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The nonlinear Klein–Gordon equation on an interval as a perturbed Sine–Gordon equation

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Abstract. We treat the nonlinear Klein–Gordon (NKG) equation as the Sine–Gordon (SG) equation, perturbed by a higher order term. It is proved that most small-amplitude finite-gap solutions of the SG equation, which satisfy either Dirichlet or Neumann boundary conditions, persist in the NKG equation and jointly form partial central manifolds, which are "Lipschitz manifolds with holes". Our proof is based on an analysis of the finite-gap solutions of the boundary problems for SG equation by means of the Schottky uniformization approach, and an application of an infinite-dimensional KAM-theory.

Introduction

The paper is devoted to small-amplitude solutions of the nonlinear Klein–Gordon equation

$$u_{tt} = u_{xx} - mu + f(u), \qquad u = u(t, x), \qquad 0 < x < \pi, \tag{1}$$

where $m > 0$ and f is an analytic function of the form

$$f(u) = \varkappa u^3 + O(|u|^5), \qquad \varkappa \neq 0, \tag{2}$$

at zero.

This assumption is fulfilled, in particular, if f is an odd function such that $f'''(0) \neq 0$ and $f'(0) = 0$ (the latter is a normalization – we absorbed a linear part of f to $-mu$).

The cases $\varkappa > 0$ and $\varkappa < 0$ can be treated similarly. Below the case

$$\varkappa > 0 \tag{2'}$$

is considered. We discuss the changes one should make to handle with negative $\varkappa$ at the end of the introduction.

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The assumptions (2), (2′) hold for many important equations of mathematical physics. In particular, for the φ^4-equation

$$u_{tt} = u_{xx} - mu + \varkappa u^3 \tag{φ^4}$$

and for the Sine–Gordon equation

$$u_{tt} = u_{xx} - \sin u, \tag{SG}$$

where now $m = 1$, $\varkappa = 1/6$.

We consider equation (1) under Dirichlet or Neumann boundary conditions:

$$u(t, 0) \equiv u(t, \pi) \equiv 0 \tag{D}$$

or

$$u_x(t, 0) = u_x(t, \pi) \equiv 0. \tag{N}$$

The results and the proof in (D)- and (N)-cases are parallel. So we mostly restrict ourselves to the Neumann problem and give a brief reformulation of the main results for the Dirichlet problem.

To simplify the formulas we suppose that $m = 1$; by a trivial rescaling of u in (1) we can achieve $\varkappa = 1/6$. So below

$$m = 1, \qquad \varkappa = 1/6. \tag{2″}$$

The equation (1) + (N) (as well as (1) + (D)) defines a dynamical system in the phase-space Z of pairs $\tilde{U}(t, x) = (u(t, x), v = \dot{u}(t, x))$[1] ($Z$ should be given some Sobolev norm $\| \cdot \|$, for example, one can take $Z = \mathring{H}^1(0, \pi) \times L_2(0, \pi)$ in the Dirichlet case). The equations (SG) + (N) and (SG) + (D) are well-known to be hamiltonian: one should supply the phase-space Z with the symplectic structure given by the 2-form ω_2,

$$\omega_2((u_1, v_1), (u_2, v_2)) = \int_0^\pi (u_1 v_2 - v_1 u_2)\, dx,$$

[1] In fact, for technical reasons in the main part of the paper we use as the phase-vector of the equation the pair $U = (u(t, x), (\partial^2/\partial x^2 + 1)^{-1/2}\dot{u}(t, x))$. In the introduction for the sake of simplicity we present trivial reformulation of the results in terms of the phase-vector $\tilde{U}$.

and consider the hamiltonian

$$\int_0^\pi \left(\frac{1}{2}(v^2 + mu^2 + u_x^2) + F(u)\right) dx,$$

where $F_u = f$.

