manuscripta math. 97, 489 – 516 (1998)

Alexander I. Bobenko · Alexander V. Kitaev

On asymptotic cones of surfaces with constant curvature and the third Painlevé equation

Received: 23 April 1998

Abstract. It is shown that for any surface in $\mathbb{R}^3$ with constant negative Gaussian curvature and two straight asymptotic lines there exists a cone such that the distances from all its points to the surface are bounded. Analytic and geometric descriptions of the cone are obtained. This cone is asymptotic also for constant mean curvature planes in $\mathbb{R}^3$ with inner rotational symmetry.

1. Introduction

We study the surfaces with constant negative Gaussian curvature (K-surfaces) and two straight asymptotic lines, introduced by Bianchi [3]. The Gauss equation of these surfaces reduces to an ordinary differential equation. This equation is a special case of the third Painlevé equation in a trigonometric form. Further study and the first plot of these surfaces are due to Amsler [1]. In his honor we call K-surfaces with two straight asymptotic lines Amsler surfaces.

Our study of the Amsler surfaces is based on recent progress in asymptotic analysis of the Painlevé equations[1] and, in particular, on results obtained in [11]. We show that the Amsler surfaces have asymptotic cones, i.e., for any Amsler surface there exists a cone such that the distances from its points to the surface are bounded. This is the main result of the paper formulated in Theorem 4. To prove it we need some advanced asymptotic results for the particular case of the third Painlevé equation, which we obtain in Section 7.

In previous studies [4] of surfaces with constant mean curvature and inner rotational symmetry there also appeared a cone which describes asymptotic properties of these surfaces. In our discussions U. Pinkall conjectured that asymptotic cones for both types of surfaces coincide. This remarkable universality property of the asymptotic cone is proven in Section 5. In Section 6

A. I. Bobenko: Fachbereich Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany. e-mail: bobenko@math.tu-berlin.de

A. V. Kitaev: Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191011, Russia

Mathematics Subject Classification (1991): 53A10, 53C42, 34E20, 35Q53

[1] For other classes of surfaces described by the Painlevé equations see [6, 7].

we give a geometric description of the asymptotic cone, in particular, in a form of the smoke-ring evolution (see [13]) of a curve.

2. Surfaces with constant negative Gaussian curvature

In this section we recall basic facts concerning a description of surfaces with constant negative Gaussian curvature (K-surfaces) in terms of quaternions (see, for example, [5] for details and missing proofs). Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the Chebyshev net parametrization of a (weakly regular) K-surface $\mathcal{F}$. It is the asymptotic line parametrization with $|F_x| = \text{const},\ |F_y| = \text{const}$, where the subscripts denote the corresponding partial derivatives. For the Gauss map $N : \mathbb{R}^2 \to S^2$ one has

$$F_x, F_y, F_{xx}, F_{yy} \perp N. \tag{2.1}$$

With each K-surface $\mathcal{F}$ one can associate a one-parameter ($\lambda > 0$) family $\{\mathcal{F}(\lambda)\}_{\lambda\in\mathbb{R}_+}$ of K-surfaces (*the associated family*) given by the following fundamental forms:

$$\mathbf{I}(dx, dy) = \hat{\rho}^2(\lambda^2 a^2 dx^2 + 2ab\cos(\phi) dx dy + \lambda^{-2} b^2 dy^2), \tag{2.2}$$

$$\mathbf{II}(dx, dy) = 2\hat{\rho} ab \sin(\phi) dx dy, \tag{2.3}$$

where $a, b, \hat{\rho} > 0$ are independent of x, y, λ and $\phi = \phi(x, y)$ is the angle between the asymptotic lines. Thus the associated family can be viewed as a deformation (with the deformation parameter λ) of the K-surface $\mathcal{F} = \mathcal{F}(1)$, which preserves the second fundamental form, the Gaussian curvature, $K = -\frac{1}{\hat{\rho}^2}$, and the angle ϕ between the asymptotic lines.

We will use the well-known isomorphism between $\mathbb{R}^3$ and $\Im\mathbb{H}$, the space of imaginary quaternions:

$$F = -i \sum_{\alpha=1}^{3} F^\alpha \sigma_\alpha \in \Im\mathbb{H} \Longleftrightarrow F = (F^1, F^2, F^3) \in \mathbb{R}^3,$$

where σ_α are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Henceforth we adopt the same notations for the vectors regardless whether they are considered as elements of $\mathbb{R}^3$ or $\Im\mathbb{H}$. Recall that, the usual scalar $\langle F,\ G\rangle$ and cross $F \times G$ products of vectors in the matrix notations (r.-h. s.'s of the following equations) read:

$$\langle F,\ G\rangle = -\frac{1}{2}\operatorname{tr} FG, \quad F \times G = FG + \langle F,\ G\rangle.$$

A moving frame on the surface $\mathcal{F}(\lambda)$ can be chosen as follows:

$$F_x = -ia\hat{\rho}\lambda\Phi^{-1}\begin{pmatrix} 0 & e^{-\frac{i\phi}{2}} \\ e^{\frac{i\phi}{2}} & 0 \end{pmatrix}\Phi,\ F_y = -\frac{ib\hat{\rho}}{\lambda}\Phi^{-1}\begin{pmatrix} 0 & e^{\frac{i\phi}{2}} \\ e^{-\frac{i\phi}{2}} & 0 \end{pmatrix}\Phi,$$

$$N = -i\Phi^{-1}\sigma_3\Phi, \tag{2.4}$$

where Φ is a map, $\Phi = \Phi(x,\ y,\ \lambda) : \mathbb{R}^2 \times \mathbb{R}_+ \to SU(2)$.

Theorem 1. *The function $\Phi(x,\ y,\ \lambda)$ is a solution of the system:*

$$\Phi_x = U\Phi, \quad \Phi_y = V\Phi, \tag{2.5}$$

$$U = \begin{pmatrix} \frac{i\phi_x}{4} & -\frac{ia}{2}\lambda e^{-\frac{i\phi}{2}} \\ -\frac{ia}{2}\lambda e^{\frac{i\phi}{2}} & -\frac{i\phi_x}{4} \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{i\phi_y}{4} & \frac{ib}{2\lambda}e^{\frac{i\phi}{2}} \\ \frac{ib}{2\lambda}e^{-\frac{i\phi}{2}} & \frac{i\phi_y}{4} \end{pmatrix}, \tag{2.6}$$

$\phi = \phi(x,\ y)$ solves the sine-Gordon equation

$$\phi_{xy} = ab\sin\phi. \tag{2.7}$$

The asymptotic line parametrization of $\mathcal{F}$ is given by the Sym-formula (see [16])

$$F(x,\ y,\ \lambda) = 2\hat{\rho}\lambda\Phi^{-1}\frac{\partial}{\partial\lambda}\Phi. \tag{2.8}$$

Conversely, if $\phi(x,\ y)$ is a solution of the sine-Gordon equation (2.7), then there exists a function $\Phi(x,\ y,\ \lambda) \in SU(2)$, which for $\lambda \in \mathbb{R}_+$ defines via (2.8) *the asymptotic line parametrization of an associated family of K-surfaces with the Gaussian curvature $K = -\frac{1}{\hat{\rho}^2}$.*

Corollary 1. *The Gauss map $N = N(x,\ y,\ \lambda)$ of a K-surface is Lorentz-harmonic:*

$$N_{xy} = ab\cos(\phi)N \tag{2.9}$$

and $|N_x| = a\lambda,\ |N_y| = b/\lambda$.

Corollary 2. *Principle curvatures of $\mathcal{F}(\lambda)$ are as follows:*

$$k_1 = \frac{1}{\hat{\rho}}\tan\frac{\phi(x,\ y)}{2}, \quad k_2 = -\frac{1}{\hat{\rho}}\cot\frac{\phi(x,\ y)}{2}. \tag{2.10}$$

3. The Amsler surface

In this section we give an analytic description of K-surfaces with two straight asymptotic lines. These surfaces were first studied by Bianchi [3], who proved their existence and has shown that in this case the sine-Gordon equation reduces to an ordinary differential equation. A more detailed treatment of these surfaces is due to Amsler [1], who produced a first plot. We call a K-surface with two non-parallel straight asymptotic lines ($\mathcal{L}^x$ and $\mathcal{L}^y$) *an Amsler surface*.

Lemma 1. *The angle $\phi(x, y)$ between asymptotic lines for the Amsler surface is constant along the straight asymptotic lines:*

$$\phi(x, y)\big|_{\mathcal{L}^x} = \phi(x, y)\big|_{\mathcal{L}^y} \equiv \varphi(0), \tag{3.1}$$

where $\varphi(0)$ is the angle between $\mathcal{L}^x$and $\mathcal{L}^y$. In a proper parametrization condition (3.1) *reads as*

$$\phi(x, 0) = \phi(0, y) \equiv \varphi(0). \tag{3.2}$$

Proof. Since $\mathcal{L}^x$, $\mathcal{L}^y$ are the straight lines and $|F_x| = \hat{\rho}a\lambda$, $|F_y| = \frac{\hat{\rho}b}{\lambda}$, we get that

$$F_{xx}\big|_{\mathcal{L}^x} = F_{yy}\big|_{\mathcal{L}^y} = 0. \tag{3.3}$$

Conditions (2.1) and Eq. (2.9) imply:

$$\langle F_x, N_{xy}\rangle = \langle F_y, N_{xy}\rangle = 0. \tag{3.4}$$

Now (2.3), (3.3), and (3.4) yield:

$$\frac{\partial}{\partial x}\sin(\phi)\bigg|_{\mathcal{L}^x} = \frac{1}{ab}\frac{\partial}{\partial x}\langle F_x, N_y\rangle = 0 = \frac{1}{ab}\frac{\partial}{\partial y}\langle F_y, N_x\rangle = \frac{\partial}{\partial y}\sin(\phi)\bigg|_{\mathcal{L}^y}.$$

Additionally, we know that $\phi\big|_{\mathcal{L}^x}$ $\left(\phi\big|_{\mathcal{L}^y}\right)$ is independent of y (of x), since the latter variables are fixed on corresponding asymptotic lines. Finally, using that $\phi(x, y)$ is supposed to be a smooth function of x and y, one obtains (3.1). □

Lemma 2. *Any two intersecting straight lines, $\mathcal{L}^x$ and $\mathcal{L}^y$ uniquely define an Amsler surface $\mathcal{F}$ with the asymptotic lines $\mathcal{L}^x$ and $\mathcal{L}^y$. Vice versa, given an Amsler surface $\mathcal{F}$ the pair of its straight asymptotic lines, $\mathcal{L}^x$ and $\mathcal{L}^y$, is uniquely defined.*

