*Geometriae Dedicata* **68:** 187–227, 1997.



# Surfaces with Harmonic Inverse Mean Curvature and Painlevé Equations

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(Received: 23 February 1996; in final form: 10 February 1997)

**Abstract.** In this paper we study surfaces immersed in $\mathbb{R}^3$ such that the mean curvature function $H$ satisfies the equation $\Delta(1/H) = 0$, where $\Delta$ is the Laplace operator of the induced metric. We call them HIMC surfaces. All HIMC surfaces of revolution are classified in terms of the third Painlevé transcendent. In the general class of HIMC surfaces we distinguish a subclass of $\theta$-isothermic surfaces, which is a generalization of the isothermic HIMC surfaces, and classify all the $\theta$-isothermic HIMC surfaces in terms of the solutions of the fifth and sixth Painlevé transcendents.

**Mathematics Subject Classifications (1991):** 53A10, 53C42, 53A05, 34A12, 34E05, 35Q58.

**Key words:** immersions, Painlevé, asymptotics, similarity, integrable systems.

## 1. Introduction

In [1] a new class of surfaces was introduced, defined by the property

$$\Delta\left(\frac{1}{H}\right) = 0, \tag{1}$$

where $H$ is the mean curvature of the surface and $\Delta$ is the Laplace operator of the induced metric. We call these surfaces the harmonic inverse mean curvature surfaces (HIMC surfaces).

HIMC surfaces can be considered as a natural generalization of surfaces with constant mean curvature (CMC surfaces). The latter class has been intensively studied (see [2–4]) by the methods of the theory of integrable equations (the solition theory). The starting point of this theory is a representation of the nonlinear differential equation (in differential geometry these are the Gauss–Codazzi equations) in a form of compatibility condition (Lax or Zakharov–Shabat representation in the theory of solitons)

$$U_y(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)] = 0. \tag{2}$$

The parameter $\lambda$, which is called a spectral parameter in the theory of solitons, describes some special one-parametric deformation of surfaces, which is called an associated family in geometry. The representation (2) for HIMC surfaces was found in [1], where the associated families of HIMC surfaces were described as one-parametric conformal deformations preserving the ratio of the principle curvatures. The conformal factor is the modulus of a holomorphic function. The mean curvature function of a HIMC surface is given by

$$H(w,\overline{w}) = \frac{1}{w+\overline{w}},$$

where $w(z)$ is a holomorphic function of the complex coordinate $z$ on the surface. We assume that $w$ itself is a local complex coordinate. In this coordinate one can characterize HIMC surfaces via the solutions of the nonlinear system of partial differential equations for the Hopf differential $Q(\mathrm{d}w)^2$

$$\left(\frac{Q_{w\bar{w}}}{Q_{\bar{w}}}\right)_{\bar{w}} - Q_{\bar{w}} = \frac{1}{(w+\overline{w})^2}\left(2 - \frac{|Q|^2}{\overline{Q}_w}\right), \qquad Q_{\overline{w}} = \overline{Q}_w. \tag{3}$$

By the substitution $Q = f_w$ this system can be reduced to one equation for one real-valued function $f(w,\overline{w})$

$$\Delta^2 f - \frac{|\nabla\Delta f|^2}{\Delta f} - (\Delta f)^2 = \frac{4}{(w+\overline{w})^2}\,(2\Delta f - |\nabla f|^2),$$

where $|\nabla f| = 2|f_w|$, $\Delta = 4\partial_w\partial_{\bar{w}}$. Note, that this equation is integrable from the point of view of the soliton theory. This equation is the compatibility condition (Lax representation) for the system (11) of linear differential equations* with an additional parameter $\tau$.

Another confirmation of the fact that HIMC surfaces are natural generalization of CMC surfaces was obtained in [5], where it was shown that only these two classes admit a Lie-point group of transformations of a certain type.

The integrability allows us to study HIMC surfaces by analytical methods of the soliton theory. By using one of these techniques, namely, the so-called dressing procedure, a Bäcklund transformation for HIMC surfaces was constructed in [6]. In spite of the mentioned analytical results no concrete examples of nontrivial (not CMC) HIMC surfaces are constructed up to nowadays. This is the problem we address in the present publication. We exclude the CMC case from our consideration and assume $H \neq \text{const}$.

Following [1] we use the quaternionic representation of surfaces in $\mathbb{R}^3$, which is explained in Section 2.

The notion of the associated family and the corresponding Lax representation play an important role in our study. Therefore we recall the results concerning them in Section 3.

* To express all the coefficients of (11) in terms of $f$ one should substitute $h = w$, $Q = f_w$, $H = 1/(w+\overline{w})$, $\mathrm{e}^u = -2(w+\overline{w})^2 f_{w\bar{w}}$.

Since there are no harmonic functions without zeros on a compact Riemann surface, there are no compact nontrivial HIMC immersions. The simplest case to study are the surfaces of revolution. Surfaces of revolution are isothermic, i.e. they allow conformal curvature line parametrizations. The duals of isothermic HIMC surfaces (for the definition of the dual isothermic surface see, for example, [1]) are the Bonnet surfaces, i.e. the surfaces admitting isometries preserving the mean curvature function. In [7] all Bonnet surfaces are described in terms of the third and sixth Painlevé transcendents. Hence, all isothermic HIMC surfaces can be also described in terms of the same Painlevé transcendents. In particular, in Section 4 we classify all HIMC surfaces of revolution in terms of the third Painlevé transcendent and employ its asymptotic properties to analyze their embeddedness.

Isothermic surfaces admit isothermic coordinates, i.e. conformal coordinates in which $\mathrm{Im}(Q) = 0$. A generalization of the isothermic surfaces is suggested in Section 5. In this section we consider the surfaces which allow a conformal parametrization, such that the Hopf differential has constant imaginary part

$$\mathrm{Im}(Q) = \theta. \tag{4}$$

We call these surfaces $\theta$-*isothermic* and classify their associated families in terms of the solutions of special third-order ordinary differential equation, which can be obtained by imposing (4) into (3). These equations can be viewed as generalizations of the Hazzidakis equations [8]. We find the following geometrical property of $\theta$-isothermic HIMC surfaces: the associated family of each $\theta$-isothermic HIMC surface acts on it just by scaling. We conjecture that this self-similarity property is a characterization of the $\theta$-isothermic HIMC surfaces.

In Sections 6 and 7, the general solutions of the generalized Hazzidakis equations corresponding to $\theta$-isothermic HIMC surfaces in terms of the fifth and sixth Painlevé transcendents are obtained. These Painlevé equations are more general than those in the isothermic case. For instance, for $\mathrm{Im}(Q) \neq 0$ the fifth Painlevé equation cannot be reduced to the third Painlevé equation as it is in the isothermic case. Note, that the solutions of the Painlevé equations are recognized as nonlinear special functions [9], [10], and their properties are rather well known. Therefore the $\theta$-isothermic HIMC surfaces can be investigated as well as the isothermic ones.

In Section 8 we study the exceptional families of HIMC surfaces, which are not related with the solutions of the Painlevé equations. We call them Cartan cones, since Cartan studied these surfaces but from a different point of view: as the class of applicable Bonnet surfaces [11].

In the Appendix, we establish the relation between the Hamiltonian functions for the Painlevé equations and the Hopf differentials of $\theta$-isothermic HIMC surfaces.

## 2. Quaternionic Description of Surfaces in Euclidean 3-Space

For analytical researches of surfaces in $\mathbb{R}^3$ it is convenient to describe them in terms of $2 \times 2$ matrices (for more details see [1]). In Sections 6 and 7, this description allows us to identify the equations for the moving frame of HIMC surfaces, which satisfy some special symmetry reduction, with the zero-curvature representation for the Painlevé equations.

Let $F : \mathcal{R} \to \mathbb{R}^3$ be a conformal parametrization of an orientable surface $\mathcal{F}$

$$\langle F_z, F_z\rangle = \langle F_{\bar z}, F_{\bar z}\rangle = 0, \quad \langle F_z, F_{\bar z}\rangle = \tfrac{1}{2}\mathrm{e}^u.$$

Here $\mathcal{R}$ is a Riemann surface with the induced complex structure,

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 + v_3 w_3,$$

$z$ is a complex coordinate, and

$$F_z = \frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)F, \qquad F_{\bar z} = \frac{\partial F}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F.$$

The vectors $F_z$, $F_{\bar z}$, and $N$ define a moving frame on the surface. The fundamental forms are as follows

$$\langle \mathrm{d}F, \mathrm{d}F\rangle = \mathrm{e}^u\, \mathrm{d}z\, \mathrm{d}\overline{z},$$
$$-\langle \mathrm{d}F, \mathrm{d}N\rangle = Q\, \mathrm{d}z^2 + H\, \mathrm{e}^u\, \mathrm{d}z\, \mathrm{d}\overline{z} + \overline{Q}\, \mathrm{d}\overline{z}^2,$$

where $Q = \langle F_{zz}, N\rangle$ is the Hopf differential and $H$ the mean curvature function on $\mathcal{F}$.

The compatibility conditions of the moving frame equations are

$$\begin{aligned} &\text{Gauss equation} && u_{z\bar z} + \frac{H^2}{2}\,\mathrm{e}^u - 2Q\overline{Q}\,\mathrm{e}^{-u} = 0,\\ &\text{1. Codazzi equation} && Q_{\bar z} = \frac{H_z}{2}\mathrm{e}^u,\\ &\text{2. Codazzi equation} && \overline{Q}_z = \frac{H_{\bar z}}{2}\mathrm{e}^u. \end{aligned} \tag{5}$$

Let us denote the algebra of quaternions by $\mathbb{H}$ and the standard basis by $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$

$$\mathbf{ij} = -\mathbf{ji}, \qquad \mathbf{jk} = -\mathbf{kj}, \qquad \mathbf{ki} = -\mathbf{ik}.$$

We will use the standard matrix representation of $\mathbb{H}$

$$\begin{aligned} &\mathbf{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\\ &\mathbf{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{6}$$

We identify the three-dimensional Euclidean space with the space of imaginary quaternions,

$$\operatorname{Im}\mathbb{H} = \mathrm{su}(2) = \mathrm{span}(\mathbf{i}, \mathbf{j}, \mathbf{k}),$$

by

$$X = (x_1, x_2, x_3)^t = x_1\mathbf{i} + x_2\mathbf{j} + x_3\,\mathbf{k} \in \mathrm{su}(2). \tag{7}$$

The scalar product of vectors in terms of matrices is then

$$\langle X, Y\rangle = -\tfrac{1}{2}\mathrm{tr}(XY).$$

Let us take $\Phi \in \mathrm{SU}(2)$ which transforms the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into the frame $F_x$, $F_y$, $N$

$$F_x = \mathrm{e}^{u/2}\Phi^{-1}\mathbf{i}\Phi, \quad F_y = \mathrm{e}^{u/2}\Phi^{-1}\mathbf{j}\Phi, \quad N = \Phi^{-1}\mathbf{k}\Phi.$$

One can prove [1] that $\Phi$ satisfies the following linear system

$$\begin{aligned} \Phi_z\Phi^{-1} &= \begin{pmatrix} \dfrac{u_z}{4} & -Q\,\mathrm{e}^{-u/2} \\ \dfrac{H}{2}\mathrm{e}^{u/2} & -\dfrac{u_z}{4} \end{pmatrix}, \\ \Phi_{\bar z}\Phi^{-1} &= \begin{pmatrix} -\dfrac{u_{\bar z}}{4} & -\dfrac{H}{2}\,\mathrm{e}^{u/2} \\ \overline{Q}\,\mathrm{e}^{-u/2} & \dfrac{u_{\bar z}}{4} \end{pmatrix}. \end{aligned} \tag{8}$$

## 3. Associated Family of HIMC Surfaces

Since, to the best of our knowledge, attention to HIMC surfaces was paid only in the recent works [1], [5], and [6] not too many results about them are known. The most important properties of HIMC surfaces can be summarized as follows

(1) Such surfaces are included in an associated family with conformally equivalent induced metric; and
(2) There is a *Sym*-formula which gives the surface as a left-translation of the tangent vector field on the $\mathrm{SU}(2)$-representation of the moving frame of the associated family.

Let us start with the Lax pair for an arbitrary surface with the special condition

$$H(z, \bar z) = \frac{1}{h(z) + \bar h(\bar z)}, \tag{9}$$

where $h(z)$ is an arbitrary holomorphic function and $z$ is a local conformal coordinate. Note that, $h(z)$ is not supposed to be a local coordinate as opposed to $w$ in (3).

The main results concerning HIMC surfaces are formulated in the following.

