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# Helicoids with handles and Baker-Akhiezer spinors

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## 1 Introduction

For more then 200 years the helicoid was the only known infinite total curvature embedded minimal surface of finite topology. The situation changed in 1993, when Hoffman, Karcher and Wei [9] discovered the genus one helicoid - a minimal torus with one end, which has a form of the helicoid at infinity. The genus one helicoid was constructed using the Weierstrass representation. Karcher, Wei and Hoffman have solved the corresponding period problem [11] and were able to produce detailed plots of the surface, strongly suggesting that it is embedded.

The Gauss map of this surface has an essential singularity at the puncture. This makes the problem familiar to specialists in the theory of integrable systems, where the Baker-Akhiezer functions – functions with essential singularities on compact Riemann surfaces – have become a basic tool of the finite-gap integration theory (see for example [13], [1]).

In the present paper we describe all immersed minimal surfaces of finite topology with just one helicoidal end. Using the spinor Weierstrass representation [19, 3, 15] these immersions are described in terms of holomorphic spinors with essential singularities at the puncture, which we call the Baker-Akhiezer spinors. Those are described explicitly, as well as the Gauss map, which is a meromorphic function with an essential singularity at the puncture. Further we discuss the periodicity conditions for the immersion and show how they yield the Riemann surfaces, which are two-sheeted ramified coverings. This motivates the following

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**Conjecture 1** *Any immersed minimal surface of finite topology with one helicoidal end has normal symmetry: it is invariant with respect to a* $180^0$*-rotation about a line orthogonal to the surface.*

In Sect. 6 we present a new description of the Karcher-Hoffman-Wei genus one helicoid. Our description contains only one integration (one integration less then in [11]) and we hope can be used to prove the embeddedness. Chosing a different spin structure we construct a twisted genus one helicoid. We complete the paper with the plots of the genus one helicoids produced by W. Hornauer, who visualised the formulas of the present paper with the help of the software Mathematica and the Mathematica package `smallucd.m` by U. Pinkall.

The class of immersions under consideration is rather rich and we believe contains many embedded examples. Beside the genus one immersed helicoids only the helicoids with two vertically displaced handles can be described in elliptic functions. To investigate other surfaces one should deal with Riemann surfaces of higher genus and therefore can not rely on Mathematica. A similar problem of visualisation of the theta functional formulas for all constant mean curvature tori [2] has been solved recently by M. Heil [7], who developed a software for calculations on hyperelliptic Riemann surfaces. To visualise the formulas of the present paper a modification of this software is required. We hope that these problems will be overcome and we will see embedded helicoids with many handles soon[1].

## 2 Spinor representation of minimal surfaces

The Weierstarss representation of minimal surfaces in terms of spinors was formulated in [19]. Independently it was reinvented in [3] as a special case of the spinor description for general not necessarily minimal conformal immersions into $\mathbf{R}^3$. These ideas were developed further [14,18,17,12] into a general theory of the spinor representation of surfaces in $\mathbf{R}^3$ [15,10]. For the most elaborated treatment of the minimal surface case we refer the reader to [15].

Let $\mathcal{R}$ be a Riemann surface and $S$ a spin structure on it, i.e. a complex line bundle over $\mathcal{R}$ satisfying $S \otimes S \cong K$, where $K$ is the holomorphic cotangent bundle. The spin structures are parametrized by quadratic forms

$$s : H_1(\mathcal{R}, \mathbf{Z}_2) \to \mathbf{Z}_2, \tag{1}$$

that is

$$s(\gamma_1 + \gamma_2) = s(\gamma_1) + s(\gamma_2) + \gamma_1 \circ \gamma_2, \tag{2}$$

[1] M. Traizet using a different approach has produced a plot of a probably embedded helicoid of genus 2

where $\gamma_1 \circ \gamma_2$ is the intersection number of $\gamma_1$ with $\gamma_2$. For a compact Riemann surface of genus $g$ with 0 or 1 puncture there are $2^{2g}$ different spin structures.

**Theorem 1** *(Spinor Weierstrass representation). Let $a, b$ be two not simultaneously vanishing holomorphic sections of a spin bundle $S$ over a Riemann surface $\mathcal{R}$, i.e. $|a|^2 + |b|^2 \neq 0$ and for any holomorphic coordinate chart $(U, z)$ with simply connected $U \subset \mathcal{R}$ there exist holomorphic functions $A, B : U \to \mathbf{C}$ such that*

$$a = A\sqrt{dz}, \qquad b = B\sqrt{dz}$$

*on $U$. Then*

$$F(P) := \mathrm{Re} \int_Q^P (-a^2 + b^2, i(a^2 + b^2), 2ab) \tag{3}$$

*is a conformal minimal immersion $F : \mathcal{R} \to \mathbf{R}^3$ with the metric*

$$e^u dz d\bar{z} = (|a|^2 + |b|^2)^2. \tag{4}$$

*The Gauss map of the surface is the stereographic projection*

$$N_1 + iN_2 = \frac{2g}{|g|^2 + 1}, \qquad N_3 = \frac{|g|^2 - 1}{|g|^2 + 1} \tag{5}$$

*of the meromorphic function*

$$g = \frac{a}{b}. \tag{6}$$

*All oriented minimal immersions are described in this way.*

*The spin structure of the immersion coincides with the spin structure $S$.*

The last claim of the theorem needs explanation. It is known [16] that immersions into $\mathbf{R}^3$ come with the corresponding quadratic form (1) (i.e. with the corresponding spin structure), which can be described entirely in geometrical terms (see below). The theorem allows us to reformulate this geometrical characteristics in terms of holomorphic data.

There are various equivalent descriptions of the spin structures. For the construction of concrete surfaces it is convenient to use the description of [3], which allows us to compute easily the spin structure of the immersion starting with the Weierstrass data. Let $\gamma$ be a closed smooth contour on $\mathcal{R}$ without self-intersections. Let $z : U_\gamma \to V_\gamma$ be a coordinate chart from an annular neighborhood $U_\gamma$ of $\gamma$ to an annular domain $V_\gamma \subset \mathbf{C}$. Representing the holomorphic differential $a^2 = D(z)dz$ associated to the spinor $a$ on $V_\gamma$ in terms of this coordinate one obtains a holomorphic function $D : V_\gamma \to \mathbf{C}$.

The square root $A = \sqrt{D}$ may have the monodromy $\pm 1$ by analytical continuation along $\gamma$

$$A \xrightarrow{\gamma} (-1)^{p(\gamma)} A. \tag{7}$$

We call so defined $p(\gamma) \in \mathbf{Z}_2$ the flip number of the spinor $a$ along $\gamma$. The flip numbers of the spinors $a$ and $b$ in (3) coincide.