Proof. The intersection point $\mathcal{O}$ of $\mathcal{L}^x$ and $\mathcal{L}^y$ divide them on the rays $\mathcal{L}^x_\pm$ and $\mathcal{L}^y_\pm$ correspondingly. In parametrization (3.2) $\mathcal{L}^x_+,\ \mathcal{L}^y_+$ are the images of the coordinate semi-axes $x > 0,\ y > 0$. Let $\varphi(0)$ be an angle between the rays $\mathcal{L}^x_+$ and $\mathcal{L}^y_+$. It is well known that the boundary value problem (3.1) for the hyperbolic sine-Gordon equation (2.7) is uniquely solvable (see [3]). The solution of this problem $\phi(x,\ y)$ defines via (2.5), (2.6), and (2.8) a part of $\mathcal{F}$ bounded by $\mathcal{L}^x_+$ and $\mathcal{L}^y_+$. Another part of $\mathcal{F}$ bounded by $\mathcal{L}^x_-$ and $\mathcal{L}^y_+$ is defined by the unique solution of the boundary value problem (3.1) for Eq. (2.7) but with the angle $\varphi(0)$ changed to $\varphi(0) - \pi$. This gives us a half of the surface. The remaining part of $\mathcal{F}$ results from a revolution of the constructed part with respect to the axis $F_x(x,\ 0) \times F_y(0,\ y)$ by angle π.
Conversely, let $\mathcal{L}^x_1$ *and* $\mathcal{L}^y_1$ be another pair of straight asymptotic lines on $\mathcal{F}$. One of them, say $\mathcal{L}^x_1$, is intersecting $\mathcal{L}^y$ with the angle $\varphi(0)$. Thus, the angle between $\mathcal{L}^x_1$ *and* $\mathcal{L}^y_1$ is also equal to $\varphi(0)$. Using this and denoting a value of the parameter y (in the parametrization (3.2)) corresponding to $\mathcal{L}^x_1$ as y_1 one finds: $\varphi(x,\ 0) = \varphi(x,\ y_1) = \varphi(0)$. Thus, $\varphi_x(0,\ 0) = \varphi_x(0,\ y_1) = 0$. Hence, if $y_1 \neq 0$ (i.e. $\mathcal{L}^x_1 \neq \mathcal{L}^x$), then $\varphi_{xy}(0,\ y_2) = 0 \Rightarrow \sin(\varphi(0,\ y_2)) = \sin(\varphi(0)) = 0$ for $y_2 : 0 < y_2 < y_1$, which proves that $\mathcal{L}^x_1 \equiv \mathcal{L}^x$. In analogous manner one finds that $\mathcal{L}^y_1 \equiv \mathcal{L}^y$. □

Lemma 3. *The angle function $\phi(x,\ y)$ of an Amsler surface is a similarity solution*

$$\phi(x,\ y) = \varphi(r), \quad r = \sqrt{-4abxy}, \tag{3.5}$$

of the sine-Gordon equation (2.7). *The function $\varphi(r)$ is the unique solution of the following initial value problem*

$$\varphi(r)\big|_{r=0} = \varphi(0), \quad \frac{d}{dr}\varphi(r)\bigg|_{r=0} = 0, \tag{3.6}$$

for the third Painlevé equation in the trigonometric form

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + \sin(\varphi(r)) = 0. \tag{3.7}$$

Proof. If $\phi(x,\ y)$ is the solution of the boundary value problem discussed in Lemma 2, then $\phi(\alpha x,\ y/\alpha)$ for arbitrary $\alpha > 0$ is also the solution of this problem. Due to the uniqueness of the solution we have $\phi(x,\ y) = \phi(\alpha x,\ y/\alpha)$. Differentiating this equation with respect to α and putting $\alpha = 1$, one gets

$$x\phi_x(x,\ y) = y\phi_y(x,\ y), \tag{3.8}$$

which means that $\phi(x,\ y)$ is a function of xy. By introducing the similarity variables (3.5) into Eq. (2.7) we obtain (3.7). The second condition in (3.6) follows from the fact that $\phi_x(x,\ 0)$ and $\phi_y(0,\ y)$ are bounded. □

Since the function $\varphi(r)$ plays an important role in our studies we summarize its properties in the following

Proposition 1. *Denote* $\varphi(r,\ \varphi(0)) := \varphi(r)$*, the solution of the initial value problem* (3.6) *for Eq.* (3.7)*. For* $r,\ \varphi(0) \in \mathbb{R}$*, it is a real-analytic function of these variables with the following symmetry properties:*

$$\varphi(r,\ \varphi(0)) = \varphi(-r,\ \varphi(0)),\ \varphi(r,\ -\varphi(0)) = -\varphi(r,\ \varphi(0)),$$
$$\varphi(r,\ \varphi(0) \pm 2\pi) = \pm 2\pi + \varphi(-r,\ \varphi(0)),$$

$$\varphi(ir,\ \varphi(0)) = \pi + \varphi(r,\ \varphi(0) - \pi). \tag{3.9}$$

Proof. Identities (3.9) follow from the unique solvability of the boundary value problem (3.2) for the sine-Gordon equation (2.7). Since $\varphi(r)$ and $\varphi'(r)$ are bounded, the Cauchy theorem from the analytic theory for ordinary differential equations [10] ensure that $\varphi(r)$ is an analytic function of $r \in \mathbb{R} \setminus 0$. The proof of the analyticity of $\varphi(r)$ at $r = 0$ is analogous to the one for the Cauchy theorem, i.e., one can prove the convergence of the corresponding Taylor series in some neighborhood of the origin. □

Theorem 2. *An Amsler surface is uniquely characterized by the angle* $\varphi(0)$ *between its straight asymptotic lines, where* $0 < \varphi(0) \le \frac{\pi}{2}$.

Proof. If an Amsler surface is given, then according Lemma 2 the angle $\varphi(0)$: $0 < \varphi(0) \le \frac{\pi}{2}$ is uniquely defined, and using Lemma 3 the corresponding function $\varphi(r,\ \varphi(0))$ can be obtained. The inverse map is given by Lemma 3 and Theorem 1. □

Corollary 3. *The associated family of an Amsler surface consists of one surface.*

For the analytic studies presented in the next sections it is convenient to specify the mapping described in Theorem 2 in a more explicit form.

Lemma 4. *Let* $\mathcal{F}(\lambda)$ *be an associated family of Amsler surfaces. Consider the system* (2.5)*, in which the matrices* U *and* V *are given by (2.6) with* $\phi(x,\ y) = \varphi(r,\ \varphi(0))$*. There exists a solution* $\Phi = \Phi(x,\ y,\ \lambda) \in SU(2)$ *of this system which satisfies the following equation*

$$\frac{\partial}{\partial \lambda}\Phi = \left(\frac{x}{\lambda}U - \frac{y}{\lambda}V\right)\Phi. \tag{3.10}$$

Conversely, if a real-analytic function $\varphi(x,\ y)$ *is such that the system of three equations* (2.5), (2.6)*, and* (3.10) *is compatible, then its solution,* $\Phi = \Phi(x,\ y,\ \lambda) \in SU(2)$*, defines via Eq.* (2.8) *the Amsler family* $\mathcal{F}(\lambda)$.

Proof. Lemma 3 states that the angle function $\phi(x, y)$ of $\mathcal{F}(\lambda)$ is a similarity solution of the sine-Gordon equation (2.7). The similarity condition for $\phi(x, y)$ (3.8) in terms of the matrices U and V reads as:

$$\lambda V_\lambda + V = xV_x - yV_y, \qquad \lambda U_\lambda - U = xU_x - yU_y. \tag{3.11}$$

Since the system (2.5), (2.6) is compatible, Eqs. (3.11) are equivalent to the compatibility condition of Eq. (3.10) with this system. Conversely, the compatibility means that the function $\phi(x, y)$ has the properties (3.5), (3.6), and (3.7), so that we can use Theorem 2. □

Proposition 2. *Let $\mathcal{F} \equiv \mathcal{F}(\lambda)$ be the Amsler surface defined by a pair of straight lines $\mathcal{L}^x$ and $\mathcal{L}^y$ with the intersection angle $\varphi(0)$. Choose a coordinate system in $\mathbb{R}^3$ such that parametrization of $\mathcal{L}^x$ and $\mathcal{L}^y$ reads:*

$$\mathcal{L}^x : F(x, 0, \lambda) = -i\hat{\rho}ax\lambda\sigma_3, \qquad \mathcal{L}^y : F(0, y, \lambda) = -\frac{i\hat{\rho}by}{\lambda}Q^{-1}\sigma_3 Q, \tag{3.12}$$

where

$$Q = \begin{pmatrix} \cos(\frac{\varphi(0)}{2}) & i\sin(\frac{\varphi(0)}{2}) \\ i\sin(\frac{\varphi(0)}{2}) & \cos(\frac{\varphi(0)}{2}) \end{pmatrix}. \tag{3.13}$$

Then for $y \neq 0$ a parametrization of $\mathcal{F}$ is given by the following equations:

$$F(x, y, \lambda) = 2\hat{\rho}\mu\Psi^{-1}(r, \mu)A\Psi(r, \mu), \quad r = \sqrt{-4abxy}, \quad \mu = -\frac{2\lambda}{by}, \tag{3.14}$$

where $\Psi = \Psi(r, \mu)$ is the solution of the system:

$$\Psi_\mu = A\Psi, \qquad \Psi_r = W\Psi, \tag{3.15}$$

$$A = -\frac{ir^2}{16}\sigma_3 - \frac{ir\varphi_r(r)}{4\mu}\sigma_1 + \frac{1}{\mu^2}\begin{pmatrix} i\cos(\varphi(r)) & -\sin(\varphi(r)) \\ \sin(\varphi(r)) & -i\cos(\varphi(r)) \end{pmatrix}, \tag{3.16}$$

$$W = -\frac{ir\mu}{8}\sigma_3 - \frac{i\varphi_r(r)}{2}\sigma_1, \tag{3.17}$$

with the asymptotic expansion

$$\Psi \underset{\mu\to\infty}{=} \left(I + O\left(\frac{1}{\mu}\right)\right)\exp\left(-\frac{ir^2\mu}{16}\sigma_3\right). \tag{3.18}$$

The function $\varphi(r) = \varphi(r, \varphi(0))$ in (3.16) and (3.17) *is defined in Proposition 1.*

Proof. The coordinate system is chosen such that the intersection point $\mathcal{O}$ of $\mathcal{L}^x$ and $\mathcal{L}^y$ is at the origin, $\mathcal{L}^x$ coincides with the third axis, the first axis is a normal to the plane spanned on $\mathcal{L}^x$ and $\mathcal{L}^y$, and the angle between the second axis and $\mathcal{L}^y$ is equal to $\frac{\pi}{2} + \varphi(0)$. Note that $\mathcal{L}^x$ and $\mathcal{L}^y$ are independent of λ.

