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Discrete isothermic surfaces

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1. Introduction

Surfaces in an Euclidean 3-space studied by the theory of integrable systems are usually described as immersions of $\mathbb{R}^2$ or of some domain in $\mathbb{R}^2$

$$F: \mathbb{R}^2 \to \mathbb{R}^3 . \tag{1}$$

The list includes surfaces with constant negative Gaussian and mean curvature and some others (see the survey [2]). In this case $\mathbb{R}^2$ is treated as a space-time of the corresponding integrable system. One can describe topological planes, cylinders and tori in this way. To study surfaces of more complicated topology one should deal with immersions

$$F: \mathcal{R} \to \mathbb{R}^3 , \tag{2}$$

where $\mathcal{R}$ is some Riemann surface. The traditional theory of integrable systems needs essential improvements to be applied to this case and up to now there exists no real progress.

This paper is part of our general program on the discretization of surfaces described by integrable systems. By such a surface we mean a map

$$F: G \to \mathbb{R}^3 , \tag{3}$$

where G is some graph, which is treated as a discrete space-time of the corresponding discrete integrable system. When the size of the edges tends to zero such a surface approximates a corresponding smooth surface (2). The case closest to (1)

$$F: \mathbb{Z}^2 \to \mathbb{R}^3 \tag{4}$$

is relatively well understood. In [3], [4] the discrete surfaces (1) with constant Gaussian and mean curvature were defined and their geometrical properties were investigated (for

[1]) Partially supported by the Sonderforschungsbereich 288.

the loop group factorization interpretation of these surfaces see [10]). We hope that the integrable discretization (3) may become a method to understand the structure of its corresponding smooth limit (2). Besides this, the elementary geometry of the case (4) itself is very nice and worth studying.

The present paper was initiated by the recent preprint of J. Cieśliński, P. Goldstein and A. Sym [7]. These authors, based on classical results of P. Calapso [6] and W. Blaschke [1], give a characterization of isothermic surfaces as "soliton surfaces" by introducing a spectral parameter. The geometrical meaning of this spectral parameter was clarified in [5], where it was shown how pairs of isothermic surfaces are given by curved flats in a pseudo-Riemannian symmetric space and vice versa.

Here, using the spinor representation of $SO(4,1)$, in Section 3 we rewrite the zero-curvature representation for the associated family of isothermic surfaces in terms of 2×2 matrices with quaternionic coefficients. A natural discretization of this representation yields the following definition of discrete isothermic surfaces.

Definition 1. A discrete isothermic surface is a map $F: \mathbb{Z}^2 \to \mathbb{R}^3$ such that the cross-ratios $q_{n,m} = q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ of all elementary quadrilaterals are real $q_{n,m} \in \mathbb{R}$ and satisfy the factorization condition

$$q_{n,m} q_{n+1,m+1} = q_{n+1,m} q_{n,m+1} . \tag{5}$$

This definition coincides with the definition (35) in the Remark after Definition 6 in Section 4.

Geometrical properties of discrete isothermic surfaces as well as the analytical description of the corresponding associated family are studied in Sections 4, 5. In Section 7 we define discrete isothermic minimal surfaces, derive a Weierstrass type representation for them and construct some simple examples.

2. Isothermic surfaces and their properties

Let $\mathscr{F}$ be a smooth surface in a 3-dimensional Euclidean space and $F(x,y)$

$$F = (F_1, F_2, F_3): \mathscr{R} \to \mathbb{R}^3 ,$$

a conformal parametrization of $\mathscr{F}$

$$\langle F_x, F_x \rangle = \langle F_y, F_y \rangle = e^u, \quad \langle F_x, F_y \rangle = 0 . \tag{6}$$

Here the brackets denote the scalar product

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 ,$$

and F_x and F_y are the partial derivatives $\partial F / \partial x$ and $\partial F / \partial y$.

The vectors F_x, F_y as well as the normal N,

$$(7) \qquad \langle F_x, N\rangle = \langle F_y, N\rangle = 0, \quad \langle N, N\rangle = 1\,,$$

define a moving frame on the surface. We consider a local theory: $\mathscr{R}$ is a domain in $\mathbb{R}^2$.

If the second fundamental form is diagonal

$$(8) \qquad -\langle dF, dN\rangle = e^u(k_1 dx^2 + k_2 dy^2)$$

with respect to the induced metric

$$(9) \qquad \langle dF, dF\rangle = e^u(dx^2 + dy^2)\,,$$

then the parametrization is called isothermic. In this case k_1 and k_2 are the principal curvatures and the preimages of the curvature lines are the lines $x = \text{const}$ and $y = \text{const}$ in the parameter domain.

Definition 2. Surfaces which admit isothermic coordinates are called isothermic.

Surfaces of revolution, quadrics and constant mean curvature surfaces without umbilics are examples of isothermic surfaces. We consider isothermic immersions, i.e. suppose that $u(x, y)$ is finite.

Due to (6, 7) the moving frame

$$\sigma = (F_x, F_y, N)^T$$

satisfies the following Gauss-Weingarten equations:

$$(10) \qquad \sigma_x = \begin{pmatrix} u_x/2 & -u_y/2 & k_1 e^u \\ u_y/2 & u_x/2 & 0 \\ -k_1 & 0 & 0 \end{pmatrix} \sigma, \quad \sigma_y = \begin{pmatrix} u_y/2 & u_x/2 & 0 \\ -u_x/2 & u_y/2 & k_2 e^u \\ 0 & -k_2 & 0 \end{pmatrix} \sigma.$$

The compatibility conditions of this system are the Gauss-Codazzi equations

$$(11) \qquad \begin{aligned} &u_{xx} + u_{yy} + 2k_1 k_2 e^u = 0\,, \\ &2k_{2x} - (k_1 - k_2)\,u_x = 0\,, \\ &2k_{1y} + (k_1 - k_2)\,u_y = 0\,. \end{aligned}$$

Our goal is to find a proper discrete version of isothermic surfaces. Let us mention two important geometrical properties of isothermic surfaces which we want to retain in the discrete case.

Property 1 (Möbius invariance). *Let $F: \mathscr{R} \to \mathbb{R}^3$ be an isothermic immersion and $\mathscr{M}$ a Möbius transformation of Euclidean* 3*-space. Then $\tilde{F} \equiv \mathscr{M} \circ F: \mathscr{R} \to \mathbb{R}^3$ is also isothermic.*

Proof. Isothermic coordinates (8), (9) can be characterized in terms of F only

$$\|F_x\| = \|F_y\|, \quad \langle F_x, F_y\rangle = 0, \quad F_{xy} \in \operatorname{span}\{F_x, F_y\}. \tag{12}$$

Clearly, it is enough to prove Property 1 for the case of an inversion $\mathscr{M}$

$$\tilde{F} = \frac{F}{\langle F, F\rangle}.$$

Calculating $\tilde{F}_x, \tilde{F}_y$ we see the conformality of $\tilde{F}$. A direct calculation proves the third property of (12) of $\tilde{F}$

$$\tilde{F}_{xy} = \left(\alpha - 2\frac{\langle F, F_y\rangle}{\langle F, F\rangle}\right)\tilde{F}_x + \left(\beta - 2\frac{\langle F, F_x\rangle}{\langle F, F\rangle}\right)\tilde{F}_y,$$

where α and β are defined by

$$F_{xy} = \alpha F_x + \beta F_y.$$

Property 2 (Dual surface). *Let $F: \mathscr{R} \to \mathbb{R}^3$ be an isothermic immersion. Then the immersion $F^*: \mathscr{R} \to \mathbb{R}^3$ defined by the formulas*

$$F_x^* = e^{-u}F_x, \quad F_y^* = -e^{-u}F_y \tag{13}$$

is isothermic. The Gauss maps of F and F^* are antipodal

$$N = -N^*.$$

The fundamental forms of F^* are as follows

$$\begin{aligned} \langle dF^*, dF^*\rangle &= e^{-u}(dx^2 + dy^2), \\ -\langle dF^*, dN^*\rangle &= -k_1 dx^2 + k_2 dy^2. \end{aligned} \tag{14}$$

Definition 3. The immersion $F^*: \mathscr{R} \to \mathbb{R}^3$ defined above is called dual to F.

Proof of Property 2. The definition (13) of F^* makes sense since the equality $F_{xy}^* = F_{yx}^*$ is equivalent to $(e^{-u}F_x)_y = \frac{1}{2}e^{-u}(u_x F_y - u_y F_x) = (-e^{-u}F_y)_x$. Here we use (13) and the Gauss-Weingarten equation for F_{xy}. The conformality of F^* is evident. The expressions (14) are obtained by straightforward calculation.

Remark. $F^{**} = F$.

To discretize isothermic surfaces (see Sect. 5) and for investigation by analytical methods it is more convenient to use 2×2 matrices instead of 3×3 matrices (10), therefore first we rewrite equations (10) for the moving frame in terms of quaternions.

Let us denote the algebra of quaternions by $\mathbf{H}$, the multiplicative quaternion group by $\mathbf{H}_* = \mathbf{H}\setminus\{0\}$ and the standard basis of $\mathbf{H}$ by $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$,

(15) $$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}.$$

Using the standard matrix representation of $\mathbf{H}$ the Pauli matrices σ_α are related to this basis as follows:

(16) $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\mathbf{i}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\mathbf{j},$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\mathbf{k}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We identify 3-dimensional Euclidean space with the space of imaginary quaternions $\operatorname{Im}\mathbf{H}$

(17) $$X = -i\sum_{\alpha=1}^{3} X_\alpha \sigma_\alpha \in \operatorname{Im}\mathbf{H} \leftrightarrow X = (X_1, X_2, X_3) \in \mathbb{R}^3.$$

The scalar product of vectors in terms of matrices is then

(18) $$\langle X, Y \rangle = -\frac{1}{2}\operatorname{tr} XY.$$

We also denote by F and N the matrices obtained in this way from the vectors F and N.