These flip numbers have a simple geometrical interpretation. Let us consider the image $\Gamma = F(\gamma)$ of $\gamma$ on the surface. The contour $\Gamma$ together with the normal field $N$ of the surface on $\Gamma$ define a closed orientable strip in space. Let $M_\Gamma \in \mathbf{Z}$ be the number of twists of this strip, or equaivalently, the winding number of the contours $\Gamma$ and $(\Gamma + \epsilon N)_\Gamma$, where $\epsilon$ small. The parity of twists $M_\Gamma (mod\, 2)$ equals to the flip number of $a, b$

$$p(\gamma) = M_{F(\gamma)} \qquad \mod 2. \tag{8}$$

Moreover these numbers by $s(\gamma) = p(\gamma)$ define the quadratic form (1). One can show that so defined $s$ is indeed a quadratic form on $H_1(\mathcal{R}, \mathbf{Z}_2)$.

Let $\mathcal{R}$ be a compact Riemann surface of genus $g$ with 0 or 1 puncture and a canonical basis $a_1, b_1, \ldots, a_g, b_g$ of $H_1(\mathcal{R}, \mathbf{Z})$ on it. The spin structure (1) can be characterized by the values of the quadratic form on the basic cycles

$$\alpha_i = s(a_i), \beta_i = s(b_i) \in \mathbf{Z}_2. \tag{9}$$

So defined vector

$$\epsilon = [\alpha, \beta] \in \mathbf{Z}_2^{2g}, \qquad \alpha = (\alpha_1, \ldots, \alpha_g), \beta = (\beta_1, \ldots, \beta_g) \tag{10}$$

is called the theta characteristics. The theta characteristics is even if $\alpha \bullet \beta = \alpha_1\beta_1 + \ldots \alpha_g\beta_g = 0 \mod 2$ and odd if $\alpha \bullet \beta = 1 \mod 2$. Embeddings have even theta characteristics.

Usually one has in mind a ”qualitatively correct” picture of the expected minimal surface. Due to (8) the spin structure $S$ of the holomorphic spinors can be computed by using the ”qualitatively correct” picture.

**Corollary 1** *(Relation to the standard Weierstrass representation)*

*The spinor representation (3) is equivalent to the standard Weierstrass representation for minimal surface. Introducing the differentials*

$$\eta = b^2, \qquad \xi = ab, \tag{11}$$

*one obtains two versions of the standard Weierstrass representation*

$$\begin{aligned} F(P) &:= \operatorname{Re} \int_Q^P (1 - g^2, i(1 + g^2), 2g)\eta, \\ F(P) &:= \operatorname{Re} \int_Q^P (\frac{1}{g} - g, i(\frac{1}{g} + g), 2)\xi, \end{aligned} \tag{12}$$

*where $g$ is the meromorphic Gauss map (6). The spin structure of the immersion is determined by the holomorphic section $\sqrt{\eta}$ (or equivalently by $\sqrt{\xi}$) of $S$.*

## 3 Helicoid and helicoidal end singularity

The minimal immersion of the plane $F : \mathbf{C} \to \mathbf{R}^3$ with the data

$$g = e^z; \qquad \xi = idz$$

is the helicoid

$$F(x, y) = 2(\sinh x \sin y, -\sinh x \cos y, -y),$$

where $z = x + iy$. We study immersions with the helicoidal asymptotics at infinity. To analyse the singularity of the holomorphic data one should treat $\mathbf{C}$ here as the punctured Riemann sphere $\bar{\mathbf{C}} \setminus \{\infty\}$. Chosing the pucture at the origin $w = 1/z$, the helicoid is described as the minimal immersion of the punctured sphere $\bar{\mathbf{C}} \setminus \{0\}$ with the holomorphic data

$$g = e^{\frac{1}{w}}, \qquad \xi = -i\frac{dw}{w^2}. \tag{13}$$

The corresponding holomorphic spinors $a, b$ on $\bar{\mathbf{C}} \setminus \{0\}$ are

$$\begin{aligned} a &= e^{-\frac{\pi i}{4}} e^{\frac{1}{2w}} \frac{\sqrt{dw}}{w}, \\ b &= e^{-\frac{\pi i}{4}} e^{-\frac{1}{2w}} \frac{\sqrt{dw}}{w}. \end{aligned} \tag{14}$$

Note that here $w$ is a local parameter, which vanishes at the puncture.

## 4 Baker-Akhiezer spinors as the Weierstrass data for helicoids with handles

Let us assume that a minimal conformal immersion $F : \mathcal{R} \to \mathbf{R}^3$ with finite topology and a helicoidal end exists. The corresponding Riemann surface is a compact Riemann surface $C$ with a puncture $\mathcal{R} = C \setminus \{P_0\}$. The spinors $a, b$ in the Weierstrass representation (3) are holomorphic on $\mathcal{R}$. Motivated by (14) we give the following

**Definition 1** *We call a minimal immersion $F : \mathcal{R} = C \setminus \{P_0\} \to \mathbf{R}^3$ a helicoid of genus $g$ if $C$ is of genus $g$ and the Weierstrass spinors $a, b$ are holomorphic on $\mathcal{R}$ and have the asymptotics*

$$\begin{aligned} a &= e^{-\frac{\pi i}{4}} (1 + o(1)) e^{\frac{1}{2w}} \frac{\sqrt{dw}}{w}, \\ b &= e^{-\frac{\pi i}{4}} (1 + o(1)) e^{-\frac{1}{2w}} \frac{\sqrt{dw}}{w}, \qquad w \to 0 \end{aligned} \tag{15}$$

*at the puncture $P_0$.*

These analytical properties bring us naturally to the notion of the Baker-Akhiezer spinors. Let $C$ be a compact Riemann surface of genus $g$, $P_0$ a point on it, and $w$ a local parameter at $P_0$ with $w(P_0) = 0$. Let $S_\epsilon$ be a spin bundle on $C$. Provided a canonical homology basis $a_1, b_1, \ldots, a_g, b_g$ of $C$, the latter can be characterized by $\epsilon \in \mathbf{Z}_2^{2g}$ as in (10).

**Definition 2** *Let $C, P_0, w, S_\epsilon$ be as above and $q(z) = \sum_{n=1}^{N} q_n z^n$ a polynomial. We will call a section $\psi$ of $S_\epsilon$ which is holomorphic on $C \setminus \{P_0\}$ and has an essential singularity of the form*

$$\psi(P) = (1 + o(1)) e^{q(\frac{1}{w})} \frac{\sqrt{dw}}{w}, \qquad w \to 0 \tag{16}$$

*at $P_0$ a Baker-Akhiezer spinor.*

To justify the name, let us note that in the theory of finite-gap integration [13, 4, 1] functions with essential singularities on compact Riemann surfaces are called the Baker-Akhiezer functions.