LEMMA 1. *Let $\mathcal{F}$ be a surface with local conformal parametrization $F(z,\overline{z})$ such that condition* (9) *is valid. Denote by* $\mathrm{e}^u$ $(u = u(z,\overline{z}))$ *its induced metric and $Q = Q(z,\overline{z})$ its Hopf differential. There exists a one-parametric $(\tau \in \mathbb{R})$ family of surfaces $\mathcal{F}_\tau$ (called the* associated family *of $\mathcal{F} = \mathcal{F}_0$) which have the property* (1). *Each $\mathcal{F}_\tau$ is given locally by the conformal parametrization* (*Sym*-formula)

$$F(\tau, z, \overline{z}) = 2\,\Psi(\tau, z, \overline{z})^{-1}\,\frac{\partial}{\partial\,\tau}\Psi(\tau, z, \overline{z}), \tag{10}$$

*where $\Psi = \Psi(\tau, z, \overline{z}) \in \mathrm{SU}(2)$ solves the following system*

$$\begin{aligned}
\Psi_z\Psi^{-1} &= \begin{pmatrix} \dfrac{u_z}{4} & -Q\,\mathrm{e}^{-u/2} \\ \Lambda(\tau, z, \overline{z})\,\dfrac{H}{2}\mathrm{e}^{u/2} & -\dfrac{u_z}{4} \end{pmatrix}, \\
\Psi_{\overline{z}}\Psi^{-1} &= \begin{pmatrix} -\dfrac{u_{\overline{z}}}{4} & -\dfrac{H}{2\,\Lambda(\tau, z, \overline{z})}\,\mathrm{e}^{u/2} \\ \overline{Q}\,\mathrm{e}^{-u/2} & \dfrac{u_{\overline{z}}}{4} \end{pmatrix}.
\end{aligned} \tag{11}$$

*Here*

$$\Lambda = \frac{1 - \mathrm{i}\tau\overline{h}(\overline{z})}{1 + \mathrm{i}\tau h(z)} \tag{12}$$

*and $F(z,\overline{z}) = F(0, z, \overline{z})$. The functions of the fundamental forms of the associated family $F_\tau$ in parametrization $F(\tau, z, \overline{z})$ are as follows*

$$\begin{aligned}
&\mathrm{e}^{u_\tau(z,\overline{z})} = \frac{\mathrm{e}^{u(z,\bar{z})}}{|1 + \mathrm{i}\tau h(z)|^4}, \qquad Q_\tau(z,\overline{z}) = \frac{Q(z,\overline{z})}{(1 + \mathrm{i}\tau h(z))^2}, \\
&H(\tau, z, \overline{z}) = H(z,\overline{z})|1 + \mathrm{i}\tau h(z)|^2.
\end{aligned} \tag{13}$$

*Proof*. See [1].

## 4. All HIMC Surfaces of Revolution

The goal of this section is to derive explicit formulas for all analytical HIMC surfaces of revolution. The properties of analytical surfaces of revolution which we need here are summarized in the following lemma:

LEMMA 2. *Let $\mathcal{F}$ be a surface of revolution different from a plane. Then there exist local coordinates $x$, $y$ such that the parametrization*

$$F(x,y) = \frac{1}{a}\begin{pmatrix} \mathrm{e}^{u(x)/2}\cos(ay) \\ \mathrm{e}^{u(x)/2}\sin(ay) \\ c(x) \end{pmatrix}, \quad a \in \mathbb{R}\setminus\{0\}, \tag{14}$$

*of $\mathcal{F}$ is isothermic, i.e.*

$$1 = \frac{1}{a^2}\left(\left(\frac{u'(x)}{2}\right)^2 + c'(x)^2\,\mathrm{e}^{-u(x)}\right). \tag{15}$$

*Moreover, the mean curvature function $H = H(x)$ is given by the following equation*

$$H(x) = \frac{4a^2 - u'(x)^2 - 2u''(x)}{8c'(x)} \tag{16}$$

*and*

$$c''(x) = H(x)u'(x)\,\mathrm{e}^{u(x)}. \tag{17}$$

*Proof*. Equation (14) is just a curvature line parametrization of $\mathcal{F}$. The coordinates are isothermic iff they are conformal curvature line coordinates. The conformity condition for the above representations yields (15). Thus the isothermic coordinates $(\tilde{x},\tilde{y})$ read

$$\tilde{x}(x) = \frac{1}{a}\int_{x_0}^{x}\sqrt{\left(\frac{u'(\xi)}{2}\right)^2 + c'(\xi)^2\,\mathrm{e}^{-u(\xi)}}\,\mathrm{d}\xi, \quad \tilde{y} = y.$$

To derive (16) one uses the definition of the mean curvature in terms of the functions $u(x)$ and $c(x)$, formula (15), and that $c'$ does not identically vanish in some neighborhood of $x$. Finally, differentiating (15) and using (16) we arrive at (17).

COROLLARY 1. *For non-CMC HIMC surfaces of revolution there exist isothermic coordinates $(x, y)$ such that the mean curvature function is of the form*

$$H(x) = \frac{1}{x}.$$

*Proof.* According to Lemma 2 the mean curvature is a function of $x$, $H = H(x)$. Since $H(x)$ is a solution of (1) and we are considering non-CMC surfaces, $H(x) = 1/x$.

For the rest of this section we fix $a = 2$.

THEOREM 1. *Let $\mathcal{F}$ be a HIMC surface of revolution with* (14) *as a local parametrization. Then there exists a real-valued smooth function $\phi(x)$ such that*

$$\mathrm{e}^{u(x)} = \frac{x^2\,(\phi'(x) + 2\sin(\phi(x)))^2}{4}, \tag{18}$$

$$c(x) = -\frac{x^2}{4}(\phi'(x)^2 - 4\sin^2(\phi(x)). \tag{19}$$

*Moreover, $\phi$ solves the Painlevé III equation* (*in trigonometric form*),

$$x(\phi''(x) - 2\sin(2\,\phi(x)) + \phi'(x) + 2\sin(\phi(x)) = 0. \tag{20}$$

*Inversely, let $\phi(x)$ be any arbitrary solution of* (20) *with $\phi'(x) + 2\sin(\phi(x)) \not\equiv 0$. Then with $u(x)$ and $c(x)$ defined by* (18) *and* (19), (14) *we define an isothermic parametrization of a HIMC surface of revolution*.

*Proof.* Using (15), introduce a function $\phi$ defined as follows

$$\sin(\phi(x)) = \tfrac{1}{2}c'(x)\,\mathrm{e}^{-u(x)/2}, \qquad \cos(\phi(x)) = \tfrac{1}{4}u'(x). \tag{21}$$

Putting these equations into (16) we get (18). Taking the logarithm of (18), differentiating it with respect to $x$, and substituting the result into the second equation of (21), we find (20).

Finally, integration of (17) gives us

$$c(x) = xc'(x) - \mathrm{e}^{u(x)} + C_0,$$

where $C_0$ can be chosen to be zero. Inserting $\mathrm{e}^{u(x)}$ (18) and $c'(x)$ (16), in which $u'(x)$ and $u''(x)$ are defined by the second equation of (21), we get (19).

To prove the inverse statement one should just substitute (18) and (19), in which $\phi(x)$ solves (20), into (15) and (16) and find that these equations are valid identically.

It is interesting to notice that the functions important for the geometrical description of HIMC surfaces of revolution can be presented in a lucid form by using a formulation of the Painlevé III equation as a time-dependent Hamiltonian system. Actually, by introducing the mechanical notations

$$\begin{aligned} &\tau \;\; (\text{time}) = x, \qquad q \;\; (\text{coordinate}) = \phi, \\ &p \;\; (\text{impulse}) = \tfrac{1}{2}x(\phi'(x) + 2\sin(\phi(x))), \end{aligned}$$

we arrive at the following proposition:

PROPOSITION 1. *The Painlevé III equation* (20) *can be considered as the Hamiltonian system,*

$$\dot{p} = \{\mathcal{H}, p\}, \qquad \dot{q} = \{\mathcal{H}, q\},$$

*with the canonical Poisson structure,*

$$\{p, p\} = \{q, q\} = 0, \qquad \{p, q\} = 1,$$

*and the time-dependent Hamiltonian,*

$$\mathcal{H}(\tau, p, q) = \frac{p^2}{\tau} - 2p\sin(q).$$

*The immersion function in these notations reads as*

$$F(x, y) = \frac{1}{2}\begin{pmatrix} p\cos(2\,y) \\ p\sin(2\,y) \\ -\tau\mathcal{H}(\tau, p, q) \end{pmatrix}.$$

*The functions of the fundamental forms are given by*

$$\mathrm{e}^{u(x)} = p^2 = -\tau\mathcal{L}(\tau, p, q), \qquad H(x) = \frac{1}{\tau}, \qquad Q(x) = \tfrac{1}{2}\mathcal{H}(\tau, p, q),$$

*where* $\mathrm{e}^u$ *denotes the induced metric,* $H$ *is the mean curvature function, and* $Q$ *is the Hopf differential. Moreover,* $\mathcal{L}$ *is the Lagrangian function defined via the Legendre transformation,*

$$\mathcal{L}(\tau, p, q) = \mathcal{H}(\tau, p, q) - p\frac{\mathrm{d}}{\mathrm{d}\tau}q.$$

*Proof*. Direct verification.

LEMMA 3. *Let* $\phi$ *be a solution of* (20). *Then*

(i) *If there is* $x_0 \in (0, \infty)$ *with*

$$\phi'(x_0) + 2\sin(\phi(x_0)) \neq 0, \tag{22}$$

*then inequality* (22) *is valid for all* $x \in \mathbb{R}^+$;

(ii) *The solution* $\phi$ *is a real analytical function for* $x \in \mathbb{R}^+$.

*Proof*. To prove (i), one notices that any solution of

$$\phi'(x) + 2\sin(\phi(x)) = 0 \tag{23}$$

solves (20) as well. So, from the uniqueness theorem for solutions of ordinary differential equations the first part of the lemma follows. The statement (ii) is a result of the Painlevé property for $p(x) = \mathrm{e}^{\mathrm{i}\phi(x)}$. The function $p(x)$ solves the Painlevé III equation in the canonical form and, therefore, its possible singularities for $x \neq 0, \infty$ are poles of the first order [12] (the Painlevé property). That is why the real solution (20), $\phi(x)$, is an analytical function without singularities for $x \in \mathbb{R}^+$.

DEFINITION 1. The solutions of Equation (20) which are different from that defined by (23), are said to be transcendental.

*Remark* 1. Actually, while all solutions of (23) are elementary functions defined by the equation

$$\cos(\phi(x)) = \frac{1 - C\,\mathrm{e}^{-4x}}{1 + C\,\mathrm{e}^{-4x}} \quad \text{or} \quad \phi(x) \equiv \pi k, \quad k \in \mathbb{Z}, \tag{24}$$

one can show that the transcendental solutions, in the above sense, can not be 'expressed in a certain way' in terms of elementary functions. See the work [13] for the mathematical details. So, from Theorem 1, we see that it is the Painlevé transcendent which is responsible for the geometry of HIMC surfaces of revolution.

COROLLARY 2. *Any analytically immersed HIMC surface of revolution can be globally parametrized by*

$$F(x,y) \;=\; \frac{x\,(\phi'(x) + 2\,\sin(\phi(x)))}{8} \times$$
$$\times \begin{pmatrix} 2\cos(2y) \\ 2\sin(2y) \\ -x(\phi'(x) - 2\sin(\phi(x))) \end{pmatrix}, \;\; x \in \mathbb{R}^{+}, \tag{25}$$

*where* $\phi(x)$ *is a transcendental solution of* (20).

*Proof.* The statement is a result of Lemma 2 and Theorem 1. Formula (25) becomes singular at $x = 0$, where the mean curvature has a pole.

For further discussions of the geometry of HIMC surfaces of revolution, we need some asymptotic results for the solutions of (20).

LEMMA 4. *Solutions of* (20) *are transcendental iff they are not holomorphic functions in a neighborhood of* $x = 0$.

*Proof.* Let $\phi_0(x)$ be a solution of (20) holomorphic in some neighborhood of $x = 0$. Substituting the corresponding Taylor expansion for $\phi_0(x)$ into (20) one gets that there exists only one one-parameter family of such solutions of (20). On the other hand a one-parameter family of (20) is given by (24). Comparing the Taylor expansions of $\phi_0(x)$ and $\phi(x)$, one completes the proof.

Now we are ready to formulate.

THEOREM 2. *For each real transcendental solution of* (20) *there exist real numbers* $k_0$, $p_0$, $\sigma_0$ *and* $\beta_0$, *and an* $n \in \mathbb{Z}$ *such that the following asymptotic expansions are valid*

$$\pm\phi(x) \underset{x\to+\infty}{=} 2\pi n + \tfrac{1}{2}\pi + \frac{2k_0}{\sqrt{2\,x}}\sin(\varphi_\infty) - \frac{1}{2x} + \frac{1}{(2x)^{3/2}} \times$$
$$\times(k_0^3(\sin(\varphi_\infty) + \tfrac{1}{6}\sin(3\varphi_\infty)) +$$

$$+\tfrac{3}{2}k_0(\tfrac{1}{2}+k_0^4)\cos(\varphi_\infty)) + \mathrm{O}\left(\frac{1}{x^2}\right), \tag{26}$$

$$\phi(x) \underset{x\to+0}{=} \varphi_0 - \frac{2\,x}{1+\sigma_0^2}\sin(\varphi_0+\kappa) + \left(\frac{x}{1+\sigma_0^2}\right)^2 \times$$
$$\times\left(-\sigma_0+\sqrt{1+\frac{\sigma_0^4}{4}}\sin(2\varphi_0+2\kappa+2\kappa_2)\right) + \mathrm{O}(x^3), \tag{27}$$

*where*

$$\varphi_\infty = 2x - k_0^2\log(2x) + p_0, \qquad \varphi_0 = \sigma_0\log(x) + \beta_0,$$

*and*

$$\cos(\kappa) = \frac{1-\sigma_0^2}{1+\sigma_0^2}, \qquad \sin(\kappa) = -\frac{2\sigma_0}{1+\sigma_0^2},$$

$$\cos(2\kappa_2) = \frac{2-\sigma_0^2}{\sqrt{4+\sigma_0^4}}, \qquad \sin(2\kappa_2) = \frac{2\sigma_0}{\sqrt{4+\sigma_0^4}}.$$

*The asymptotic expansions* (26) *and* (27) *are differentiable, i.e., the asymptotics of* $\phi'(x)$ *are just the derivatives of* (26) *and* (27) but with the same error estimations *as in* (26) *and* (27). *Vice-versa, an arbitrary pair of real numbers can be taken as the pair* $(k_0, p_0)$ *or* $(\sigma_0, \beta_0)$, *in the last case* $\sigma_0 \neq 0$, *to define, via asymptotic expansions* (26) *or correspondingly* (27), *a unique real transcendental solution of* (20).