Let us take a unitary quaternion

(19) $$\Phi \in \mathbf{H}_*, \quad \det \Phi = 1,$$

which transforms the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into the basis F_x, F_y, N:

(20) $$F_x = e^{u/2}\Phi^{-1}\mathbf{i}\Phi, \quad F_y = e^{u/2}\Phi^{-1}\mathbf{j}\Phi, \quad N = \Phi^{-1}\mathbf{k}\Phi.$$

A simple calculation proves the following form of the Gauss-Weingarten system.

Theorem 1. *The moving frame F_x, F_y, N of an isothermic surface is described by the formulas* (20), *where the unitary quaternion Φ satisfies the equations*

(21) $$\Phi_x = A\Phi, \quad \Phi_y = B\Phi$$

with A, B of the form

(22) $$A = \frac{u_y}{4}\mathbf{k} + \frac{k_1}{2}e^{u/2}\mathbf{j}, \quad B = -\frac{u_x}{4}\mathbf{k} - \frac{k_2}{2}e^{u/2}\mathbf{i}.$$

The compatibility condition of (21)

(23) $$A_y - B_x + [A, B] = 0$$

is the Gauss-Codazzi system (11).

3. Lax representation for smooth isothermic surfaces

To put a class of surfaces into frames of the theory of integrable equations one should "insert" a spectral parameter λ into the Gauss-Codazzi system

$$A_y(\lambda) - B_x(\lambda) + [A(\lambda), B(\lambda)] = 0 .$$

Geometrically λ describes some deformation of surfaces preserving their properties (one can find many examples in [2]). In some sense, the case of isothermic surfaces is more complicated. A characterization of isothermic surfaces as "soliton surfaces" by introducing a spectral parameter was given in a short note by J. Cieśliński, P. Goldstein and A. Sym [7]. They described a Lax pair with coefficients in $so(4,1)$. The Lax representation (24), (25) we present below results by using a 4-dimensional spinor representation of $so(4,1)$ (see, for example, the Appendix in [8]). By direct calculation one can check the following statement:

Theorem 2. *The system*

$$\Psi_x = U(\lambda)\Psi, \quad \Psi_y = V(\lambda)\Psi, \tag{24}$$

$$U(\lambda) = \begin{pmatrix} A & \lambda \mathbf{i} e^{u/2} \\ \lambda \mathbf{i} e^{-u/2} & A \end{pmatrix}, \quad V(\lambda) = \begin{pmatrix} B & \lambda \mathbf{j} e^{u/2} \\ -\lambda \mathbf{j} e^{-u/2} & B \end{pmatrix}, \tag{25}$$

where A, B are the matrices (22), *is compatible if and only if the Gauss-Codazzi equations of isothermic surfaces* (11) *are satisfied.*

Whereas for $\lambda = 0$ the system (24), (25) is just a doubled Gauss-Weingarten system (21), (22), the geometrical meaning of the Lax representation with $\lambda \neq 0$ is more complicated, and we do not discuss it in this paper. For this interpretation we refer to the recent preprint [5] by F. Burstall, U. Hertrich-Jeromin, F. Pedit and U. Pinkall, where they showed how pairs of isothermic surfaces are given by curved flats in a pseudo-Riemannian symmetric space and vice versa.

Let us note the following symmetries of the system (24), (25). If $\Psi(x, y, \lambda)$ is a solution of the Cauchy problem with $\Psi(x_0, y_0, \lambda) = I$, then

$$\Psi(-\lambda) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Psi(\lambda) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{26}$$

and

$$\Psi(\lambda = 0) = \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \tag{27}$$

where $\Phi \in H_*$, solves (21), (22).

Theorem 3 (Sym formula). *Let $\Psi(x, y, \lambda)$ be a solution of* (24), (25) *satisfying* (26), (27), *then the isothermic immersion F and its dual F^* are given by the formula*

$$\Psi^{-1} \frac{\partial \Psi}{\partial \lambda} \Big|_{\lambda = 0} \equiv \hat{F} = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}. \tag{28}$$

The corresponding Gauss maps are

$$\Psi^{-1}\begin{pmatrix} 0 & \mathbf{k} \\ -\mathbf{k} & 0 \end{pmatrix}\Psi|_{\lambda=0} \equiv \hat{N} = \begin{pmatrix} 0 & N \\ N^* & 0 \end{pmatrix}.$$

Proof. For $\hat{F}_x$ formulas (27), (28) imply

$$\hat{F}_x = \Psi^{-1}\frac{\partial U}{\partial \lambda}\Psi|_{\lambda=0} = \begin{pmatrix} 0 & e^{u/2}\Phi^{-1}\mathbf{i}\Phi \\ e^{-u/2}\Phi^{-1}\mathbf{i}\Phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & F_x \\ F_x^* & 0 \end{pmatrix},$$

where one should use (20), (13). The proof for $\hat{F}_y$ and $\hat{N}$ is the same.

4. Discrete isothermic surfaces

To define discrete isothermic surfaces we need a notion of a cross-ratio of a quadrilateral (X_1, X_2, X_3, X_4) in 3-dimensional Euclidean space. One can easily extend the notion of the cross-ratio of complex numbers to points in $\mathbb{R}^3$ identifying a sphere[2]) S, passing through X_1, X_2, X_3, X_4 with the Riemann sphere $\mathbb{CP}^1$.

It turns out that there is a nice quaternionic description of the cross-ratio if one uses the isomorphism (17).

Definition 4. Let $X_1, X_2, X_3, X_4 \in \operatorname{Im}\mathbf{H}$ be 4 points in $\mathbb{R}^3$ (see Fig. 1). The unordered pair $\{q, \bar{q}\}$ of eigenvalues of the quaternion

(29) $$Q = (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}$$

is called the cross-ratio of the quadrilateral (X_1, X_2, X_3, X_4).

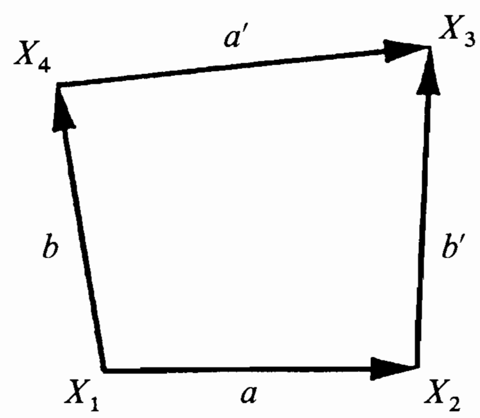

Figure 1. Vertices and edges of a quadrilateral

Q is invariant with respect to translations and dilatation of $\mathbb{R}^3$; rotations of $\mathbb{R}^3$ act on Q as $Q \to RQR^{-1}$ and therefore preserve $\{q, \bar{q}\}$.

[2]) A plane is a special case.

We usually will just speak of the "cross-ratio $q \in \mathbb{C}$". One has to keep in mind that q is only well defined up to complex conjugation. Denoting the quaternions corresponding to the vector-edges as in Fig. 1 one gets

$$Q = aa'^{-1}b'^{-1}b\,.$$

If the quadrilateral (X_1, X_2, X_3, X_4) is planar, it can be put to the $\mathbf{i}$, $\mathbf{j}$ plane and its edges

$$a_1\mathbf{i} + a_2\mathbf{j} = (a_1\mathbf{1} + a_2\mathbf{k})\mathbf{i}$$

can be identified with complex numbers

$$a = a_1 + ia_2 \in \mathbb{C}\,. \tag{30}$$

For the cross-ratio this implies

$$q = \frac{aa'}{bb'} \in \mathbb{C}\,. \tag{31}$$

Lemma 1. *The cross-ratio is invariant with respect to the Möbius transformations of* $\mathbb{R}^3$.

Proof. Only for inversions the proof needs some calculation. Describing the inversion by the inverse matrix

$$X \to \tilde{X} = \frac{X}{\langle X, X\rangle} = -X^{-1} \tag{32}$$

we get

$$\tilde{Q} = (X_1^{-1} - X_2^{-1})(X_2^{-1} - X_3^{-1})^{-1}(X_3^{-1} - X_4^{-1})(X_4^{-1} - X_1^{-1})^{-1} = X_1^{-1}QX_1\,.$$

The eigenvalues of Q and $\tilde{Q}$ coincide.

Definition 5. Let $F: \mathscr{R} \to \mathbb{R}^3$ be an immersion and $F, F_x, \ldots, F_{yy}$ the values of the immersion function and its derivatives at some point $(x, y) \in \mathscr{R}$. A one-parametric family of quadrilaterals $F^\varepsilon = (F_1, F_2, F_3, F_4)$ with vertices

$$F_1 = F + \varepsilon(-F_x - F_y) + \frac{\varepsilon^2}{2}(F_{xx} + F_{yy} + 2F_{xy})\,,$$

$$F_2 = F + \varepsilon(F_x - F_y) + \frac{\varepsilon^2}{2}(F_{xx} + F_{yy} - 2F_{xy})\,,$$

$$F_3 = F + \varepsilon(F_x + F_y) + \frac{\varepsilon^2}{2}(F_{xx} + F_{yy} + 2F_{xy})\,,$$

$$F_4 = F + \varepsilon(-F_x + F_y) + \frac{\varepsilon^2}{2}(F_{xx} + F_{yy} - 2F_{xy})$$

is called an infinitesimal quadrilateral at (x, y).

