Discrete Conformal Maps and Surfaces

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1 Definition of a Discrete Conformal Map

Definition 1.1. A map $f: \mathbb{Z} \to \mathbb{C}$ is called *discrete conformal (discrete holomorphic)* if the cross-ratios of all its elementary quadrilaterals are equal to -1:

$$q_{n,m} := q(f_{n,m},\, f_{n+1,m},\, f_{n+1,m+1},\, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \tag{1}$$

This definition appeared [1] in 1991 and is motivated by the following properties:

- $f: D \subset \mathbb{C} \to \mathbb{C}$ is a (smooth) conformal (holomorphic or antiholomorphic) map if and only if $\forall (x, y) \in D$

$$\lim_{\epsilon \to 0} q\big(f(x, y),\, f(x+\epsilon, y),\, f(x+\epsilon, y+\epsilon),\, f(x, y+\epsilon)\big) = -1. \tag{2}$$

- Definition 1.1 is Möbius invariant:

$$f \qquad \text{and} \qquad \tilde{f} = \frac{af + b}{cf + d} \tag{3}$$

  are discrete conformal simultaneously.

*Partially supported by the SFB288 and by NSF grant DMS93–12087
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- The dual, $f^*\colon \mathbb{Z}^2 \to \mathbb{C}$, to a discrete conformal map f, is defined in [2] by

$$f^*_{n+1,m} - f^*_{n,m} = \frac{1}{\overline{f_{n+1,m} - f_{n,m}}}, \; f^*_{n,m+1} - f^*_{n,m} = -\frac{1}{\overline{f_{n,m+1} - f_{n,m}}}. \tag{4}$$

The smooth limit of this duality is $(f^*)' = 1/\bar{f}'$ where f is holomorphic and f^* is antiholomorphic.

- Equation (1) is integrable. The Lax pair

$$\begin{aligned} \Psi_{n+1,m} &= U_{n,m}\Psi_{n,m} \\ \Psi_{n,m+1} &= V_{n,m}\Psi_{n,m} \end{aligned} \tag{5}$$

found by Nijhoff and Capel in [3] is of the form

$$U_{n,m} = \begin{pmatrix} 1 & -u_{n,m} \\ \dfrac{\lambda}{u_{n,m}} & 1 \end{pmatrix}, \quad V_{n,m} = \begin{pmatrix} 1 & -v_{n,m} \\ -\dfrac{\lambda}{v_{n,m}} & 1 \end{pmatrix}, \tag{6}$$

where

$$u_{n,m} = f_{n+1,m} - f_{n,m}, \qquad v_{n,m} = f_{n,m+1} - f_{n,m} \tag{7}$$

Let us mention also that all the properties are preserved [2],[3] if $q = -1$ is replaced by

$$q_{n,m} = \frac{\alpha_n}{\beta_m}.$$

The discrete conformal maps defined above are quadrilateral patterns with the combinatorics of the square grid. Ramified coverings can be modelled by quadrilateral patterns with more complicated combinatorics when N edges [1] may meet at a vertex. In this case $\mathbb{Z}^2$ in the definition should be replaced by a quad–graph G [4].

2 Examples

- $Z :=$ discrete z

$$Z(n, m) := n + im.$$

[1] usually one assumes that N is even and $N \geq 4$.

- EXP := discrete e^z

$$\mathrm{EXP}_\gamma(n,\, m) := \exp(2n\ \mathrm{arcsinh}\gamma + 2im \arcsin\gamma),\ \gamma \in \mathbb{R}.$$

- Various discrete rational, trigonometric and hyperbolic functions (for example TANH := discrete $\tanh z$) can be obtained from the first two examples by various combinations of the transformations (3), (4) (the Bäcklund–Darboux transformations).

- Z^γ: = discrete z^γ.

