutions to the Cauchy problems for the systems of linear differential and difference equations (74) and (80), respectively. Since the zero curvature conditions (79), (73) are satisfied, the existence of Ψ^ϵ, Φ is guaranteed by standard ODE theory. Furthermore, at any point (x, y), $\Psi^\epsilon(\lambda)$ and $\Phi(\lambda)$ are analytic functions of $\lambda \in D$, where D is some closed disc in the complex plane that contains 1 in its interior, but does not contain 0. The matrices $\mathcal{U}^\epsilon$, $\mathcal{V}^\epsilon$, and U, V are bounded uniformly with respect to $\lambda \in D$ and $(x, y) \in \mathcal{B}(\bar{r})$.
A natural norm $|\cdot|$ on the space of λ-dependent 2×2-matrices $A = A(\lambda)$ is given by

$$|A| = \sup_{\lambda \in D} \max_{v \in \mathbb{R}^2,\, |v|=1} |A(\lambda) \cdot v|.$$

The norms $\|\cdot\|_s$ are introduced according to (37). Define

$$U^\epsilon = (\mathcal{U}^\epsilon - \mathbf{1})/\epsilon, \quad V^\epsilon = (\mathcal{V}^\epsilon - \mathbf{1})/\epsilon.$$

Theorems 2.2 and 3.1 imply

$$\|U^\epsilon - U\|_s = \mathcal{O}(\epsilon), \qquad \|V^\epsilon - V\|_s = \mathcal{O}(\epsilon) \tag{91}$$

for all $s = 0, 1, 2, \dots$ Under the hypothesis of Theorem 5.2, one has additionally

$$U^\epsilon(x, y; \lambda) = U(x + \tfrac{\epsilon}{2}, y; \lambda) + \tfrac{\epsilon}{2} U^2(x + \tfrac{\epsilon}{2}, y; \lambda) + \mathcal{O}(\epsilon^2), \tag{92}$$

$$V^\epsilon(x, y; \lambda) = V(x, y + \tfrac{\epsilon}{2}; \lambda) + \tfrac{\epsilon}{2} V^2(x, y + \tfrac{\epsilon}{2}; \lambda) + \mathcal{O}(\epsilon^2) \tag{93}$$

by Theorem 2.7. From the definition of Φ and Ψ^ϵ,

$$\Phi(x + \epsilon, y) = \Phi(x, y) + \epsilon U(x, y)\Phi(x, y) + \mathcal{O}(\epsilon^2) \tag{94}$$

$$\Psi^\epsilon(x + \epsilon, y) = \Psi^\epsilon(x, y) + U^\epsilon(x, y)\Psi^\epsilon(x, y). \tag{95}$$

follows, So by the elementary properties of matrix multiplication,

$$|\Phi - \Psi^\epsilon|(x + \epsilon, y) = (1 + \epsilon\|U - U^\epsilon\|_0)|\Phi - \Psi^\epsilon|(x, y) + \mathcal{O}(\epsilon^2). \tag{96}$$

A similar formula holds with V and V^ϵ. The Gronwall estimate yields

$$\|\Phi - \Psi^\epsilon\|_0 = \mathcal{O}(\epsilon). \tag{97}$$

Under the hypothesis of Theorem 5.2 ...
Choose $\Lambda > 1$ so that $I_\Lambda := [\Lambda^{-1}, \Lambda]$ lies in the interior of D, and has distance $\mu > 0$ from ∂D. Since $\Phi(\lambda)$ und $\Psi^\epsilon(\lambda)$ are analytic functions of $\lambda \in D$, the Cauchy formula implies

$$\begin{aligned} &\sup_{\lambda \in I_\Lambda} \|\partial_\lambda \Psi^\epsilon(x, y; \lambda) - \partial_\lambda \Phi(x, y; \lambda)\| \\ &\quad \le \mu^{-1} \sup_{\lambda \in D} \|\Psi^\epsilon(x, y; \lambda) - \Phi(x, y; \lambda)\| = \mathcal{O}(\epsilon^p). \end{aligned} \tag{98}$$