One can rewrite Eq. (3.10) as $\lambda\Phi_\lambda = x\Phi_x - y\Phi_y$. The last equation implies that $\Phi(x, y, \lambda)$ is a similarity solution of the system (2.5), (2.6), which depends on the similarity variable r, and on μ (see the two last Eqs. in (3.14)). It enables us to introduce a new function

$$\Psi(r, \mu) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \exp\left(\frac{i\phi(x, y)}{4}\sigma_3\right) \Phi(x, y, \lambda), \qquad (3.19)$$

where $\phi(x, y)$ is the similarity solution from Lemma 3. Now, using the system (2.5), (2.6) we find that for $y \neq 0$ function Ψ satisfies the system (3.15)–(3.17), and the Sym formula 2.8) implies the first equation in (3.14). The asymptotics (3.18) follows from the parametrization (3.12) of the straight lines $\mathcal{L}^x$ and $\mathcal{L}^y$. $\square$

It can be checked by direct computation that Eq. (3.7) gains a Hamiltonian form

$$\frac{dq}{dr} = \{\mathcal{H}, q\}, \qquad \frac{dp}{dr} = \{\mathcal{H}, p\}$$

with respect to the canonical Poisson structure:

$$\{p, p\} = \{q, q\} = 0, \quad \{p, q\} = 1$$

and the Hamiltonian

$$\mathcal{H}(p, q) = \frac{p^2}{2r} + r(1 - \cos(q))$$

as soon as p and q are defined as follows:

$$p = r\varphi_r(r), \quad q = \varphi(r).$$

Corollary 4. *The square of the distance from the origin of $\mathcal{F}(\lambda)$ to its point with the coordinates $F(x, y, \lambda)$ is given by the following formula,*

$$|F(x, y, \lambda)|^2 = \frac{\hat{\rho}^2 r^2}{4}\left(\left(\frac{4}{\nu} - \frac{\nu}{4}\right)^2 + \frac{2}{r}H(r)\right), \quad \nu = r\mu,$$

where $H(r)$ is a Hamiltonian function $H(r) = \mathcal{H}(p(r), q(r))$,

$$H(r) = r\left(\frac{\varphi_r^2(r)}{2} + 1 - \cos\varphi(r)\right),$$

of Eq. (3.7).

Proof. According (3.14) $|F(x, y, \lambda)|^2 = 4\hat{\rho}^2\mu^2 \det A$. $\square$

4. Asymptotic cone

In this section we carry out an asymptotic analysis of system (3.15) in order to investigate asymptotic properties of the Amsler surface. This system was studied from the point of view of Isomonodromy Deformation Method in [11] and, in particular, we use results obtained in this book.

For general solutions of Eq. (3.7) the function $\Psi(r,\ \mu)$ (3.18) is defined on the universal covering of the complex sphere $\overline{\mathbb{C}} \ni \mu$ punctured at the points $\mu = 0$ and $\mu = \infty$. It means that $\Psi(r,\ \mu)$ on $\mathbb{C} \setminus 0$ gains a non-trivial monodromy, while analytical continuation around circles centered at $\mu = 0$ and $\mu = \infty$:

$$\Psi(r,\ \mu e^{2\pi i})|_{\mu=0} = \Psi(r,\ \mu)M_0,\ \ \Psi(r,\ \mu e^{-2\pi i})|_{\mu=\infty} = \Psi(r,\ \mu)M_\infty.$$

Here the l.h.s.'s are considered the analytical continuation of the functions in the r.h.s.'s along the circles mentioned above. Moreover, in a general situation the monodromy matrices M_0 and M_∞ are factorized in products of the Stokes multipliers (see [11] for details). But in our case the monodromy properties of the function $\Psi(r,\ \mu)$ become trivial.

Proposition 3. *The function $\Psi = \Psi(r,\ \mu)$ (3.15), (3.18) is defined on $\mathbb{C} \setminus 0$, i.e. all its Stokes and monodromy matrices are trivial, $M_0 = M_\infty = I$. In the neighborhood of $\mu = 0$, the function Ψ has the following asymptotic expansion:*

$$\Psi \underset{\mu\to 0}{=} Q(r)^{-1}(I + O(\mu)) \exp\left(-\frac{i}{\mu}\sigma_3\right) Q, \tag{4.1}$$

where

$$Q(r) = \begin{pmatrix} \cos(\frac{\varphi(r)}{2}) & i\sin(\frac{\varphi(r)}{2}) \\ i\sin(\frac{\varphi(r)}{2}) & \cos(\frac{\varphi(r)}{2}) \end{pmatrix}, \tag{4.2}$$

and Q = Q(0) is defined in (3.13).

Proof. According the second equation in (3.15) the monodromy data are independent of r so that we can calculate them just by setting $r = 0$ in the first Eq. (3.15). Thus we get

$$\Psi(0,\ \mu) = Q^{-1} \exp\left(-\frac{i}{\mu}\sigma_3\right) Q. \tag{4.3}$$

Here, the right factor Q corresponds to the asymptotic expansion (3.18). All the Stokes multipliers and monodromy matrices for the function (4.3) are, of course, trivial.

Substituting the asymptotic expansion (4.1) in system (3.15) we get for $Q(r)$ the expression (4.2) up to a right diagonal matrix independent of r. Eq. (4.3) implies that this factor is equal to I. □

Remark. In [11] Q is called the connection matrix.

Now we are ready to formulate symmetry properties of the Amsler surface in terms of the function Ψ, which generates the corresponding immersion by the Sym-formula (3.14). Henceforth, we call it for brevity the Ψ-*parametrization* of a surface.

Corollary 5. *Let $\mathcal{F}$ be an Amsler surface in the parametrization of Proposition 2. Then the following Ψ-parametrizations describe $\mathcal{F}$:*

$$\begin{array}{lll} \Psi(r,\mu,\varphi(r,\varphi(0)-\pi)) & \text{between} & \mathcal{L}^x_+ \text{ and } \mathcal{L}^y_+, \\ (-i\sigma_1)\Psi(r,\mu,\varphi(r,\varphi(0)-\pi))i\sigma_1 & \text{between} & \mathcal{L}^x_- \text{ and } \mathcal{L}^y_-, \\ \Psi(r,\mu,\varphi(r,\varphi(0))) & \text{between} & \mathcal{L}^x_+ \text{ and } \mathcal{L}^y_-, \\ (-i\sigma_1)\Psi(r,\mu,\varphi(r,\varphi(0)))i\sigma_1 & \text{between} & \mathcal{L}^x_- \text{ and } \mathcal{L}^y_+. \end{array}$$

Here $r,\ \mu > 0$. The notation $\Psi(r,\ \mu,\ \varphi(r,\ \varphi(0)-\pi)))$ means that the function $\Psi(r,\ \mu)$ is defined by (3.15)–(3.18) with $\varphi(r)=\varphi(r,\ \varphi(0)-\pi)$. The Ψ-parametrization of the rays $\mathcal{L}^y_\pm$ is defined by Eq. (4.3) and of the rays $\mathcal{L}^x_\pm$ is given by $\Psi(r,\ \mu,\varphi(r)\equiv 0)=\exp\left(-ir^2\mu\sigma_3/16\right)$, in which (+)-ray corresponds to $\mu\ge 0$ and (-)-ray to $\mu\le 0$.

Proof. If a point $(ax,\ by)$ belongs to quadrant I of $\mathbb{R}^2$, then we write $r=i\hat{r},\ \mu=-\hat{\mu}$, where $\hat{r},\ \hat{\mu}>0$. Now using Propositions 1, 2, and 3 one finds: $\Psi(i\hat{r},\ -\hat{\mu},\ \varphi(i\hat{r},\ \varphi(0)))=\Psi(i\hat{r},\ -\hat{\mu},\ \pi+\varphi(\hat{r},\ \varphi(0)-\pi))=\Psi(\hat{r},\ \hat{\mu},\ \varphi(\hat{r},\ \varphi(0)-\pi))$. To justify these equations one proves that Ψ-functions solve the same system (3.15) and have the same asymptotics (3.18). When the point $(ax,\ by)$ is in quadrant II of $\mathbb{R}^2$, we put $r=i\hat{r}$, then $\hat{r},\ \mu>0$. Instead of previous calculation now we have: $\Psi(i\hat{r},\mu,\varphi(i\hat{r},\varphi(0)))=\Psi(i\hat{r},\ \mu,\ \pi+\varphi(\hat{r},\ \varphi(0)-\pi))=\sigma_1\Psi(\hat{r},\ \mu,\ \varphi(\hat{r},\ \varphi(0)-\pi))\sigma_1$.
In quadrant IV we can directly use Proposition 2. The proof for quadrant III is analogous to the one for quadrant II. □

As a result of Corollary 5 we see that to construct the parametrization F (3.14) for Amsler surface it is enough to use the function $\Psi(r,\ \mu)$ only for positive values of r and μ. Its asymptotic properties are summarized in the following theorem, the proof of which is given in Chapter 6 of [11].