Equation (1) can be supplemented with the following nonautonomous constraint:

$$\begin{aligned}\gamma(f_{n,m}+\delta) &= 2(n-\alpha)\frac{(f_{n+1,m}-f_{n,m})(f_{n,m}-f_{n-1,m})}{f_{n+1,m}-f_{n-1,m}} \\ &\quad +2(m-\beta)\frac{(f_{n,m+1}-f_{n,m})(f_{n,m}-f_{n,m-1})}{f_{n,m+1}-f_{n,m-1}}.\end{aligned} \tag{8}$$

Theorem 2.1. *$f\colon \mathbb{Z}^2 \to \mathbb{C}$ is a solution to the system* (1), (8) *if and only if there exists a solution to* (5), (6), *which satisfies the following differential equation in λ:*

$$\frac{d}{d\lambda}\Psi_{n,m} = A\Psi_{n,m}, \quad A = \frac{1}{\lambda}A_0 + \frac{1}{\lambda-1}A_1 + \frac{1}{\lambda+1}A_{-1}, \tag{9}$$

where the matrices A_0, A_1, A_{-1} are λ-independent. The constraint (8) *is compatible with* (1).

In the case $\gamma = 1$ the constraint (8) and the corresponding monodrony problem (9) were obtained in [5]. The calculation of the coefficients of A is rather tedious. Correcting misprints in the monodrony problem presented in [5] and generalizing it to the case $\gamma \neq 1$ important for us, we get

$$\begin{aligned}
A_0 &= \begin{pmatrix} -\frac{\gamma}{4} & (\alpha-n)\frac{u_{n,m}u_{n-1,m}}{u_{n,m}+u_{n-1,m}} + (\beta-m)\frac{v_{n,m}v_{n,m-1}}{v_{n,m}+v_{n,m-1}} \\ 0 & \frac{\gamma}{4} \end{pmatrix}, \\
A_1 &= \frac{m-\beta}{v_{n,m}+v_{n,m-1}}\begin{pmatrix} v_{n,m} & v_{n,m}v_{n,m-1} \\ 1 & v_{n,m-1}\end{pmatrix} + \frac{\beta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\
A_{-1} &= \frac{n-\alpha}{u_{n,m}+u_{n-1,m}}\begin{pmatrix} u_{n,m} & u_{n,m}u_{n-1,m} \\ 1 & u_{n-1,m}\end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.
\end{aligned} \tag{10}$$

Remark 1. The constraint (8) is not Möbius invariant and can be easily generalized by applying a general Möbius transformation to $f_{n,m}$. The generalized constraint is similarly to (8), but with a quadratic polynomial of $f_{n,m}$ in the left hand side. So defined generalized class is invariant with respect to the Möbius and dual transformations.

Remark 2. The monodromy problem (9) coincides with the one of the Painlevé VI equation [6], which shows that the system (1) and (8) can be solved in terms of the Painlevé transcendents.

Let us assume $\gamma < 2$ and denote $\mathbb{Z}^2_+ = \{(n, m) \in \mathbb{Z}^2 : n, m \geq 0\}$. Motivated by the asymtotics of the constraint (8) at $n, m \to \infty$ and the properties

$$z^\gamma(\mathbb{R}_+) \in \mathbb{R}_+, z^\gamma(i\mathbb{R}_+) \in e^{\gamma\pi i/2}\mathbb{R}_+,$$

of the holomorphic z^γ it is natural to give the following definition of the "discrete z^γ" which we denote by Z^γ.

Definition 2.1. $Z^\gamma \colon \mathbb{Z}^2 \to \mathbb{C}$ is the solution of (1), (8) with $\alpha = \beta = \delta = 0$ and with initial conditions

$$Z^\gamma(0, 0) = 0, \quad Z^\gamma(1, 0) = 1, \quad Z^\gamma(0, 1) = e^{\gamma\pi i/2}.$$

It is easy to see that $Z^\gamma(n, 0) \in \mathbb{R}_+, Z^\gamma(0, m) \in e^{\gamma\pi i/2}\mathbb{R}_+, \quad \forall n, m \in \mathbb{N}$.