*Proof*. See the papers [14–18].

*Remark* 2. Theorem 2 and (24) imply: a solution $\phi$ of (20) is nontranscendental iff

$$\phi(x) \xrightarrow[x\to+\infty]{} \pi k, \quad k\in\mathbb{Z}.$$

*Remark* 3. Under the notation $g(x) =_{x\to a} \mathrm{O}(f(x))$ we mean that $|g(x)/f(x)| < C$ for some $C > 0$ as $x \to a$.

Since the case when $k_0 = 0$ in Equation (26) plays an important role in our classification of HIMC surfaces of revolution, we formulate the following

THEOREM 3. *There exists the unique real-valued solution of Equation* (20), $\phi(x)$, *which has the following asymptotic expansion as* $x \to \infty$,

$$\phi(x) = \tfrac{1}{2}\pi - \frac{1}{2x} + \frac{5}{6(2x)^3} + \sum_{m=2}^{\infty}\frac{a_m}{(2x)^{2m+1}}, \tag{28}$$

*where the coefficients $a_m$ are real numbers which can be obtained recursively by substituting expansion* (28) *into* (20). *The asymptotic behavior of this solution as* $x \to 0$ *is given by Equation* (27) *with the parameters*

$$\sigma_0 = \frac{1}{\pi}\ln(3+\sqrt{8}) = \frac{2}{\pi}\ln(1+\sqrt{2}) = 0,5610998522..., \tag{29}$$

$$\beta_0 = \pi - 2\sigma_0 \ln 2 + 2\arg\Gamma\left(\frac{i\sigma_0}{2}\right)\Gamma^2\left(\frac{1}{2}-\frac{i\sigma_0}{2}\right) = 0,907634370..., \tag{30}$$

*where* $\Gamma(\cdot)$ *is the standard notation for the* $\Gamma$*-function* [22] *and the function* arg *is fixed by its principle value* $-\pi < \arg(\cdot) < \pi$.

*Proof.* See [19].

*Remark* 4. Actually, the work [19] was done after the preprint version of this article was completed. Our previous numerical calculations yielded the following values for the parameters $\sigma_0 = 0,56108 \pm 0,00005$ and $\beta_0 = 0,908 \pm 0,005$.

COROLLARY 3. *The asymptotics of the immersion function is given by* (14) $(a = 2)$, *where*

$$\begin{aligned} e^{u/2} \underset{x\to+\infty}{=} \; & x + k_0\sqrt{2x}\cos(\varphi_\infty) - k_0^2\sin^2(\varphi_\infty) + \\ & + \frac{1}{\sqrt{2x}}\left(\left(\frac{k_0}{8} - \frac{3}{4}k_0^5\right)\sin(\varphi_\infty) - \right. \\ & \left. - \frac{k_0^3}{2}\cos(\varphi_\infty) + \frac{k_0^3}{4}\cos(3\varphi_\infty)\right) + O\left(\frac{1}{x}\right), \end{aligned} \tag{31}$$

$$\begin{aligned} e^{u/2} \underset{x\to+0}{=} \; & \frac{\sigma_0}{2} + \frac{\sigma_0\, x}{\sqrt{1+\sigma_0^2}}\cos(\varphi_0 + \tfrac{1}{2}\kappa) + \frac{\sigma_0 x^2}{(1+\sigma_0^2)^2} \times \\ & \times \left(1 - \frac{\sigma_0\sqrt{1+\sigma_0^2}}{2}\sin(2\varphi_0 + \tfrac{3}{2}\kappa)\right) + O(x^3), \end{aligned} \tag{32}$$

$$\begin{aligned} c(x) \underset{x\to+\infty}{=} \; & x^2 - 2k_0^2 x + k_0\sqrt{2x}\sin(\varphi_\infty) - \tfrac{1}{4} + \\ & + \frac{k_0^4}{2} + \frac{k_0^2}{2}\sin(2\varphi_\infty) + O\left(\frac{1}{\sqrt{x}}\right), \end{aligned} \tag{33}$$

$$\begin{aligned} c(x) \underset{x\to+0}{=} \; & -\frac{\sigma_0^2}{4} + \frac{\sigma_0 x}{\sqrt{1+\sigma_0^2}}\sin(\varphi_0 + \tfrac{1}{2}\kappa) + \frac{\sigma_0^2(3+\sigma_0^2)x^2}{2(1+\sigma_0^2)^2} + \\ & + \frac{\sigma_0^2 x^2}{2(1+\sigma_0^2)^{3/2}}\cos(2\varphi_0 + \tfrac{3}{2}\kappa) + O(x^3), \end{aligned} \tag{34}$$

*where the notations are the same as in Theorem* 2 *and*

$$\cos(\tfrac{1}{2}\kappa) = \frac{1}{\sqrt{1+\sigma_0^2}}, \qquad \sin(\tfrac{1}{2}\kappa) = -\frac{\sigma_0}{\sqrt{1+\sigma_0^2}}.$$

*Proof.* The formulas (31) to (34) are the result of the substitution of (26) and (27) into (18) and (19) correspondingly.

COROLLARY 4. *For any HIMC surface of revolution there exists* $x_0 > 0$ *such that equation* (25) *for* $0 < x < x_0$ *and* $1/x_0 < x$ *is an embedding.*

*Proof.* As it follows from (33) and Theorem 2, $c'(x)$ cannot vanish for large $x$. This proves embeddedness of the HIMC surfaces of revolution for large $x$.

To prove the embeddedness for small $x$, consider the logarithmic spiral defined via the first two terms of (32) and (34). This spiral is an embedding. For $x \to +0$ the curve $(\mathrm{e}^{u(x)/2}, c(x))$ can be interpreted as a deformation of $\mathrm{O}(x^2)$ of this spiral. But it is easy to see that such a deformation of the spiral cannot destroy its property of embeddedness.

LEMMA 5. *For arbitrary* $x_0 > 0$ *the meridian curve* $(\mathrm{e}^{u(x)/2}, c(x))$ *for* $x_0 < x < 1/x_0$ *has a finite number of points of self-intersection.*

*Proof.* Suppose that the number of intersections is not finite. Then using the formula for the immersion (25) and the properties of the function $\phi$ stated in Lemma 3, one proves that there exists a point $x^* \neq 0, \infty$, such that $(\mathrm{d}/\mathrm{d}x)^k \phi(x^*) = 0$ holds for any integer $k$. Now the real analyticity of $\phi$ implies $\phi(x) = \mathrm{const}$ for all $x$.

COROLLARY 5. *The meridian curve* $(0 < x < \infty)$ *has a finite number of self-intersections iff the point with the coordinates* $(\sigma_0/2, -\sigma_0^2/4)$ *does not belong to it.*

*Proof.* The statement about the finite number of self-intersections ensue from Corollary 4 and Lemma 5. If the point $(\sigma_0/2, -\sigma_0^2/4)$ belongs to the meridian curve, then the infinite number of self-intersections occur according the asymptotics given in Corollary 3.

PROPOSITION 2. *Intersection of every HIMC surface of revolution* $\mathcal{F}$ *in parametrization* (25) *with*

(1) $P^+$, *the paraboloid of revolution* (*with respect to the same axis as for* $\mathcal{F}$) *of the curve* $F_3 = 2F_1^2$, *is an infinite set of circles* (*we exclude those HIMC surfaces of revolution, which correspond to the special solutions of Theorem* 3 *and Remark* 4). *A point belongs to one of these circles, iff it corresponds to an extremum value* $(\phi'(x) = 0)$ *for the related Painlevé function* (20).

(2) $P^-$, *the paraboloid of revolution of the curve* $F_3 = -2F_1^2$, *is an infinite set of circles. A point belongs to one of these circles, iff it corresponds to a solution of the equation* $\sin(\phi(x)) = 0$.

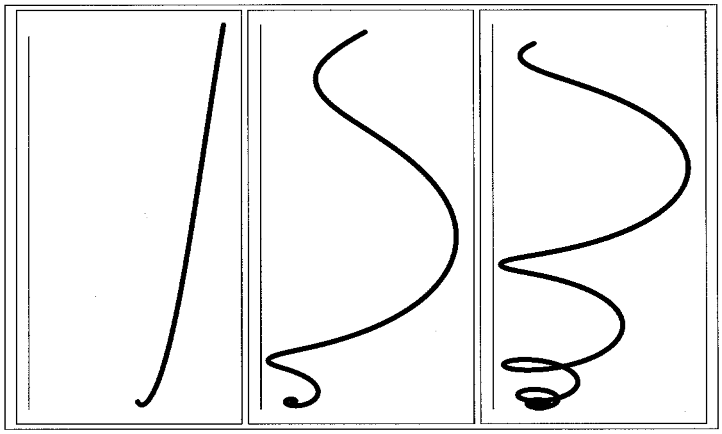

*Figure 1.* Types of possible meridian curves with their axis of rotation.

(3) *The plane $F_3 = 0$ is an umbilic circle. All umbilic points of $\mathcal{F}$ belong to this circle.*

*A point belongs to the intersection of $\mathcal{F}$ with the paraboloids $P^+$, $P^-$, iff the Gaussian curvature $K$ of $\mathcal{F}$ vanishes there.*

*The part of $\mathcal{F}$ lying above the umbilic circle is embedded.*

*Proof.* The Gaussian curvature of $\mathcal{F}$ parametrized by (25) is given by the following equation

$$K = \frac{8\sin(\phi(x))\phi'(x)}{x^2(\phi'(x) + 2\sin(\phi(x)))^2}.$$

To prove the assertions use the immersion formula (25) and the properties of the function $\phi(x)$, established in Lemma 3 and Proposition 1.

*Remark* 5. The analytical description of the global embeddedness of HIMC surfaces of revolution in terms of the solutions of (20) seems to be a complicated problem. Obviously, there are two possibilities: According to our numerical investigations both of them are realized, moreover, the existence of HIMC surfaces whose meridian curves have infinitely many self-intersections, is quite probable.

The results of our numerical simulations are presented in Figure 1.

The first plot in Figure 1 is the meridian curve of a HIMC surface of revolution described by the special solution of Theorem 3 and Remark 4. The second plot is numerically generated by the solution of (20) with $\sigma_0 = 3.0$ and $\beta_0 = -1.41$, and the third with $\sigma_0 = 10$ and $\beta_0 = 0$. Evidently, the third curve describes a nonembedded HIMC surface, while the first two should correspond to embedded ones. Considering our asymptotic results and numerical simulations we come to the

CONJECTURE 1. All HIMC surfaces of revolution for $|\sigma_0| > 4$ are not embedded, while for rather small modulus of $\sigma_0$ ($|\sigma_0| \leqslant 3$) all of them are embedded.

## 5. Reduction to Ordinary Differential Equations

Let us assume a (local) conformal parametrization of a surface with $(1/H)_{z\bar{z}} = 0$, which implies that $H$ has the representation given by (9). Now, by inserting (9) into the Codazzi equations (5), we get

$$h_z(z)\overline{Q}_z(z,\overline{z}) = \overline{h}_{\bar{z}}(\overline{z})Q_{\bar{z}}(z,\overline{z}).$$

Then solving the first Codazzi equation with respect to $u$ and substituting it into the Gauss equation, we arrive at the following third order partial differential equation for $Q$

$$h_z(z)\left(\frac{Q_{z\bar{z}}}{Q_{\bar{z}}}\right)_{\bar{z}} - Q_{\bar{z}} = \frac{|h_z(z)|^2}{(h(z)+\overline{h}(\overline{z}))^2}\left(2h_z(z) - \frac{|Q|^2}{\overline{Q}_z}\right).$$

As long as $h'(z) \neq 0$, we can assume $w = h(z)$ as a proper coordinate. Thus, we obtain system (3). In the following part of this section we study some properties of (3).

*Remark* 6. In the following it will sometimes be convenient for us to use $w$ as the original coordinate on the surface and $z = z(w)$ for its reparametrization.

It seems to be a rather complicated problem to study the general solution of system (3). Actually, in the previous section, we studied some special real solutions of (3) for $w = z/2$. In general, we can formulate the following

LEMMA 6. *Any isothermic HIMC surface is dual to a Bonnet surface*.

*Proof.* We recall that, for an isothermic surface $Q(w,\overline{w}) = f(w)q(w,\overline{w})$, where $q(w,\overline{w}) \in \mathbb{R}$ and $f(w)$ is holomorphic. If $w$ is an isothermic coordinate, then $f(w) \equiv 1$. For this coordinate, the coefficients of the fundamental forms of the dual surface (given with *) read as follows

$$\begin{aligned} &\mathrm{e}^{u^*(w,\bar{w})} = \mathrm{e}^{-u(w,\bar{w})}, \qquad H^*(w,\overline{w}) = 2q(w,\overline{w}), \\ &Q^*(w,\overline{w}) = \tfrac{1}{2}H(w,\overline{w}). \end{aligned} \tag{35}$$

So from the last equation, we get that the Hopf differential of the dual surface is real and $(1/Q^*)_{w\bar{w}} = 0$, which is an equivalent characterization for Bonnet surfaces as first found by Graustein [20].

COROLLARY 6. *All isothermic HIMC surfaces are describable in terms of Painlevé* III *and* VI *equations with the same set of coefficients as for Bonnet surfaces*.

*Proof*. As follows from (35), any solution for $H^*$ generates a solution for $Q$ and vice-versa.