Let us consider the linear Klein–Gordon equation, which describes infinitesimal oscillations in (1):

$$u_{tt} = u_{xx} - u. \tag{KG}$$

The equation (KG) + (N) is a linear oscillating system with the frequencies $0^*, 1^*, 2^*, \ldots$, where we denote

$$j^* = \sqrt{j^2 + 1}$$

(if in (1) $m \neq 1$, then the frequencies j^* will change. In the main text below we discuss how this affects our results). The solutions with frequency j^* have the form (u_j, v_j), where $v_j = \dot{u}_j$ and

$$u_j(t, x) = I_j \sin j^*(t + \varphi_j) \cos jx, \qquad I_j \geq 0.$$

Fix any $n \geq 1$ wave-numbers j,

$$j \in \mathbf{V} = \{V_1^0, \ldots, V_n^0\} \subset \mathbb{N} \cup \{0\}, \tag{3}$$

and consider superpositions (= sums) $\tilde{U}^n = (u^n, v^n)$ of solutions (u_j, v_j) with $j \in \mathbf{V}$, $u^n = u_1 + \cdots + u_n$, $v^n = v_1 + \cdots + v_n$. They are time-quasiperiodic solutions[2] of (KG) + (N) with the frequency vector $\omega = (V_1^{0*}, \ldots, V_n^{0*})$. Altogether the solutions $\tilde{U}^n$ fill the $2n$-dimensional linear subspace E^{2n} of Z,

$$E^{2n} := \operatorname{span}\{(\cos V_j^0 x, 0), (0, \cos V_j^0 x) \mid j = 1, \ldots, n\}. \tag{4}$$

Each solution $\tilde{U}^n$ lies in an invariant torus $T^n(I)$, where $\dim T^n(I) = n$ if all $I_j > 0$. So the space E^{2n} is foliated into invariant tori and

$$E^{2n} \simeq \mathbb{R}_+^n \times \mathbb{T}^n. \tag{5}$$

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[2] We recall that a solution $\tilde{U} : \mathbb{R} \to Z$ is called quasiperiodic with n frequencies if there exists a continuous map $\Sigma : \mathbb{T}^n \to Z$ and an n-vector ω, called the frequency vector of the solution, such that $\tilde{U}(t) \equiv \Sigma(\omega t)$. So the solution $\tilde{U}$ lies in the invariant n-torus $\Sigma(\mathbb{T}^n)$.

We are going to attack the following problem: do the small-amplitude solutions $\tilde{U}^n$ and the invariant tori $T^n(I)$ of the linearized equation persist in the equation $(1)+(N)$? How do solutions of $(1)+(N)$ behave near the tori? The question looks rather naïve – even in the finite-dimensional situation the behavior of the perturbed linear hamiltonian system can be very complicated (see e.g. [M]). Still, the purpose of our paper is to prove that the answer to the first question is "mostly affirmative" and that the surviving quasiperiodic solutions are linearly stable. In fact, the persistence of the quasiperiodic solutions $\tilde{U}^n$ has the natural explanation: under the assumptions (2), (2″) we have

$$-u+f(u)=-\sin u+O(|u|^5),$$

so small-amplitude solutions of (1) can be approximated by solutions of the (SG)-equation, which is known to be integrable!

The final results of our analysis are given in Theorem 6.2. In a somewhat simplified form they can be stated as follows:

THEOREM. *For each invariant subspace E^{2n} as in* (4) *there exists a subset $\tilde{E}\subset E^{2n}\simeq\mathbb{R}^n_+\times\mathbb{T}^n$ of the form $\tilde{E}\simeq\tilde{M}\times\mathbb{T}^n$; a Lipschitz map $\tilde{\Phi}:\tilde{E}\simeq\tilde{M}\times\mathbb{T}^n\to Z$, analytic in $q\in\mathbb{T}^n$, and a Lipschitz map $\tilde{W}:\tilde{M}\to\mathbb{R}^n$ such that*

(i) *the subset $\tilde{E}\subset E^{2n}$ has unit density at zero*[3];

(ii) *the curves $t\mapsto\tilde{\Phi}(\mu,D+t\tilde{W}(\mu))$, where $(\mu,D)\in\tilde{E}$, are quasiperiodic solutions of* $(1)+(N)$. *All Lyapunov exponents of these solutions are zero*;

(iii) *the set $\tilde{\mathscr{T}}^{2n}=\tilde{\Phi}(\tilde{E})$ has a tangent space at zero, coinciding with the space E^{2n}.*

By the last assertion of the Theorem one can treat $\mathscr{T}^{2n}$ as a partial central manifold of $(1)+(\mathrm{N})$, corresponding to the invariant subspace E^{2n} of the linearized equation $(\mathrm{KG})+(\mathrm{N})$.