Theorem 3. *If $\varphi(0):0<\varphi(0)<\pi$, then*

$$\varphi(r,\ \varphi(0)) \underset{r\to+\infty}{=} \frac{\alpha}{\sqrt{r}}\cos(\theta(r))+o\left(\frac{1}{\sqrt{r}}\right),\quad \theta(r)=r-\beta\ln r+\gamma, \tag{4.4}$$

where $\alpha>0$:

$$\beta=\frac{\alpha^2}{16}=-\frac{1}{\pi}\ln\cos\left(\frac{\varphi(0)}{2}\right),\quad \gamma=\frac{3\pi}{4}-\arg\Gamma\left(-i\beta\right)-2\beta\ln 2, \tag{4.5}$$

and $\Gamma(\cdot)$ *is the gamma function [2].*
Denote

$$4\xi = (\nu - 4)\sqrt{r}, \quad \nu = r\mu > 0. \tag{4.6}$$

For

$$r \to +\infty, \qquad |\xi| < O\left(r^{1/6}\right)$$

the following asymptotics holds:

$$\Psi(r,\ \mu,\ \varphi(r,\ \varphi(0))) = (1 + o(1)) \exp\left(-\frac{\pi\beta}{4} - \frac{i\theta}{2}\sigma_3\right) \mathbf{D}(\xi)\mathbf{E}, \tag{4.7}$$

$$\mathbf{E} = \exp\left(\frac{i\pi}{8}\sigma_3 - \frac{i}{2}\arg\Gamma(-i\beta)\sigma_3\right), \tag{4.8}$$

$$\mathbf{D}(\xi) = \begin{pmatrix} D_{-i\beta}\left(\xi e^{\frac{i\pi}{4}}\right) & \frac{\alpha}{4} e^{\frac{i\pi}{4}} D_{i\beta-1}\left(\xi e^{-\frac{i\pi}{4}}\right) \\ -\frac{4}{\alpha} e^{-\frac{i\pi}{4}} \nabla D_{-i\beta}\left(\xi e^{\frac{i\pi}{4}}\right) & -\nabla D_{i\beta-1}\left(\xi e^{-\frac{i\pi}{4}}\right) \end{pmatrix}. \tag{4.9}$$

Here $D_\bullet(\cdot)$ *is the parabolic cylinder function [2], and*

$$\nabla = e^{\frac{i\pi}{4}} \frac{\partial}{\partial \xi} - e^{-\frac{i\pi}{4}} \frac{\xi}{2}. \tag{4.10}$$

For

$$r \to +\infty, \qquad \xi \geq O\left(r^{\varepsilon}\right) > 0, \qquad \varepsilon > 0,$$

and also for $\nu \to +\infty$ *and bounded* r *the asymptotics of the* Ψ*-parametrization is given by*

$$\Psi(r,\ \mu,\ \varphi(r,\ \varphi(0))) = \left(T(r,\ \nu) + O\left(\frac{1}{\xi}\right)\right) \exp(-i\sigma_3 J(r,\ \nu)), \tag{4.11}$$

where

$$T(r,\ \nu) = \left(I - \frac{16}{\nu^2} Q^{-1}(r) \cos\frac{\varphi(r,\ \varphi(0))}{2}\right) \frac{\nu^2}{\nu^2 - 16},$$

$$J(r,\ \nu) = r\left(\frac{\nu}{16} + \frac{1}{\nu}\right) + \frac{\alpha^2}{16} \ln\frac{|\nu - 4|}{\nu + 4},$$

and $Q(r)$ *is defined in (4.2).*
For

$$r \to +\infty, \qquad \xi \leq O\left(r^{\varepsilon}\right) < 0, \qquad \varepsilon > 0,$$

and also for $\nu \to +0$ *and bounded* r *one has the following asymptotic expansion,*

$$\Psi(r,\ \mu,\ \varphi(r,\ \varphi(0))) = \left(\frac{T(r,\ \nu)}{\cos(\varphi(r,\ \varphi(0))/2)} + O\left(\frac{\nu}{\xi}\right)\right) \times \exp(-i\sigma_3 J(r,\ \nu)) Q,$$

where $Q = Q(0)$ *is defined in (3.13).*

Corollary 6. *Between any pair of the rays* $(\mathcal{L}^x_\pm,\ \mathcal{L}^y_\pm)$ *the Amsler surface* $\mathcal{F}(\lambda)$ *has infinite number of cusps,* $\mathcal{E}^{\pm\pm}_n$*, i.e., the curves on which one of the principle curvatures is collapsing. In the parametrization of Corollary 5:* $\mathcal{E}^{++}_n$ *and* $\mathcal{E}^{--}_n$ *are images of the curves* $-4abxy = (r^+_n)^2$, $n \in \mathbb{N}$*;* $\mathcal{E}^{+-}_n$ *and* $\mathcal{E}^{+-}_n$ *are images of* $-4abxy = (r^-_n)^2$, $n \in \mathbb{N}$*, where* $\{r^+_n\}_{n\in\mathbb{N}}$ *and* $\{r^-_n\}_{n\in\mathbb{N}}$ *are sequences of zeroes of the functions* $\varphi(r,\ \varphi(0)-\pi)$ *and* $\varphi(r,\ \varphi(0))$*, respectively. These sequences have the following asymptotics:*

$$r^\pm_n \underset{n\to\infty}{=} \frac{\pi}{2} + \pi n + \beta^\pm \ln(\pi n) - \gamma^\pm + o(1), \tag{4.12}$$

where $\beta^-,\ \gamma^-$ *are given by (4.5),* $\beta^+,\ \gamma^+$ *are given by the same equations where* $\varphi(0)$ *is replaced by* $\varphi(0)-\pi$*.*

Proof. The existence of the infinite sequence of zeroes of $\varphi(r,\ \cdot)$ and their distribution (4.12) follows from the asymptotics (4.4). The appearance of the edges is clear from the formulas for principle curvatures (2.10). □

Remark. Each edge $\mathcal{E}^{\pm\pm}_n$ is winding around both corresponding rays $\mathcal{L}^x_\pm$, $\mathcal{L}^y_\pm$ approaching them as x or $y \to 0$.

Asymptotic results of Theorem 3 combined with the Sym formula (2.8) motivate the following

Definition 1. *Let* $\mathcal{F}(\lambda)$ *be an Amsler surface parametrized as in Proposition 2. We call the cone* $\mathcal{A}F(\lambda)$ *with* Ψ*-parametrization given by*

$$\Psi_A = \exp\left(-\frac{\pi\beta}{4} - \frac{i\theta}{2}\sigma_3\right)\mathbf{D}(\xi)\mathbf{E} \tag{4.13}$$

the asymptotic cone of the surface $\mathcal{F}(\lambda)$*. Here* $\mathbf{D}(\xi)$ *and* $\mathbf{E}$ *are defined in Eqs.(4.9) and (4.8) for* $x > 0$ *and* $y < 0$*. For other signs of* x *and* y Ψ_A *is defined by the same formulas as the* Ψ*-parametrization for* $\mathcal{F}$ *in Corollary 5.*

Corollary 7. *The immersion* $F_A = F_A(x,\ y,\ \lambda)$ *of the asymptotic cone* $\mathcal{A}F$ *in the notations of Proposition 2 and Theorem 3 is given by*

$$F_A = \hat{\rho}\sqrt{r}\mathbf{E}^{-1}\mathbf{D}^{-1}(\xi)\left(-\xi i\sigma_3 - \frac{\alpha i}{2}\sigma_2\right)\mathbf{D}(\xi)\mathbf{E}, \tag{4.14}$$

Remark. Note that: 1) The asymptotic cone is independent of the parameter λ so that, we designate it also as $\mathcal{A}\mathcal{F}$; 2) The origin of $\mathcal{A}\mathcal{F}$ coincides with the intersection point $\mathcal{O}$ of the straight lines $\mathcal{L}^x$ and $\mathcal{L}^y$; 3) It is natural to consider asymptotic lines $\mathcal{L}^x$ and $\mathcal{L}^y$ as belonging to $\mathcal{A}\mathcal{F}$. In this case by considering the limits $\xi \to \pm\infty$ one finds that sections of $\mathcal{A}\mathcal{F}$ by spheres are compact.

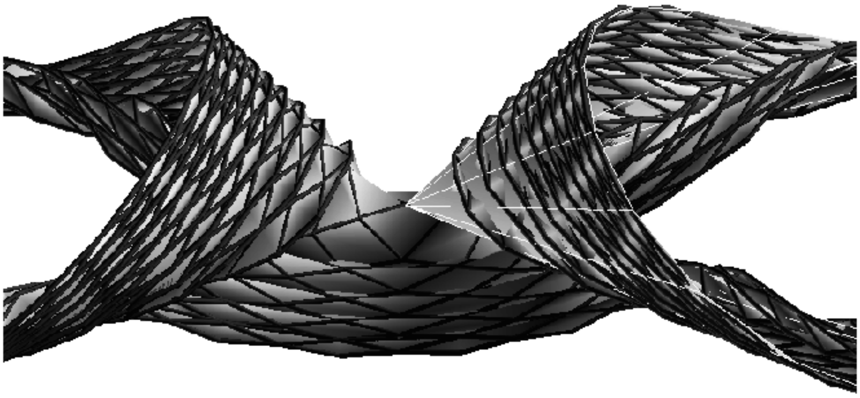

Fig. 1. An Amsler surface with its asymptotic cone

Definition 2. *Any cone defined as $\mathcal{AF}$ (see Definition 1) but with $|\xi| < \xi_0$ for some $\xi_0 > 0$ is called a proper sub-cone of $\mathcal{AF}$. It is designated further as $\mathcal{PAF}$.*

Theorem 4. *Any $\mathcal{PAF}$ is lying in a finite neighborhood of the the surface $\mathcal{F}(\lambda)$.*

Proof. The proof is given in Section 7. □

Remark. It means that distances from points of $\mathcal{PAF}$ to the surface $\mathcal{F}(\lambda)$ are bounded by a constant depending on ξ_0. Note that the assignment of Theorem 4 does not necessarily imply that $\mathcal{F}(\lambda)$ lies in a finite neighborhood of $\mathcal{PAF}$.

An example of the Amsler surface and its asymptotic cone is presented in Fig. 1. Actually, the figure depicts a *discrete* Amsler surface introduced and studied in [14]. Surfaces with constant negative Gaussian curvature have a natural discrete version – discrete K-surfaces [8]. The latter are the maps $F : \mathbb{Z}^2 \to \mathbb{R}^3$ with special geometric properties. Every image point together with all its nearest neighbors lie in a plane. In addition to this property the distance between neighboring image points in each of the two coordinate directions is constant. Discrete Amsler surfaces are defined (exactly as in the smooth case) as the discrete K-surfaces with two straight lines. Starting with this geometrical definition a discrete third Painlevé equation is derived in [13]. In addition to the assertion of Theorem 4, Fig. 1 demonstrates that an Amsler surface can be approximated by discrete Amsler surfaces.

5. Universality of the asymptotic cone. Constant mean curvature surfaces with internal symmetry

As mentioned in the introduction, the cone defined in Sec. 4 also describes asymptotic properties of some special surfaces with constant mean curvature (CMC). These surfaces are CMC immersions of $\mathbb{R}^2$ with internal isometry (actually, with the inner rotational symmetry, see below). There are several papers where these surfaces are studied [15,18,4,9]. In particular, in [18] it is proved that these immersions are proper. A more detailed description is obtained in [4], where the asymptotics of these surfaces at infinity are computed. Each surface of the family is asymptotic to a cone[2]. Our main goal in this section is to prove that the cone $\mathcal{A}F$ of Sec. 4 and the one introduced in [4] coincide.