The next natural step to continue the study of system (3) is to consider the following generalization of the isothermic property (given in the proof of Lemma 6). In any conformal parametrization the Hopf differential reads as follows

$$Q(w,\overline{w}) = f(w)(q(w,\overline{w}) + \mathrm{i}\theta), \tag{36}$$

where $f(w)$ is holomorphic, $q(w,\overline{w})$ is a real-valued function, and $\theta$ is a real constant. It is convenient for us to make the following.

DEFINITION 2. A HIMC surface is said to be $\theta$-isothermic if there exists a local conformal coordinate $z$ such that the Hopf differential of the surface is of the form (36).

*Remark* 7. Note that, $\theta$, of course, is not an invariant of the surface as it is dependent on $f$ and on the choice of the conformal coordinate. If $\theta \neq 0$ then it can always be set equal to 1. But we find it convenient in our study to keep $\theta$ as a free parameter.

PROPOSITION 3. *A $\theta$-isothermic surface with $\theta \neq 0$ is isothermic iff it is a Bonnet or CMC surface*.

*Proof*. If a $\theta$-isothermic surface with $\theta \neq 0$ is isothermic, then the following equation holds

$$\widetilde{f}(z)\widetilde{q}(z,\overline{z}) = f(z)(q(z,\overline{z}) + \mathrm{i}\theta), \tag{37}$$

where $\widetilde{f}(z)$ and $f(z)$ are holomorphic functions, and $\widetilde{q}(z,\overline{z})$ and $q(z,\overline{z})$ are real-valued functions. Putting $h(z) = -\mathrm{i}\widetilde{f}(z)/(2\theta f(z))$ one finds that

$$\widetilde{q}(z,\overline{z}) = \frac{1}{h(z) + \overline{h}(\overline{z})}, \tag{38}$$

which means that for $h(z) \equiv$ const., the surface is a CMC surface; otherwise, it is a Bonnet surface [20].

Conversely, if the surface is a CMC or a Bonnet surface, then for any conformal coordinate $z$, $Q = \widetilde{f}(z)\widetilde{q}(z,\overline{z})$, where $\widetilde{q}(z,\overline{z})$ is of the form (38). Define

$$q(z,\overline{z}) = -\theta\frac{\mathrm{Im}(h(z))}{\mathrm{Re}(h(z))}, \qquad f(z) = \frac{\widetilde{f}(z)}{2\theta\,\mathrm{i}h(z)},$$

and find that (37) holds.

LEMMA 7. *Let $Q$ be given by* (36). *With respect to the new variable*

$$z = \int f(w)\,\mathrm{d}w, \tag{39}$$

*Equation* (3) *reduces to the ordinary differential equation*

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = s(t)\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right), \tag{40}$$

*where* $t = z + \overline{z}$, *and* $s(t)$ *is a real analytical function of* $t \in \mathbb{R}\backslash D$, *with* $D$ *a discrete set.*

*Proof.* Let us insert the ansatz (36) into the second equation of (3), then we get

$$\overline{f}(\overline{w})q_w(w,\overline{w}) = f(w)q_{\bar{w}}(w,\overline{w}),$$

which implies that for $z$ defined in (39) we have $q_z = q_{\bar{z}}$. As a consequence, we get that $q(z,\overline{z}) \equiv q(t)$.

Suppose that there is a solution of (3) such that both sides of this equation vanish identically. This would mean $\overline{Q}_w > 0$. OO, the Codazzi equations yield for the metric

$$\mathrm{e}^u = -\overline{Q}_w(w+\overline{w})^2.$$

So such a solution could not be realized on a surface in $\mathbb{R}^3$.

Now using this fact and inserting

$$Q(w,\overline{w}) = f(w)(q(t) + \mathrm{i}\theta)$$

into the first equation of (3) we can define the function

$$s(t) = \frac{1}{|f\circ w(z)|^2(w(z)+\overline{w}(\overline{z}))^2} \tag{41}$$

which had the properties stated in the Lemma.

Using the coordinate (39), we can rewrite the fundamental functions as follows:

$$\begin{aligned} Q(z,\overline{z}) &= \frac{q(t)+\mathrm{i}\,\theta}{f\circ w(z)},\\ \mathrm{e}^{u(z,\bar{z})} &= -2q'(t)(w(z)+\overline{w}(\overline{z}))^2, \\ H(z,\overline{z}) &= \frac{1}{w(z)+\overline{w}(\overline{z})}. \end{aligned} \tag{42}$$

COROLLARY 7. *Let* $f(w)$ *be as in Lemma* 7. *Then*

$$f(w) = \frac{1}{aw^2 + \mathrm{i}bw + c}, \tag{43}$$

*with* $a, b$, *and* $c \in \mathbb{R}$.

*Proof.* Denote $g(w) = 1/f(w)$: then differentiate $s(t)$ (41) with respect $z$ and then to $\overline{z}$ to get $s_z(t) = s_{\bar{z}}(t)$, which, by means of (39), is equivalent to

$$2g(w) - g_w(w)(w + \overline{w}) = 2\overline{g}(\overline{w}) - \overline{g}_{\bar{w}}(\overline{w})(w + \overline{w}). \tag{44}$$

Differentiation of (44) with respect to $w$ and $\overline{w}$ gives

$$g_{ww}(w) = \overline{g}_{\bar{w}\bar{w}}(\overline{w}) \equiv 2a \in \mathbb{R}. \tag{45}$$

The solution of (45) is a polynomial of second degree. Substituting it into (44) yields the denominator of (43).

LEMMA 8. *Let* $\mathcal{F}$ *be a* $\theta$*-isothermic HIMC surface. Then, up to scaling and analytical reparametrization, the functions of its fundamental forms are given by* (42), *where* $f(w)$ *and* $w(z)$ *is one of the following forms*

$$\begin{array}{lll}
(\mathrm{A}) & f(w) = \dfrac{1}{4\mathrm{i}w}, & w(z) = -\mathrm{i}\,\mathrm{e}^{4\mathrm{i}z}, \\
(\mathrm{B}) & f(w) = \dfrac{1}{-w^2+4}, & w(z) = 2\coth(2z), \\
(\mathrm{C}) & f(w) = 2, & w(z) = \frac{1}{2}z, \\
(\mathrm{D}) & f(w) = \dfrac{1}{2\,w^2}, & w(z) = -\dfrac{1}{2z}, \\
(\mathrm{E}) & f(w) = \dfrac{1}{w^2+4}, & w(z) = -2\cot(2z).
\end{array} \tag{46}$$

*Proof.* To normalize the coefficients in (43) let us apply the following transformations

- $w \to w + \mathrm{i}\alpha$, $\alpha \in \mathbb{R}$ (reparametrization of the surface);
- $f \to \beta f$, $\beta \in \mathbb{R}$, $\beta \neq 0$ ($f$ in (36) is defined up to a factor only);
- for the immersion $F$, $F \to \gamma F$, $\quad \gamma \in \mathbb{R}^+$, which yields the transformation $w \to \gamma w$.

Under the action of these transformations the coefficients in (43) are changed as follows

$$a \to \frac{a}{\beta}, \qquad b \to \frac{1}{\beta}(b\gamma + 2a\alpha), \qquad c \to \frac{1}{\beta}(c\gamma^2 - a\alpha^2 - b\alpha\gamma).$$

One can choose $\alpha$, $\beta$, $\gamma$ to fix the coefficients in the cases

$$
\begin{aligned}
&(\mathrm{A}) \qquad a = 0, \quad b \neq 0,\\
&(\mathrm{B}) \qquad a \neq 0, \quad \frac{b^2}{4a^2} + \frac{c}{a} < 0,\\
&(\mathrm{C}) \qquad a = 0, \quad b = 0,\\
&(\mathrm{D}) \qquad a \neq 0 \quad \frac{b^2}{4a^2} + \frac{c}{a} = 0,\\
&(\mathrm{E}) \qquad a \neq 0 \quad \frac{b^2}{4a^2} + \frac{c}{a} > 0,
\end{aligned}
\tag{47}
$$

as is indicated in (46).

COROLLARY 8. *For each of the cases* A, B, C, D, and E, *and each choice of initial conditions in* (40) *one obtains an associated family of $\theta$-isothermic HIMC surfaces. Their fundamental and immersion functions are given by Equations* (10)–(13) *in Lemma* 1 *in which one should substitute $w(z)$ for $h(z)$, and* (42).

*Proof.* Actually, we can choose five different pairs for $w(z)$ and $f(w)$ involved in the equation for the immersion function (see formulas (10) and (11)) for the associated families. Moreover, these equations are parametrized via a general solution of the third order ordinary differential equation (40).

LEMMA 9. A $\equiv$ E *and* C $\equiv$ D.

*Proof.* Calculating the coefficients of the fundamental forms for the cases A and C (correspondingly, for D and E) we get that these functions are the same (up to scaling) if one assumes the following relationships between the family parameters $(\tau := \tau_A,\ \tau_B,\ \tau_C,\ \tau_D,\ \tau_E$; see Lemma 1$)$:

$$
\tau_A = \frac{2\tau_E + 1}{2\tau_E - 1}, \qquad \tau_C = \frac{4}{\tau_D}. \tag{48}
$$

Each surface of a family of the set A (of the set C) corresponds to a surface of the set E (of the set D) as follows:

$$
F_A(\tau_A) = \frac{(1 - 2\tau_E)^2}{4} F_E(\tau_E), \qquad -\tau_D^2 F_C(\tau_C) = F_D(\tau_D),
$$

where $F_\nu$ denotes a parametrization of a surface in the set $\nu$, $\nu = \mathrm{A, C, D, E}$.

*Remark* 8. Actually, we see that in each family from the sets E and C, one surface is not included in the corresponding family from the sets A and D since we see that equations (48) have singular points at $\tau_E = \frac{1}{2}$ and $\tau_C = 0$. To include these special surfaces of the E and C families into the A and D families, we have to enlarge the domain of $\tau$ by adding the point $\tau = \infty$. So, in Lemma 9, one should understand the family parameter $\tau$ as an element of $\mathbb{R}P^1$.

Table I. Functions of the Fundamental Forms of the Associated Family.

| | A | B | C |
|---|---|---|---|
| $\mathrm{e}^{u(\tau,z,\bar{z})}$ | $-\dfrac{q'(t)\sin^2(2t)}{2\tau^2\lvert\sin(2(z-\zeta))\rvert^4}$ | $-\dfrac{8q'(t)\sinh^2(2t)}{(1+4\tau^2)^2\lvert\sinh(2(z-\zeta))\rvert^4}$ | $-\dfrac{q'(t)t^2\zeta^4}{2\lvert z-\zeta\rvert^4}$ |
| $Q(\tau,z,\overline{z})$ | $\dfrac{(q(t)+\mathrm{i}\theta)}{\tau\sin^2(2(z-\zeta))}$ | $\dfrac{4(q(t)+\mathrm{i}\theta)}{(1+4\tau^2)\sinh^2(2(z-\zeta))}$ | $\dfrac{\zeta^2(q(t)+\mathrm{i}\theta)}{2(z-\zeta)^2}$ |
| $H(\tau,z,\overline{z})$ | $-\dfrac{2\tau\lvert\sin(2(z-\zeta))\rvert^2}{\sin(2t)}$ | $-\dfrac{(1+4\tau^2)\lvert\sinh(2(z-\zeta))\rvert^2}{2\sinh(2t)}$ | $-\dfrac{2\lvert z-\zeta\rvert^2}{\zeta^2 t}$ |
| $\zeta=\zeta(\tau)$ | $\frac{1}{2}\mathrm{i}\log(\sqrt{-\tau})$ | $-\frac{1}{2}\log\left(\sqrt{\dfrac{1+2\mathrm{i}\tau}{1-2\mathrm{i}\tau}}\right)$ | $-\dfrac{2\mathrm{i}}{\tau}$ |
| $s(t)$ | $\dfrac{4}{\sin^2(2t)}$ | $\dfrac{4}{\sinh^2(2t)}$ | $\dfrac{1}{t^2}$ |
| $h(z)$ | $-\mathrm{i}\,\mathrm{e}^{4\mathrm{i}z}$ | $2\coth(2z)$ | $\frac{1}{2}z$ |

Now, we are ready to formulate the main result of this section.

THEOREM 4. *There exist three sets, which we call* A, B, *and* C, *of associated families of* $\theta$*-isothermic HIMC surfaces. The immersion function of each family is given by equations* (10) *and* (11), *where the functions of the fundamental forms are presented in Table* 1, *with* $q(t)$ *an arbitrary real solution of* (40) *with* $s(t)$ *specified in Table* 1 *and* $q'(t) < 0$. *All the sets* A, B, *and* C *depend on* 3 *arbitrary parameters and any* $\theta$*-isothermic HIMC surface belong to one of these sets*.

*Proof.* The first and the last statements follow from Lemmas 8 and 9. The statement about the immersion function is the result of Corollary 8. The functions of the fundamental forms given in Table 1 are calculated via the substitution of $f(w)$ and $w(z)$ presented in Lemma 8 into the formulas (42), and using Lemma 1. To compute $s(t)$, one substitutes the results of Lemma 8 into (41). The properties of $q(t)$ are a consequence of Lemma 7 and the positiveness of the metrics (see Table I).

*Remark* 9. Equation (40) with $s(t)$ given in Table I is a generalization of the *Hazzidakis* equation [8].

DEFINITION 3. We call two parametrized surfaces $F_1$ and $F_2$ equivalent up to scaling if $F_2 = cF_1$ holds with $c \in \mathbb{R}\backslash 0$.