Conjecture 2.1. $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ is an embedding, i.e. different open elementary quadrilaterals of the pattern $Z^\gamma(\mathbb{Z}^2_+)$ do not intersect.

Computer experiments made by Tim Hoffmann confirm this conjecture.

Conjecture 2.2. Z^γ is the only embedded discrete conformal map $f : \mathbb{Z}^2_+ \to \mathbb{C}$ with

$$f(0, 0) = 0, \quad f(n, 0) \in \mathbb{R}_+, \quad f(0, m) \in e^{\gamma\pi i/2}\mathbb{R}_+ \quad \forall n, m \in \mathbb{N}.$$

We hope to prove these conjectures by combining geometrical methods with the modern theory of the Painlevé equations [7], [6].

In the discrete as well as in the smooth case (up to constant factor) one has

$$(Z^\gamma)^* = \overline{Z^{2-\gamma}}.$$

3 Discrete Surfaces and Coordinate Systems

Almost all notions of this section belong to the conformal (Möbius) geometry. One can easily extend the notion of the cross-ratio (1) to points in $\mathbb{R}^3$ identifying a sphere S passing through $X_1, X_2, X_3, X_4 \in \mathbb{R}^3$ with the Riemann sphere $\mathbb{CP}^1$. The cross-ratio is real when the four points are concircular. A direct generalization of the definitions of Section 1 to $\mathbb{R}^3$ yields the following definition [2].

Definition 3.1. A *discrete I–surface* (discrete isothermic surface) is a map $F: \mathbb{Z}^2 \to \mathbb{R}^3$ for which

$$q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = \frac{\alpha_n}{\beta_m}, \qquad \alpha, \beta: \mathbb{Z} \to \mathbb{R}. \tag{11}$$

All the properties of the discrete conformal maps listed in Section 1 hold (for simplicity we set $q_{n,m} = -1$):

- Infintesimal quadrilaterals of the smooth isothermic surfaces satisfy (2).
- Definition 3.1 is Möbius invariant (now with respect to the Möbius transformations in $\mathbb{R}^3 \cup \{\infty\}$).
- The dual discrete I–surface is defined by

$$F^*_{n+1,m} - F^*_{n,m} = \frac{F_{n+1,m} - F_{n,m}}{\|F_{n+1,m} - F_{n,m}\|^2},$$
$$F^*_{n,m+1} - F^*_{n,m} = -\frac{F_{n,m+1} - F_{n,m}}{\|F_{n,m+1} - F_{n,m}\|^2}.$$

There exists a Lax pair [2] for (11). Special classes of the discrete I–surfaces can be characterized as follows:

- Discrete M–surfaces (minimal) [2] : The dual surface F^* lies on a sphere. The Gauss map of F is F^*.
- Discrete H–surfaces (constant mean curvature) [1],[8]: A dual surface F^* is "parallel" to F, i.e. it lies in constant distance of F

$$\|F_{n,m} - F^*_{n,m}\| = \frac{1}{H} = \text{const.}$$

  Then H is the mean curvature for both F and F^*.

Discrete integrable systems are closely related to the Bäcklund–Darboux (BD) transformations of their smooth analogues. The loop group interpretation (see for example [9]) of the BD-transformation naturally yields the permutability theorem: given two BD-transformations D_1, D_2, there exist transformations D_1', D_2' such that

$$D_1' D_2 = D_2' D_1 \tag{12}$$

holds. Here the D's lie in the corresponding loop group. Equation (12) becomes the Lax representation of the discrete 2-dimensional net. Moreover the commuting diagram can be generalized to the N–dimensional case for arbitrary $D_1, ..., D_N$: the edges of an N–dimensional cube can be completed with the BD transformations so that the diagram commutes.