For generic data $\{C, P_0, w, \epsilon, q\}$ the Baker-Akhiezer spinors exist and can be described explicitly. For this description we need some preparation. More detailed presentation of the facts from the theory of compact Riemann surfaces and theta functions, for what follows, can be found in [6, 4, 1]. The Riemann theta function corresponding to the Riemann surface $C$ of genus $g$ is an entire function of $g$ complex variables $z = (z_1, \ldots, z_g)$ defined by the formula

$$\theta[\alpha, \beta](z) = \sum_{m \in \mathbf{Z}^g} \exp(\frac{1}{2}(B(m + \frac{\alpha}{2}), m + \frac{\alpha}{2}) + (z + \pi i \beta, m + \frac{\alpha}{2})),$$

where $\alpha, \beta \in \mathbf{Z}_2^g$ is the characteristics of the theta function, $B = B_{n,m}$ is the period matrix

$$B_{n,m} = \int_{b_n} v_m$$

of the normalized holomorphic differentials $v_m$ on $C$

$$\int_{a_n} v_m = 2\pi i \delta_{n,m}.$$

The theta function is quasi-periodic with respect to the lattice $\Lambda$ generated by the vectors $2\pi i N + BM,\ N, M \in \mathbf{Z}^g$

$$\begin{gathered}\theta[\alpha, \beta](z + 2\pi i N + BM) = \\ \theta[\alpha, \beta](z) \exp(-\tfrac{1}{2}(BM, M) - (z, M) + \pi i((\alpha, N) - (\beta, M))).\end{gathered} \tag{17}$$

The torus

$$J(C) = \mathbf{C}^g / \Lambda$$

is called the Jacobian variety of $C$ and the map $\mathcal{A}: C \to J(C)$, defined by

$$\mathcal{A}(P) = \int_{P_0}^{P} v, \qquad v = (v_1, \dots, v_g)$$

is called the Abel map.

Let $\Omega^q$ be the unique differential holomorphic on $C \setminus \{P_0\}$, which has the form

$$\Omega^q = d(q(\frac{1}{w}) + o(1))$$

near $P_0$ and is normalized by the condition

$$\int_{a_n} \Omega^q = 0, \qquad n = 1, \dots, g.$$

Its vector of $b$-periods $W = (W_1, \dots, W_g)$ has coordinates

$$W_n = \int_{b_n} \Omega^q.$$

Theta functions with odd characteristics $\delta$ vanish at the origin $\theta[\delta](0) = 0$. An odd characteristics is called non-singular if there exists $i \in \{1, \dots, g\}$ such that

$$\frac{\partial \theta[\delta]}{\partial z_i}(0) \neq 0.$$

For a generic Riemann surface all odd theta characteristics are non-singular. For a non-singular odd theta-characteristics there exists a holomorphic section $h_\delta$ of the spin bundle $S_\delta$ on $C$ given by

$$h_\delta^2(P) = \sum_{i=1}^{g} \frac{\partial \theta[\delta]}{\partial z_i}(0) v_i(P). \tag{18}$$

A meromorphic section of $S_\gamma$ for $\gamma$ even and non-singular (i.e. $\theta[\gamma](0) \neq 0$) with a simple pole at $P_0 \in C$ is given by

$$k_\gamma(P) = \frac{\theta[\gamma](\int_{P_0}^{P} v)}{\theta[\delta](\int_{P_0}^{P} v)} h_\delta(P). \tag{19}$$

All zeros of the theta function in the denominator of this formula except the zero at $P = P_0$ are cancelled by the zeros of $h_\delta(P)$.

A modification of the last formula provides us with a formula for the Baker-Akhiezer spinors.

**Theorem 2** *For generic (i.e.* $\theta[\epsilon](W) \neq 0$*) data* $\{C, P_0, w, s, q\}$ *there exists exactly one Baker-Akhiezer spinor. It is given by the formula*

$$\psi(P) = c\frac{\theta[\epsilon](\int_{P_0}^P v + W)}{\theta[\delta](\int_{P_0}^P v)\theta[\epsilon](W)} e^{\int_{P_0}^P \Omega^q} h_\delta(P), \tag{20}$$

*where* $\delta$ *is a non-singular odd theta characteristics,* $\epsilon$ *is the theta characteristics (9) of the spin structure* $s$*. The Abelian integral of the second kind* $\int_{P_0}^P \Omega^q$ *is defined to have the asymptotics*

$$\int_{P_0}^P \Omega^q = q(\frac{1}{w(P)}) + o(1), \qquad P \to P_0$$

*at* $P_0$ *and all the integration paths in (20) coincide. The normalization constant* $c$ *equals*

$$c = \sqrt{\sum_{i=1}^g \frac{\partial\theta[\delta]}{\partial z_i}(0) V_i}, \tag{21}$$

*where V is the derivative of the Abel map at* $P_0$

$$v_k(P_0) = V_k dw. \tag{22}$$

*Proof* is standard for the theory of the finite-gap integration. The existence is proven by the formula (20). As in (19) all zeros but the one at $P = P_0$ of the theta function $\theta[\delta](\int_{P_0}^P v)$ are cancelled by $h_\delta(P)$. The shift $W$ in the argument of the theta function in the numerator implies that $\psi$ is a section of $S_\delta$. The constant $c$ is chosen to satisfy the normalization (16). To prove the uniqueness let us assume that $\tilde{\psi}$ is a section of $S_\epsilon$ with the singularity (16) at $P_0$. The quotient $\tilde{\psi}/\psi$ is a meromorphic function whose pole divisor coincides with the divisor of $\theta[\epsilon](\int_{P_0}^P v + W)$, thus it is a non-special divisor of degree $g$. Due to the Riemann-Roch theorem such a function must be constant $\tilde{\psi}/\psi = 1$.

*Remark.* The uniqueness implies that $\psi$ is independent of the choice of $\delta$.

The Weierstrass spinors $a, b$ are a special case of the spinors defined above.