Up to terms of order $O(\varepsilon^3)$ the vertices F_1, F_2, F_3, F_4 of the infinitesimal quadrilateral coincide with $F(x-\varepsilon, y-\varepsilon)$, $F(x+\varepsilon, y-\varepsilon)$, $F(x+\varepsilon, y+\varepsilon)$, $F(x-\varepsilon, y+\varepsilon)$ respectively. The following remark is trivial:

Lemma 2. (1) $q(F^\varepsilon) = -1 + O(\varepsilon) \Leftrightarrow Q(F^\varepsilon) = -I + O(\varepsilon)$,

(2) $q(F^\varepsilon) = -1 + O(\varepsilon^2) \Leftrightarrow Q(F^\varepsilon) = -I + O(\varepsilon^2)$.

Theorem 4. *Conformal and isothermic immersions F are characterized in terms of infinitesimal quadrilateral as follows*:

(1) $Q(F^\varepsilon) = -I + O(\varepsilon) \Leftrightarrow$ *F is conformal*,

(2) $Q(F^\varepsilon) = -I + O(\varepsilon^2) \Leftrightarrow$ *F is isothermic*.

Proof. To calculate the cross-ratio of the infinitesimal quadrilateral we note, that F^ε up to scaling is a translation of the quadrilateral with vertices at

$$X_1 = 0, \quad X_2 = F_x - \varepsilon F_{xy}, \quad X_3 = F_x + F_y, \quad X_4 = F_y - \varepsilon F_{xy}.$$

Inverting it by the transformation (32) we send one of the points to infinity

$$\tilde{X}_1 = \infty, \quad \tilde{X}_2 = \frac{F_x - \varepsilon F_{xy}}{\|F_x - \varepsilon F_{xy}\|^2}, \quad \tilde{X}_3 = \frac{F_x + F_y}{\|F_x + F_y\|^2}, \quad \tilde{X}_4 = \frac{F_y - \varepsilon F_{xy}}{\|F_x - \varepsilon F_{xy}\|^2}.$$

The condition

$$Q(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4) = -I + O(\varepsilon^2)$$

is equivalent to the equality of vectors

$$\overrightarrow{\tilde{X}_2\tilde{X}_3} = \overrightarrow{\tilde{X}_3\tilde{X}_4} + O(\varepsilon^2),$$

or equivalently

$$(33) \qquad 2\frac{F_x + F_y}{\|F_x + F_y\|^2} = \frac{F_x - \varepsilon F_{xy}}{\|F_x - \varepsilon F_{xy}\|^2} + \frac{F_y - \varepsilon F_{xy}}{\|F_y - \varepsilon F_{xy}\|^2} + O(\varepsilon^2)$$

$$= \frac{F_x - \varepsilon F_{xy}}{\|F_x\|^2}\left(1 + 2\varepsilon\frac{\langle F_x, F_{xy}\rangle}{\|F_x\|^2}\right) + \frac{F_y - \varepsilon F_{xy}}{\|F_y\|^2}\left(1 + 2\varepsilon\frac{\langle F_y, F_{xy}\rangle}{\|F_y\|^2}\right) + O(\varepsilon^2).$$

The zero order term in (33) yields (6) and the term of order ε implies that F_{xy} lies in the tangential plane, i.e. the third condition in (12).

The following geometrical properties of the cross-ratio are standard and can be easily checked:

Lemma 3. *If the cross-ratio of a quadrilateral* (X_1, X_2, X_3, X_4) *is real* $q \in \mathbb{R}$, $Q = qI$, *then* X_1, X_2, X_3, X_4 *lie on a circle. The cross-ratio of a square is* $q = -1$.

Now we are in a position to give a definition of discrete isothermic surfaces. By a discrete surface we mean a map $F: \mathbb{Z}^2 \to \mathbb{R}^3$. We use the following notations for the elements of discrete surfaces (n, m are integer labels):

$F_{n,m}$ – for the vertices,

$[F_{n+1,m}, F_{n,m}]$, $[F_{n,m+1}, F_{n,m}]$ – for the edges,

$(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ – for the elementary quadrilaterals.

Motivated by Theorem 4 above we define discrete isothermic surfaces as follows:

Definition 6. A discrete isothermic surface is a map $F: \mathbb{Z}^2 \to \mathbb{R}^3$, for which all elementary quadrilaterals have cross-ratio -1:

$$(34) \qquad Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = -I \quad \text{for all} \quad n, m \in \mathbb{Z}.$$

Remark. More generally discrete isothermic surfaces can be defined by the property that $Q_{n,m}$ is a product of two factors

$$(35) \qquad Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = -\frac{\beta_m}{\alpha_n} I \quad \text{for all} \quad n, m \in \mathbb{Z},$$

where α_n does not depend on m and β_m not on n. This condition is reformulated in another way in Definition 1. Mostly we restrict ourself in this paper to the simplest symmetric case (34).

Lemma 1 implies

Theorem 5 (Möbius invariance). *Let $F: \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete isothermic surface and $\mathcal{M}$ a Möbius transformation of Euclidean 3-space. Then $\tilde{F} \equiv \mathcal{M} \circ F: \mathbb{Z}^2 \to \mathbb{R}^3$ is also isothermic.*

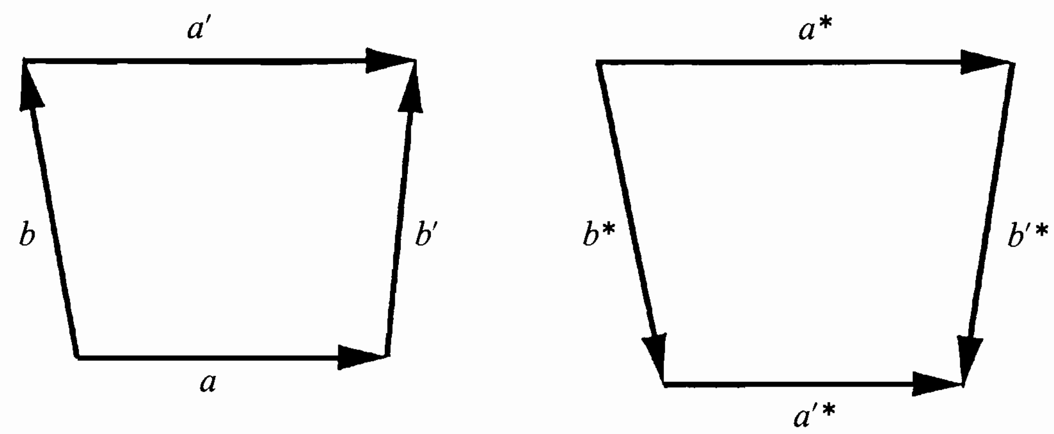

Figure 2. An elementary quadrilateral and its dual

Property 2 of smooth isothermic surfaces also persists in the discrete case:

Theorem 6 (Dual surface). *Let $F: \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete isothermic surface. Then the discrete surface $F^*: \mathbb{Z}^2 \to \mathbb{R}^3$ defined (up to translation) by the formulas*

(36) $$F^*_{n+1,m} - F^*_{n,m} = \frac{F_{n+1,m} - F_{n,m}}{\|F_{n,m} - F_{n+1,m}\|^2}, \quad F^*_{n,m+1} - F^*_{n,m} = -\frac{F_{n,m+1} - F_{n,m}}{\|F_{n,m} - F_{n,m+1}\|^2}$$

is isothermic.

Proof. It is convenient to use the complex language (30) (elementary quadrilaterals are planar). Thus we assume $a, b, a', b' \in \mathbb{C}$.

For the elementary quadrilateral $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ we have

(37) $$a + b' = b + a', \quad aa' = -bb'.$$

Formulas (36) imply

(38) $$a^* = \frac{1}{\bar{a}}, \quad a'^* = \frac{1}{\bar{a}'}, \quad b^* = -\frac{1}{\bar{b}}, \quad b'^* = -\frac{1}{\bar{b}'}$$

for the edges of the dual quadrilateral. Combining (37) with (38) one gets

$$a^* + b'^* = b^* + a'^*, \quad a^* a'^* = -b^* b'^*,$$

the dual quadrilateral closes up and has a cross-ratio -1.

Remark. For a discrete isothermic surface in the sense of definition (35) the dual surface is defined by

(39) $$F^*_{n+1,m} - F^*_{n,m} = \frac{1}{\alpha_n^2} \frac{F_{n+1,m} - F_{n,m}}{\|F_{n,m} - F_{n+1,m}\|^2},$$

$$F^*_{n,m+1} - F^*_{n,m} = -\frac{1}{\beta_m^2} \frac{F_{n,m+1} - F_{n,m}}{\|F_{n,m} - F_{n,m+1}\|^2}.$$

The cross-ratios of the corresponding quadrilaterals of F and F^* coincide

(40) $$Q(F^*_{n,m}, F^*_{n+1,m}, F^*_{n+1,m+1}, F^*_{n,m+1}) = -\frac{\beta_m}{\alpha_n} I.$$

5. Lax representation for discrete isothermic surfaces

Definition 6 of discrete isothermic surfaces is geometrically motivated and looks natural. In this section we explain how it was found. Namely, we discretize the Lax representation (24), (25) in a natural way preserving all its symmetries and show how Definition 6 appears in this approach.