The case $N = 3$ is geometrically interesting. Motivated by the interpretation of (11) as the permutability theorem for the Bäcklund–Darboux transformations, we suggest

Definition 3.2. A *discrete I–system* is a map $F : \mathbb{Z}^3 \to \mathbb{R}^3$ for which:

$$\begin{aligned} q(F_{n,m,l}, F_{n+1,m,l}, F_{n+1,m+1,l}, F_{n,m+1,l}) &= \frac{\alpha_n}{\beta_m}, \\ q(F_{n,m,l}, F_{n+1,m,l}, F_{n+1,m,l+1}, F_{n,m,l+1}) &= \frac{\alpha_n}{\gamma_l}, \\ q(F_{n,m,l}, F_{n,m+1,l}, F_{n,m+1,l+1}, F_{n,m,l+1}) &= \frac{\beta_n}{\gamma_l} \end{aligned}$$

hold for some $\alpha, \beta, \gamma \colon \mathbb{Z} \to \mathbb{R}$.

All the coordinate surfaces of a discrete I-system are discrete I-surfaces. An analytical description of the discrete I-systems via the BD transformations for isothermic surfaces is given in [10].

A discrete I-system is uniquely determined by its Cauchy data

$$F(\bullet, 0, 0,), \quad F(0, 0, \bullet), \quad F(0, \bullet, 0,) \colon \mathbb{Z} \to \mathbb{R}^3, \qquad \alpha, \beta, \gamma \colon \mathbb{Z} \to \mathbb{R}.$$

A direct geometrical proof of this is presented in [8].

The following generalization is motivated by the smooth limit and the Möbius invariance of the curvature line parametrization.

Definition 3.3. A *discrete C–surface* (discrete curvature line parametrized surface) is a map $F \colon \mathbb{Z}^2 \to \mathbb{R}^3$ such that all elementary quadrilaterals have negative cross-ratios (i.e. they are concircular and embedded).

A 2–parametric family of spheres is called a Ribeaucour sphere congruence if the curvature lines of the two enveloping surfaces do correspond [11].

Definition 3.4. Two discrete C–surfaces $F, \tilde{F} \colon \mathbb{Z}^2 \to \mathbb{R}^3$ envelope a *discrete R–sphere congruence* if $\forall\, n, m \in \mathbb{Z}$ the vertices of the elementary hexahedron $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}, \tilde{F}_{n,m}, \tilde{F}_{n+1,m}, \tilde{F}_{n+1,m+1}, \tilde{F}_{n,m+1})$ lie on a sphere.

The discrete R–sphere congruences allow a natural quaternionic description, which in the special case of the discrete I–surfaces yields their Lax representation.

Definition 3.5. A *discrete O–system* (discrete triply–orthogonal coordinate system) is a map $F \colon \mathbb{Z}^3 \to \mathbb{R}^3$ for which all elementary quadrilaterals have negative cross-ratios.

This definition is motivated by the Dupin theorem [11], which claims that the coordinate surfaces of a smooth triply–orthogonal coordinate system intersect along their curvature lines.

Theorem 3.1. *$F\colon \mathbb{Z}^3 \to \mathbb{R}^3$ is a non–degenerate discrete O–system if and only if all its elementary hexahedra*

$$\begin{aligned} H_{n,m} &= (F_{n,m,l},\, F_{n+1,m,l},\, F_{n+1,m+1,l}, F_{n,m+1,l}, \\ &\quad F_{n,m,l+1},\, F_{n+1,m,l+1},\, F_{n+1,m+1,l+1},\, F_{n,m+1,l+1}) \end{aligned}$$

lie on spheres and are embedded.

Remark 3. A less restrictive version of Definitions 3.3, 3.4, 3.5 and Theorem 3.1 includes the condition $q \in \mathbb{R}$ only (i.e. does not assume the embeddedness).

The discrete O–systems are sphere packings with the combinatorics of the cube grid. Since the spheres comprise the "dual lattice" it is natural to label them by half–integer numbers: the vertices of the hexahedreon $H_{n,m}$ lie on the sphere $S_{n+\frac{1}{2},m+\frac{1}{2},l+\frac{1}{2}}$, the vertex $F_{n,m}$ is the intersection of 8 spheres $S_{n\pm\frac{1}{2},m\pm\frac{1}{2},l\pm\frac{1}{2}}$.