**Corollary 2** *For generic (i.e.* $\theta[\epsilon](V/2) \neq 0$*) data* $\{C, P_0, w, s\}$ *there exist unique spinors* $a, b$*, holomorphic on* $C \setminus \{P_0\}$*, with the asymptotics (15) at the puncture* $P_0$*. These spinors are given by the formulas*

$$a(P) = ce^{-\frac{\pi i}{4}} \frac{\theta[\epsilon](\int_{P_0}^P v - \frac{1}{2}V)}{\theta[\delta](\int_{P_0}^P v)\theta[\epsilon](\frac{1}{2}V)} e^{\frac{1}{2}\int_{P_0}^P \Omega} h_\delta(P),$$

$$b(P) = ce^{-\frac{\pi i}{4}} \frac{\theta[\epsilon](\int_{P_0}^P v + \frac{1}{2}V)}{\theta[\delta](\int_{P_0}^P v)\theta[\epsilon](-\frac{1}{2}V)} e^{-\frac{1}{2}\int_{P_0}^P \Omega} h_\delta(P), \tag{23}$$

*where* $\int_{P_0}^P \Omega$ *is the normalized* $(\int_{a_n} \Omega = 0)$ *Abelian integral of the second kind with the asymptotics*

$$\int_{P_0}^P \Omega = \frac{1}{w(P)} + o(1), \qquad P \to P_0,$$

$c, V$ *are defined in (21, 22) and* $\delta, \epsilon$ *are as in Theorem 2.*

To identify formulas (23) and (20) for $\Omega^q = \frac{1}{2}\Omega$ one should apply the Riemann bilinear relation for the normalized Abelian differentials of the first and the second kind:

$$\int_{b_n} \Omega = -V_n.$$

**Corollary 3** *The minimal immersion with the Weirstrass spinors (38) is given by the formula*

$$\begin{aligned} F_1(P) &= \mathrm{Re}\Big(\frac{c^2}{\theta^2[\epsilon](\frac{1}{2}V)} \int_Q^P (i\theta^2[\epsilon](\int_{P_0}^{\tilde{P}} v - \frac{1}{2}V) e^{\int_{P_0}^{\tilde{P}} \Omega} - \\ &\quad i\theta^2[\epsilon](\int_{P_0}^{\tilde{P}} v + \frac{1}{2}V) e^{-\int_{P_0}^{\tilde{P}} \Omega}) \frac{h_\delta^2(\tilde{P})}{\theta^2[\delta](\int_{P_0}^{\tilde{P}} v)}, \\ F_2(P) &= \mathrm{Re}\Big(\frac{c^2}{\theta^2[\epsilon](\frac{1}{2}V)} \int_Q^P (\theta^2[\epsilon](\int_{P_0}^{\tilde{P}} v - \frac{1}{2}V) e^{\int_{P_0}^{\tilde{P}} \Omega} + \\ &\quad \theta^2[\epsilon](\int_{P_0}^{\tilde{P}} v + \frac{1}{2}V) e^{-\int_{P_0}^{\tilde{P}} \Omega}) \frac{h_\delta^2(\tilde{P})}{\theta^2[\delta](\int_{P_0}^{\tilde{P}} v)}, \\ F_3(P) &= -2\,\mathrm{Im}\Big(\int_Q^P \Omega - \sum_{i,j=1}^{g} \frac{\partial^2 \log\theta[\epsilon]}{\partial z_i \partial z_j}(\frac{1}{2}V) V_i \int_Q^P v_j\Big). \end{aligned} \qquad (24)$$

*The Gauss map is the stereographic projection (5) of*

$$g(P) = (-1)^{<\epsilon>} \frac{\theta[\epsilon](\int_{P_0}^P v - \frac{1}{2}V)}{\theta[\epsilon](\int_{P_0}^P v + \frac{1}{2}V)} \exp^{\int_{P_0}^P \Omega}, \qquad (25)$$

*where* $<\epsilon>$ *is the parity of the spin structure* $<\epsilon> = \alpha \bullet \beta \in \mathbf{Z}_2$.

*Proof.* Only the formula for the third coordinate of the immersion has to be proven. The differential

$$ab(P) = \frac{-ic^2(-1)^{<\epsilon>}}{\theta^2[\epsilon](\frac{1}{2}V)} \frac{\theta[\epsilon](\int_{P_0}^P v - \frac{1}{2}V)\theta[\epsilon](\int_{P_0}^P v + \frac{1}{2}V)}{\theta^2[\delta](\int_{P_0}^P v)} h_\delta^2(P)$$

is meromorphic. To integrate it we use the identity (39) of Fay's book [6], which in our notations reads as follows:

$$\frac{c^2}{\theta^2(e)} \frac{\theta(\int_{P_0}^P v - e)\theta(\int_{P_0}^P v + e)}{\theta^2[\delta](\int_{P_0}^P v)} h_\delta^2(P) =$$
$$-\Omega + \sum_{i,j=1}^{g} \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) V_i v_j(P)$$

for any $e \in \mathbf{C}$. Rewriting the theta functions with characteristics $[\epsilon] = [\alpha, \beta] \in \mathbf{Z}_2^2$

$$\theta[\epsilon](z) = \theta(z + \pi i \beta + B\frac{\alpha}{2}) \exp(\frac{1}{8}(B\alpha, \alpha) + (z + \pi i \beta, \frac{\alpha}{2})),$$

and applying the identity of Fay and the transformation properties (17) we get

$$ab(P) = i\Omega - i \sum_{i,j=1}^{g} \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) V_i v_j(P),$$

with

$$e = \frac{1}{2}V + \pi i \beta + B\frac{\alpha}{2}.$$

The last representation is equivalent to (24).

The Gauss map (25) is a special case of the Baker-Akhiezer functions in the theory of finite-gap integration [13, 4, 1].

*Remark.* We call two local parameters $w$ and $\tilde{w}$ at $P_0$ with $w(P_0) = \tilde{w}(P_0) = 0$ equivalent if $d\tilde{w}/dw(0) = 1$, since they yield the same spinors $a, b$. The Weierstrass spinors (23) are parametrized by the corresponding equivalence classes $[w]$, i.e. by elements of the tangent space $T_{P_0}C$.

## 5 Periodicity conditions and two-sheeted coverings

For generic data $\{C, P_0, [w], s\}$, the formulas (24) define a minimal immersion with translation periods, whose Gauss map is well defined on $\mathcal{R}$. To obtain a helicoid of genus $g$ one should kill all the translation periods.

**Lemma 1** *The formulas (24) define a helicoid of genus $g$ iff*

$$\mathrm{Re} \int_\gamma (-a^2 + b^2, i(a^2 + b^2), 2ab) = 0 \qquad (26)$$

*for any $\gamma \in H_1(\mathcal{R}, \mathbf{Z})$.*

Let us compare the number of free parameters of the data $\{C, P_0, [w], s\}$ with the number of constraints (26). The moduli space of compact Riemann surfaces of genus $g \geq 2$ has complex dimension $3g-3$; $P_0$ and $[w]$ add complex dimension 1 each. This yields for the real dimension of the parameter space

$$\dim_{\mathbf{R}}\{C, P_0, [w], s\} = 6g - 2. \tag{27}$$

On the other hand (26) is equivalent to $2g$ independent vector valued periodicity constraints (26) with

$$\gamma = a_1, b_1, \ldots, a_g, b_g,$$

thus $6g$ independent real scalar constraints. The number of independent constraints exeeds the dimension of the parameter space, which shows that there exist no helicoid of genus $g \geq 2$ with generic conformal structure.