An integrable discretization of the system (24), (25) looks as follows. To each point (n, m) of the $\mathbb{Z}^2$-lattice one associates a matrix $\Psi_{n,m}$. These matrices at two neighbouring vertices are related by

(41) $$\Psi_{n+1,m} = \mathcal{U}_{n,m} \Psi_{n,m}, \quad \Psi_{n,m+1} = \mathcal{V}_{n,m} \Psi_{n,m},$$

where the matrices $\mathcal{U}_{n,m}$ and $\mathcal{V}_{n,m}$ are associated to the edges connecting the points $(n+1,m)$, (n,m) and $(n,m+1)$, (n,m) respectively. Having in mind the continuum limit (ε is a characteristic size of edges)

$$\mathcal{U} = I + \varepsilon U + \cdots, \quad \mathcal{V} = I + \varepsilon V + \cdots$$

with U, V of the form (25), and preserving the group structure and the dependency of λ in the continuous case, it is natural to set

$$(42) \qquad \mathcal{U}_{n,m} = \begin{pmatrix} \mathcal{A}_{n,m} & \lambda p'_{n,m}\mathbf{i} \\ \lambda p''_{n,m}\mathbf{i} & \mathcal{A}_{n,m} \end{pmatrix}, \quad \mathcal{A}_{n,m} = a_{n,m}\mathbf{1} + b_{n,m}\mathbf{k} + c_{n,m}\mathbf{j},$$

$$\mathcal{V}_{n,m} = \begin{pmatrix} \mathcal{B}_{n,m} & \lambda q'_{n,m}\mathbf{j} \\ -\lambda q''_{n,m}\mathbf{j} & \mathcal{B}_{n,m} \end{pmatrix}, \quad \mathcal{B}_{n,m} = d_{n,m}\mathbf{1} + e_{n,m}\mathbf{k} + f_{n,m}\mathbf{j},$$

where the fields p, q, a, b, c, d, e, f live on the corresponding edges.

These matrices can be simplified

$$\mathcal{U}_{n,m} \to \frac{g_{n+1,m}}{g_{n,m}} \mathcal{U}_{n,m}, \quad \mathcal{V}_{n,m} \to \frac{g_{n,m+1}}{g_{n,m}} \mathcal{V}_{n,m}$$

by a λ-independent gauge transformation

$$\Psi_{n,m} \to g_{n,m} \Psi_{n,m}$$

with $g_{n,m} \in \mathbb{R}$ at vertices. By such a transformation, which preserves the structure (42) of $\mathcal{U}$ and $\mathcal{V}$, one can achieve the normalization $p'_{n,m} = 1/p''_{n,m} \equiv p_{n,m}$, $q'_{n,m} = 1/q''_{n,m} \equiv q_{n,m}$ for all n, m. Then $\mathcal{U}$ and $\mathcal{V}$ are of the form

$$(43) \qquad \mathcal{U}_{n,m} = \begin{pmatrix} \mathcal{A}_{n,m} & \lambda p_{n,m}\mathbf{i} \\ \lambda p_{n,m}^{-1}\mathbf{i} & \mathcal{A}_{n,m} \end{pmatrix}, \quad \mathcal{V}_{n,m} = \begin{pmatrix} \mathcal{B}_{n,m} & \lambda q_{n,m}\mathbf{j} \\ \lambda q_{n,m}^{-1}\mathbf{j} & \mathcal{B}_{n,m} \end{pmatrix}.$$

Let us take four adjacent vertices (n,m), $(n+1,m)$, $(n+1,m+1)$, $(n,m+1)$ and consider the compatibility condition

$$(44) \qquad \mathcal{V}'\mathcal{U} = \mathcal{U}'\mathcal{V},$$

changing notations for the matrices $\mathcal{U}_{n,m}$, $\mathcal{U}_{n,m+1}$, $\mathcal{V}_{n,m}$, $\mathcal{V}_{n+1,m}$ and their coefficients as it is shown in Fig. 3.

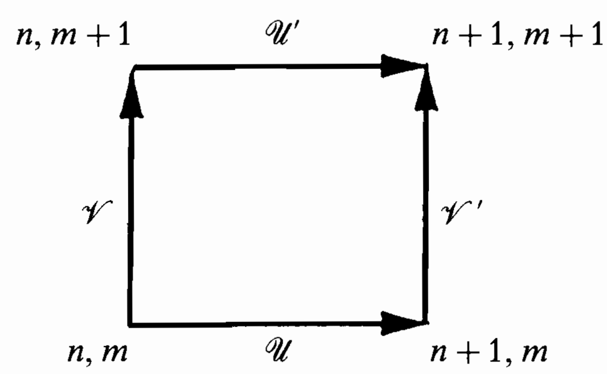

Figure 3. Compatibility condition

Then for the coefficients of $\mathcal{U}, \mathcal{U}', \mathcal{V}, \mathcal{V}'$ one gets

$$pp' = qq', \tag{45}$$

$$\mathcal{B}'\mathcal{A} = \mathcal{A}'\mathcal{B}, \tag{46}$$

$$p\mathcal{B}'\mathbf{i} + q'\mathbf{j}\mathcal{A} = q\mathcal{A}'\mathbf{j} + p'\mathbf{i}\mathcal{B}, \tag{47}$$

$$\frac{1}{p}\mathcal{B}'\mathbf{i} - \frac{1}{q'}\mathbf{j}\mathcal{A} = -\frac{1}{q}\mathcal{A}'\mathbf{j} + \frac{1}{p'}\mathbf{i}\mathcal{B}. \tag{48}$$

With the notation

$$u_{n,m}(\lambda) = \det \mathcal{U}_{n,m}, \quad v_{n,m}(\lambda) = \det \mathcal{V}_{n,m}$$

(49) implies

$$v_{n+1,m}(\lambda)\, u_{n,m}(\lambda) = u_{n,m+1}(\lambda)\, v_{n,m}(\lambda). \tag{49}$$

Now note that the zeros of both sides of (49) (considered as functions of λ) should coincide. This implies

$$(\lambda^2 + \det \mathcal{B}_{n+1,m})^2 (\lambda^2 + \det \mathcal{A}_{n,m})^2 = (\lambda^2 + \det \mathcal{A}_{n,m+1})^2 (\lambda^2 + \det \mathcal{B}_{n,m})^2 .$$

We suppose that the zeros of $u_{n,m+1}(\lambda)$ and $u_{n,m}(\lambda)$ coincide. This is equivalent to

$$\det \mathcal{A}_{n,m} = \alpha_n, \quad \det \mathcal{B}_{n,m} = \beta_m, \tag{50}$$

where we have incorporated into the notation that α_n does not depend on m and β_m not on n.

Now let us *define* discrete surfaces $F_{n,m}$ and $F^*_{n,m}$ in $\mathbb{R}^3$ by the Sym formula (28)

$$\Psi^{-1}_{n,m} \frac{\partial \Psi_{n,m}}{\partial \lambda}\Big|_{\lambda=0} \equiv \hat{F}_{n,m} = \begin{pmatrix} 0 & F_{n,m} \\ F^*_{n,m} & 0 \end{pmatrix}, \tag{51}$$

where $\Psi_{n,m}(\lambda)$ is the solution of the initial value problem (41), (43) with $\Psi_{0,0}(\lambda) = I$.

Lemma 4. *The discrete surface $F_{n,m}$ defined by* (51) *is isothermic in the sense of the generalized definition* (35):

$$Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = -\frac{\beta_m}{\alpha_n} I .$$

The surface $F^*_{n,m}$ in (51) *is its dual* (39).

Proof. Formula (51) implies for the edges

$$\begin{pmatrix} 0 & F_{n+1,m} - F_{n,m} \\ F^*_{n+1,m} - F^*_{n,m} & 0 \end{pmatrix} = \Psi^{-1}_{n,m} \mathcal{U}^{-1}_{n,m} \frac{\partial \mathcal{U}_{n,m}}{\partial \lambda}\Big|_{\lambda=0} \Psi_{n,m}$$

$$= \begin{pmatrix} 0 & p_{n,m}\Phi^{-1}_{n,m}\mathcal{A}^{-1}_{n,m}\mathbf{i}\Phi_{n,m} \\ p^{-1}_{n,m}\Phi^{-1}_{n,m}\mathcal{A}^{-1}_{n,m}\mathbf{i}\Phi_{n,m} & 0 \end{pmatrix},$$

where the matrices $\mathcal{U}_{n,m}$ and $\mathcal{V}_{n,m}$ are associated to the edges connecting the points $(n+1, m)$, (n, m) and $(n, m+1)$, (n, m) respectively. Having in mind the continuum limit (ε is a characteristic size of edges)

$$\mathcal{U} = I + \varepsilon U + \cdots, \quad \mathcal{V} = I + \varepsilon V + \cdots$$

with U, V of the form (25), and preserving the group structure and the dependency of λ in the continuous case, it is natural to set

$$(42) \qquad \mathcal{U}_{n,m} = \begin{pmatrix} \mathcal{A}_{n,m} & \lambda p'_{n,m}\mathbf{i} \\ \lambda p''_{n,m}\mathbf{i} & \mathcal{A}_{n,m} \end{pmatrix}, \quad \mathcal{A}_{n,m} = a_{n,m}\mathbf{1} + b_{n,m}\mathbf{k} + c_{n,m}\mathbf{j},$$

$$\mathcal{V}_{n,m} = \begin{pmatrix} \mathcal{B}_{n,m} & \lambda q'_{n,m}\mathbf{j} \\ -\lambda q''_{n,m}\mathbf{j} & \mathcal{B}_{n,m} \end{pmatrix}, \quad \mathcal{B}_{n,m} = d_{n,m}\mathbf{1} + e_{n,m}\mathbf{k} + f_{n,m}\mathbf{j},$$

where the fields p, q, a, b, c, d, e, f live on the corresponding edges.