Using this equivalence relation, we can formulate

THEOREM 5. *Any associated family of the set* A *consists of four equivalence classes of surfaces, any family of the set* B *consists of one equivalence class, and any family of the set* C *consists of two equivalence classes.*

*Proof.* The scaling of the surface $F \to \eta F$ yields the following transformation of the coefficients of the fundamental forms

$$\mathrm{e}^u \to \eta^2\,\mathrm{e}^u, \qquad Q \to \eta Q, \qquad H \to \frac{1}{\eta}H.$$

To define the different (up to scaling) surfaces (which generate the corresponding equivalence classes), one has to consider the following values for the family parameter $\tau$

$$\tau_A = -1, 0, 1, \infty, \qquad \tau_B = 0, \qquad \tau_C = 0, \infty.$$

It is easy to see from Table I that these surfaces generate, via scaling, the whole corresponding associated family.

*Remark* 10. In the A and C cases, one needs to consider also the asymptotic limits as $\tau \to 0$ and $\infty$ of the functions of the fundamental forms to cover the whole family. To handle the limits $\tau \to 0$ and $\infty$ in the A case, one has to use the *Euler* formula for $\sin(\cdot)$, the formula for $\zeta(\tau)$ given in Table I, and, for the case $\tau \to \infty$, to scale the surface by a factor of $O(\tau^{\pm 2})$. To light this we will discuss the limit for $\tau \to \infty$ in the A case. Using the *Euler* formula for $\sin(\cdot)$, we get

$$\lim_{\tau\to\infty} \mathrm{e}^{u(\tau,z,\bar{z})}\tau^4 = -\frac{8q'(t)\sin^2(2t)}{|\,\mathrm{e}^{2\mathrm{i}z}|^4},$$

$$\lim_{\tau\to\infty} Q(\tau,z,\bar{z}))\tau^2 = \begin{cases} -\dfrac{4(q(t)+\mathrm{i}\theta)}{\mathrm{e}^{4\mathrm{i}z}} & \tau < 0, \\[2ex] \dfrac{4(q(t)+\mathrm{i}\theta)}{\mathrm{e}^{4\mathrm{i}z}} & \tau > 0, \end{cases}$$

$$\lim_{\tau\to\infty} \frac{H(\tau,z,\bar{z})}{\tau^2} = \begin{cases} \dfrac{|\mathrm{e}^{2\mathrm{i}z}|^2}{2\sin(2t)} & \tau < 0, \\[2ex] -\dfrac{|\mathrm{e}^{2\mathrm{i}z}|^2}{2\sin(2t)} & \tau > 0. \end{cases}$$

Scaling the asymptotical surfaces by $\pm 1/\tau^2$ gives coinciding regular limit surfaces for both, $\tau \to \pm\infty$.

*Remark* 11. The surfaces which belong to the associated A, B, and C families of Theorem 4 in the isothermic case $(\theta = 0)$ are dual to the A, B, and C families of Bonnet surfaces originally introduced by E. Cartan [11].

COROLLARY 9. *The HIMC surfaces of revolution are included in the* C *case for* $\theta = \tau = 0$.

*Proof*. Let us compute the coefficients of the fundamental forms for a C case surface with $\theta = \tau = 0$. According to Table I we have

$$\mathrm{e}^{u(0,z,\bar{z})} = -\tfrac{1}{2}q'(t)t^2, \qquad Q(0,z,\overline{z}) = \tfrac{1}{2}q(t), \qquad H(0,z,\overline{z}) = \frac{2}{t},$$

where $t = z + \overline{z} = 2\,\mathrm{Re}(z)$. For $x = \mathrm{Re}(z)$ we find that this reads as follows:

$$\begin{aligned} &\mathrm{e}^{u(0,z,\bar{z})} = -x^2 \frac{\mathrm{d}}{\mathrm{d}x}(q \circ t)(x), \qquad Q(0,z,\overline{z}) = \tfrac{1}{2}(q \circ t)(x), \\ &H(0,z,\overline{z}) = \frac{1}{x}. \end{aligned} \tag{49}$$

On the other hand let us take a surface of revolution parametrized by (25) with $\phi(x)$ a solution of the Painlevé III (20). Define

$$(q \circ t)(x) = \tfrac{1}{4}x(\phi'(x)^2 - 4\sin^2(\phi(x)), \;\; t = 2x.$$

It follows that $q(t) \equiv q \circ t(x)$ solves the Hazzidakis equation (40) with $s(t) = 1/t^2$ and $\theta = 0$ since $\phi(x)$ solves the Painlevé III equation (20). The functions of the fundamental forms of this surface expressed in $(q \circ t)(x)$ are just (49), which proves the corollary.

*Remark* 12. According to Lemma 9 and Remarks 8 and 10 one could equivalently say that the HIMC surfaces of revolution are included in D families for $1/\theta = \tau = \infty$.

LEMMA 10. *Let $F\colon M^2 \to \mathbb{R}^3$ be a conformally parametrized surface in $\mathbb{R}^3$ with the Hopf differential*

$$Q(z,\overline{z}) = q(z,\overline{z}) + \mathrm{i}\,\theta, \qquad q(z,\overline{z}) \in \mathbb{R}, \quad \theta \in \mathbb{R} \textit{ fixed}.$$

*The functions defined via* (35) *are the coefficients of the fundamental forms of a conformally immersed surface*

$$F^{*}\colon M^2 \to \begin{cases} S^3_r, & \textit{if } \theta \neq 0, \\ \mathbb{R}^3, & \textit{if } \theta = 0, \end{cases} \tag{1}$$

*where $r = 1/(2|\theta|)$ is the radius of the sphere.*

*Proof*. $\theta = 0$ can be treated as a limit. The Codazzi equations for conformal immersions $F\colon M^2 \to S^3_{(1/(2|\theta|))}$ are independent of $\theta$. Inserting the functions (35) into the Codazzi equations and the Gauss equation

$$u_{z\bar{z}} + \tfrac{1}{2}(H^2 + (2\theta)^2)\,\mathrm{e}^u - 2|Q|^2\,\mathrm{e}^{-u} = 0.$$

proves the claim.

DEFINITION 4. Let $\mathcal{F}$ be a $\theta$-isothermic surface. $\mathcal{F}^*$ defined above is called a dual surface.

THEOREM 5. *Let* $F\colon M^2 \to \mathbb{R}^3$ *be a* $\theta$*-isothermic HIMC surface. Then its dual surface is a Bonnet surface in the* $S^3_{(1/(2|\theta|))}$ ($\mathbb{R}^3$ *can be considered as a limit*), *i.e. there exists a* 1*-parametric family of isometric surfaces with the same principle curvatures. This associated family is given by the dual surfaces of the associated family of the* $\theta$*-isothermic HIMC surface.*

*Any Bonnet surface* (*isothermically parametrized*) *into* $S^3_{(1/(2|\theta|))}$ *is dual to a* $\theta$*-isothermic HIMC surface with fundamental forms*.

$$\mathrm{e}^{u^*(w,\bar{w})} = \mathrm{e}^{-u(w,\bar{w})}, \qquad H^*(w,\overline{w}) = 2Q(w,\overline{w}) = 2q(w,\overline{w}) \in \mathbb{R},$$

$$Q^*(w,\overline{w}) = \tfrac{1}{2}H(w,\overline{w}) \pm \mathrm{i}\,\theta.$$

*Proof*. According to [25] Bonnet surfaces in $S^3_{(1/(2|\theta|))}$ are characterized by the harmonicity of the inverse of the Hopf differential in isothermic coordinates. Combining this characterization with (35) and the formula above, one proves the theorem.

We finish this section by the following

CONJECTURE 2. Iff the scaling of a HIMC surface generates the associated family, then this surface is $\theta$-isothermic.

## 6. A and B Families and the Painlevé VI Equation

In this section we solve the following generalization of the Hazzidakis equation (see Remark 9),

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \frac{4}{\sin^2(2t)}\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right), \quad t = z + \overline{z}, \tag{50}$$

in terms of the sixth Painlevé equation [12],

$$\begin{aligned} y''(s) \;=\; & \frac{1}{2}\left(\frac{1}{y(s)} + \frac{1}{y(s)-1} + \frac{1}{y(s)-s}\right) y'^2(s) - \\ & - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{y(s)-s}\right) y'(s) + \\ & + \frac{y(s)(y(s)-1)(y(s)-s)}{s^2(s-1)^2} \times \\ & \times \left(\alpha + \beta\,\frac{s}{y^2(s)} + \gamma\,\frac{(s-1)}{(y(s)-1)^2} + \delta\,\frac{s(s-1)}{(y(s)-s)^2}\right), \end{aligned} \tag{51}$$

where $\alpha, \beta, \gamma$, and $\delta \in \mathbb{C}$.

LEMMA 10. *Equation* (50) *has the first integral*

$$2\left(\frac{q''}{4\,q'}+\cot(2t)\right)^2-$$
$$-\frac{1}{q'}\left(\theta^2\left(\cot^2(2t)+1\right)+q^2+\left(q\cot(2t)+\frac{q'}{2}\right)^2\right)=\mu^2. \qquad (52)$$

*Remark* 13. One can easily prove Lemma 10 by verifying, that the derivative of the left-hand side of (52) is a product of the difference between the left- and right-hand sides of (50) with a factor which is finite for the solutions of (50). But we choose another way to prove this lemma since we need later the notations which we introduce in its proof and we want to explain how to get this first integral via the Lax representation for the differential equation (50).

*Proof of Lemma* 10. Let us introduce the similarity variables,

$$s=\mathrm{e}^{4\mathrm{i}t},\qquad \lambda=-\tau\,\mathrm{e}^{4\mathrm{i}z},\qquad \Psi(\lambda,s)=\Psi(\tau,z,\overline{z}). \qquad (53)$$

Substituting them into the moving frame equations (11), in which the coefficients of the fundamental forms are given in Table I for $\tau=-1$, we get the following system

$$\begin{aligned}\Psi_\lambda\Psi^{-1}&=\frac{1}{\lambda}A_0(s)+\frac{1}{\lambda-1}A_1(s)+\frac{1}{\lambda-s}A_s(s),\\ \Psi_s\Psi^{-1}&=\frac{1}{s-\lambda}A_s(s)+B(s),\end{aligned} \qquad (54)$$

where

$$A_0(s)=\begin{pmatrix}a(s)&\varphi(s)\\-\varphi^*(s)&-a(s)\end{pmatrix},\qquad A_1(s)=-\frac{\sigma(s)}{s}\begin{pmatrix}0&0\\1&0\end{pmatrix},$$
$$A_s(s)=\sigma(s)\begin{pmatrix}0&1\\0&0\end{pmatrix},$$
$$B(s)=\begin{pmatrix}-\dfrac{a(s)}{2s}+\dfrac{1}{4s}&-\dfrac{\sigma(s)}{(s-1)}\\[2mm]\dfrac{\varphi^*(s)}{s}+\dfrac{\sigma(s)}{s(s-1)}&\dfrac{a(s)}{2s}-\dfrac{1}{4s}\end{pmatrix},$$

and

$$
\begin{aligned}
a(s) &= -\tfrac{1}{2}\mathrm{i}\left(\frac{q''(t)}{4q'(t)} + \cot(2t)\right),\\
\varphi(s) &= -\frac{\mathrm{i}\sqrt{-2\,q'(t)}}{8} + \frac{\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t}(q(t)+\mathrm{i}\theta)}{2\sin(2t)\sqrt{-2q'(t)}},\\
\sigma(s) &= \frac{\mathrm{e}^{2\mathrm{i}t}}{4}\sin(2t)\sqrt{-2q'(t)},\\
\varphi^*(s) &= \frac{\mathrm{i}\sqrt{-2q'(t)}}{8} - \frac{\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}t}(q(t)-\mathrm{i}\theta)}{2\sin(2t)\sqrt{-2q'(t)}}.
\end{aligned}
\tag{55}
$$

Define the matrix $A_\infty = -A_0 - A_1 - A_s$. Denote the eigenvalues of $A_\nu$ by $\pm\theta_\nu/2$ $(\nu = 0, 1, s, \infty)$. The compatibility condition for system (54) yields that the $\theta_\nu$'s are constants. Since $\det(A_\nu) = -\theta_\nu^2/4$, we get $\theta_1 = \theta_s = 0$ and

$$
\begin{aligned}
\left(\frac{\theta_0}{2}\right)^2 &= a(s)^2 - \varphi(s)\varphi^*(s),\\
\left(\frac{\theta_\infty}{2}\right)^2 &= a(s)^2 - (\varphi(s)+\sigma(s))\left(\varphi^*(s) + \frac{\sigma(s)}{s}\right).
\end{aligned}
\tag{56}
$$

Substituting (55) into (56), one finds the first integral (52), where

$$\mu = -\theta_0^2 - \theta_\infty^2, \qquad \theta = \theta_\infty^2 - \theta_0^2.$$

*Remark* 14. Although we start from the geometrical situation for which $q'(t) < 0$ and get the Lax representation for (50) by using the moving frame equations, this representation (54) is valid for any complex solution of (50). For most of this section, we consider the relation between the complex solutions of (50) and the Painlevé VI functions since it seems to be difficult to describe the geometrical solutions in terms of the Painlevé VI functions. On the other hand for the analysis of the corresponding surfaces one can work with the representation (54) directly and parametrize it in terms of $q$.

LEMMA 11. *Let $q(t)$ be a solution of* (50) *with the first integral* (52) *and $\theta_\infty = \sqrt{(\theta-\mu)/2} \neq 0$. Then*

$$y(s) = 1 + \frac{(s-1)}{1 + s^2\left(\dfrac{a - \theta_\infty/2}{\sigma + \varphi}\right)^2}, \tag{57}$$

*where $a(s)$, $\varphi(s)$, and $\sigma(s)$ are defined in* (55)*, solves the Painlevé VI equation* (51) *with*

$$\alpha = \tfrac{1}{2}(\theta_\infty - 1)^2, \qquad \beta = -\frac{\mu+\theta}{4}, \qquad \gamma = 0, \qquad \delta = \tfrac{1}{2}. \tag{58}$$

*If* $\theta \in \mathbb{R}$ *and* $q'(t) < 0$, *then* $\mu > \max(0, \theta)$.