The situation changes for Riemann surfaces, which admit a conformal involution with fixed points. We show that in this case, natural from both geometrical and analytical points of view, one has exactly as many free parameters as periodicity conditions.

Let $C \to C_0$ be a ramified double covering of genus $g = 2g_0 + N - 1$ of a compact Riemann surface $C_0$ of genus $g_0$ with $2N$ branch points at $Q_1, \ldots, Q_{2N} \in C_0$. If $\pi : C \to C$ is the conformal involution with fixed points at $Q_1, \ldots, Q_{2N}$, a canonical homology basis of $H_1(C, \mathbf{Z})$

$$a_1, b_1, \ldots, a_{g_0}, b_{g_0}, a_{g_0+1}, b_{g_0+1}, \ldots, a_{2g_0}, b_{2g_0},$$
$$a_{2g_0+1}, b_{2g_0+1}, \ldots, a_{2g_0+N-1}, b_{2g_0+N-1}$$

can be chosen [6] such that $a_1, b_1, \ldots, a_{g_0}, b_{g_0}$ is a canonical basis of $H_1(C_0, \mathbf{Z})$ and

$$a_n + \pi(a_{g_0+n}) = b_n + \pi(b_{g_0+n}) = 0, \qquad 1 \leq n \leq g_0,$$
$$a_i + \pi(a_i) = b_i + \pi(b_i) = 0, \qquad 2g_0 + 1 \leq i \leq 2g_0 + N - 1.$$

The involution $\pi$ acts on the spin structure characterized by $\epsilon = [\alpha, \beta] \in \mathbf{Z}_2^{2g}$ in the chosen basis of $H_1(C, \mathbf{Z})$ as follows:

$$\pi^*[\alpha, \beta] = \pi^*[(\alpha, \tilde{\alpha}, \hat{\alpha}), (\beta, \tilde{\beta}, \hat{\beta})] = [(\tilde{\alpha}, \alpha, \hat{\alpha}), (\tilde{\beta}, \beta, \hat{\beta})],$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_{g_0}), \tilde{\alpha} = (\alpha_{g_0+1}, \ldots, \alpha_{2g_0}), \hat{\alpha} = (\alpha_{2g_0+1}, \ldots, \alpha_{2g_0+N-1}),$$
$$\beta = (\alpha_1, \ldots, \beta_{g_0}), \tilde{\beta} = (\beta_{g_0+1}, \ldots, \beta_{2g_0}), \hat{\beta} = (\beta_{2g_0+1}, \ldots, \beta_{2g_0+N-1}).$$

Let us call the data $\{C, P_0, [w], s\}$ *admissible* iff

– $C \to C_0$ is a two-sheeted ramified covering as described above

– $\pi P_0 = P_0$
– $\pi^*[w] = -[w]$
– $\pi^* s = s \Leftrightarrow \pi^* \epsilon = \epsilon$.

**Theorem 3** *Minimal immersions with admissible data* $\{C, P_0, [w], s\}$ *possess normal symmetry, i.e. a* $180^0$ *rotation about the first coordinate axis*

$$(F_1, F_2, F_3) \to (F_1, -F_2, -F_3).$$

*The real dimension*

$$6g_0 + 4N - 4$$

*of the space of admissible data coincides with the number of independent periodicity conditions*

$$\begin{aligned} &\mathrm{Re}\int_\gamma (-a^2+b^2, i(a^2+b^2), 2ab) = 0, \\ &\quad \gamma = a_1, b_1, \dots, a_{g_0}, b_{g_0}, \\ &\mathrm{Re}\int_{\hat\gamma} (i(a^2+b^2), 2ab) = 0, \\ &\quad \hat\gamma = a_{2g_0+1}, b_{2g_0+1}, \dots, a_{2g_0+N-1}, b_{2g_0+N-1}. \end{aligned} \tag{28}$$

*Proof.* $\pi^* a$ and $\pi^* b$ are sections of the same spin bundle $S_\epsilon$ on $\mathcal{R}$. Comparing the asymptotics of $\pi^* a, \pi^* b$ at $P_0$ with (15) and using the uniqueness part of Corollary 2 we get

$$\pi^* a = \mp ib,\ \pi^* b = \pm ia. \tag{29}$$

In addition

$$ab = -i(1+o(1))\frac{dw}{w^2}, \qquad w \to 0,$$

which, combined with (29), implies the following identities for the holomorphic differentials in the Weierstrass representation

$$\pi^* a^2 = -b^2,\ \pi^* b^2 = -a^2,\ \pi^* ab = -ab. \tag{30}$$

Let us label $P_0 = Q_{2N}$ and chose another fixed point $Q_1$ as the starting integration point in (3). Denoting by $l$ a path from $Q_1$ to $P \in \mathcal{R}$ one has

$$(F_1, F_2, F_3)(P) := \mathrm{Re}\int_l (-a^2+b^2, i(a^2+b^2), 2ab) =$$
$$\mathrm{Re}\int_{\pi l} (-a^2+b^2, -i(a^2+b^2), -2ab) =: (F_1, -F_2, -F_3)(\pi P).$$

As one can see from (30) for any $\hat\gamma \in H_1(C, \mathbf{Z})$ with $\pi\hat\gamma = -\hat\gamma$ the first coordinate of the corresponding period vanishes

$$\int_{\hat\gamma} dF_1 = 0.$$

Thus for admissible data the periodicity condition (26) is equivalent to $6g_0 + 4N - 4$ scalar equations (28). The moduli space of $C_0$ of genus $g_0 \geq 2$ has complex dimension $3g_0 - 3$, the points $Q_1, \ldots, Q_{2N}$ and the local parameter $[w]$ add complex dimension $2N + 1$. This yields the same number $2(3g_0 + 2N - 2)$ for the real dimension of the parameter space of admissible data. In case $g_0 = 1$ a slightly different counting yields the same result.