Let $F : \mathbb{C} \to \mathbb{R}^3$ be a conformal parametrization of a topological plane $\mathcal{F}$ with the constant mean curvature $H = 1$ and $N : \mathbb{C} \to S^2$ its Gauss map. The Hopf differential is holomorphic and the conformal metric $e^u = 2\langle F_z, F_{\bar z}\rangle$ satisfies the Gauss equation

$$u_{z\bar{z}} + \frac{1}{2}e^u - 2|Q|^2 e^{-u} = 0. \tag{5.1}$$

The set of umbilic points is discrete. Assume that $\mathcal{F}$ has a continuous group of internal isometries (i.e., the conformal factor e^u is invariant with respect to the action of a vector field on $\mathbb{C}$) and at least one unbilic point P_0. One can explicitly describe all surfaces of this class.

Actually, in a neighborhood $U \ni P_0$ one can introduce a conformal coordinate $z : U \to V \in \mathbb{C}$ such that $z(P_0) = 0$ and $Q\,dz^2 = \frac{1}{2}z^m dz^2$, where $m \geq 1$ is the order of the umbilic point P_0. The level sets of $|Q|$ are preserved by internal isometries, therefore, in the chosen parametrization of U, the conformal factor e^u depends on $|z|$ only. The Gauss equation (5.1) on U becomes an ordinary differential equation, which by the transformation

$$e^v := e^u \left(\frac{4}{(m+2)\rho}\right)^{\frac{2m}{m+2}}, \quad \rho := \frac{4}{m+2}|z|^{1+\frac{m}{2}}$$

is reduced to the following form of the third Painlevé equation:

$$v_{\rho\rho} + \frac{1}{\rho}v_\rho + \sinh v = 0. \tag{5.2}$$

[2] Note that using the results of [11] only the leading term of the asymptotics of the Smith surfaces is computed in [4]. In particular, it is not yet proven that the asymptotic cone lies in a finite neighborhood of the corresponding surface. To prove this statement one should use technically more advanced results similar to those of Section 7.

The solution of Eq. (5.2) which corresponds to the surface U has the following behavior as $\rho \to 0$,

$$v(\rho) = -\frac{2m}{m+2}\ln\rho + \frac{2m}{m+2}\ln\frac{4}{m+2} + u(0) + o(\rho).$$

It is uniquely determined by the value $u(0)$ of the metric at the central point $z = 0$. According to [18] this solution can be smoothly continued for all $\rho \in (0, \infty)$. The latter solution of Eq. (5.2) and $Q = \frac{1}{2}z^m$ defines the CMC immersion $F_\Pi : \mathbb{C} \to \mathbb{R}^3$ on the whole complex plane. We denote the latter CMC surface by $\Pi_m(u(0))$. The surface $\Pi_m(u(0))$ has a common part U with the original surface $\mathcal{F}$ we started with and, due to the uniqueness (CMC surfaces are real analytic) of CMC surfaces, coincides with $\mathcal{F}$. The previous discussion can be summarized in the following

Proposition 4. *Up to Euclidian motion there exists one and only one family of the CMC planes $\Pi_m(u(0))$ with continuous internal isometries and at least one umbilic point of order $m \geq 1$. This family is parametrized by the value $u(0)$ of the metric*[3] *at the central point $z = 0$.*

Theorem 5. *(see [18,4]). The surface $\Pi_m(u(0))$ possesses $m + 2$ symmetry planes which intersect along the axis l passing through the central point $z = 0$. This central point is umbilic*[4] *of order m.*
The $\frac{1}{2(m+2)}$-part of the surface $\Pi_m(u(0))$ lying between consequent symmetry planes is asymptotically (see [4] for details) described by the cone $\mathcal{A}\Pi_m(u(0))$ given by the following

Definition 3. *The cone $\mathcal{A}\Pi_m(u(0))$ given by the parametrization*

$$(r, \xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \Longrightarrow F_{\mathcal{A}\Pi} = \frac{i}{2} r \tilde{P}^{-1} \begin{pmatrix} 0 & \alpha_\Pi \\ \frac{4\xi^2}{\alpha_\Pi} + \alpha_\Pi & 0 \end{pmatrix} \tilde{P}, \tag{5.3}$$

is called the asymptotic cone of the CMC surface $\Pi_m(u(0))$. The matrix $\tilde{P}$ is defined as follows:

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ -\frac{2\xi}{\alpha_\Pi} & 1 \end{pmatrix} \begin{pmatrix} D^-_{-s_\Pi} & -\frac{\alpha_\Pi}{4} e^{\frac{i\pi}{4}} D^+_{s_\Pi - 1} \\ -\frac{4}{\alpha_\Pi} 4 e^{\frac{i\pi}{4}} D'^+_{-s_\Pi} & -D'^+_{s_\Pi - 1} \end{pmatrix}$$

[3] The coordinate is normalized by the Hopf differential.
[4] in the case $m = 0$ we obtaine a surface with the intrinsic rotational isometry without umbilics. The central point is a fixed point of the isometry.

with

$$\alpha_\Pi = \sqrt{-\frac{16}{\pi}\ln\frac{\sin\frac{\pi}{m+2}}{\cosh\sqrt{\beta_\Pi}}} > 0, \quad s_\Pi = -\frac{i\alpha_\Pi^2}{16},$$
$$p = \tanh\sqrt{\beta_\Pi}\sin\frac{\pi}{m+2} + i\cos\frac{\pi}{m+2},$$
$$\sqrt{\beta}_\Pi = \frac{2}{\pi}(4m+8)^{\frac{m}{m+2}}e^{-\frac{u(0)}{2}}\sin\frac{\pi}{m+2}\Gamma^2\left(\frac{m+1}{m+2}\right),$$

where $D^{\pm}_{(\cdot)}$ *and* $D'^{\pm}_{(\cdot)}$ *are the parabolic cylinder function and its derivatives by the argument whose values are taken at the points* $\xi e^{\pm\frac{i\pi}{4}}$*, i.e.,* $D^{\pm}_{(\cdot)} = D_{(\cdot)}\left(e^{\pm\frac{i\pi}{4}}\right)$ *and* $D'^{\pm}_{(\cdot)} = \frac{d}{dr}D_{(\cdot)}(r)\Big|_{r=e^{\pm\frac{i\pi}{4}}}$*, respectively.*

Remark. The cone (5.3) differs from the one defined in [4] by non-essential rotations.

Proposition 5. *Under the condition*

$$\cos\frac{\phi(0)}{2} = \frac{\sin\frac{\pi}{m+2}}{\cosh\sqrt{\beta_\Pi}} \tag{5.4}$$

the cone $\mathcal{A}F$ *coincides, up to the rotation* $\sigma_1 e^{-\sigma_3\left(\frac{i\pi}{8}+\frac{i}{2}\arg\Gamma(s_\Pi)\right)}$*, with the cone* $\mathcal{A}\Pi_m(u(0))$.

Proof. By using the following identities for the parabolic cylinder functions (see [2])

$$D'^{-}_{-s_\Pi} = -s_\Pi D^{-}_{-s_\Pi-1} + \frac{\xi}{2}e^{-\frac{i\pi}{4}}D^{+}_{-s_\Pi}, \tag{5.5}$$

$$D'^{+}_{s_\Pi-1} = -D^{+}s_\Pi + \frac{\xi}{2}e^{\frac{i\pi}{4}}D^{+}_{s_\Pi-1}, \tag{5.6}$$

one finds that

$$\tilde{P} = e^{\frac{i\pi}{4}\sigma_3}\sigma_1\mathbf{D}_\Pi\sigma_1 e^{-\frac{i\pi}{4}\sigma_3}, \tag{5.7}$$

where

$$\mathbf{D}_\Pi = \begin{pmatrix} D^{+}_{s_\Pi} & \frac{\alpha}{4}e^{\frac{i\pi}{4}}D^{-}_{-s_\Pi-1} \\ -\frac{\alpha}{4}e^{-\frac{i\pi}{4}}D^{+}_{s_\Pi-1} & D^{-}_{-s_\Pi} \end{pmatrix}. \tag{5.8}$$

The parametrization of the immersion (5.3) by means of Eq. (5.7) can be rewritten as follows:

$$F_{\mathcal{A}\Pi} = \sqrt{\rho}e^{\frac{i\pi}{4}\sigma_3}\sigma_1\mathbf{D}_\Pi^{-1}\left(-i\xi\sigma_3 - \frac{\alpha_\Pi i}{2}\sigma_3\right)\mathbf{D}_\Pi\sigma_1 e^{-\frac{i\pi}{4}\sigma_3}. \tag{5.9}$$

On the other hand, by applying the identities (see [2])

$$\nabla D^{+}_{-i\beta} \equiv \beta D^{+}_{-i\beta-1} \quad \text{and} \quad \nabla D^{-}_{i\beta-1} \equiv -D^{-}_{i\beta},$$

where the operator ∇ is defined in Eq. (4.10) in Eq. (4.9) one arrives at

$$\mathbf{D}(\xi) = \mathbf{D}_{\Pi}$$

provided that $-i\beta = s_{\Pi}$. The latter equation implies (5.4). Comparing Eqs. (4.14) and (5.3) one proves that $\mathcal{A}\Pi_m(u(0))$ is a rotation of $\mathcal{A}F$ defined by the matrix

$$e^{-\frac{i\pi}{8}\sigma_3+\frac{i}{2}\sigma_3 \arg\Gamma(s_{\Pi})}\sigma_1 e^{-\frac{i\pi}{4}\sigma_3} = \sigma_1 e^{-\sigma_3\left(\frac{i\pi}{8}+\frac{i}{2}\arg\Gamma(s_{\Pi})\right)}. \qquad \square$$

6. Geometry of the asymptotic cone

The analytic description of the asymptotic cone $\mathcal{A}F$ given in Sec. 5 is not quite satisfactory since the formulas are rather complicated and do not reveal much about the geometry of the cone. To describe any cone it is enough to fix a curve lying on it. In this section we consider two different curves generating $\mathcal{A}F$. Both these curves have interesting geometric properties.

Consider the curve,

$$\gamma(\xi) := \mathbf{D}^{-1}\left(-i\xi\sigma_3 - \frac{\alpha i}{2}\sigma_2\right)\mathbf{D}, \tag{6.1}$$

where $\mathbf{D}$ is given in (5.8). Note that as shown in Section 4 this curve is visible on the Amsler surface. After an appropriate scaling the cusp curves $\mathcal{E}_n$ for $n \to \infty$ are close to $\gamma(\xi)$.

One can check that

$$\frac{d\mathbf{D}}{d\xi}\mathbf{D}^{-1} = \frac{1}{2}\left(-i\xi\sigma_3 - \frac{\alpha i}{2}\sigma_2\right). \tag{6.2}$$

Proposition 6. *The generating curve (6.1) is arclength parametrized and governed by the equation*

$$\gamma'' = \gamma' \times \gamma, \tag{6.3}$$

where the primes denote the derivative with respect to ξ. Its curvature and torsion are equal to

$$\kappa = \frac{\alpha}{2}, \quad \tau = \xi.$$

Proof. Using (6.2) one finds that

$$\gamma^{'} = \mathbf{D}^{-1}(-i\sigma_3)\mathbf{D}, \quad \gamma^{''} = \frac{\alpha}{2}\mathbf{D}^{-1}(i\sigma_1)\mathbf{D}.$$

Thus, $\gamma^{'}$ and $\frac{2}{\alpha}\gamma^{''}$ build a Frenet frame of the curve. The expressions for κ and τ follow directly.