Hence, by the Sym formulas (75) and (81),

$$F^\epsilon_\lambda - F_\lambda = 2\lambda(\Psi^\epsilon(\lambda))^{-1}\partial_\lambda \Psi^\epsilon(\lambda) - 2\lambda(\Phi(\lambda))^{-1}\partial_\lambda \Phi(\lambda) = \mathcal{O}(\epsilon).$$

for all $\lambda \in I_\Lambda$ uniformly on the $\mathcal{B}^\epsilon(\bar{r})$. It remains to prove the approximation of the higher order partial derivatives of F. it is easy to see that for corresponding solutions (a^ϵ, b^ϵ) and (a, b) one has discrete C^k-approximation for all $k > 0$ and all $\lambda \in D$:

$$\|\mathbf{U}^\epsilon - U\|_k \to 0 \text{ and } \|\mathbf{V}^\epsilon - V\|_k \to 0$$

as $\epsilon \to 0$. We find for $m + n = k + 1$ with $m > 0$:

$$\begin{aligned} |(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n(\Psi^\epsilon - \Phi)| &= |(\delta_x^\epsilon)^{m-1}(\delta_y^\epsilon)^n(\mathbf{U}^\epsilon\Psi^\epsilon - U\Phi)| + \mathcal{O}(\epsilon) \\ &\le C\|U\|_k \cdot \|\Psi^\epsilon - \Phi\|_k + C\|\mathbf{U}^\epsilon - U\|_k \cdot \|\Psi^\epsilon\|_k + \mathcal{O}(\epsilon). \end{aligned}$$

Here we used that for discrete C^k-norms of matrix products

$$\|A \cdot B\|_k \le C_k\|A\|_k \cdot \|B\|_k \tag{99}$$

holds (cf. the remark after the proof of Lemma 3.2 in the Appendix). If $m = 0$, we can do the same calculations with the roles of x and y interchanged and V, $\mathbf{V}^\epsilon$ in place of U, $\mathbf{U}^\epsilon$. From this estimate, we conclude by induction in k that

$$\|\Psi^\epsilon - \Phi\|_k \to 0,$$

and therefore

$$\lim_{\epsilon\to 0} \sup_{\mathcal{B}^\epsilon(r)} |(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n\Psi^\epsilon - \partial_x^m\partial_y^n\Phi| = 0.$$

Again by the Cauchy estimate, we get also the similar result for the respective λ-derivatives for all values $\lambda \in [\Lambda^{-1}, \Lambda]$. From the Sym formulas, we get (90). Finally, the statement about the approximation of Bäcklund transformed surfaces follows in a completely similar way with the refence to Theorem 4.2. □

6. Multi–dimensional systems

The developed approximation theory generalizes to higher dimensions without difficulties. Before the most general situation is described, a three–dimensional example is given and will serve as a guiding principle.

Example 2. Consider the equation

$$\partial_x\partial_y\partial_z u = F(u, \partial_x u, \partial_y u, \partial_z u, \partial_x\partial_y u, \partial_x\partial_z u, \partial_y\partial_z u). \tag{100}$$

and the Goursat problem obtained by prescribing the values of

$$u(x, y, 0),\ u(x, 0, z),\ u(0, y, z) \quad \text{for} \quad 0 \le x, y, z \le r.$$

Equation (100) can be rewritten as a hyperbolic system,

$$\begin{cases} \partial_x u = a, & \partial_y u = b, \quad \partial_z u = c, \\ \partial_y a = h, & \partial_z b = f, \quad \partial_x c = g, \\ \partial_z a = g, & \partial_x b = h, \quad \partial_y c = f, \\ \partial_x f = \partial_y g = \partial_z h = F(u, a, b, c, f, g, h). \end{cases}$$

Replacing all partial derivatives ∂_x etc. by the corresponding difference quotients δ_x^ϵ etc. yields an approximating discrete hyperbolic system. In this difference system it is natural to assume that the variables a^ϵ, b^ϵ, c^ϵ live on the edges of the cubic lattice starting from the point (x, y, z) in the direction of the x–, y– and z–axis, respectively. The variables f, g, h are associated to two–cells (elementary squares) adjacent to the point (x, y, z) and orthogonal to the x–, y– and z–axes, respectively.

Recall the notations (1) and (2) for continuous and discrete d-dimensional domains. Denote by $\mathbf{e}_i$ the i-th canonical basis vector of $\mathbb{R}^d$.
A general discrete hyperbolic Goursat problem is defined by the following elements:

(1) Functions $\vec{a} : \mathcal{B}^{\boldsymbol{\epsilon}}(\mathbf{r}) \to \mathcal{X}$ are considered. The components $\vec{a} = (a_1, \dots, a_N) \in \mathcal{X}_1 \times \dots \mathcal{X}_N$ play the role of *dependent variables.*

(2) A set $\mathcal{E}_k \subset \{1, \dots, d\}$ is given for each $1 \le k \le N$, and $\mathcal{S}_k$ is its complement,

$$\mathcal{E}_k \hat{\cup} \mathcal{S}_k = \{1, \dots, d\}.$$

The elements of $\mathcal{E}_k$ and $\mathcal{S}_k$ are called the *evolutionary* and *stationary* directions of the component a_k, respectively.