*Remark.* It might seem that there exist other possible symmetries of the data, which reduce the number of periodicity conditions to the number of free parameters. It is tempting to analyse other symmetries of the Karcher-Hoffman-Wei genus one helicoid (see Sect. 6) in this context. This surface like the helicoid contains a "vertical" line. A $180^0$ rotation $\tau$ about this line is a non-trivial symmetry of the surface corresponding to an anti-holomorphic involution $\tau : \mathcal{R} \to \mathcal{R}$ of the Riemann surface. Fixed points of $\tau$ must comprise exactly one real oval with the puncture $P_0$ on it. One can easily see that this symmetry reduces the numbers of parameters (27) and constraints by half to $3g - 1$ and $3g$ respectively, which shows that the immersion with the symmetry $\tau$ and without additional symmetries do not exist. Theorem 3 and this observation motivate the conjecture formulated in the introduction of this paper.

*Remark.* When describing embeddings one can chose a basis of $H_1(C, \mathbf{Z})$ such that $\epsilon = [0, 0]$.

Now we have reached the end of our analytic calculations for the case $g \geq 2$ (the case $g = 1$ is considered in Sect. 6). To solve effectively the periodicity conditions we need the help of a computer (see discussion in the introduction). We expect that there exisit many solutions to (28). Depending on the number of the fixed points, the admissible data include the hyperelliptic case $g_0 = 0$, the coverings with only two branch points $(N = 1)$ and all imtermediate cases. In the hyperelliptic case one should expect helicoidal surfaces with handles displaced along the normal symmetry line. In the case $N = 1$ the normal symmetry line intersects the surface at the origin only, the handles are displaced in vertical direction. Probably all the intermediate cases $g_0 \neq 0, N > 1$ are also realizable.

## 6 Genus one helicoids

If $C$ has genus 1 the Abel map establishes an isomorphism between $C$ and its Jacobian $J(C)$. This allows us to rewrite the formulas of Sect. 4 in terms of the Jacobi theta functions. We prefer however in this section not to refer to these formulas (23). Instead we describe $C$ as the factor

$$C = \mathbf{C}/L, \tag{31}$$

where $L$ is the lattice

$$2\omega_1 n + 2\omega_3 m;\ n, m \in \mathbf{Z},\ \mathrm{Im}\frac{\omega_3}{\omega_1} > 0$$

and based on the analytical properties of the Weiestrass spinors $a, b$ formulated in Definition 1, obtain a representation for them in terms of the Weierstrass elliptic functions. Thus the presentation in this section is an independent, simplified version of the one in Sect. 4.

Denote by $w$ the coordinate on **C** in (31) and chose the puncture $P_0$ at the origin $w(P_0)$. Let us denote the cycles on $C$ corresponding to the shifts

$$w \to w + 2\omega_1, \qquad w \to w + 2\omega_3$$

by $\gamma_1$ and $\gamma_3$ respectively. A spin structure $s$ on $C$ is characterized by its values on the generators of $L$

$$\alpha = s(\gamma_1),\ \beta = s(\gamma_3) \in \mathbf{Z}_2. \tag{32}$$

The only holomorphic differential on $C$ is $dw$ and

$$h = \sqrt{dw}$$

is a holomorphic spinor with the odd spin structure

$$\delta = [\alpha, \beta] = [1, 1].$$

*Remark.* The last statement agrees with the definition of the flip number in Sect. 2. Indeed not $w$ but

$$z_\mu = \exp(\pi i w/\omega_\mu), \qquad \mu = 1, 3$$

are annular coordinates along $\gamma_\mu$. In terms of $z_\mu$ the differential $(\sqrt{dw})^2$ reads as follows

$$dw = \frac{\omega_\mu}{\pi i z_\mu} dz_\mu.$$

On the $z_\mu$-plane $\gamma_\mu$ is the simple loop $|z_\mu| = 1$. For the flip number defined by (7) this implies $p(\gamma_\mu) = 1$

$$\sqrt{dw} \stackrel{w \to w + 2\omega_\mu}{\longrightarrow} -\sqrt{dw}, \qquad \mu = 1, 3. \tag{33}$$

Let us denote by $\omega_2$ the half-period

$$\omega_2 = -\omega_1 - \omega_3.$$

The same arguments show that under the shift on $2\omega_2$ the flip number of $\sqrt{dw}$ is also 1

$$\sqrt{dw} \stackrel{w \to w + 2\omega_2}{\longrightarrow} -\sqrt{dw},$$

which agrees with (2).

*Remark*. It is easy to see that for $g = 1$ there is a bijection

$$\{C, P_0, [w]\} \leftrightarrow L.$$

Let us note also that any data $\{C, P_0, [w]\}$ is admissable in the case $g = 1$. The involution $\pi$ is given by

$$\pi z = -z.$$

The torus $C = \mathbf{C}/L$ is a two-sheeted ramified covering of the Riemann sphere $C/\pi$.

Introduce the Weierstrass functions (see [5])

$$\begin{aligned}
&\sigma(w) := w \prod{}'(1 - \frac{w}{\omega}) e^{\frac{w}{\omega} + \frac{w^2}{2\omega^2}}, \qquad \omega = 2n\omega_1 + 2m\omega_3, \\
&\zeta(w) := \frac{d}{dw} \log \sigma(w) = \frac{1}{w} + \sum{}'(\frac{1}{w-\omega} + \frac{1}{\omega} + \frac{w}{\omega^2}), \\
&\wp(w) := -\frac{d}{dw}\zeta(w) = \frac{1}{w^2} + \sum{}'(\frac{1}{(w-\omega)^2} + \frac{1}{\omega^2}).
\end{aligned} \tag{34}$$

They have the following periodicity properties:

$$\begin{aligned}
&\sigma(w + 2\omega_\mu) = -e^{2\eta_\mu(w+\omega_\mu)}\sigma(w), \\
&\zeta(w + 2\omega_\mu) = \zeta(w) + 2\eta_\mu \\
&\wp(w + 2\omega_\mu) = \wp(w),
\end{aligned} \tag{35}$$

where $\eta_\mu$ are the periods

$$\eta_\mu = \zeta(\omega_\mu), \qquad \eta_1 + \eta_2 + \eta_3 = 0.$$

The only pole of $\zeta(w)$, $\wp(w)$ and the only zero of $\sigma(w)$ on $C$ are at the origin

$$\begin{aligned}
&\sigma(w) = w + O(w^5), \\
&\zeta(w) = \frac{1}{w} + O(w^3) \\
&\wp(w) = \frac{1}{w^2} + O(w^2), \qquad w \to 0.
\end{aligned} \tag{36}$$

To describe different spin structures it is convenient to introduce the $\sigma$-functions with characteristics

$$\sigma_\nu(w) := -e^{\eta_\nu w} \frac{\sigma(w - \omega_\nu)}{\sigma(\omega_\nu)}, \qquad \nu = 1, 2, 3,$$

which have the following periodicity properties:

$$\begin{aligned}
&\sigma_\nu(w + 2\omega_\mu) = -e^{2\eta_\mu(w+\omega_\mu)}\sigma_\nu(w), \ \mu \neq \nu, \\
&\sigma_\nu(w + 2\omega_\nu) = e^{2\eta_\nu(w+\omega_\nu)}\sigma_\nu(w).
\end{aligned} \tag{37}$$

The functions $\sigma, \zeta$ are odd, the functions $\wp, \sigma_\nu$ are even. The $\sigma$-functions correspond to the theta functions and the $\zeta$-function corresponds to the Abelian differential $\Omega$ of Sect. 4 for a different normalization.