These matrices can be simplified

$$\mathcal{U}_{n,m} \to \frac{g_{n+1,m}}{g_{n,m}} \mathcal{U}_{n,m}, \quad \mathcal{V}_{n,m} \to \frac{g_{n,m+1}}{g_{n,m}} \mathcal{V}_{n,m}$$

by a λ-independent gauge transformation

$$\Psi_{n,m} \to g_{n,m} \Psi_{n,m}$$

with $g_{n,m} \in \mathbb{R}$ at vertices. By such a transformation, which preserves the structure (42) of $\mathcal{U}$ and $\mathcal{V}$, one can achieve the normalization $p'_{n,m} = 1/p''_{n,m} \equiv p_{n,m}$, $q'_{n,m} = 1/q''_{n,m} \equiv q_{n,m}$ for all n, m. Then $\mathcal{U}$ and $\mathcal{V}$ are of the form

$$(43) \qquad \mathcal{U}_{n,m} = \begin{pmatrix} \mathcal{A}_{n,m} & \lambda p_{n,m}\mathbf{i} \\ \lambda p^{-1}_{n,m}\mathbf{i} & \mathcal{A}_{n,m} \end{pmatrix}, \quad \mathcal{V}_{n,m} = \begin{pmatrix} \mathcal{B}_{n,m} & \lambda q_{n,m}\mathbf{j} \\ \lambda q^{-1}_{n,m}\mathbf{j} & \mathcal{B}_{n,m} \end{pmatrix}.$$

Let us take four adjacent vertices (n, m), $(n+1, m)$, $(n+1, m+1)$, $(n, m+1)$ and consider the compatibility condition

$$(44) \qquad \mathcal{V}'\mathcal{U} = \mathcal{U}'\mathcal{V},$$

changing notations for the matrices $\mathcal{U}_{n,m}$, $\mathcal{U}_{n,m+1}$, $\mathcal{V}_{n,m}$, $\mathcal{V}_{n+1,m}$ and their coefficients as it is shown in Fig. 3.

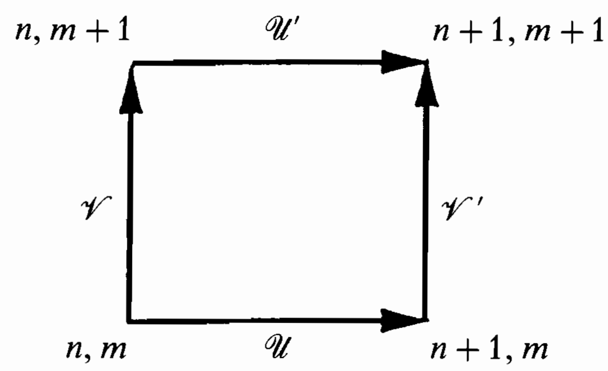

Figure 3. Compatibility condition

Then for the coefficients of $\mathscr{U}$, $\mathscr{U}'$, $\mathscr{V}$, $\mathscr{V}'$ one gets

$$pp' = qq', \tag{45}$$

$$\mathscr{B}'\mathscr{A} = \mathscr{A}'\mathscr{B}, \tag{46}$$

$$p\mathscr{B}'\mathbf{i} + q'\mathbf{j}\mathscr{A} = q\mathscr{A}'\mathbf{j} + p'\mathbf{i}\mathscr{B}, \tag{47}$$

$$\frac{1}{p}\mathscr{B}'\mathbf{i} - \frac{1}{q'}\mathbf{j}\mathscr{A} = -\frac{1}{q}\mathscr{A}'\mathbf{j} + \frac{1}{p'}\mathbf{i}\mathscr{B}. \tag{48}$$

With the notation

$$u_{n,m}(\lambda) = \det \mathscr{U}_{n,m}, \quad v_{n,m}(\lambda) = \det \mathscr{V}_{n,m}$$

(49) implies

$$v_{n+1,m}(\lambda)\, u_{n,m}(\lambda) = u_{n,m+1}(\lambda)\, v_{n,m}(\lambda). \tag{49}$$

Now note that the zeros of both sides of (49) (considered as functions of λ) should coincide. This implies

$$(\lambda^2 + \det \mathscr{B}_{n+1,m})^2 (\lambda^2 + \det \mathscr{A}_{n,m})^2 = (\lambda^2 + \det \mathscr{A}_{n,m+1})^2 (\lambda^2 + \det \mathscr{B}_{n,m})^2.$$

We suppose that the zeros of $u_{n,m+1}(\lambda)$ and $u_{n,m}(\lambda)$ coincide. This is equivalent to

$$\det \mathscr{A}_{n,m} = \alpha_n, \quad \det \mathscr{B}_{n,m} = \beta_m, \tag{50}$$

where we have incorporated into the notation that α_n does not depend on m and β_m not on n.

Now let us *define* discrete surfaces $F_{n,m}$ and $F^*_{n,m}$ in $\mathbb{R}^3$ by the Sym formula (28)

$$\Psi_{n,m}^{-1} \frac{\partial \Psi_{n,m}}{\partial \lambda}\Big|_{\lambda=0} \equiv \hat{F}_{n,m} = \begin{pmatrix} 0 & F_{n,m} \\ F^*_{n,m} & 0 \end{pmatrix}, \tag{51}$$

where $\Psi_{n,m}(\lambda)$ is the solution of the initial value problem (41), (43) with $\Psi_{0,0}(\lambda) = I$.

Lemma 4. *The discrete surface $F_{n,m}$ defined by* (51) *is isothermic in the sense of the generalized definition* (35):

$$Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = -\frac{\beta_m}{\alpha_n} I.$$

The surface $F^*_{n,m}$ in (51) *is its dual* (39).

Proof. Formula (51) implies for the edges

$$\begin{pmatrix} 0 & F_{n+1,m} - F_{n,m} \\ F^*_{n+1,m} - F^*_{n,m} & 0 \end{pmatrix} = \Psi_{n,m}^{-1} \mathscr{U}_{n,m}^{-1} \frac{\partial \mathscr{U}_{n,m}}{\partial \lambda}\Big|_{\lambda=0} \Psi_{n,m}$$

$$= \begin{pmatrix} 0 & p_{n,m} \Phi_{n,m}^{-1} \mathscr{A}_{n,m}^{-1} \mathbf{i} \Phi_{n,m} \\ p_{n,m}^{-1} \Phi_{n,m}^{-1} \mathscr{A}_{n,m}^{-1} \mathbf{i} \Phi_{n,m} & 0 \end{pmatrix},$$

and a similar formula for the edges $[F_{n,m}, F_{n,m+1}]$, $[F^*_{n,m}, F^*_{n,m+1}]$. Here as in the smooth case $\Phi_{n,m}$ denotes the components of

$$\Psi_{n,m}(\lambda = 0) = \begin{pmatrix} \Phi_{n,m} & 0 \\ 0 & \Phi_{n,m} \end{pmatrix}.$$

Considering the local geometry of the quadrilaterals $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ and $(F^*_{n,m}, F^*_{n+1,m}, F^*_{n+1,m+1}, F^*_{n,m+1})$ we can neglect a general rotation $\Phi^{-1}_{n,m} \ldots \Phi_{n,m}$. Then the edges of these quadrilaterals in notations of Fig. 3 are given by the vectors in Fig. 4.

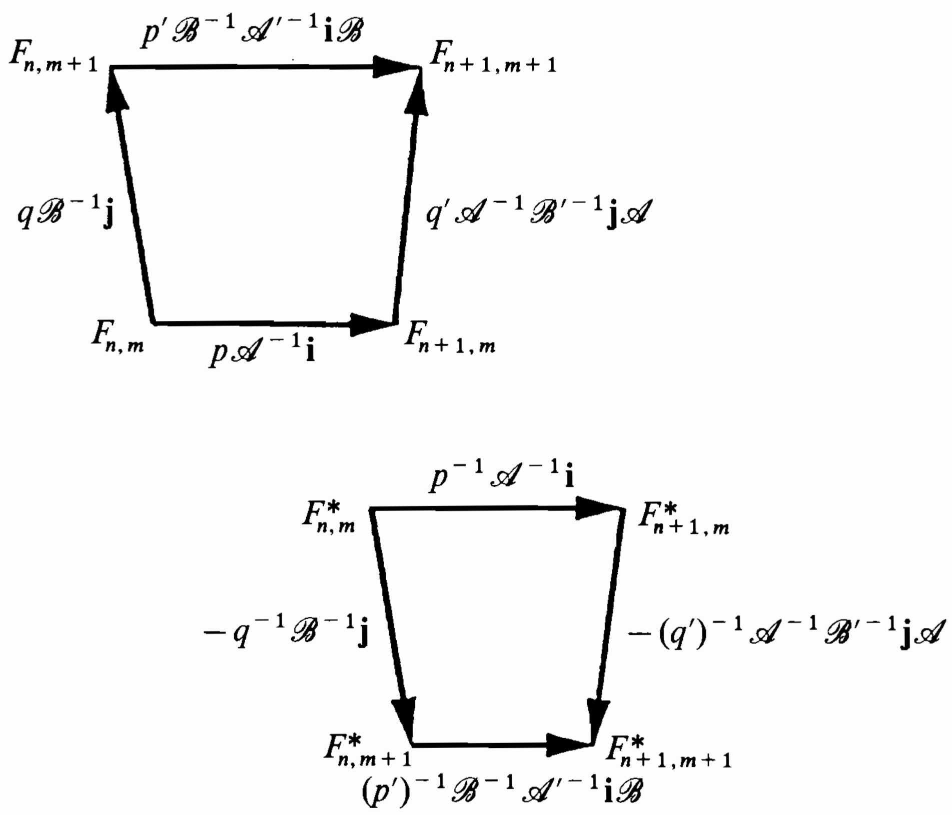

Figure 4. Quadrilaterals $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ and $(F^*_{n,m}, F^*_{n+1,m}, F^*_{n+1,m+1}, F^*_{n,m+1})$ obtained from the Sym formula