Recall the well-known (see [13]) smoke-ring evolution, $\tilde{\gamma}(t,\xi)$, of a curve $\tilde{\gamma}(\xi) = \tilde{\gamma}(0,\xi)$, which is the evolution of the curve $\tilde{\gamma}(\xi)$ in the binormal direction with the velocity equal to the curvature. The function $\tilde{\gamma}(t,\xi)$ solves the following *smoke-ring propagation equation*,

$$\dot{\tilde{\gamma}} \equiv \frac{d\tilde{\gamma}}{dt} = \tilde{\gamma}^{'} \times \tilde{\gamma}^{''}. \tag{6.4}$$

Proposition 7. *Under the smoke-ring flow (6.4) the generating curve $\gamma(\xi)$ evolves by a homothety:*

$$\dot{\gamma} = -\gamma + \gamma^{'}\left\langle \gamma, \gamma^{'} \right\rangle. \tag{6.5}$$

Proof. The proof follows immediately from Eqs. (6.3) and (6.4). The first term in the r.-h.s. of Eq. (6.5) corresponds a homothety, whereas the second one is the tangential vector field which preserves the arclength parametrization.

Let us now investigate geometry of another curve describing the cone $\mathcal{A}F$,

$$\Upsilon(\xi) := \frac{\gamma(\xi)}{|\gamma(\xi)|} = \frac{\mathbf{D}^{-1}\left(-i\xi\sigma_3 - \frac{\alpha i}{2}\sigma_2\right)\mathbf{D}}{\sqrt{\xi^2 + \frac{\alpha^2}{4}}},$$

which is the intersection of $\mathcal{A}F$ with the unit sphere. □

Proposition 8. *The curve Υ lies on the unit sphere and has the total length π. Its geodesic curvature, as the function of the arclength parameter s, is as follows,*

$$\kappa(s) = \frac{\alpha^2}{4\cos^3 s}, \quad s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{6.6}$$

Proof. Computing $\Upsilon^{'}(\xi)$ we obtain for the arclength parametrization, s, and the unit tangent vector, T, the following expressions:

$$\frac{ds}{d\xi} = \frac{2\alpha}{4\xi^2 + \alpha^2}, \quad T := \frac{\Upsilon^{'}}{|\Upsilon^{'}|} = \frac{\mathbf{D}^{-1}\left(i\xi\sigma_2 - \frac{\alpha i}{2}\sigma_3\right)\mathbf{D}}{\sqrt{\xi^2 + \frac{\alpha^2}{4}}}.$$

The first equation can be solved,

$$\xi = \frac{\alpha}{2}\tan s, \quad s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

To compute the curvature note that the third vector of the orthogonal frame (Υ, T, N) is given by the equation

$$N = T \times \Upsilon = \mathbf{D}^{-1}(i\sigma_3)\mathbf{D}.$$

For the curvature $\kappa = -\left\langle T, \frac{dN}{ds}\right\rangle = -\left\langle T, \frac{dN}{d\xi}\right\rangle \frac{d\xi}{ds}$ this implies formula (6.6).
□

7. Proof of Theorem 4

By substituting ξ for μ in the first Eq. (3.15), where ξ is as introduced in (4.6), one arrives at the following equation

$$\frac{\partial}{\partial \xi}\Psi = \mathcal{A}(\xi, r)\Psi, \tag{7.1}$$

where the matrix $\mathcal{A}(\xi, r)$ can be developed in the Taylor series

$$\mathcal{A}(\xi, r) = \sum_{l=0}^{\infty} \frac{P_l(r)}{4} \frac{\xi^l}{\left(\sqrt{r}\right)^{l-1}}. \tag{7.2}$$

The coefficients $P_l \equiv P_l(r)$ are given by the equations:

$$P_0 = -i\sigma_1\varphi^{'}(r) - i\sigma_2 \sin(\varphi(r)) - 2i\sigma_3 \sin^2\left(\frac{\varphi(r)}{2}\right), \tag{7.3}$$

$$P_l = (-)^l \left((l+1)P - i\varphi^{'}(r)\sigma_1\right), \ P = i\sigma_3 \cos(\varphi(r)) - ii\sigma_2 \sin(\varphi(r)). \tag{7.4}$$

According to the results stated in Theorem 3 one has the following estimations as $r \to \infty$:

$$P_0 = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), \ P_l = (-)^l(l+1)\left(i\sigma_3 + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right)\right). \tag{7.5}$$

It is clear that these estimations are uniform on l and the radius of convergence for the series (7.2) is equal to $\sqrt{r}$.

Define the function $\Psi_0 \equiv \Psi_0(\xi, r)$ as a fundamental solution of the following ODE

$$\frac{\partial}{\partial \xi}\Psi_0 = (B_0 + B_1\xi)\,\Psi_0, \qquad B_0 = \frac{P_0\sqrt{r}}{4}, \ B_1 = \frac{P_1}{4}. \tag{7.6}$$

Proposition 9. *The following series*

$$\Psi = \sum_{k=0}^{\infty} Q_k(r)\xi^k\Psi_0, \tag{7.7}$$

where

$$Q_0 = I, \quad Q_1 = Q_2 = 0, \quad Q_3 = \frac{P_2}{4\sqrt{r}}, \tag{7.8}$$

and for $k \geq 4$

$$kQ_k = \left[B_1,\ Q_{k-2}\right] + \left[B_0,\ Q_{k-1}\right] + \frac{1}{4}\sum_{l=2}^{k-1} \frac{P_l Q_{k-l-1}}{\left(\sqrt{r}\right)^{l-1}} \tag{7.9}$$

is a formal fundamental solution of the equation (7.1).

Proof. Just a substitution of series (7.7) into Eq. (7.1) in which $\mathcal{A}(\xi,\ r)$ is defined in (7.2). □

Some of the following results are formulated in a slightly more general form than we actually need for the proof of Theorem 4.

Lemma 5. *Given a sequence of numbers* $f_l,\ (l \geq 0)$ *such that*

$$\|P_0(r)\sqrt{r}\| \underset{r\to\infty}{\leq} f_0,\ \ \|P_l(r)\| \underset{r\to\infty}{\leq} f_l,\ \ f_l \underset{l\to\infty}{=} o\left((1+\epsilon)^l\right),$$

the coefficients $Q_k = Q_k(r),\ (k \geq 1)$ *can be written as*

$$Q_k = \sum_{l=2}^{l=k-1} Q_k^l(r)\left(\frac{1+\epsilon}{\sqrt{r}}\right)^{l-1}, \tag{7.10}$$

where $\epsilon > 0$ *is arbitrary and for the functions* $Q_k^l(r)$ *the following estimations as* $r \to \infty$

$$\|Q_{2n}^l(r)\| < \frac{K(\epsilon)b^{2n-l-1}}{(2n-l-1)!!2n}, \quad \|Q_{2n+1}^l(r)\| < \frac{K(\epsilon)b^{2n-l}}{(2n-l)!!(2n+1)}, \tag{7.11}$$

are valid uniformly on l *and* k. *Hereafter* $\|\cdot\|$ *denotes arbitrarily fixed matrix norm. The function* $K(\epsilon) > 1$ *is independent of* r. *The parameter* b *is defined as follows*

$$b = \max\{3,\ \|B_0\|,\ \|B_1\|\},$$

which means that b *is actually a function of the coefficients of Eq.(7.1).*

Proof. Let us put $Q_0^1 = Q_0 = I$ and $Q_k^l = 0$ for all pairs $(l,\ k) \in Z^2$ which belong to the following set $\{(l \leq 0)\&(\forall k)\} \cup \{(\forall l)\&(k < 0)\} \cup \{l \geq k \geq 1\} \cup \{(l = 1)\&(k \neq 0)\} \cup \{(l \neq 1)\&(k = 0)\}$. Substituting Eq. (7.10) into Eq. (7.9) and using initial conditions (7.8) we get the following equation for $Q_k^l(r)$

$$kQ_k^l = \left[B_0,\ Q_{k-1}^l\right] + \left[B_1,\ Q_{k-2}^l\right] + \frac{1}{4}\sum_{m=2}^{k-1} \frac{P_m Q_{k-1-m}^{l+1-m}}{(1+\epsilon)^{m-1}},$$

Now the statement of the lemma can be obtained by mathematical induction. □

Remark. In our case, Eq. (7.5) shows that we can choose the numbers $f_l = (l+1)\|\sigma_3\|$ for $l \geq 1$. Thus, the function $K(\epsilon)$ is universal for all functions $\varphi(r)$ such that $\varphi(r),\ \varphi^{'}(r) \underset{r\to\infty}{=} O\left(\frac{1}{\sqrt{r}}\right)$.

Proposition 10.

$$\|Q_k^l(r)\| < \frac{\sqrt{\pi}K(\epsilon)}{bk\Gamma\left(\frac{k-l+1}{2}\right)}\left(\frac{b}{\sqrt{2}}\right)^{k-l}, \tag{7.12}$$

where $\Gamma(\cdot)$ *is the standard gamma function [2].*

Proof. Rewrite $k!!$ in Eqs. (7.11) in terms of Γ-function and apply the double argument formula [2]. □

Proposition 11.

$$\|Q_k(r)\| \underset{r\to\infty}{<} \left(\frac{k-3}{2} + \frac{\sqrt{\pi}K(\epsilon)}{\sqrt{2}k}\right)\left(\frac{1+\epsilon}{\sqrt{r}}\right)^{k-2} + y_k\left(\frac{b}{\sqrt{2}}\right)^{k-2}, \tag{7.13}$$

$$y_k = \sum_{l=2}^{l=k-2}\frac{k-1-l}{k-2}\left(\frac{\sqrt{\frac{\pi}{2}}K(\epsilon)}{k\Gamma\left(\frac{k-l+1}{2}\right)}\right)^{\frac{k-2}{k-1-l}}. \tag{7.14}$$

Proof. Apply Young's inequality, $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$, where $A > 0,\ B > 0,\ p > 1,\ q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, to the first $k-2$ entries of the sum (7.10) and use estimation (7.12). □

Proposition 12. *Let* $x(k) > 0$ *be a point of the absolute minimum of the function* $g(x,\ k) = (k\Gamma(x+1))^{\frac{k-2}{2x}}$ *on the positive semi-axis, i.e.* $g(x(k),\ k) = \min\limits_{x>0} g(x,\ k)$*, then* $\lim\limits_{k\to\infty} x(k) = +\infty$.