In particular, taking $n=1$ we obtain

COROLLARY. *The equation* $(1)+(N)$ *has time-periodic solutions, forming infinitely many families. The family number j consists of solutions with the frequencies close to j^*; these solutions are parameterized by the points of some one-dimensional set of positive Lebesgue measure.*

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[3] That is, the intersection of $\tilde{E}$ with the δ-ball centered at zero fills most part of the ball when $\delta\to0$. See Part 6 for the exact definition.

Altogether the manifolds $\tilde{\mathcal{T}}^{2n}$, $n = 1, 2, \ldots$, are "infinitesimally dense" at zero: the union of their tangent spaces at zero is dense in $T_0 Z \simeq Z$. So their union $\tilde{\mathcal{T}} = \cup \tilde{\mathcal{T}}^{2n}$ is a linearly stable set which is "dense near zero" – it intersects each open nonempty cone with the vertex at zero (see Part 7). Sufficiently small solutions of (1) + (N) are close to $\tilde{\mathcal{T}}$; for a long time they follow quasiperiodic solutions in $\tilde{\mathcal{T}}$ and look "regular". The phenomenon of regular behavior of small-amplitude solutions of (φ^4) + (N) is well-known from numeric experiments [ZIS] (for some time there was a hope that this equation is integrable).

The proof of the Theorem goes as follows. We start with an analysis of time-quasiperiodic (=finite-gap) solutions of (SG) + (N) of small amplitude $\rho \ll 1$ and prove that they form smooth submanifolds $\mathcal{T}^{2n}_{\rho}$ of the phase-space Z with the tangent spaces at zero equal to the spaces E^{2n}. Next we study linearizations of the (SG) + (N) equation on the solution in $\mathcal{T}^{2n}_{\rho}$ and show that these equations can be reduced to constant-coefficient linear equations. After this an application of the KAM-theory for infinite-dimensional systems (see [K1, K4])[4] proves persistence of most of the (SG)-tori in the equation (1) and complete the proof.

The equation (SG) has well-known finite-gap solutions, given by the theta-formula

$$u(t, x; X, D) = 2i \log \frac{\theta(i(Vx + Wt + D + \Delta))}{\theta(i(Vx + Wt + D))}, \tag{6}$$

obtained first by Kozel and Kotlyarov [KK] and Its (see in [Mat]). The solution (6) defines (and is defined by) its spectral curve X which is a hyperelliptic Riemann curve with a real involution. In general any hyperelliptic curve X with a real involution determines a solution of the SG equation. Moreover, there are usually many connected components of the solutions corresponding to the same X, which makes a general picture rather complicated (for details see [BBEIM, DN, EF]). The picture simplifies if we consider only small-amplitude solutions. In this case the genus g of the curve equals the number of nontrivial spectral branches of the corresponding L-operator (see [McK, EFM, BBEIM]); the branching points of X are $\{0, \infty\} \cup \{\lambda_1, \bar{\lambda}_1; \ldots; \lambda_g, \bar{\lambda}_g\}$, where $\lambda_j, \bar{\lambda}_j$ $(j = 1, \ldots, g)$ are the edges of the nontrivial spectral branches. The vectors $(\lambda_1, \ldots, \lambda_g) \in \mathbb{C}^g \simeq \mathbb{R}^{2g}$ and $D \in \mathbb{T}^g$ are parameters of the solution.

The analysis of the formula (6) we give in Part 1 (following [Bo] and [BiK]) shows how to single out among the g-gap solutons (6) real-valued 2π-periodic

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[4] For the classical finite-dimensional KAM-theory see e.g., [A2], [M] and [P].

solutions, which are even or odd in x. The solutions from the first group satisfy Neumann boundary conditions, and from the second group – the Dirichlet. Moreover, solutions $\tilde{U} = (u, \dot{u})$ of (SG) + (N) thus obtained form $2n$-dimensional analytic varieties $\mathscr{T}^{2n} \subset Z$, $n = [g/2] + 1$, and similar with the solutions of the Dirichlet problem. The solutions in $\mathscr{T}^{2n}$ of an amplitude $<\rho$ form a smooth analytic manifold $\mathscr{T}^{2n}_{\rho}$, foliated to invariant tori of (SG) + (N):

$$\mathscr{T}^{2n}_{\rho} = \bigcup_{X = X(\mu)} T^n(X), \tag{7}$$

where an n-dimensional μ parameterizes all the curves X giving rise to solutions (6) which satisfy (N).