The cross-ratio of the quadrilateral $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ is equal to

$$\begin{aligned} Q_{n,m} &= (-p\mathscr{A}^{-1}\mathbf{i})(-q'\mathscr{A}^{-1}\mathscr{B}'^{-1}\mathbf{j}\mathscr{A})^{-1}(p'\mathscr{B}^{-1}\mathscr{A}'^{-1}\mathbf{i}\mathscr{B})(q\mathscr{B}^{-1}\mathbf{j})^{-1} \\ &= -\mathscr{A}^{-1}\mathbf{i}\mathscr{A}^{-1}\mathbf{k}\mathscr{B}\mathbf{j}\mathscr{B} = -\frac{\beta_m}{\alpha_n} I, \end{aligned}$$

where we have used equations (45), (46), the equalities

$$\alpha_n \mathbf{i}\mathscr{A}^{-1} = \mathscr{A}\mathbf{i}, \quad \mathbf{j}\mathscr{B} = \beta_m \mathscr{B}^{-1}\mathbf{j}$$

and the notations (50). One can easily check that the edges of the surface $F^*: \mathbb{Z}^2 \to \mathbb{R}^3$ are related to the edges of $F: \mathbb{Z}^2 \to \mathbb{R}^3$ by (39).

The special case

$$\det \mathscr{A}_{n,m} = \det \mathscr{B}_{n,m} = 1,$$

leads to discrete isothermic surfaces as defined in Definition 6.

Now let us show that any discrete isothermic surface generates the Lax representation (43). Consider an elementary quadrilateral $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$. In a fixed frame its edges (see Fig. 5) are represented by the matrices X, Y, X', Y', which satisfy the following identities:

$$X + Y' = X' + Y, \tag{52}$$

$$X Y'^{-1} X' Y'^{-1} = -1 . \tag{53}$$

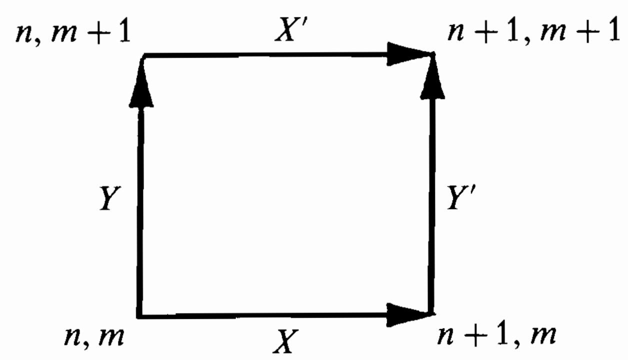

Figure 5. An elementary quadrilateral in $\mathbb{R}^3$

The dual quadrilateral is also closed:

$$X^{-1} - Y'^{-1} = X'^{-1} - Y^{-1} . \tag{54}$$

To compare these identities with the system (45)–(48) let us introduce a moving frame, defined at each vertex of the surface. This frame is described by unitary quaternions $\Phi_{n,m}$. We denote by $\tilde{X}, \tilde{Y}$ the coordinate matrices of the vectors X and Y in the frame $\Phi_{n,m}$ taken at the basic vertex of the vector

$$X = \Phi_{n,m}^{-1} \tilde{X} \Phi_{n,m}, \quad X = \Phi_{n,m}^{-1} \tilde{Y} \Phi_{n,m} .$$

Then for the vectors $\tilde{X}', \tilde{Y}'$ one has

$$\begin{aligned} X' &= \Phi_{n,m+1}^{-1} \tilde{X}' \Phi_{n,m+1} = \Phi_{n,m}^{-1} \mathscr{B}_{n,m}^{-1} \tilde{X}' \mathscr{B}_{n,m} \Phi_{n,m} , \\ Y' &= \Phi_{n+1,m}^{-1} \tilde{Y}' \Phi_{n+1,m} = \Phi_{n,m}^{-1} \mathscr{A}_{n,m}^{-1} \tilde{Y}' \mathscr{A}_{n,m} \Phi_{n,m} , \end{aligned} \tag{55}$$

where

$$\mathscr{A}_{n,m} = \Phi_{n+1,m} \Phi_{n,m}^{-1}, \quad \mathscr{B}_{n,m} = \Phi_{n,m+1} \Phi_{n,m}^{-1}$$

still have to be defined.

Theorem 7. *Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete isothermic surface,*

$$X_{n,m} = F_{n+1,m} - F_{n,m}, \quad Y_{n,m} = F_{n,m+1} - F_{n,m}$$

the vectors of its edges and

$$p_{n,m} = \| X_{n,m} \|, \quad q_{n,m} = \| Y_{n,m} \| \tag{56}$$

the lengths of these edges. Let $\Phi_{n,m}$ *be a unitary quaternion describing the frame at the vertex* (n, m) *and*

$$\tilde{X}_{n,m} = \Phi_{n,m} X_{n,m} \Phi_{n,m}^{-1}, \quad \tilde{Y}_{n,m} = \Phi_{n,m} Y_{n,m} \Phi_{n,m}^{-1} \tag{57}$$

be a representation of the vectors $X_{n,m}, Y_{n,m}$ *in this frame. Suppose that the frames at neighbouring points are related by*

$$\Phi_{n+1,m} = \mathscr{A}_{n,m} \Phi_{n,m}, \quad \Phi_{n,m+1} = \mathscr{B}_{n,m} \Phi_{n,m}, \tag{58}$$

where

$$\mathscr{A}_{n,m} = -\mathbf{i} \frac{1}{p_{n,m}} \tilde{X}_{n,m}, \quad \mathscr{B}_{n,m} = -\mathbf{j} \frac{1}{q_{n,m}} \tilde{Y}_{n,m}. \tag{59}$$

Then

$$\mathscr{A} = \mathscr{A}_{n,m}, \quad \mathscr{A}' = \mathscr{A}_{n,m+1}, \quad \mathscr{B} = \mathscr{B}_{n,m}, \quad \mathscr{B}' = \mathscr{B}_{n+1,m},$$
$$p = p_{n,m}, \quad p' = p_{n,m+1}, \quad q = q_{n;m}, \quad q' = q_{n+1,m}$$

satisfy the system (45)–(48), *which is equivalent to the Lax representation* (43), (44) *for the discrete isothermic surface F.*

Proof. Identity (45) follows immediately from (53), (56). Let us show first that the frames $\Phi_{n,m}$ are well-defined, i.e. the compatibility condition (46) for the system (58), (59) is satisfied. Taking into account (55) we see, that $\mathscr{B}'\mathscr{A} = \mathscr{A}'\mathscr{B}$ is equivalent to

$$-\mathbf{j} \frac{1}{q'} \mathscr{A} \Phi Y' = -\mathbf{j} \frac{1}{p'} \mathscr{B} \Phi X'. \tag{60}$$

Formulas (57) imply

$$\mathscr{A} \Phi = -\mathbf{i} \frac{1}{p} \Phi X, \quad \mathscr{B} \Phi = -\mathbf{j} \frac{1}{q} \Phi Y.$$

Substituting these identities into (60) and using (45) and the obvious relations

$$Y = -q^2 Y^{-1}, \quad X = -p^2 X^{-1},$$

we see that (46) follows from (53). In the same way, using

$$\mathbf{j} \mathscr{B}^{-1} = \mathscr{B} \mathbf{j}, \quad \mathbf{i} \mathscr{A}^{-1} = \mathscr{A} \mathbf{i},$$

one shows that (52), (54) imply (47), (48).

6. Construction of discrete isothermic surfaces

The discrete isothermic surfaces can be constructed in the same way as the discrete K-surfaces we discussed in our paper [3].

Using the definition of discrete isothermic surfaces and starting with some initial stairway loop

$$F_{n,m}, F_{n+1,m}, F_{n+1,m-1}, F_{n+2,m}, \dots, F_{n+N,m-N} = F_{n,m},$$

one can recursively determine the coordinates $F_{n,m}$ of all the points of the discrete isothermic surface in a unique way (see Fig. 6).