*Conversely, let* $y(s)$ *be an arbitrary solution of* (51) *with*

$$\alpha, \beta \in \mathbb{C}, \qquad \gamma = 0, \qquad \delta = \tfrac{1}{2}. \tag{59}$$

*Define*

$$\begin{aligned} &\theta_\infty = (1 + \sqrt{2\alpha}), \qquad \theta = (1 + \sqrt{2\alpha})^2 + 2\beta, \\ &\mu = 2\beta - (1 + \sqrt{2\alpha})^2, \end{aligned} \tag{60}$$

*where the branch of* $\sqrt{2\alpha}$ *is fixed such that* $\sqrt{2\alpha} > 0$ *for* $\alpha > 0$. *Then*

$$\begin{aligned} q(t) \;=\; &\mathrm{i}\,\theta \frac{s(y-1) + y - s}{y(s-1)} + \\ &+ 2\mathrm{i}\, \frac{\theta_\infty^2 (y-s)^2 (y-1)^2 - (s(s-1)y' - y(y-1))^2}{(s-1)y(y-1)(y-s)}, \end{aligned} \tag{61}$$

*with* $s = \mathrm{e}^{4\mathrm{i}t}$, *is a solution of* (50) *with the first integral* (52). *Furthermore, if* $\mathrm{i}(1 + \sqrt{2\alpha}) \in \mathbb{R}$, $\beta \geqslant 0$, *and*

$$q'(t) = -8 \frac{(\theta_\infty (y-s)(y-1) - y(y-1) + s(s-1)y')^2}{(s-1)^2 (y-1)(y-s)} < 0, \tag{62}$$

*then according to Theorem* 4, $q(t)$ *defines an* A *family of HIMC surfaces whose functions of its fundamental forms are given in Table* I.

*Proof.* Since $\theta_\infty \neq 0$, one proves that $a(s) \pm \theta_\infty/2 \neq 0$. Thus by defining $\Phi = \Omega\Psi$, where

$$\Omega = \mathrm{i} \sqrt{\frac{a}{\theta_\infty} + \frac{1}{2}} \begin{pmatrix} \dfrac{\sigma + s\varphi^*}{s(a + \theta_\infty/2)} & 1 \\ 1 & \dfrac{\sigma + \varphi}{a + \theta_\infty/2} \end{pmatrix},$$

one finds that the function $\Phi$ solves the system of the form (54) but with other matrices $A_\nu \to \widetilde{A}_\nu$ $(\nu = 0, 1, s)$ and $B \to \widetilde{B}$, where $\widetilde{B}$ is a traceless diagonal matrix. In particular, for the matrix elements we find

$$\frac{\widetilde{A}_1^{12}}{\widetilde{A}_s^{12}} = \frac{1}{s} \left( \frac{\sigma + \varphi}{a - \theta_\infty/2} \right)^2, \qquad \widetilde{A}_0^{11} = \frac{\theta}{4\theta_\infty} - \frac{\theta_\infty}{2} - \frac{\sigma^2}{s\theta_\infty}, \tag{63}$$

$$\widetilde{A}_1^{11} = -\frac{\theta}{4\theta_\infty} - \frac{\sigma\varphi^*}{\theta_\infty}, \qquad \widetilde{A}_s^{11} = -\frac{\theta}{4\theta_\infty} - \frac{\sigma}{s\,\theta_\infty\,\varphi}. \tag{64}$$

Now, to prove (57) and (58), we use the parametrization of $\widetilde{A}_\nu^{ik}$ in terms of the Painlevé VI function presented in [21]. If $\theta \in \mathbb{R}$ and $q'(t) < 0$, then $\varphi^* = \overline{\varphi}$

and $a$ is pure imaginary. Therefore, (56) yields $\theta_0, \theta_\infty \in \mathrm{i}\mathbb{R}$, which proves that $\mu > \max(0, \theta)$.

Let $y(s)$ be a solution of (51) with coefficients given by (59). Then we define the parameters $\theta_\infty \neq 0$, $\mu$, and $\theta$ according to (60). Since $\theta_\infty \neq 0$, we can use the parametrization of the system (54) $(A_\nu \to \widetilde{A}_\nu)$ in terms of Painlevé VI functions [21]. We substitute this parametrization into the left-hand side of (64) and Equations (55) into the right-hand side of (64) for $\sigma\varphi^*$ and $\sigma/\varphi$ to get a linear algebraic system for $q(t)$ and $q'(t)$. Solving this system we arrive at the formulas for $q(t)$ (61) and $q'(t)$ (62). We need to prove the correctness of these definitions as it is not clear a priori that $q'(t)$ is, in fact, the derivative of $q(t)$: to prove this, differentiate (61) with respect to $t$ and substitute for $y''(t)$ equation (51) with the constants (58). This gives (62). Now, using the compatibility conditions for (54) in terms of $a$, $\sigma$, $\varphi$, and $\varphi^*$, we get the formula for $a(s)$ (55) and that the function $q(t)$ satisfies (50).

THEOREM 6. *If* $q(t) \neq \pm\theta \cot(2t)$ *is a solution of* (50)*, then it satisfies* (52)*. The function* $y(s)$ *given by* (57) *with* $\theta_\infty = \sqrt{(\theta - \mu)/2}$ *solves the Painlevé VI equation* (51) *with the coefficients* (58)*. Moreover, if* $\theta \in \mathbb{R}$ *and* $q'(t) < 0$*, then* $\mu > \max(0, \theta)$.

*Conversely, if* $y(s)$ *is a solution of the Painlevé VI equation* (51) *with the coefficients* (59)*, then* $q(t)$ *given by* (61) *and* (60) *is a solution of* (50) *with the first integral* (52)*. Furthermore, if* $\mathrm{i}(1 + \sqrt{2\alpha}) \in \mathbb{R}$, $\beta \geqslant 0$*, and inequality* (62) *is valid, then* $q(t)$ *defines an* A *family of HIMC surfaces according to Theorem* 4.

*Proof.* Since the case $\theta_\infty \neq 0$ is considered in Lemma 11, we must deal only with the case $\theta_\infty = 0$. If $\theta_\infty = 0$ and $q(t) \neq \theta \cot(2t)$, then formulas (57) and (61) are well defined for the solutions of (50) and (51), respectively. One proves the statement of the theorem in this case by a direct calculation.

*Remark* 15. The functions $q(t) = \pm\theta \cot(2t)$ are the only two solutions of the Hazzidakis equations which are not expressible through (61) in terms of the Painlevé VI functions. The function $q(t) = \theta \cot(2t)$ for $\theta > 0$ is a geometrical solution of (50) which is investigated in Section 8.

In the rest of this section, we consider the solution of another generalization of the Hazzidakis equation (40) (B case),

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \frac{4}{\sinh^2(2t)} \left(2 - \frac{q^2(t) + \theta^2}{q'(t)}\right), \tag{65}$$

with $t = w + \overline{w}$, in terms of Painlevé VI equations. Since, as in the A case, we did not find a proper description of the geometrical solution in terms of the Painlevé VI functions, we present here only a result for the general complex solutions of (65). An explicit formula for the case B can be obtained as a direct corollary of Theorem 6 and the following:

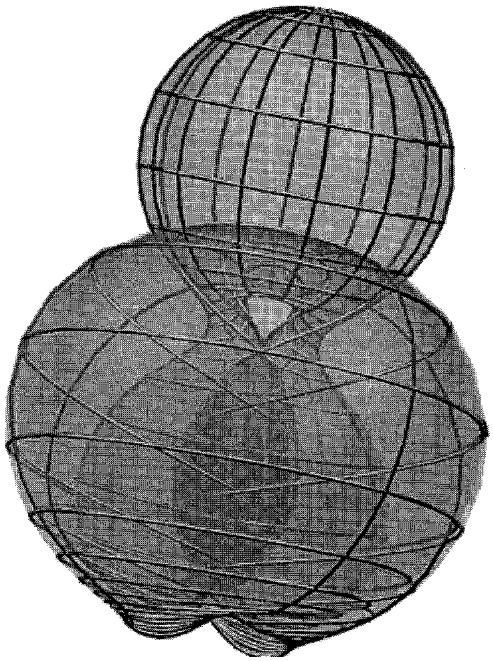

*Figure 2.* $\theta$-Isothermic HIMC surface of Type B.

LEMMA 12. *Let* $q_A(t_A)$ *be an arbitrary solution of* (50) *with* $\theta = \theta_A \in \mathbb{C}$. *Then*

$$q_B(t_B) = -iq_A(t_A), \quad \textit{where } \ t_B = it_A, \tag{66}$$

*is a solution of* (65) *with* $\theta = \pm i\theta_A$.

*Proof*. Straightforward verification.

*Remark* 16. One finds an explicit formula for $q_B(t_B)$ in terms of the Painlevé VI functions by substituting $\theta \to i\theta$, $q(t) \to iq(t)$, and $s \to \mathrm{e}^{4t}$ into (61). This formula yields all solutions of (65) except $q(t) = \pm i\theta \coth(2t)$.

Figure 2 shows a numerical produced picture of a $\theta$-isothermic surface of the B type. The initial values inserted in (40), $(s(t)$ as for the B case given in Table I) are

$$\theta = 1, \qquad \xi(\tau) = i, \qquad t_0 = 0.1,$$
$$q(t_0) = 0, \qquad q'(t_0) = -1, \qquad q''(t_0) = 1.$$

## 7. C Families and the Painlevé V Equation

In this section we consider the solution of the following generalization of the Hazzidakis equation (see Remark 9),

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \frac{1}{t^2}\left(2 - \frac{q^2(t) + \theta^2}{q'(t)}\right), \tag{67}$$

in terms of the fifth Painlevé equation [12],

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{t} + \\ + \frac{(y-1)^2}{t^2}\left(\alpha y + \beta \frac{1}{y}\right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}, \tag{68}$$

where $\alpha, \beta, \gamma$, and $\delta \in \mathbb{R}$. Let us start with the following.

LEMMA 13. *Equation* (67) *has the first integral*,

$$\left(\frac{q''(t)}{2q'(t)} + \frac{1}{t}\right)^2 - \frac{q^2(t) + \theta^2}{2q'(t)t^2} - \frac{q'(t)}{2} - \frac{q(t)}{t} = \frac{\mu^2}{4}. \tag{69}$$

*Proof.* To prove this lemma, we use the way more convenient for us rather than short: the reasons are analogous to those explained in Remark 13. We introduce the similarity variables

$$\lambda = \frac{z + \zeta}{z + \overline{z}}, \qquad t = z + \overline{z}, \qquad \Psi(\lambda, t) = \Psi(\tau, z, \overline{z}), \tag{70}$$

where $\zeta$ is given in Table I. By substituting them into the moving frame equations (11), in which the coefficients of the fundamental forms are taken from Table I for $\zeta = \infty$, we get the following system

$$\begin{aligned} \Psi_\lambda \Psi^{-1} &= t \begin{pmatrix} a(t) & \varphi(t) \\ \overline{\varphi}(t) & -a(t) \end{pmatrix} + \frac{\mathrm{e}^{u(t)/2}}{2} \begin{pmatrix} 0 & \dfrac{1}{\lambda - 1} \\ -\dfrac{1}{\lambda} & 0 \end{pmatrix}, \\ \Psi_t \Psi^{-1} &= \lambda \begin{pmatrix} a(t) & \varphi(t) \\ \overline{\varphi}(t) & -a(t) \end{pmatrix} + \begin{pmatrix} -\dfrac{a(t)}{2} & 0 \\ -\overline{\varphi}(t) & \dfrac{a(t)}{2} \end{pmatrix}, \end{aligned} \tag{71}$$

where

$$a(t) = \frac{u'(t)}{2} = \frac{q''(t)}{2q'(t)} + \frac{1}{t}, \qquad \varphi(t) = -\frac{q(t) + \mathrm{i}\theta}{\sqrt{-2q'(t)t}} + \sqrt{\frac{-q'(t)}{2}}. \tag{72}$$

It is a consequence of the compatibility condition of system (71) that the determinant of the first matrix is independent of $\lambda$ and $t$. Thus, we get the first integral of (67),

$$a^2(t) + |\varphi(t)|^2 = \frac{\mu^2}{4}, \tag{73}$$

which can be rewritten due to (72) as (69).

*Remark* 17. Note that, the case $\mu = 0$ for real solutions of (67) is possible, iff $\theta = 0$ because, otherwise, there must exists a real-valued function $q$ which solves

$$q' = -\frac{1}{t}(q + \mathrm{i}\,\theta). \tag{74}$$

The case $\mu = \theta = 0$ is investigated in Section 8: for the rest of this section $\mu \neq 0$.