(3) For each pair (k, i) such that $i \in \mathcal{E}_k$, a difference equation

$$\delta_{x_i}^{\epsilon_i} a_k = f_{(k,i)}(\vec{a}), \quad i \in \mathcal{E}_k, \tag{101}$$

is given, where the *nonlinearities* $f_{(k,i)} : \mathcal{X} \to \mathcal{X}_k$ are smooth functions.

(4) One requires that $a_k(\mathbf{x}) = a_{k0}(\mathbf{x})$ on the subsets

$$\mathcal{G}_k^{\boldsymbol{\epsilon}} = \mathrm{span}\{\mathbf{e}_j,\ j \in \mathcal{S}_k\} \cap \mathcal{B}^{\boldsymbol{\epsilon}}(\mathbf{r}).$$

The prescribed functions a_{k0} are the *Goursat data.*

On should think of the field $a_k(\mathbf{x})$ as attached to the following cell of dimension $\#\mathcal{S}_k$

$$c_k(\mathbf{x}) \quad = \quad \big\{\mathbf{x}',\ x_i' = x_i \text{ for } i \in \mathcal{E}_k,\ x_j \le x_j' \le x_j + \epsilon_j \text{ for } j \in \mathcal{S}_k\big\}.$$

The hyperbolic system admits solutions for arbitrary Goursat data only if the *compatibility condition*

$$\delta_{x_j}^{\epsilon_j} \delta_{x_i}^{\epsilon_i} a_k = \delta_{x_i}^{\epsilon_i} \delta_{x_j}^{\epsilon_j} a_k \tag{102}$$

is satisfied for any choice of $i, j \in \mathcal{E}_k$, $i \ne j$. By (101), this formal requirement is equivalent to

$$\delta_{x_j}^{\epsilon_j} f_{k,i}(\vec{a}(\mathbf{x})) \quad = \quad \delta_{x_i}^{\epsilon_i} f_{k,j}(\vec{a}(\mathbf{x})). \tag{103}$$

Compatibility is a property of the equations, and has to hold *independently* of the specific solution $\vec{a}$. Consequently, both sides of (103) have to be functions of the value $\vec{a}(\mathbf{x}) \in \mathcal{X}$ only. This implies that the function $f_{(k,i)}$ at most depends on those components a_ℓ, for which $\delta_{x_j}^{\epsilon_j} a_\ell$ is again expressible in terms of the equations (101), i.e., for which $j \in \mathcal{E}_\ell$. As (103) needs to be satisfied for all $j \in \mathcal{E}_k$, one obtains

Lemma 6.1. *Compatibility (102) of the hyperbolic equations implies that each $f_{(k,i)}$ depends only on those components a_ℓ for which $\mathcal{E}_k \setminus \{i\} \subset \mathcal{E}_\ell$. If this is the case, then compatibility is equivalent to*

$$\epsilon_i f_{(k,i)}(\vec{a}) + \epsilon_j f_{(k,j)}\big(\vec{a} + \epsilon_i \vec{f_i}(\vec{a})\big) \quad = \quad \epsilon_j f_{(k,j)}(\vec{a}) + \epsilon_i f_{(k,i)}\big(\vec{a} + \epsilon_j \vec{f_j}(\vec{a})\big),$$

where symbolically $(\vec{f_i}(\vec{a}))_\ell = f_{(\ell,i)}(\vec{a})$ for $i \in \mathcal{E}_\ell$.

It is clear that

Proposition 6.2. *A Goursat problem for a compatible discrete hyperbolic system admits a unique solution on $\mathcal{B}^{\boldsymbol{\epsilon}}(\mathbf{r})$.*