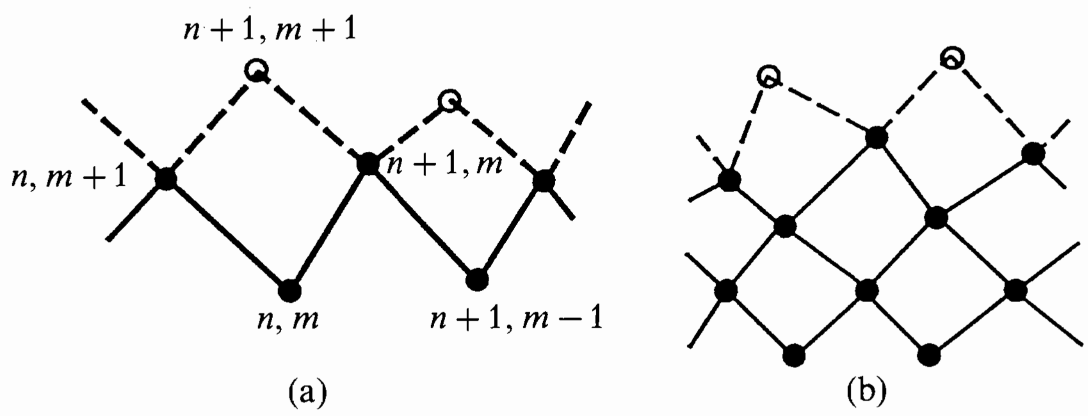

Figure 6. Discrete isothermic surfaces as evolutions of initial stairways: (a) with $Q_{n,m} = -1$, (b) with $Q_{n,m} = -\beta_m/\alpha_n$

Indeed, given the points $F_{n,m}$, $F_{n+1,m}$, $F_{n+1,m+1}$ the cross-ratio (34) uniquely determines the point $F_{n,m+1}$.

To construct the discrete isothermic surface in the sense of the generalized Definition 1 one should start with an initial stairway strip as shown in Fig. 6(b). The factorization property (35) of the cross-ratio allows us to calculate all α_n, β_m up to a common factor and, as a corollary, the cross-ratios of all the quadrilaterals of the discrete isothermic surface $F: \mathbb{Z}^2 \to \mathbb{R}^3$. After that one can proceed as before in Fig. 6(a) reconstructing the surface step by step.

As in the case of a discrete K-surface, one can construct by this elementary method various cylinders and some other interesting surfaces. We should mention here also, that the evolution of the stairway loops described above is unstable. A small variation of the initial loop yields dramatic changes of the surface. This simple geometrical method does not allow us to control the global behaviour of the surface (for example to control the periodicity of the evolution of the initial stairway loop) and to construct compact discrete isothermic surfaces.

To construct them one should apply methods from the theory of integrable equations, which are based on an analytic solution of the problem (and are described for the case of the discrete K-surface in [3]). But this could be the subject of a future paper.

7. Discrete isothermic minimal surfaces

One can define and study special classes of discrete isothermic surfaces. A mean curvature H for discrete isothermic surfaces has been defined in [4], where discrete isothermic surfaces with constant mean curvature were studied. Here we present some results about the discrete isothermic minimal surfaces[3]), which correspond to the special case $H = 0$.

Definition 7. A discrete isothermic surface $F: X^2 \to \mathbb{R}^3$ is called minimal if for any vertex $F_{n,m}$ of this surface there is a plane $\mathcal{P}_{n,m}$ such that the distances of the points $F_{n-1,m}, F_{n+1,m}, F_{n,m-1}, F_{n,m+1}$ to this plane are equal and

$$\langle F_{n+1,m} - F_{n,m}, N_{n,m}\rangle = \langle F_{n-1,m} - F_{n,m}, N_{n,m}\rangle$$
$$= -\langle F_{n,m+1} - F_{n,m}, N_{n,m}\rangle = -\langle F_{n,m-1} - F_{n,m}, N_{n,m}\rangle = \Delta_{n,m},$$

where $N_{n,m}$ is a normal vector to $\mathcal{P}_{n,m}$ (see Fig. 7).

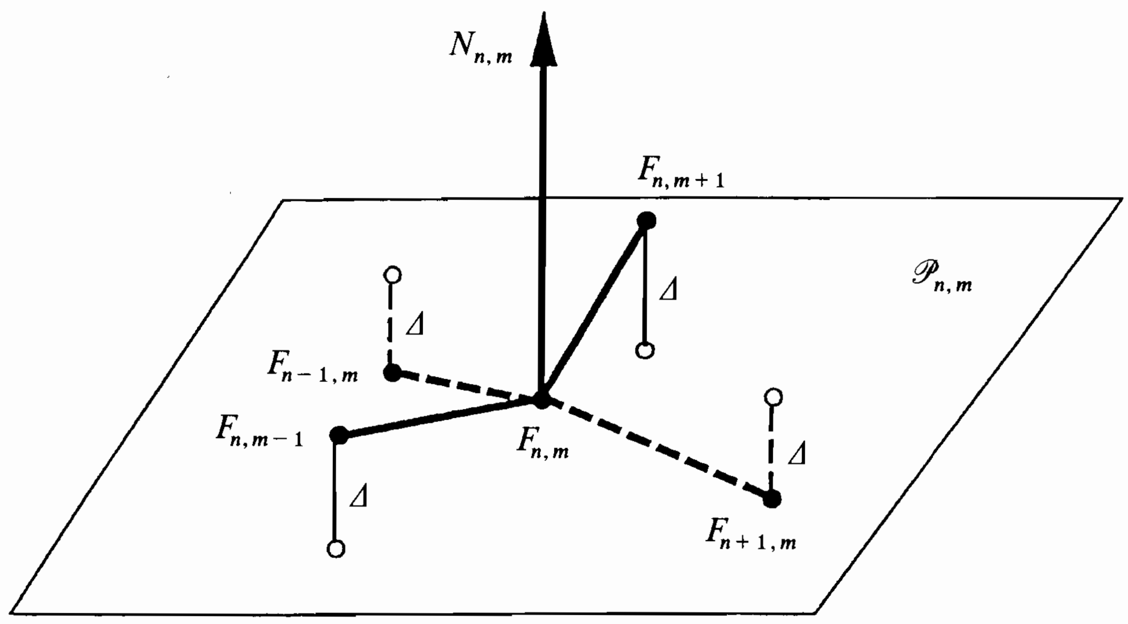

Figure 7. Definition of discrete minimal isothermic surfaces

Lemma 5. *For all points of a discrete isothermic minimal surface*

$$\Delta_{n,m} = \Delta$$

is constant on $\mathbb{Z}^2$.

Proof. Since the formulas (36) for the n- and m- edges of a dual surface differ by a sign, we introduce 4 vectors

$$H = F_{n+1,m} - F_{n,m}, \quad H' = F_{n-1,m} - F_{n,m},$$
$$V = F_{n,m} - F_{n,m+1}, \quad V' = F_{n,m} - F_{n,m-1}$$

[3]) The results of this section were obtained earlier and presented by the authors on conferences (Granada, Oberwolfach) in 1991.

with the basic point at the vertex $F_{n,m}$. Then all the endpoints of these vectors lie in a plane $\mathscr{P}^{\Delta}_{n,m}$ and $\Delta_{n,m}$ is the distance between these two planes (Fig. 8). If we identify $F_{n,m} = F^*_{n,m}$, formulas (36) show that all the points $F^*_{n-1,m}$, $F^*_{n,m-1}$, $F^*_{n,m+1}$ of the dual surface lie on a sphere $S_{n,m}$ of radius $\frac{1}{2\Delta_{n,m}}$, which is the image of $\mathscr{P}^{\Delta}_{n,m}$ under inversion of $\mathbb{R}^3$ with respect to the unit sphere with a center at $F_{n,m}$. The points

$$F^*_{n-1,m-1}, \quad F^*_{n+1,m-1}, \quad F^*_{n+1,m+1}, \quad F^*_{n-1,m+1}$$

also lie on $S_{n,m}$. Indeed, since F^* is discrete isothermic, $F^*_{n+1,m+1}$ lies on a circle determined by $F^*_{n,m}$, $F^*_{n+1,m}$, $F^*_{n,m+1}$, which lies on $S_{n,m}$. Six points

$$F^*_{n,m+1}, \quad F^*_{n+1,m+1}, \quad F^*_{n,m}, \quad F^*_{n+1,m}, \quad F^*_{n,m-1}, \quad F^*_{n+1,m-1}$$