The cross-ratios of the faces (the index labels the "center" of the corresponding face)

$$\begin{aligned} R_{n,m+\frac{1}{2},l+\frac{1}{2}} &= q(F_{n,m,l},\, F_{n,m+1,l},\, F_{n,m+1,l+1},\, F_{n,m,l+1}) \\ R_{n+\frac{1}{2},m,l+\frac{1}{2}} &= q(F_{n,m,l},\, F_{n+1,m,l},\, F_{n+1,m,l+1},\, F_{n,m,l+1}) \\ R_{n+\frac{1}{2},m+\frac{1}{2},l} &= q(F_{n,m,l},\, F_{n+1,m,l},\, F_{n+1,m+1,l},\, F_{n,m+1,l}) \end{aligned}$$

satisfy

$$\begin{gathered} R_{n+\frac{1}{2},m,l+\frac{1}{2}}\, R_{n+\frac{1}{2},m+1,l+\frac{1}{2}} = \\ R_{n,m+\frac{1}{2},l+\frac{1}{2}}\, R_{n+1,m+\frac{1}{2},l+\frac{1}{2}}\, R_{n+\frac{1}{2},m+\frac{1}{2},l}\, R_{n+\frac{1}{2},m+\frac{1}{2},l+1}. \end{gathered}$$

The last equation holds for any 8 points on a sphere and by modular transformations of the cross-ratios of the n and l faces

$$\begin{aligned} T_{n+\frac{1}{2},m,l+\frac{1}{2}} &:= R_{n+\frac{1}{2},m,l+\frac{1}{2}} \\ T_{n,m+\frac{1}{2},l+\frac{1}{2}} &:= q(F_{n,m,l}, F_{n,m+1,l+1}, F_{n,m+1,l}, F_{n,m,l+1}) = 1 - R_{n,m+\frac{1}{2},l+\frac{1}{2}} \\ T_{n+\frac{1}{2},m+\frac{1}{2},l} &:= q(F_{n,m,l}, F_{n,m+1,l+1}, F_{n,m+1,l}, F_{n,m,l+1}) = \left(1 - R^{-1}_{n+\frac{1}{2},m+\frac{1}{2},l}\right)^{-1}. \end{aligned}$$

is transformed to a gauge–invariant form of the $3D$ Hirota bilinear difference equation (see, for example, [12]) on a sublattice. Directing the n– m– and l–axes to right, front and up respectively one can write this equation as

$$T_f T_b = \frac{(1 - T_l)(1 - T_r)}{(1 - T_u^{-1})(1 - T_d^{-1})}, \tag{13}$$

where the labels denote the f(ront), b(ack), l(eft), r(ight), u(p), and d(own) faces of a hexahedron.

4 Circle Patterns of Schramm

Recently, coming from questions in approximation theory, Schramm in his fundamental paper [13] proposed a more restrictive definition of discrete conformal maps than Definition 1.1. He considers circle patterns with the combinatorics of $\mathbb{Z}^2$ in the plane with the following characteristic properties:

1. On each circle there are 4 vertices.
2. Each vertex has 2 pairs of touching circles in common, the pairs intersecting orthogonally.
3. Let C be a circle of the pattern and C_1, C_2, C_3, C_4 its neighbors, intersecting C orthogonally, then $C_i \cap C_j \in C \quad \forall i \neq j$.

This definition is obviously Möbius invariant. A pattern is embedded, provided the open discs of the circles, which are not neighbors, are disjoint. If this holds for the half–neighbors (touching circles) we say that the pattern is immersed (or planar).