Proof. The minimum value exists since as $x \to +0$ and as $x \to +\infty$ $\frac{k-2}{2x}\lim(\ln k + \ln\Gamma(x+1)) = +\infty$. The critical points of $g(x,\ k)$ are solutions of the equation

$$\frac{\Gamma^{'}(x+1)}{\Gamma(x+1)\left(\ln k + \ln\Gamma(x+1)\right)} = \frac{1}{x}.$$

If $x_0(k)$ is the point of the first local minimum, i.e. the smallest solution of this equation, then $x_0(k_2) > x_0(k_1)$ for $k_2 > k_1$ and $x_0 \to +\infty$. The assertion of the lemma ensue from the inequality $x(k) > x_0(k)$. □

Proposition 13.

$$\lim_{k\to\infty} \sqrt[k]{y_k} = 0. \tag{7.15}$$

Proof. As it is resulted from Eq. (7.14), one can estimate y_k as follows

$$y_k \le (k-4)\left(\sqrt{\frac{\pi}{2}}K(\epsilon)\right)^{k-2} \max\left\{\frac{1}{g(x(k),\ k)},\ \frac{1}{g(\frac{k-3}{2},\ k)},\ \frac{1}{g(\frac{1}{2},\ k)}\right\}.$$

Now using for the Γ-function asymptotics as $x \to \infty$ (see [2]), one finds

$$g(x(k),\ k)^{\frac{1}{k}} > \left(\frac{x(k)+1}{e}\right)^{1/3},\ g\left(\frac{k-3}{2},\ k\right)^{\frac{1}{k}} > \left(\frac{k-1}{2e}\right)^{1/3}.$$

According Proposition 12 $x(k) \to \infty$ as $k \to \infty$ and $g\left(\frac{1}{2},\ k\right)^{\frac{1}{k}} > \left(\sqrt{\frac{\pi}{2}}\,k\right)^{1/3}$ and thus Eq. (7.15) holds. □

Lemma 6. *The radius of convergence of the series*

$$Q(\xi,\ r) = \sum_{k=1}^{\infty} Q_k(r)\xi^k \tag{7.16}$$

equals $\sqrt{r}$. That means that the construction of Proposition 9 yields the classical fundamental solution of equation (7.1) in the domain $|\xi| < \sqrt{r}$.

Proof. According Proposition 11 the absolute value of partial sums of the series $\sum_{k=1}^{\infty} \|Q_k\|\xi^k$ can be estimated as the sum of two partial sums for series, where the first one has the convergence radius equal to $\sqrt{r}/(1+\epsilon)$ for arbitrary $\epsilon > 0$. The result stated in Proposition 13 means that the radius of convergence of the second series is equal to ∞. □

Lemma 7. *The function $\varphi(r,\ \varphi(0))$ admits the following asymptotic expansion*

$$\varphi(r,\ \varphi(0)) \underset{r\to\infty}{=} \sum_{k=1}^{\infty} \frac{T_k(\theta(r))}{r^{\frac{k}{2}}}, \tag{7.17}$$

where $T_k(\theta(r))$ are trigonometric polynomials of $\theta(r)$ of power k.

Proof. One proves asymptotic expansion (7.17) by changing the order of summation of the convergent double series obtained as a representation of the solution of the third Painlevé equation in [17]. Note that after this resummation one gets *asymptotic* (divergent) series. □

Remark. The first term of the sum (7.17) is given in the expansion (4.4).

Lemma 8. *The general solution of system (3.15) can be presented in the following form*

$$\Psi(\mu,\ r) = Q(\xi,\ r)\Psi_0(\xi,\ r)\Psi_0^{-1}(0,\ r)I(r)C, \tag{7.18}$$

where ξ *is defined as in (4.6),* $Q(\xi,\ r)$ *as defined in Eq. (7.16),* $\Psi_0(\xi,\ r)$ *is a fundamental solution of Eq. (7.6),* $C \in Mat(2,\ \mathbb{C})$ *is an arbitrary matrix whose entries are independent of* ξ *and* r*, and I(r) is the fundamental solution of the ODE*

$$\frac{d}{dr}I(r) = -\frac{i}{4}\{(1+\cos(\varphi(r)))\sigma_3 - \sin(\varphi(r))\sigma_2 + \varphi_r(r)\sigma_1\}\, I(r), \tag{7.19}$$

with the following asymptotic expansion

$$I(r) \underset{r\to\infty}{=} \left(I + \sum_{k=1}^{\infty} \frac{\psi_k(\theta(r))}{r^{k/2}}\right) \exp\left(-\frac{i\theta(r)}{2}\right), \tag{7.20}$$

where $\theta(r)$ *is defined as in (4.4),and* $\psi_k(\theta)$ *are periodic (bounded as* $r \to \infty$*) functions of* θ*.*

Proof. The general solution of the system (3.15) can be written as follows

$$\Psi(\mu,\ r) = Q(\xi,\ r)\Psi_0(\xi,\ r)C(r), \tag{7.21}$$

where $C(r)$ is a matrix independent of ξ which is determined by the condition that the function $\Psi(\mu,\ r)$ defined in Eq. (7.21) solves the second equation of system (3.15). Since system (3.15) is compatible for all ξ and the matrix $C(r)$ is independent of ξ, it is entirely determined by the equation (7.19) which, in fact, is the equation for the function $\Psi(4/r,\ r)$ and deduced from system (3.15). To get formula (7.18) one should notice that $\xi = 0$ means that $\mu = 4/r$ and $Q(0,\ r) = I$.
To prove the asymptotic expansion (7.20), we make the following transformation

$$I(r) = \exp\left(i\frac{\theta(r)}{2}\sigma_3\right) J(r). \tag{7.22}$$

The function $J(r)$ is a fundamental solution of the ODE

$$\frac{d}{dr}J(r) = W(r)J(r), \tag{7.23}$$

where

$$W(r) = -\frac{i}{4}(1 + 2\theta_r(r) + \cos(\varphi(r)))\sigma_3 - \\ -\frac{i}{4}\begin{pmatrix} 0 & e^{-i\theta(r)}(\varphi_r(r) + i\sin(\varphi(r))) \\ e^{i\theta(r)}(\varphi_r(r) - i\sin(\varphi(r))) & 0 \end{pmatrix},$$

Using the definition of $\theta(r)$ (see (4.4)) and the expansion for $\varphi(r)$ established in lemma 8 one finds that

$$W(r) \underset{r\to\infty}{=} -i\left(1-\frac{3\alpha^2}{32r}+\frac{\alpha^2}{16r}\cos(2\theta(r))\right)\sigma_3 + \frac{\alpha i}{4\sqrt{r}}\sigma_2 + \sum_{k=3}^{\infty}\frac{w_k(\theta(r))}{r^{\frac{k}{2}}}, \tag{7.24}$$

where $w_k(\theta(r))$ are trigonometric polynomials of $\theta(r)$. As follows from Eq. (7.24) the function J(r) has an irregular singular point at $r = \infty$ and therefore its asymptotic expansion is as follows

$$J(r) \underset{r\to\infty}{=} \left(I + \sum_{k=1}^{\infty}\frac{j_k(\theta(r))}{r^{k/2}}\right)\exp\left(-i\theta(r)\right), \tag{7.25}$$

where $j_k(\theta(r))$ are the periodic functions of $\theta(r)$. The proof of this expansion can be done in the way analogous to the one for ODE with the rational coefficients [12]. □

Remark. To prove theorem 4 we actually only need the first two terms of the expansions (7.17) and (7.20). One can find that

$$T_2(\theta(r)) = 0, \qquad j_1(\theta(r)) = -\frac{\alpha i}{8}\sigma_1. \tag{7.26}$$

Proposition 14. *The quaternionic parametrization of the Amsler surface has the following representation*

$$\begin{aligned}&\Psi(r,\ \mu,\ \varphi(r,\ \varphi(0)))\\ &= Q(\xi,\ r)\Psi_0(\xi,\ r)\Psi_0^{-1}(0,\ r)e^{i\frac{\theta(r)}{2}\sigma_3}J(r)e^{-\frac{\pi\beta}{4}}\mathbf{D}(0)\mathbf{E},\end{aligned} \tag{7.27}$$

where the matrices $\mathbf{D}(\cdot)$ *and* $\mathbf{E}$*, the functions* $\varphi(r,\ \varphi(0))$ *and* $\theta(r)$*, the variable* ξ*, and the parameter* β *are defined in Theorem 3. The function* $Q(\xi,\ r)$ *is constructed in Lemma 6,* $J(r)$ *is the fundamental solution of Eq. (7.23) with asymptotics (7.25), and* $\Psi(\cdot,\ r)$ *is a fundamental solution of Eq. (7.6).*

Proof. Set the right-hand sides of Eqs. (7.18) and (4.7) equal, then put $\xi = 0$ and take the limit $r \to \infty$. □

Proposition 15. *Fundamental solutions of Eq. (7.6) can be written as follows*

$$\Psi_0(\xi,\ r) = K(r)\left(\Psi_A(\xi,\ r) + \frac{1}{\sqrt{r}}\Xi(\xi,\ r)\right)N(r), \tag{7.28}$$

where $\Psi_A(\xi,\ r)$ *is the* Ψ*-parametrization for the asymptotic cone* $\mathcal{A}F$ *(see Eq. (4.13)),* $\Xi(\xi,\ r)$ *is an entire function of* ξ*,* $K(r)$ *and* $N(r)$ *are the functions independent of* ξ*,*

$$K(r) \underset{r\to\infty}{=} I + \mathcal{O}\left(1/\sqrt{r}\right),\ \ \Xi \underset{r\to\infty}{=} \mathcal{O}(1),\ \ \textit{and}\ \ \frac{\partial}{\partial\xi}\Xi \underset{r\to\infty}{=} \mathcal{O}(1). \tag{7.29}$$

Proof. Equation (7.6) can be solved explicitly in terms of the parabolic cylinder functions [2], namely,

$$\Psi_0(\xi,\ r) = \widetilde{K}(r)\exp\left(-\frac{i\theta(r)}{2}\sigma_3\right)\widetilde{\mathbf{D}}(\widetilde{\xi},\ r)N(r). \tag{7.30}$$

Here the function $\widetilde{K}(r)$ brings the matrix B_1 into the diagonal form

$$\begin{aligned}\widetilde{K}^{-1}(r)B_1\widetilde{K}(r) &= b_1(r)\sigma_3,\ b_1(r)\\ &= -i\sqrt{\det B_1},\ \widetilde{K}(r) = \frac{(B_1\sigma_3 + b_1(r)I)}{2b_1(r)}.\end{aligned} \tag{7.31}$$

Using Eq. (7.4) for $l = 1$ and asymptotics for $\varphi(r)$ obtained in Theorem 3 and Lemma 7 one proves that

$$b_1(r) \underset{r\to\infty}{=} -\frac{i}{2} - \frac{i\alpha^2}{16r}\sin^2(\theta(r)) + \mathcal{O}\left(\frac{1}{r^{3/2}}\right)$$

and $\widetilde{K}(r)$ has the same asymptotic expansion as $K(r)$ in Eq. (7.29).