THEOREM 7. *If* $q(t) \neq C/t$ *(*$C \in \mathbb{C}$*) is a solution of* (67), *then it satisfies* (69), *and the functions*

$$z(t) = -\frac{1}{2\mu}\varphi(t)\,\mathrm{e}^{u(t)/2}, \qquad y(t) = \frac{2a(t) + \mu}{2a(t) - \mu}, \tag{75}$$

*where* $\varphi(t)$ *and* $a(t)$ *are given by* (72), *solve the following system*

$$\begin{aligned} tz' &= z\left(-\mathrm{i}\,\frac{\theta}{\mu} + z\right)\left(y - \frac{1}{y}\right), \\ ty' &= \mu t y - 2z(y-1)^2 + \mathrm{i}\,\frac{\theta}{\mu}(y-1)^2. \end{aligned} \tag{76}$$

*Moreover, if* $q'(t) < 0$, *then* $y$ *is a negative solution of* (68) *with the coefficients*

$$\alpha = -\beta = -\frac{\theta^2}{4\mu^2}, \qquad \gamma = \mu, \qquad \delta = -\frac{\mu^2}{2}. \tag{77}$$

*Conversely, if* $y$ *is a solution of the Painlevé V equation* (68) *with the coefficients* (77), *where* $\theta$ *and* $\mu \neq 0$ *are arbitrary numbers, then*

$$q(t) = \frac{t(\mu^2 y^2 - y'^2)}{2y(y-1)^2} - \theta^2 \frac{(y-1)^2}{2\mu^2 t y} \tag{78}$$

*is a solution of* (67) *with the first integral* $\mu$ (69). *Moreover, if* $y < 0$ *and does not solve*

$$y' = \mu y \pm \mathrm{i}\,\frac{\theta}{\mu t}(y-1)^2,$$

*then* $q(t)$ *given by* (78) *is real and defines a HIMC surface* (*see Theorem* 4) *with the metric function*

$$\mathrm{e}^{u(t)} = -\theta^2 \frac{(y-1)^2}{\mu^2 y} - \frac{t^2(\mu y - y')^2}{y(y-1)^2}.$$

*Proof.* The existence of the first integral (69) is the result of Lemma 13. To find an explicit representation of $q(t)$ in terms of the fifth Painlevé function, let us consider the following gauge transformation, $\Phi = \Omega(t)\Psi$, where

$$\Omega(t) = \frac{v(t)^{\mathrm{i}\,\mathbf{k}}}{\sqrt{\mu\varphi(t)}} \begin{pmatrix} a(t) + \frac{1}{2}\mu & \varphi(t) \\ a(t) - \frac{1}{2}\mu & \varphi(t) \end{pmatrix},$$

with $\mathbf{k}$ defined in (6) and $v(t)$ a solution of the differential equation

$$v'(t) = v(t)\frac{\varphi(t)}{\mu}\left(\frac{a(t)}{\varphi(t)}\right)' + \tfrac{1}{4}\mu.$$

*Remark* 18. Note that, for $\mu \neq 0$, the function $\varphi(t)$ has no zeros. Since any zero of $\varphi(t)$ solves (74), there are no such zeros for $\theta \neq 0$. Moreover, for $\theta = 0$, any solution $\varphi(t)$ having at least one zero is identically zero: this is only possible for $\mu = 0$ (see Equation (73)).

*Remark* 19. The gauge transformation is only an analytical trick rather than a geometrical transformation of the moving frame because $\Omega(t) \notin \mathrm{SU}(2)$.

Now for $\Phi$ we get the following system

$$\begin{aligned}\Phi_\lambda\Phi^{-1} &= \frac{\mu t^{\mathrm{i}\,\mathbf{k}}}{2} + \frac{1}{\lambda}\frac{\varphi(t)\,\mathrm{e}^{u(t)/2}}{2\mu}\begin{pmatrix} -1 & v(t)^2 \\ -\dfrac{1}{v(t)^2} & 1 \end{pmatrix} + \\ &\quad + \frac{1}{\lambda-1}\frac{\mathrm{e}^{u(t)/2}}{2\mu\varphi(t)}\begin{pmatrix} |\varphi(t)|^2 & (a(t)+\frac{1}{2}\mu)^2 v(t)^2 \\ -\dfrac{(2a(t)-\mu)^2}{4v(t)^2} & -|\varphi(t)|^2 \end{pmatrix}, \\ \Phi_t\Phi^{-1} &= \frac{\mu\lambda}{2}\sigma_3 + \begin{pmatrix} -\frac{1}{4}\mu & \dfrac{v(t)^2}{2}(a(t)+\frac{1}{2}\mu) \\ \dfrac{1}{v(t)^2}(a(t)-\frac{1}{2}\mu) & \frac{1}{4}\mu \end{pmatrix} + \\ &\quad + \Omega_t\Omega^{-1}.\end{aligned} \tag{79}$$

On the one hand, the coefficients $a(t)$ and $\varphi(t)$ of the system (79) are expressed in terms of the function $q(t)$ via (72). On the other hand, it is a result of the work [21] that these coefficients can be presented in terms of a Painlevé V function (68) with the coefficients given by (77). Comparing (79) with the parameterization given in [21], we get the following system

$$\begin{aligned} a(t) &= \tfrac{1}{2}\mu\frac{y(t)+1}{y(t)-1} = \frac{q''}{2q'} + \frac{1}{t}, \\ \varphi(t)\,\mathrm{e}^{u(t)/2} &= \mu\frac{t(y'-\mu y)}{(y-1)^2} - i\theta = -q't - q - i\theta, \\ \overline{\varphi}(t)\,\mathrm{e}^{-u(t)/2} &= \frac{\mu^2 y}{\mu t(\mu y - y') + i\theta(y-1)^2} = \frac{1}{2t} + \frac{q-i\theta}{2t^2 q'}, \end{aligned} \tag{80}$$

and that the functions $z(t)$ and $y(t)$ defined by (75) satisfy system (76).

If $q'(t) < 0$, then $q(t)$ is real and, hence, $a(t)$ is real. Since $q(t)$ is real and $q(t) \neq C/t$, it is not a solution of (74), and thus $\varphi(t) \neq 0$. Now, from (73) and (75) we get that $y(t) < 0$.

Conversely, let $y(t)$ be a solution of (68) with (77). Then, we define the function $q(t)$ by using the system (80). The consistency condition for this system gives us (69). Differentiating (69) and taking into consideration that $(q''/q'+2/t) \neq 0$, since $y(t) = -1$ is not a solution of (68) with the coefficients given by (77), one arrives at (67). Solving the last two equations in (80), one finds an explicit expression for $q(t)$ (78), and

$$q'(t) = \frac{1}{y(t)}\left(\theta^2\frac{(y(t)-1)^2}{2\mu^2 t^2} + \frac{(y'(t)-\mu y(t))^2}{2(y(t)-1)^2}\right).$$

If $y(t) < 0$, then $q(t)$ is real and $q'(t) < 0$. Now, to finish the proof, we use Theorem 4 and the formula for $\mathrm{e}^{u(t)}$ from Table I.

*Remark* 20. In the case $\theta = 0$ one puts

$$y(t) = -\cot^2(\phi(x))$$

into the fifth Painlevé equation (68) with the coefficients (77) to get a third Painlevé equation in trigonometric form (20). The solutions of this Painlevé III equation are in one-to-one correspondence with the HIMC surfaces of revolution as proved in Section 4.

*Remark* 21. The parameter $\mu$ is not an essential parameter in our description of HIMC surfaces since it can be removed from the Painlevé V equation (68) with the coefficients in (77) via the scaling $t \to \mu t$. This scaling is just a reparameterization of the surfaces: so, in the case $\mu \neq 0$, one may fix $\mu = 1$.

*Remark* 22. The scaling property of the associated families stated in Theorem 5, which we establish by calculation of the functions of the fundamental forms, can be derived in another way: use the Sym formula (10) and the $\lambda$-equations (54) or

(71) obtained in Section 6 and here. Actually, by means of (54) or (71), we can rewrite the Sym formula as follows

$$F(\tau, z, \overline{z}) = \frac{\partial\lambda}{\partial\tau}\Psi^{-1}(\lambda, t)A(\lambda, t)\Psi(\lambda, t), \tag{81}$$

where $A(\lambda, t)\Psi(\lambda, t)$ denotes the right-hand side of (54) or (71). Now let us consider the A case with $\tau < 0$ (the calculations in the A case $\tau > 0$, the B case, and the C case are similar). According to (53) we have

$$\frac{\partial\lambda}{\partial\tau} = -\mathrm{e}^{4\,\mathrm{i}\,z}$$

and all other terms depend on $\tau$ only through the variable $\lambda$. For the immersion function this implies

$$F(\tau, z, \overline{z}) = -\tau F(-1, z + \tfrac{1}{4}i\, \log(-\tau), \overline{z} - \tfrac{1}{4}i\, \log(-\tau)). \tag{82}$$

Thus the associated family $F(\tau, z, \overline{z})$ can be obtained by the reparametrization $z \to z + \frac{1}{4}\mathrm{i}\, \log(-\tau) = z + \xi(\tau)$ with $\xi(\tau)$ as given in Table I of the surface $F(-1, z, \overline{z})$ and its scaling by the factor $(-\tau)$.

## 8. Special Families of HIMC Surfaces and Cartan Families

It is proved (see Remark 15), that in the A case there are two and only two special solutions of the generalized Hazzidakis equation (50), $q(t) = \pm\theta \cot(2t)$, which are not related with the Painlevé VI equation. Since we are considering $t > 0$, only one of these solutions, namely,

$$q(t) = \theta \cot(2t), \quad \theta > 0, \tag{83}$$

generates a HIMC A family. In Section 7, where we analyzed the C case, we obtained that for $\theta = \mu = 0$, there is a one-parameter family of solutions

$$q(t) = \frac{\alpha}{t}, \quad \alpha > 0 \text{ for } t > 0, \tag{84}$$

which is not related to the fifth Painlevé equation, but generates a HIMC C family. The A families produced by the solutions (83) seem to be, at first glance, non-isothermic since $\theta \neq 0$, while the C families are, of course, isothermic ones. Unexpectedly, both solutions (83) and (84) define the same families.

PROPOSITION 4. *The* A *families defined by the solutions* (83) *coincide with the* C *families generated by the solutions* (84) *if* $\theta = 2\alpha$.

*Proof.* The following equations are an explicit mapping from the special A family into the C one

$$\alpha = \tfrac{1}{2}\theta, \qquad \tau_C = \sqrt{2|\tau_A|}, \qquad z_C = \mathrm{sign}(\tau_A)\mathrm{i}\,(\mathrm{e}^{4\,\mathrm{i}\,z_A} - 1),$$

where $\tau_A$ and $\tau_C$ are the family parameters, and $z_A$ and $z_C$ are local coordinates.

DEFINITION 4. The special C families of HIMC surfaces corresponding to solution (84) are called *Cartan families*.

The motivation for this definition might be clarified by the following.

PROPOSITION 5. *Iff an isothermic HIMC surface is dual to itself and is not a CMC surface, then it belongs to a Cartan family*.

*Proof.* If a non-CMC HIMC surface is dual to itself, then $1/H$ and $1/Q$ are non-constant harmonic functions. By using the Codazzi equations (5), one proves that there is a local conformal parameterization of this surface in which the coefficients of the fundamental forms read as follows

$$Q(z,\overline{z}) = bH(z,\overline{z}) = \frac{a}{z+\overline{z}}, \qquad \mathrm{e}^{u(z,\bar{z})} = 2\frac{a^2}{b}, \quad |a|b > 0,$$

which means that it is an applicable Bonnet (and HIMC) surface. Now, scaling its local coordinate $z \to a/(2b)z$, we get exactly the same functions of the fundamental forms as for the special HIMC surface defined by solution (84) with $\alpha = a^2/b$ and $\tau \to 0$ (see Table I and Theorem 4).

*Remark* 23. Due to this Proposition, the special C families (84) of HIMC surfaces are simultaneously special C families of Bonnet surfaces. Cartan was the first who distinguished these special C families as being the only kind of families of applicable Bonnet surfaces [7, 11].

COROLLARY 10. *All $\theta$-isothermic HIMC surfaces with $\theta \neq 0$, which do not belong to the Cartan families, are nonisothermic*.

*Proof.* This statement is a consequence of Propositions 3 and 5.

PROPOSITION 6. *The immersion function for the Cartan families is*

$$F(\tau,z,\overline{z}) = \frac{1}{t}\begin{pmatrix} \mathrm{Re}\left(\dfrac{\omega^2}{1-\lambda}G^2(\lambda) - \lambda G'^2(\lambda)\right) \\ \mathrm{Im}\left(\dfrac{\omega^2}{1-\lambda}G^2(\lambda) - \lambda G'^2(\lambda)\right) \\ 2\omega\,\mathrm{Re}(G'(\lambda)G(1-\lambda)) \end{pmatrix}, \tag{85}$$

*where*

$$G(\lambda) = C_1\,{}_2F_1(\mathrm{i}\omega,-\mathrm{i}\omega,1;\lambda) + C_2(1-\lambda)\,{}_2F_1(1+\mathrm{i}\omega,1-\mathrm{i}\omega,2;1-\lambda),$$

$C_1, C_2 \in \mathbb{R}$, *and*

$$\lambda = \frac{\tau\overline{z}+2\,\mathrm{i}}{\tau(z+\overline{z})}, \qquad t = z+\overline{z}, \qquad \omega = \tfrac{1}{2}\sqrt{\tfrac{1}{2}\alpha}.$$

*Here* ${}_2F_1(a,b,c;t)$ *is the standard notation for the Gaussian hypergeometric function* [22].