There are 4 different spin structures on $C$, which we label by the index $\nu \in \{0,1,2,3\}$ as follows:

| $\nu$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\alpha$ | 1 | 0 | 0 | 1 |
| $\beta$ | 1 | 1 | 0 | 0 |

Here $\alpha, \beta$ are defined by (32). It is convenient to denote

$$\sigma_0(w) := \sigma(w), \qquad \omega_0 := 0.$$

**Theorem 4** *For any $L$ and any spin structure $\nu \in \{0,1,2,3\}$ with*

$$\frac{1}{2} + \alpha\omega_1 + \beta\omega_3 \notin L$$

*where $\alpha, \beta$ and $\nu$ are related as in the table above, there exist unique spinors $a, b$ with the spin structure $\nu$ holomorphic on $\mathbf{C}/L\setminus\{P_0\}$ and the asymptotics (15) at the puncture $P_0 = \{w = 0\}$. These spinors are given by the formulas*

$$a(w) = e^{-\frac{\pi i}{4}} \frac{\sigma_\nu(w - \frac{1}{2})}{\sigma(w)\sigma_\nu(-\frac{1}{2})} e^{\frac{1}{2}\zeta(w)}\sqrt{dw},$$

$$b(w) = e^{-\frac{\pi i}{4}} \frac{\sigma_\nu(w + \frac{1}{2})}{\sigma(w)\sigma_\nu(\frac{1}{2})} e^{-\frac{1}{2}\zeta(w)}\sqrt{dw}, \tag{38}$$

*Proof.* The periodicity properties follow from (33, 35, 37). The asymptotics at $w = 0$ follows from (36).

**Theorem 5** *The minimal immersion with the Weierstrass spinors (38) is given by the formula*

$$F_1(z) = \mathrm{Re}(\frac{1}{\sigma_\nu^2(\frac{1}{2})} \int_{z_0}^{z} (i\sigma_\nu^2(w - \frac{1}{2})e^{\zeta(w)} - i\sigma_\nu^2(w + \frac{1}{2})e^{-\zeta(w)}) \frac{dw}{\sigma_\nu^2(w)}),$$

$$F_2(z) = \mathrm{Re}(\frac{1}{\sigma_\nu^2(\frac{1}{2})} \int_{z_0}^{z} (\sigma_\nu^2(w - \frac{1}{2})e^{\zeta(w)} + \sigma_\nu^2(w + \frac{1}{2})e^{-\zeta(w)}) \frac{dw}{\sigma_\nu^2(w)}),$$

$$F_3(z) = -2\,\mathrm{Im}(\zeta(z) + \wp(\frac{1}{2} + \omega_\nu)z). \tag{39}$$

*The Gauss map is the stereographic projection (5) of*

$$g(z) = -\frac{\sigma(z - \frac{1}{2})}{\sigma(z + \frac{1}{2})} e^{\zeta(z)}, \qquad \nu = 0,$$

$$g(z) = \frac{\sigma_\nu(z - \frac{1}{2})}{\sigma_\nu(z + \frac{1}{2})} e^{\zeta(z)}, \qquad \nu \neq 0.$$

*These formulas describe an immersed genus one helicoid if the following periodicity conditions are satisfied:*

$$\mathrm{Re}\left(\frac{1}{\sigma_\nu^2(\frac{1}{2})}\int_{z_0}^{z_0+2\omega_\mu}(\sigma_\nu^2(w-\frac{1}{2})e^{\zeta(w)}+\sigma_\nu^2(w+\frac{1}{2})e^{-\zeta(w)})\frac{dw}{\sigma_\nu^2(w)}\right)=0, \tag{40}$$

$$\mathrm{Im}(\eta_\mu+\wp(\frac{1}{2}+\omega_\nu)\omega_\mu)=0, \qquad \mu=1,3. \tag{41}$$

*Proof*. The differential $ab$ is meromorphic and can be easily integrated. Indeed, $ab$ is holomorphic everywhere on $C$ except at $w=0$, where it has a pole

$$ab(w)=(-\frac{i}{w^2}+O(1))dw, \qquad w\to 0. \tag{42}$$

In addition,

$$ab(\omega_\nu+\frac{1}{2})=0,$$

which, combined with (42), yields

$$ab(w)=i(\wp(\omega_\nu+\frac{1}{2})-\wp(w))dw.$$

Integrating one obtains the formula for the third coordinate in (39). The periodicity conditions (40),(41) follow from the formulas for $F_2(z), F_3(z)$. Using the involution $\pi z=-z$ one can easily check (like in Theorem 3) that $F_1(z)$ is always doubly periodic.

The integrals in (39) are along two independent cycles on $C$. We have 4 real conditions (40),(41) on 2 complex parameters $\omega_1,\omega_3\in\mathbf{C}$.

The number of conditions (and parameters) can be reduced if one assumes the real symmetry

$$\omega_1=\bar{\omega_3}, \qquad \omega_2=-2\mathrm{Re}\omega_1. \tag{43}$$

**Theorem 6** *In the case of the rhombic torus (43) minimal surfaces described by formulas (39) with the spin structures*