Adding temporarily the midpoints of the circles one obtains a refinement of the lattice with the following properties:

- There are two kinds of vertices which alternate: at the original vertices of the pattern there are 2 perpendicular outgoing pairs of edges, whereas the outgoing edges at the added vertices have the same length. This nicely relates the properties of the smooth case: $f_x \perp f_y$ and $|f_x| = |f_y|$ respectively.
- The quadrilaterals of the refined lattice are of the "kite" shape, in particular, they have cross-ratio -1, thus providing a discrete conformal map in the sense of Definition 1.1.
- Take an immersed Schramm circle pattern, construct its refinement as above, build the dual (4) of it, delete the added points. This recipe provides us with another Schramm immersion, which we call *dual*.

We return to the lattice formed by the intersections of circles, which we denote by $f\colon \mathbb{Z}^2 \to \mathbb{C}$. Define a function on faces by

$$T_{n+\frac{1}{2},m+\frac{1}{2}} := q(f_{n+1,m},\, f_{n+1,m+1},\, f_{n,m+1},\, f_{n,m})$$

and another function S on vertices by

$$S_{n,m} := q(f_{n,m-1}, f_{n+1,m}, f_{n,m+1}, f_{n-1,m}).$$

It is easy to see that both these functions are negative-valued. A more elaborated calculation [13] proves that they satisfy the discrete Cauchy–Riemann equations

$$\frac{S_u}{S_d} = \left(\frac{1-T_r}{1-T_l}\right)^2, \qquad \frac{S_r}{S_l} = \left(\frac{1-T_u^{-1}}{1-T_d^{-1}}\right)^2. \tag{14}$$

Here we use the notation of Section 3, where

$$\begin{aligned} S_u = S_{n,m+1},\ S_d = S_{n,m},\ T_r = T_{n+\frac{1}{2},m+\frac{1}{2}},\ T_l = T_{n-\frac{1}{2},m+\frac{1}{2}},\\ S_r = S_{n+1,m},\ S_l = S_{n,m},\ T_u = T_{n+\frac{1}{2},m+\frac{1}{2}},\ T_d = T_{n+\frac{1}{2},m-\frac{1}{2}}. \end{aligned}$$

Taking a quadrilateral formed by the 4 vertices on a circle, the cross-ratios of the 4 neighboring quadrilaterals satisfy the compatibility condition of (14)

$$T^2 = \frac{(1-T_r)(1-T_l)}{(1-T_u^{-1})(1-T_d^{-1})}, \tag{15}$$

where T is the cross-ratio of the center quadrilateral. This is exactly the $3D$ Hirota equation with a translational symmetry in the front–back direction (cf. (13)).

The negative solutions of the discrete Cauchy–Riemann equations (14) are in one to one correspondence with the Möbius equivalence classes of the Schramm circle patterns [13]. Moreover, for a negative solution of (15) there exists a one–parametric family of negative solutions of (14). (S is unique up to a multiplication by a positive constant $S \mapsto \lambda S$, $\lambda \in \mathbb{R}_+$) and consequently a one–parametric family (associated family) of circle patterns. Obviously, λ in this construction plays the role of a spectral parameter. Equation (15) possesses a maximum principle, which allows proof of global results. In particular it was proven in [13], that the only embedding of the whole $\mathbb{Z}^2$ is the standard circle pattern (where all circles have constant radius).

Let us mention also an alternative description of this geometry. The equation for the radii of the neighboring circles is

$$r^2(r_u + r_d + r_l + r_r) = r_u r_d r_l r_r (r_u^{-1} + r_d^{-1} + r_l^{-1} + r_r^{-1}), \tag{16}$$

which is probably also integrable.

It is possible to generalize Schramm's patterns replacing $\mathbb{Z}^2$ by a quad–graph G (see Section 1). Instead of having 4 vertices on every circle, one allows various numbers N of vertices (and as a consequence N neighboring and N half–neighboring circles). Such a singular point is natural to call a *branch point* of order $N/4-1$. In case of even $N=2M$ cross-ratios can be prescribed to faces and there is a generalization of (15), which takes various cross-ratios of the central circle into account.