The tangent spaces to the manifolds $\mathscr{T}^{2n}_{\rho}$ at zero are exactly the spaces E^{2n} as in (4). So the spaces E^{2n} (or, equivalently, the vectors $\mathbf{V}$ as in (3)) parameterize the manifolds $\mathscr{T}^{2n}_{\rho}$.

The manifolds $\mathscr{T}^{2n}_{\rho}$ are symplectic submanifolds of Z and (SG) + (N) restricted to $\mathscr{T}^{2n}_{\rho}$ is an integrable hamiltonian vectorfield with a singularity at zero. We prove (with some efforts) the following statement which substitutes the Liouville–Arnold theorem for systems with singularities: in $\mathscr{T}^{2n}_{\rho}$ there exist analytic Darboux coordinates (p, q) such that the hamiltonian of the system on $\mathscr{T}^{2n}_{\rho}$ depends only on the actions $p_j^2 + q_j^2$, $j = 1, \ldots, n$.

Next we study linearization of the equation (SG) about the solution (6):

$$v_{tt} = v_{xx} - (\cos u(t, x))v. \tag{LSG}$$

The integrability of the (SG)-equation exhibits itself in the linearized equation in the following way: the equation (LSG) has infinitely many complex x-periodic "Bloch-like" solutions $v^j_+(t, x)$, $v^j_-(t, x)$ of the form

$$(v^j_{\pm}, \dot{v}^j_{\pm})(t, x) = e^{\pm i w_j t}\tilde{\Psi}^j_{\pm}(W^n t + D^n)(x), \qquad j = n+1, n+2, \ldots, \tag{8}$$

where W^n and D^n are the vectors formed by the first n components of the vectors W and D from (6); the frequencies w_j and the functions $\tilde{\Psi}^j_{\pm}(D^n)(x)$ depend on the curve $X(\mu)$. The even in x parts of (8) give solutions of (LSG) + (N) of the same form but with $\tilde{\Psi}^j_{\pm}$ replaced by $\Psi^j_{\pm}(x) = (\tilde{\Psi}^j_{\pm}(x) + \tilde{\Psi}^j_{\pm}(-x))/2 \in Z$.

Critical for the perturbation techniques we are going to apply to the manifolds $\mathscr{T}^{2n}_{\rho}$, as well as for the subsequent investigation of the manifolds, is the following nonresonance property:

$$W^n \cdot s \pm w_j \not\equiv 0, \qquad W^n \cdot s \pm w_j \pm w_k \not\equiv 0 \tag{9}$$

as functions of the curve X, for all $s \in \mathbb{Z}^n$ and all $j \neq k$ (see [K4, Part 4] for a discussion of the relations (9)).

Relations (9) as well hold for the (SG)-equation under Dirichlet boundary conditions, but not under the periodic ones! In the latter case the frequencies w_j go in pairs $w_{j\pm}$ in such a way that $|w_{j+} - w_{j-}| \leq \exp - j/C$. So the periodic boundary conditions are *asymptotically resonant* and our techniques can not be applied there.

Our calculations also prove the *nondegenerate amplitude-frequency modulation* for solutions forming the manifold $\mathscr{T}_\rho^{2n}$:

$$\det \partial W^n / \partial \mu \,|_{\mu = 0} \neq 0. \tag{10}$$

Thus, the vectors W^n, corresponding to the solutions (6) of (SG) + (N), form an n-dimensional domain.

The nonresonance and nondegeneracy relations (9), (10) jointly with asymptotics for the solutions (8) as $j \to \infty$, allow us to prove that for fixed D^n, μ the vectors $\{\Psi^j_\pm(D^n, \mu) \mid j \geq n+1\}$ forms a skew-orthogonal basis of the skew-orthogonal complement in Z to the tangent space to $\mathscr{T}_\rho^{2n}$. Next an application of an abstract theorem from [K2–K4] supplies us with a symplectic coordinate system (q, p, y) in a neighborhood of $\mathscr{T}_\rho^{2n}$ in Z, such that y varies in a symplectic subspace $Y \subset Z$ of codimension $2n$; the manifold $\{(q, p, 0)\}$ equals $\mathscr{T}_\rho^{2n}$ with the Darboux coordinates (q, p) in it, and the hamiltonian of (SG) + (N) in these variables equals