$$\nu=0 \text{ and } \nu=2$$

*have additional symmetries: a* $180^0$ *rotation*

$$\tau_h:(F_1,F_2,F_3)\to(-F_1,F_2,-F_3)$$

*about the horizontal line* $l_h$ *and a* $180^0$ *rotation*

$$\tau_v:(F_1,F_2,F_3)\to(-F_1,-F_2,F_3)$$

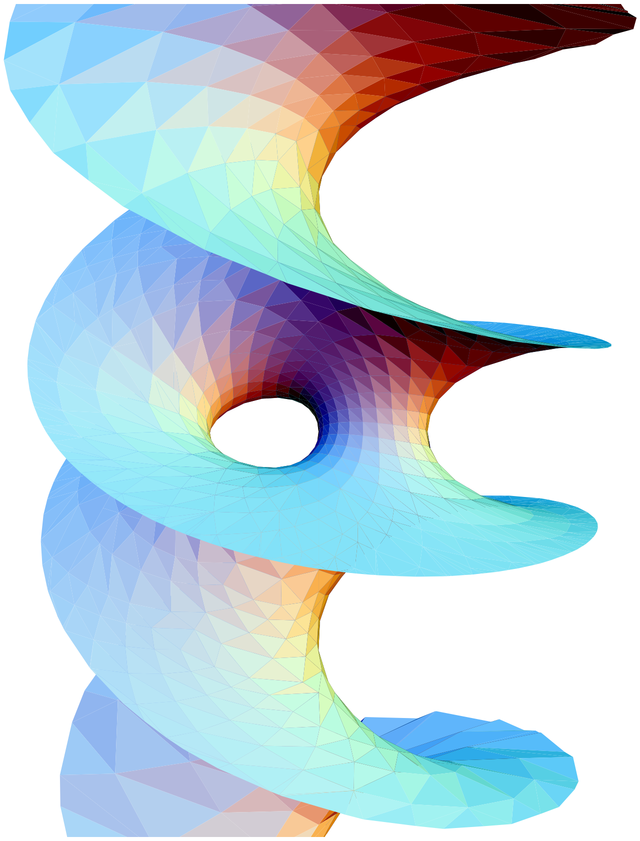

**Fig. 1.** The Hoffman-Karcher-Wei genus one helicoid, corresponding to $\omega_3 = 0.61529 + i\,0.53298$ and $\nu = 2$. The surface contains one vertical and one horizontal straight line. The normal strips along $\gamma_1$ and $\gamma_3$ are both untwisted. The surface is probably embedded

*about the vertical line $l_v$. Both $l_h$ and $l_v$ lie on the surface. The surfaces are immersed genus one helicoids if the following periodicity conditions are satisfied:*

$$\begin{aligned} \mathrm{Re}\Big(\frac{\omega_1}{\sigma_\nu^2(\frac{1}{2})}\int_{-1}^{1}(\sigma_\nu^2(\omega_2 - \frac{1}{2} + t\omega_1)e^{\zeta(\omega_2+t\omega_1)} + \\ \sigma_\nu^2(\omega_2 + \frac{1}{2} + t\omega_1)e^{-\zeta(\omega_2+t\omega_1)})\frac{dt}{\sigma_\nu^2(\omega_2+t\omega_1)}\Big) = 0, \\ \mathrm{Im}(\eta_1 + \wp(\frac{1}{2}+\omega_\nu)\omega_1) = 0. \end{aligned} \tag{44}$$

*Proof.* For the rhombic torus and $\nu = 0$ or $\nu = 2$ one has

$$\overline{\sigma_\nu(w)} = \sigma_\nu(\bar{w}),\ \wp(\frac{1}{2}+\omega_\nu) \in \mathbf{R}.$$

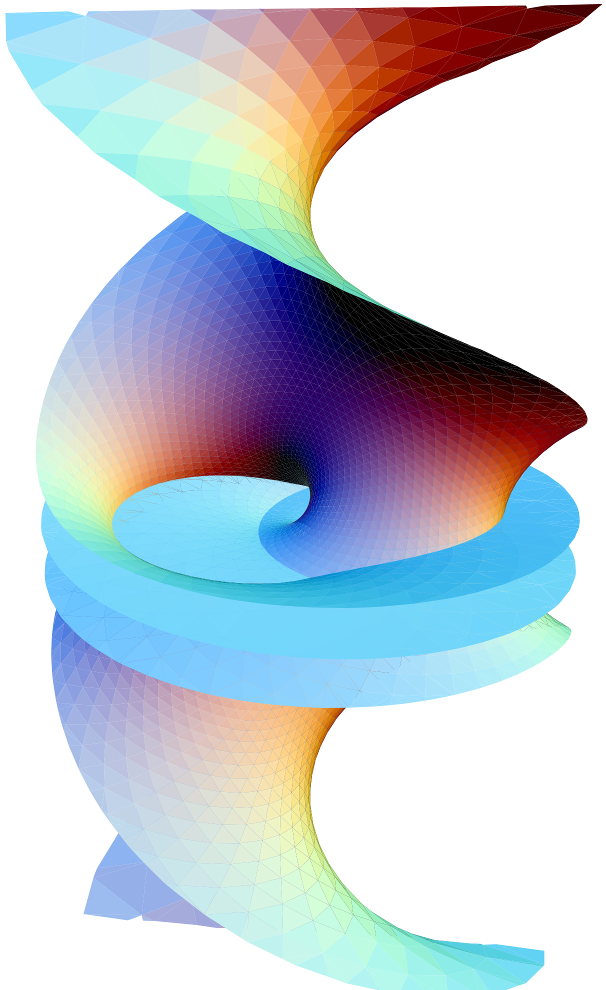

**Fig. 2.** The twisted genus one helicoid, corresponding to $\omega_3 = 0.20651 + i\,0.26238$ and $\nu = 0$. The surface has the same symmetry group as the surface in Fig. 1 but a different spin structure. The normal strips along $\gamma_1$ and $\gamma_3$ are both twisted

Chosing $z_0 = \omega_2$ in the integrals (39) one obtains

$$(F_1, F_2, F_3)(\bar{z}) = (-F_1, F_2, -F_3)(z),$$
$$(F_1, F_2, F_3)(-\bar{z}) = (-F_1, -F_2, F_3)(z).$$

The anti-holomorphic involutions

$$\tau_h z = \bar{z}, \qquad \tau_v z = -\bar{z}$$

of $\mathbf{C}/L$ induce the corresponding symmetries of the surface. The preimages of the straight lines $l_h$ and $l_v$ on the surfaces are the sets $\mathrm{Im} z = 0$ and

$\mathrm{Re} z = \omega_2$ respectively. The periodicity conditions (40, 41) for $\mu = 1$ and $\mu = 3$ coincide. Finally one has 2 real periodicity conditions (44) on one complex parameter $\omega$.

Numerical experiments with Mathematica show that in both cases ($\nu = 0$ and $\nu = 2$) the solutions of (44) are unique:

$$\omega_3 = 0.61529 + i\,0.53298 \qquad \text{for } \nu = 2,$$
$$\omega_3 = 0.20651 + i\,0.26238 \qquad \text{for } \nu = 0$$

The case $\nu = 2$ gives a new representation for the famous Hoffman-Karcher-Wei genus one helicoid found and investigated in [9], [11]. We hope that the representation found in the present paper can help in the proof of the embeddedness of the surface[2].

The twisted genus one helicoid presented in Fig. 2 has the same symmetries as the helicoid of Hoffman, Karcher and Wei but the odd spin structure $\nu = 0$. The surface has selfintersections.

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