The function $\widetilde{\mathbf{D}}(\widetilde{\xi},\ r)$ is given by the same equation as $\mathbf{D}(\xi)$ (see Eq. (4.9)) but with the functions:

$$\begin{aligned}\alpha(r) &= \frac{4iexp(-i\theta(r))}{\sqrt{2ib_1(r)}}\left(\widetilde{K}^{-1}(r)B_1\widetilde{K}(r)\right)_{21},\quad \beta(r) = \frac{\alpha^2(r)}{16},\\ \textit{and}\ \ \widetilde{\xi} &= \sqrt{2ib_1(r)}\xi + \frac{2i}{\sqrt{2ib_1(r)}}\left(\widetilde{K}^{-1}(r)B_1\widetilde{K}(r)\right)_{11}\end{aligned}$$

substituted for the variables $\alpha,\ \beta$, and ξ respectively. The asymptotic properties of $\varphi(r)$ yield the following estimations as $r \to \infty$:

$$\begin{aligned}\alpha(r) = \alpha + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), &\quad \beta(r) = \beta + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right),\\ \widetilde{\xi} = \xi\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right) &+ \mathcal{O}\left(\frac{1}{\sqrt{r}}\right).\end{aligned}$$

Now, one proves representation (7.28), (7.29) using the analytical properties of the parabolic cylinder functions or directly their integral representations [2].

To finish the proof of Theorem 4 let us rewrite the Sym formula (3.14) in terms of the variable ξ,

$$F(x,\ y,\ \lambda) = 2\hat{\rho}\sqrt{r}\left(1 + \frac{\xi}{\sqrt{r}}\right)\Psi^{-1}\mathcal{A}(\xi,\ r)\Psi, \tag{7.32}$$

according to the Eqs. (7.2)–(7.4), (4.4), and (7.17)

$$\mathcal{A}(\xi,\ r) = -\frac{i}{2}\xi\sigma_3 - \frac{\alpha i}{4}e^{-\frac{i}{2}\theta(r)\sigma_3}\sigma_2 e^{\frac{i}{2}\theta(r)\sigma_3} + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right). \qquad (7.33)$$

Now Eqs. (7.32) and (7.33) together with Eqs. (7.27)–(7.29), (7.25) and the results stated in Lemmas (5), (6) yield the following estimation

$$F(x,\ y,\ \lambda) - F_A(x,\ y,\ \lambda) \underset{r\to\infty,\ |\xi|\le\xi_0}{=} \mathcal{O}(1), \qquad (7.34)$$

where $F_A(x,\ y,\ \lambda) = F_A(\xi,\ r)$ is the parametrization of the $\mathcal{AF}(\lambda)$ (see (4.14)) in which ξ and r for the given $x,\ y,\ \lambda$ are calculated via formulae (3.14). □

Estimation (7.34) proves Theorem 4 since the distance between a point on $\mathcal{PAF}(\lambda)$ and Amsler surface $\mathcal{F}(\lambda)$ is smaller or equal

$$\sqrt{\det(F(x,\ y,\ \lambda) - F_A(x,\ y,\ \lambda))}.$$

According Eq. (7.34) the last magnitude is uniformly bounded for all points on $\mathcal{PAF}$.

Remark. Although the above construction shows that

$$\det(F(x, y, \lambda) - F_A(x, y, \lambda)) \to \infty$$

as $\xi \to \infty$, it does not mean that the whole asymptotic cone $\mathcal{AF}$ does not lie in a finite neighborhood of the corresponding Amsler surface $\mathcal{F}$. Quite possibly, $\det(F(x,\ y,\ \lambda) - F_A(\hat{x},\ \hat{y},\ \lambda))$ is uniformly bounded for the appropriately chosen functions $\hat{x} = \hat{x}(x),\ \hat{y} = \hat{y}(y)$.

Actually, the results stated above allow the distance betweens the points on $\mathcal{PAF}$ and the surface $\mathcal{F}$ to be estimated more precisely. As an example, an estimation is made below of the distance between the points lying on $\mathcal{F}$ and the *bisectrix* of $\mathcal{AF}$, which is the straight line $F_A(0,\ r)$ (see (4.14)).

It is also natural to call an infinite curve (i.e. a curve which is not contained inside a sphere) an *asymptotically straight line*, if there exists a straight line such that any cylinder coaxial to this line contains all the curve besides, possibly, its finite part (the part which is contained in a sphere).

Lemma 9. *The curve $F\left(x, -\frac{\lambda^2}{16}\frac{a}{b}x,\ \lambda\right)$ on the surface $\mathcal{F}(\lambda)$ is asymptotically parallel to the bisectrix of $\mathcal{AF}(\lambda)$. More exactly,*

$$F\left(x, -\frac{\lambda^2}{16}\frac{a}{b}x,\ \lambda\right) = F_A(0,\ r) + \frac{\alpha^2}{8}\ \vec{e}\ + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), \qquad (7.35)$$

where

$$\vec{e} = -i\mathbf{E}^{-1}\mathbf{D}^{-1}(0)\sigma_3\mathbf{D}(0)\mathbf{E} \tag{7.36}$$

is the unit vector orthogonal to the bisectrix $F_A(0,\ r)$. *Moreover the plane spanned on* $F_A(0,\ r)$ *and* $\vec{e}$ *is tangent to* $\mathcal{AF}$.

Proof. To prove Eq. (7.35) we substitute $\xi = 0 \Rightarrow y = -\frac{\lambda^2}{16}\frac{a}{b}x$ (see Eqs. (4.6), (3.14)) into Eq. (7.32) and make calculation analogous to the one used for the derivation of Eq. (7.34). Since in this case we are concerned about a more explicit approximation than that obtained in Eq. (7.34), we must use longer asymptotic expansions for $\varphi(r)$ and $J(r)$, namely, we need the terms calculated in Eq. (7.26). Differentiating Eq. (4.14) one finds that the unit vector,

$$\vec{e}\ (\xi) \equiv \frac{1}{\hat{\rho}\sqrt{r}}\frac{\partial}{\partial\xi}F_A(\xi,\ r) = -i\mathbf{E}^{-1}\mathbf{D}^{-1}(\xi)\sigma_3\mathbf{D}(\xi)\mathbf{E},$$

is tangent to $\mathcal{AF}$. Moreover, if $\xi = 0$, then $\vec{e} = \vec{e}\ (0) \perp F(0,\ r)$. □

Acknowledgements. We thank Ulrich Pinkall for his universality conjecture and useful discussions. We are also grateful to Tim Hoffmann for producing the figure. A. V. Kitaev was supported by Sonderforschungsbereich 288 and the Alexander von Humboldt Foundation.

References

[1] Amsler, M.-H.: Des surfaces à courbure négative constante dans l'espace à trois dimensions et de leurs singularities. Math. Annalen **130**, 234–256 (1955)
[2] Bateman, H., Erdélyi, A.: Higher transcendental functions, volumes 1, 2. New York: McGraw–Hill Book Co., Inc, 1953
[3] Bianchi, L.:Vorlesungen über Differentialgeometrie. Leipzig Berlin: Druck und Verlag von B. G. Teubner, 1910
[4] Bobenko, A.I.: Constant mean curvature surfaces and integrable equations. Russian Math. Surveys **46**(4), 1–45 (1991)
[5] Bobenko, A.I.: Surfaces in Terms of 2 by 2 Matrices. Old and New Integrable Cases. In: Fordy, A. P., Wood, J.(eds.) *Harmonic Maps and Integrable Systems*, Braunschweig Weisbaden: Vieweg 1994, pp. 83–127
[6] Bobenko, A. and Eitner, U.: Bonnet Surfaces and Painlevé Equations. J. reine angew. Math. **499**, 47–79 (1998)
[7] Bobenko, A., Eitner, U, Kitaev, A.: Surfaces with Harmonic Inverce Mean Curvature and Painlevé Equations. Geometriae Dedicata **68**, 187–227 (1997)
[8] Bobenko, A. and Pinkall, U.: Discrete Surfaces with Constant Negative Gaussian Curvature and the Hirota Equation. J. Diff. Geom. **43**, 527–611 (1996)
[9] Dorfmeister, J, Pedit, F., and Wu, H.: Weiertrass type representations of harmonic maps into symmetric spaces. Communications in Analysis and Geometry (to appear)
[10] Ince, E.L.: *Ordinary Differential Equations*. New York: Dover, 1956
[11] Its, A.R., Novokshenov, V.Yu.: *The isomonodromic deformation method in the theory of Painlevé equations*. Lect. Notes in Math., **1191**, Berlin–Heidelberg–New York–Tokyo: Springer-Verlag, 1986

[12] Fedorjuk, M.V.: Asymptotic methods for linear ordinary differential equations. Moscow: Nauka, 1983 (in Russian)
[13] Hasimoto, H.: A soliton on a vortex filament. J. Fluid Mech. **51**, 477–485 (1972)
[14] Hoffmann, T.: Discrete Amsler surfaces and a discrete Painlevé III equation. In: A. Bobenko and R. Seiler (eds.), *Discrete Integrable geometry and Physics*. Oxford: Oxford Univ. Press, 1999
[15] Smyth, B.: A generalization of a theorem of Delaunay on constant mean curvature surfaces. In: Davis, H. Ted (ed.) et al., *Statistical thermodynamics and differential geometry of microstructured materials*. New York: Springer-Verlag, IMA Vol. Math. Appl. **51**, pp. 123–130, 1993
[16] Sym, A.: *Soliton surfaces and their application*. Lecture Notes in Physics **239**, Berlin: Springer, 1985, pp. 154–231
[17] Takano, K.: Reduction for Painlevé equations at the fixed singular points of the second kind. J. Math. Soc. Japan **42**(3), 423–443 (1990)
[18] Timmreck, M., Pinkall, U., and Ferus, D.: Constant mean curvature planes with inner rotational symmetry in Euclidean 3-space. Math. Z. **215**, No. 4, 561–568 (1994)