*Proof.* To derive formula (85) one substitutes into the Sym formula (10) $\Psi_\tau(\tau,z,\overline{z}) = \lambda_\tau\Psi_\lambda(\lambda,t)$, where $\Psi_\lambda(\lambda,t)$ is given by the first equation of (71) with $\lambda$ changed to $1-\lambda$. Since in our case $a(t) = \varphi(t) = \theta = \mu = 0$, (71) as follows

$$\Psi_\lambda(\lambda,t)\Psi^{-1}(\lambda,t) = \omega\begin{pmatrix} 0 & \dfrac{1}{\lambda} \\ \dfrac{1}{1-\lambda} & 0 \end{pmatrix}.$$

The general solution of this equation is given by

$$\Psi(\lambda,t) = \begin{pmatrix} G(\lambda) & -\dfrac{1}{\omega}(1-\lambda)G'(1-\lambda) \\ \dfrac{1}{\omega}\lambda G'(\lambda) & G(1-\lambda) \end{pmatrix},$$

where $G$ solves the hypergeometric differential equation

$$\lambda(1-\lambda)G''(\lambda) + (1-\lambda)G'(\lambda) - \omega^2 G(\lambda) = 0:$$

since $\Psi(\lambda,t)\in \mathrm{SU}(2)$, $C_1, C_2 \in \mathbb{R}$. One uses the Sym formula (10) to get the immersion function $F(\tau,z,\overline{z})$ in the quaternionic representation. Since the Cartan surfaces are cones, one simplifies $F(\tau,z,\overline{z})$ by scaling to get, after identification (7), the final result (85).

*Remark* 24. The variation of the constants $C_1$ and $C_2$ just means a rotation of the surface as a whole. Therefore, they can be fixed arbitrarily, but not both equal to zero.

According to Proposition 6, each Cartan family consists of two different surfaces: a cone $(\tau \neq 0)$, and a cylinder $(\tau = 0)$. Actually, from the point of view of projective geometry, there is only one surface in each family: we call it the Cartan cone.

In Figures 3–5, we present the parts of different Cartan cones inside spheres, centered at their vertices. The black curves are the sections of the cones by the spheres. In all pictures these spherical curves wind around two centers whose positions depends on the parameter $\alpha$ in Equation (84); therefore, $\alpha$ regulates the number of self-intersections of the cone. Since the Cartan cones depend smoothly on $\alpha$, there should be a discrete set of values of $\alpha$ such that the corresponding Cartan cones have infinitly many lines of self-intersection when the centers of the spherical curves belong to them.

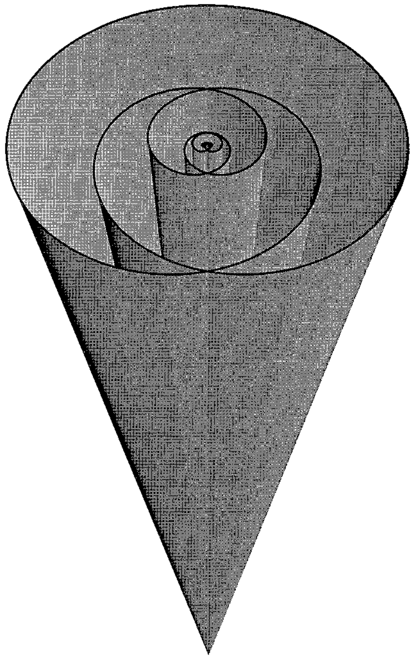

*Figure 3.* $(\alpha = 32)$.

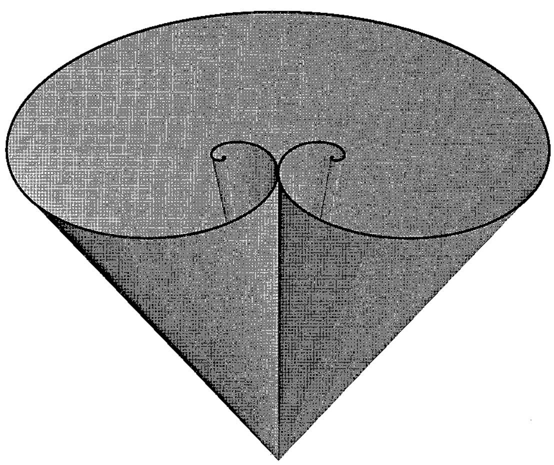

*Figure 4.* $(\alpha = 8)$.

## Appendix. The Hopf Differential as a Hamiltonian for the Painlevé Equations

It is well known that the Painlevé equations admit Hamiltonian formulations [21, 23, 24]. Since, as was shown in Section 4, Proposition 1, geometrical objects acquire a simple formulation in Hamiltonian notation, we consider here Hamiltonian structures associated with the Painlevé equations arising in our study of

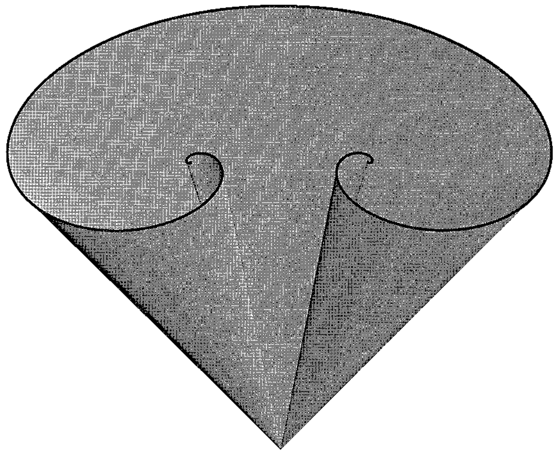

*Figure 5.* $(\alpha = 2)$.

HIMC surfaces. In the A and B cases, a reference to the Hamiltonian structure allows us to get another representation of solutions of the generalized Hazzidakis equations (50) and (65) in terms of the sixth Painlevé functions then that obtained in Section 6.

PROPOSITION 7. *Let* $q(t)$ *be a solution of* (50) *with the first integral* (52) *and* $q(t) \neq \pm\theta\cot(2t)$. *Denote by* $\{i, j, k, l\}$ *a permutation of the set* $\{1, 2, 3, 4\}$ *and put*

$$b_k = b_l = 0, \qquad b_{i,j} = \tfrac{1}{2}\sqrt{-\mu \pm \sqrt{\mu^2 - \theta^2}}.$$

*Then*

$$H_6(s) = \frac{1}{s(s-1)}\left(\frac{q(t)}{8i} - (b_1b_3 + b_1b_4 + b_3b_4)s + \frac{1}{2}\sum_{1\leqslant m<n\leqslant 4} b_mb_n\right), \qquad (86)$$

*where* $s = \frac{1}{2} - \frac{1}{2}\mathrm{i}\cot(2t)$, *is the Hamiltonian function corresponding to the Hamiltonian* $\mathcal{H}_6(q_6, p_6, s) : H_6(s) = \mathcal{H}_6(q_6(s), p_6(s), s)$

$$\mathcal{H}_6(q_6, p_6, s) = \frac{1}{s(s-1)}(q_6(q_6-1)(q_6-s)p_6^2 - $$

$$-((b_1+b_2)(q_6-1)(q_6-s) +$$

$$+(b_1-b_2)(q_6-s)q_6+(b_3+b_4)(q_6-1)q_6)p_6+$$
$$+(b_1+b_3)(b_1+b_4)(q_6-s)),$$

*where* $(q_6(s), p_6(s))$ *is a solution of the Hamiltonian system*

$$\frac{\mathrm{d}q_6}{\mathrm{d}s}=\frac{\partial\mathcal{H}_6(q_6,p_6,s)}{\partial p_6},\qquad \frac{\mathrm{d}p_6}{\mathrm{d}s}=-\frac{\partial\mathcal{H}_6(q_6,p_6,s)}{\partial q_6}. \tag{87}$$

*The function* $y(s)=q_6(s)$ *solves the Painlevé VI equation* (51) *with the coefficients*

$$\begin{aligned}\alpha&=\tfrac{1}{2}(b_3-b_4)^2, &\beta&=-\tfrac{1}{2}(b_1+b_2)^2,\\ \gamma&=\tfrac{1}{2}(b_1-b_2)^2, &\delta&=\tfrac{1}{2}(1-(b_3+b_4+1)^2).\end{aligned} \tag{88}$$

*Conversely, if* $y(s)=q_6(s)$ *is a solution of* (51) *with the coefficients* (88)*, then one finds from the first equation of* (87) *the function* $p_6(s)$ *and, thereby, from Equation* (86)*, the function* $q(t)\neq\theta\cot(2t)$.

*Proof.* The Hamiltonian formulation for the Painlevé VI equation is due to Okamoto [23]: he also proved that the Hamiltonian function for the Painlevé VI equation is the general solution of a second-order differential equation quadratic with respect to the second derivative. We just notice that the first integral (52) can be transformed into the Okamoto equation by the change of variables presented in the Proposition.

COROLLARY 11. *Let* $Q_A(\tau,z,\overline{z})$ *be a Hopf differential in the parametrization given in Theorem* 4 *for* A *families of HIMC surfaces defined by solutions of the sixth Painlevé equation* (51) *with the coefficients*

$$\alpha=0,\qquad \beta=\frac{\mu-\theta}{4},\qquad \gamma=-\frac{\mu+\theta}{4},\qquad \delta=0.$$

*Then*

$$Q_A(\tau,z,\overline{z})=\frac{8\mathrm{i}s(s-1)H_6(s)}{\tau\sin^2(2(z-\zeta(\tau)))}, \tag{89}$$

*where* $H_6(s)$ *is defined in Proposition* 7 *for*

$$b_3=b_4=0,\qquad b_{1,2}=\tfrac{1}{2}\sqrt{-\mu\pm\sqrt{\mu^2-\theta^2}}.$$

*Proof.* Direct consequence of Theorem 4 and Proposition 7.

COROLLARY 12. *Let* $q(t)$ *be a solution of* (65)*; then, it has the first integral*

$$2\left(\frac{q''}{4q'}+\coth(2t)\right)^2-$$
$$-\frac{1}{q'}\left(\theta^2(\coth^2(2t)-1)-q^2+\left(q\coth(2t)+\frac{q'}{2}\right)^2\right)=\mu.$$

*Let* $\{i,j,k,l\}$ *be a permutation of the set* $\{1,2,3,4\}$. *Define*

$$b_k = b_l = 0, \qquad b_{i,j} = \tfrac{1}{2}\sqrt{\mu \pm \sqrt{\mu^2+\theta^2}};$$

*then, for* $q(t) \neq \pm i\theta \coth(2t)$, *the function*

$$H_6(s) = \frac{1}{s(s-1)}\left(\frac{q(t)}{8} - (b_1b_3 + b_1b_4 + b_3b_4)s + \frac{1}{2}\sum_{1\leqslant m<n\leqslant 4} b_m b_n\right),$$

*where* $s = \frac{1}{2}(1+\coth(2t))$, *is a Hamiltonian function corresponding to the Hamiltonian* $\mathcal{H}_6(q_6,p_6,s)$ *defined in Proposition* 7.

*For* $\{i,j,k,l\} = \{3,4,2,1\}$ *and* $q'(t) < 0$, *the Hopf differential is as follows*

$$Q_B(\tau,z,\overline{z}) = \frac{32s(s-1)H_6(s)}{(1+4\tau^2)\sinh^2(2(z-\zeta(\tau)))}. \tag{90}$$

*The function* $y(s) = q_6(s)$ *solves a Painlevé VI equation with the coefficients*

$$\alpha = 0, \qquad \beta = -\frac{\mu+i\theta}{2}, \qquad \gamma = \frac{\mu-i\theta}{2}, \qquad \delta = 0.$$

*Proof.* This statement is an implication of Theorem 4, Lemma 12, and Proposition 7.

PROPOSITION 8. *Let* $y(t)$ *be a solution of the fifth Painlevé equation* (68) *with the coefficients* (77). *Then the system* (76) *with respect to the canonical variables* $q_5 = y$ *and* $p_5 = z/y$ *gains a Hamiltonian form*

$$\frac{\mathrm{d}q_5}{\mathrm{d}t} = \frac{\partial\mathcal{H}_5(q_5,p_5,t)}{\partial p_5}, \qquad \frac{\mathrm{d}p_5}{\mathrm{d}t} = -\frac{\partial\mathcal{H}_5(q_5,p_5,t)}{\partial q_5},$$

*with the Hamiltonian*

$$\mathcal{H}_5(q_5,p_5,t) = -\frac{1}{t}\left((q_5-1)^2q_5p_5^2 - \mu t p_5 q_5 - i\frac{\theta}{\mu}(q_5-1)^2p_5\right).$$

*The Hopf differential* $Q_C(\tau,z,\overline{z})$ *for the non-Cartan* C *families* (*see Theorem* 4 *and Section* 8) *is, up to a holomorphic function, equal to the Hamiltonian function. In the parametrization of Theorem* 4.

$$Q_C(\tau,z,\overline{z}) = \frac{\zeta^2(\tau)}{(z-\zeta(\tau))^2}\mathcal{H}_5\left(y(t),\frac{z(t)}{y(t)},t\right). \tag{91}$$

*Proof*. For the Hamiltonian formulation of the fifth Painlevé equation, see the works [21] and [24]. The proof follows from the fact that

$$\operatorname*{Res}_{\lambda=1}(\mathrm{tr}(\Psi_\lambda\Psi^{-1})^2) = \operatorname*{Res}_{\lambda=1}(\mathrm{tr}(\Phi_\lambda\Phi^{-1})^2),$$

where $\Psi$ and $\Phi = \Omega(t)\Psi$ are the solutions of (71) and (79), respectively, and the definition of the Hamiltonian in the framework of the Isomonodromy Deformation Method [21].

*Remark* 25. As it is obvious from (91), $Q_C(\tau, w, \overline{w}) = H_5(t)$ for a proper local conformal reparametrization of the surfaces in this case. In the A and B cases, one can not find such reparametrization for (89) and (90) since the factor $s(s-1)$ is not a holomorphic function: but, possibly, the richness of the group of canonical transformations for the sixth Painlevé Hamiltonian system [23] may allow one to find, instead of (89) and (90), relations between the Hopf differentials and Hamiltonians for these cases with holomorphic factors.

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