Typically, discrete hyperbolic systems appear as dicretizations of a continuous hyperbolic system, probably with Bäcklund transformations. Consider the situation where the uniform continuum limit $\epsilon_i \equiv \epsilon \to 0$ is performed in the first $n \leq d$ directions, $i = 1, \ldots, n$, and the remaining $n' = d - n$ directions are kept discrete, $\epsilon_i \equiv 1$ for $i = n+1, \ldots, d$. For simplicity, assume that the components of $\mathbf{r}$ are $r_i = r$ for $i \leq n$ and $r_i = 1$ for $i > n$.
The functions $f_{(k,i)} = f^{\epsilon}_{(k,i)}$ are now ϵ-dependent. At least formally, the discrete equations (101) turn into a system of differential and difference equations,

$$\begin{array}{rclll} \partial_{x_i} a_k & = & f^0_{(k,i)}(\vec{a}), & i \in \mathcal{E}_k, & 1 \leq i \leq n, \\ \delta^1_{x_i} a_k & = & f^0_{(k,i)}(\vec{a}), & i \in \mathcal{E}_k, & n < i \leq d. \end{array} \tag{104}$$

The limiting Goursat problem is posed on

$$\mathcal{B}^0(r) = [0, r]^n \times \{0, 1\}^{n'} \subset \mathbb{R}^d \times \mathbb{Z}^{d'}.$$

Theorem 6.3. *Let an ϵ–family of Goursat problems for compatible discrete hyperbolic systems (101) be given. Denote their solutions by $\vec{a}^{\,\epsilon}$. Suppose that the nonlinearities f^ϵ converge as*

$$f^{\epsilon}_{(k,i)}(\vec{a}) \quad = \quad f^0_{(k,i)}(\vec{a}) + \mathcal{O}(\epsilon)$$

uniformly on any compact subset of $\mathcal{X}$, and that the discrete Goursat data a^ϵ_0 approximate Lipschitz-continuous functions a_0

$$a^\epsilon_{k0}(\mathbf{x}) \quad = \quad a^0_{k0}(\mathbf{x}) + \mathcal{O}(\epsilon)$$

uniformly on $\mathcal{G}^0_k$.
Then there exist a positive $\bar{r} \leq r$ and a Lipschitz–continuous function $\vec{a}^{\,0} : \mathcal{B}^0(\bar{r}) \to \mathcal{X}$, such that

$$\vec{a}^{\,\epsilon} \quad = \quad \vec{a}^{\,0} + \mathcal{O}(\epsilon). \tag{105}$$

Moreover, $\vec{a}^{\,0}$ constitutes the unique classical solution of the continuous Goursat problem for the system (104) with the Goursat data a_0.
Assume, in addition, that the convergence of the f^ϵ is locally uniform in C^{s+1}, with an error $\mathcal{O}(\epsilon)$,

$$D^{s+1} f^\epsilon_{k,i} \quad = \quad D^{s+1} f_{k,i} + \mathcal{O}(\epsilon),$$

and that the continuous Goursat data is C^s-smooth and is respectively approximated by the discrete data,

$$\delta^{\epsilon}_{x_{i_1}} \cdots \delta^{\epsilon}_{x_{i_s}} a^{\epsilon}_{k0} = \partial_{x_{i_1}} \cdots \partial_{x_{i_s}} a_{k0} + \mathcal{O}(\epsilon)$$

on $\mathcal{B}^{\epsilon}(r)$, where $i_1, \ldots, i_s \in \mathcal{S}_k$ and $i_1, \ldots, i_s \leq n$, then the convergence (105) is in C^s,

$$\delta^{\epsilon}_{x_{i_1}} \cdots \delta^{\epsilon}_{x_{i_s}} \vec{a}^{\epsilon} = \partial_{x_{i_1}} \cdots \partial_{x_{i_s}} \vec{a} + \mathcal{O}(\epsilon)$$

holds uniformly on $\mathcal{B}^{\epsilon}(\bar{r})$, for arbitrary $i_1, \ldots, i_s \leq n$.

Proof. The proof of this theorem is the multi-dimensional extension of the proofs given in two and three dimensions. Technical care is needed, but no essentially new ideas enter. A detailed version of proof can be found in [Mat]. □

A typical application of Theorem 6.3 is the derivation of smooth permutability theorems from discrete ones.

Corollary 6.4 (Bianchi Permutability). *Given a smooth K–surface F and two smooth Bäcklund transformations F_1, F_2, there exists a unique smooth K–surface F_{12} which is a Bäcklund transformation of F_1 and of F_2.*

Proof. A two-parameter family of discrete K-surfaces is a four-dimensional discrete hyperbolic system in the sense above. The first two directions correspond to individual K-surfaces, the remaining two to parameters of Bäcklund transformations. The systems compatibility follows from the discrete permutability Theorem, which is identical to Corollary 6.4 upon replacing "smooth" by "discrete" everywhere. A proof of the latter can be found in [].
The classical Bianchi Permutability Theorem above follows by taking the continuum limit in the first two directions. □

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