are common for $S_{n,m}$ and $S_{n+1,m}$, therefore these spheres coincide

$$S_{n,m} = S_{n+1,m}, \quad \Delta_{n,m} = \Delta_{n+1,m}.$$

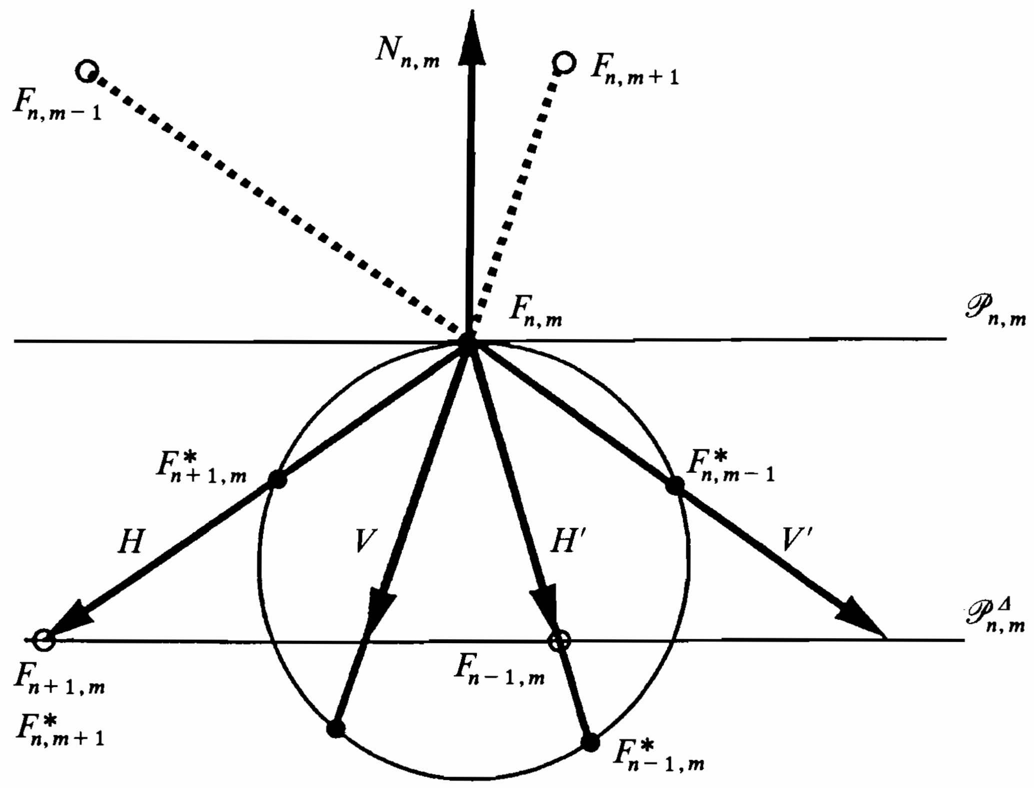

Figure 8. Dual discrete minimal isothermic surface

Actually we proved already that the dual surface lies on a sphere. Without loss of generality one can assume the normalization

$$(61) \qquad \Delta = 1/2.$$

Then the dual surface lies on the unit sphere. Moreover the Gauss map N of F coincides with F^*.

Theorem 8. *The following statements are equivalent*:

(i) $F: \mathbb{Z}^2 \to \mathbb{R}^3$ *is a discrete isothermic minimal surface normalized by* (61).

(ii) *The dual surface $F^* : \mathbb{Z}^2 \to \mathbb{R}^3$ lies on a sphere and without loss of generality one can assume*

$$F^* = N : \mathbb{Z}^2 \to S^2,$$

where N is the Gauss map of F.

This theorem allows us to parametrize discrete minimal surfaces by "holomorphic" data N.

Definition 8. A map $g : \mathbb{Z}^2 \to \mathbb{R}^2 = \mathbb{C}$ is called discrete holomorphic if the cross-ratios of all its elementary quadrilaterals are equal to -1

$$q(g_{n,m}, g_{n+1,m}, g_{n+1,m+1}, g_{n,m+1}) = \frac{(g_{n+1,m} - g_{n,m})(g_{n,m+1} - g_{n+1,m+1})}{(g_{n+1,m+1} - g_{n+1,m})(g_{n,m} - g_{n,m+1})} = -1,$$

$g_{n,m}$ are complex numbers here.

Let us mention that in a different context the discrete holomorphic map and its Lax pair different from (58) is presented in the recent paper [9].

A discrete isothermic surface $N : \mathbb{Z}^2 \to S^2$ in S^2 can be obtained from a discrete holomorphic function $g : \mathbb{Z}^2 \to \mathbb{C}$ by stereographic projection $\mathbb{C} \to S^2$

$$(N_1 + iN_2, N_3) = \left(\frac{2g}{1+|g|^2}, \frac{|g|^2-1}{|g|^2+1}\right).$$

Combining this formula with (36) one gets an analogue of the Weierstrass representation in the discrete case.

Theorem 9. *Let $g : \mathbb{Z}^2 \to \mathbb{C}$ be discrete holomorphic. Then the formulas*

$$F_{n+1,m} - F_{n,m} = \frac{1}{2}\operatorname{Re}\left(\frac{1}{g_{n+1,m} - g_{n,m}}(1 - g_{n+1,m}g_{n,m}, i(1 + g_{n+1,m}g_{n,m}), g_{n+1,m} + g_{n,m})\right),$$

$$F_{n,m+1} - F_{n,m} = -\frac{1}{2}\operatorname{Re}\left(\frac{1}{g_{n,m+1} - g_{n,m}}(1 - g_{n,m+1}g_{n,m}, i(1 + g_{n,m+1}g_{n,m}), g_{n,m+1} + g_{n,m})\right)$$

describe a discrete minimal isothermic surface. All discrete minimal isothermic surfaces are described in this way.

The identical map

$$g_{n,m} = n + im$$

(which is obviously discrete holomorphic) generates the discrete Enneper surface (Fig. 9). The discrete exponential map

$$g_{n,m} = \exp(\varrho n + i\alpha m), \quad \alpha = 2\pi/N,$$

where

$$\varrho = 2\,\mathrm{arcsinh}\left(\sin\frac{\alpha}{2}\right),$$

generates the discrete catenoid (Fig. 10).

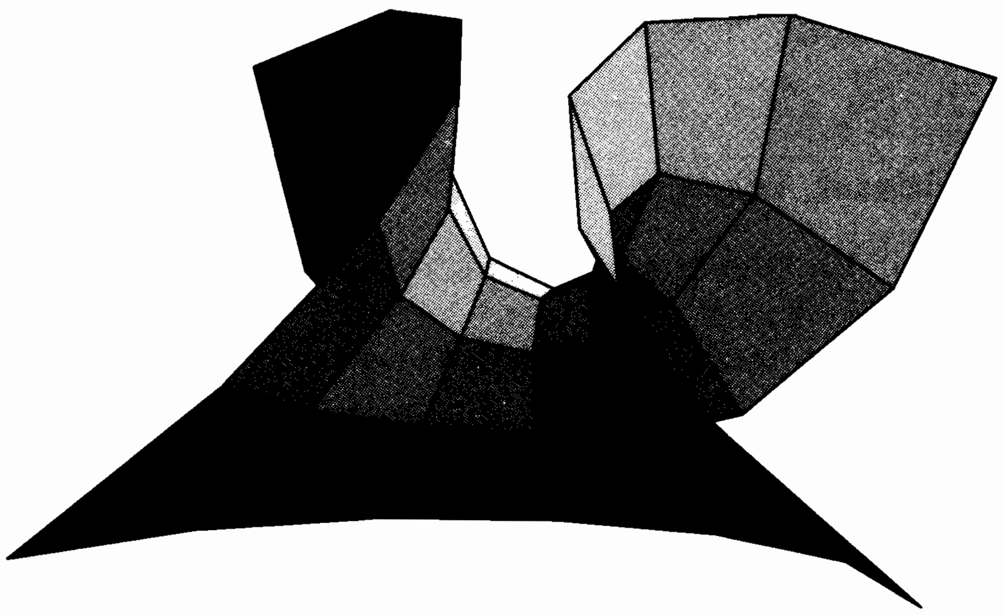

Figure 9. Discrete Enneper surface

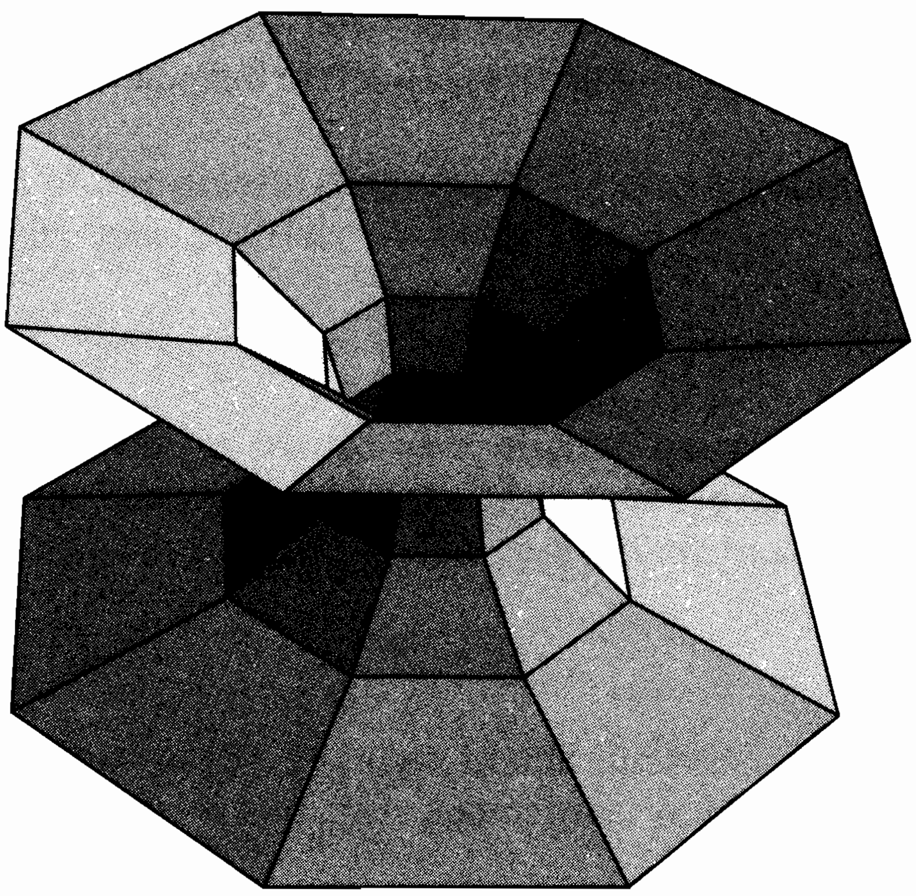

Figure 10. Discrete catenoid

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