Let $F_1, \ldots F_{2M}$ be the consequtive vertices on a circle C labeled counterclockwise. Denote by $C_1, \ldots, C_{2M}$ the neighboring circles of C with the intersection points F_i, $F_{i+1} = C \cap C_i$. Denote by H_i, and G_i the vertices on C_i neighboring F_i and F_{i+1} such that the vertices G_i, F_i, F_{i+1}, H_i are consequtive on C_i. The cross-ratios [2] around C

$$R_i: = q(F_i,\, F_{i+2},\, F_{i+1},\, F_{i-1}), \qquad \tilde{R}_i: = q(G_i,\, F_{i+1},\, H_i,\, F_i)$$

satisfy the following equation

$$\frac{R_1 R_3 \ldots R_{2M-1}}{R_2 R_4 \ldots R_{2M}} = \frac{\tilde{R}_1 \tilde{R}_3 \ldots \tilde{R}_{2M-1}}{\tilde{R}_2 \tilde{R}_4 \ldots \tilde{R}_{2M}}. \tag{17}$$

In addition R's are subject to constraints

$$\begin{aligned} R_i &> 1, \\ R_i m(R_{i+1} m(R_{i+2} \ldots R_{i+k-1} m(R_{i+k}) \ldots) &> 1, \qquad k < 2M-3 \\ R_i m(R_{i+1} m(R_{i+2} \ldots R_{i+2M-4} m(R_{i+2M-3}) \ldots) &= 1, \quad i = 1, \ldots 2M, \end{aligned} \tag{18}$$

where $m(R) = 1 - \frac{1}{R}$ and the indices are taken $mod(2M)$. Note that only three constraints of (18) are independent. Given a solution to the equation and constraints above one can define the field S on vertices by using (14) and so get an associated family (S is defined up to a multiplication by a positive constant λ) of the generalized Schramm circle patterns. For $M=2$ the system (17),(18) is equivalent to (15).

The discrete conformal map Z^γ of Section 2 with $\gamma = 4/N$, $N \in \mathbb{N}$, $N > 4$ is an example of such a generalized Schramm circle pattern. (Recall that the central points of the circles are also included). In this case the only branch point is at the origin. We call the combinatorics of this pattern combinatorics of the plane with one branch point of order $N/4-1$.

Conjecture 4.1. Up to a similarity $Z^{4/N}$ is the only embedded Schramm circle pattern with the combinatorics of the plane with one branch point of order $N/4-1$.

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[2] which differ from the choice in (15) by the modular transformation $R = 1 - T$. Thus $R > 1$.

5 Concluding remarks: discrete conformal surfaces and coverings

One can try to globalize the above ideas. Take a topological surface and pack it with simply closed loops, which model the topology of the generalized Schramm's circle patterns. If, to the combinations of four vertices described above, one can assign negative numbers R, which satisfy (17) and (18) one can define local coordinate charts, which are local Schramm's circle patterns and then talk about a *discrete conformal covering.*

It is tempting to suggest also the following generalization of Schramm's circle patterns for surfaces. Let $\mathbf{S}$ be the space of spheres in $\mathbb{R}^3$ and $S : \mathbb{Z}^2 \to \mathbf{S}$ a discrete sphere congruence such that: the neighbouring spheres intersect orthogonally $S_{n,m} \perp S_{n+1,m}$, $S_{n,m} \perp S_{n,m+1}$ $\forall n, m$ and the half-neighbouring spheres are tangent $S_{n,m} || S_{n+1,m+1}$, $S_{n,m} || S_{n+1,m-1}$ $\forall n, m$. Then the touching points of half-neighbours build a net $F : \mathbb{Z}^2 \to \mathbb{R}^3$, which is natural to call a *discrete conformal surface.*

Acknowledgment

I would like to thank Tim Hoffmann and Ulrich Pinkall for many fruitful discussions.

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