$$h(I) + \frac{1}{2}\langle A(I)y, y\rangle + h^3(q, p, y). \tag{11}$$

Here $I_j = \frac{1}{2}(p_j^2 + q_j^2), j = 1, \ldots, n$, are functions of μ only; $h^3 = O(\|y\|^3)$, the operators $A(I)$ are diagonal in an I-independent basis of Y and the hamiltonian linear operator in Y with the hamiltonian $\frac{1}{2}\langle A(I)y, y\rangle$ has the frequencies $\{w_j(I)\}$, where w_j are the same as in (8).

Now an infinite-dimensional version of the KAM-theory from [K1] can be applied to prove that most of the tori $\{I = \text{const}, y = 0\}$ (which are exactly the tori $T^n(X(\mu))$ written in the new variables) persist under perturbing the equation by higher-order terms, thus proving the Theorem.

In fact, the invariant Lipschitz manifolds $\tilde{\mathscr{T}}^{2n}$ from the Theorem "remember" that they are perturbations of the manifolds $\mathscr{T}_\rho^{2n}$ (not the spaces E^{2n} only):

AMPLIFICATION. *At the set* $\{(p, q) \in E^{2n} \mid p_j^2 + q_j^2 < 2\rho^2\} \cap \tilde{E}$ *the map* $\tilde{\Phi}$ *is close to the map* Φ_0 *parameterizing the manifold* $\mathscr{T}_\rho^{2n}$: $\|\tilde{\Phi}(p, q) - \Phi_0(p, q)\| = O|(p, q)|^{3-\varepsilon}$ *for each* $\varepsilon > 0$. *Thus, at zero the Lipschitz manifold* $\tilde{\mathscr{T}}^{2n}$ *has a second-order tangency with* $\mathscr{T}_\rho^{2n}$.

The analytic manifold $\mathscr{T}_\rho^{2n}$ is a partial central manifold of the integrable equation (SG) + (N), corresponding to the invariant subspace E^{2n} of the linearized equation (KG) + (N). The Theorem states that the equation (1) + (N) has a partial central manifold which is a "Lipschitz manifold with holes" and the Amplification states that at zero this manifold is well-approximated by $\mathscr{T}_\rho^{2n}$.

Now we briefly discuss equation (1) with $\varkappa < 0$. Suppose for simplicity that $m = 1$. We can rescale u to achieve $\varkappa = -1/6$. Then

$$-u + f(u) = -\sinh(u) + O(|u|^5),$$

and (1) is a higher-order perturbation of the Sinh–Gordon equation

$$u_{tt} = u_{xx} - \sinh u.$$

This is again an integrable equation similar to (SG) but simpler than the latter (because the L-operator for this equation – not for the (SG)! – is selfadjoint). So we can proceed exactly as above to construct the finite-gap manifolds filled with solutions of the equation under (N) or (D) boundary conditions; to put the equation into the normal form (11) in the vicinities of the manifolds and to apply the infinite-dimensional KAM-theory. As a final result of the analysis we obtain that both the Theorem and the Amplification also hold for $\varkappa < 0$.

Now we turn to a comparison of our theorem with the known results. In our work we study persistence of *small-amplitude* finite-gap solutions of an integrable equation under higher-order at zero perturbations of the equation. Persistence of finite-gap solutions of *order one* under small perturbations of the corresponding integrable equation was proved before. See [K2] for an abstract theorem and its application to nonresonant families of finite-gap solutions of the KdV equation and see [BoK1] for a proof that in the KdV case all the finite-gap families are nonresonant; see [BiK] for the perturbed (SG) equation

$$u_{tt} = u_{xx} - \sin u + \varepsilon\varphi(u).$$

The results of the present paper essentially depend on the local (near zero) theory of finite-gap manifolds $\mathscr{T}_\rho^{2n}$, based on the Schottky uniformization. It turns out that zero is a rather complicated point of the finite-gap manifolds (as far as we know, even smoothness of the manifolds $\mathscr{T}_\rho^{2n}$ at zero has not been proved before our work). Still, large-amplitude finite-gap solutions of the (SG)-equation possess some additional properties with respect to the ones of small-amplitude solutions. To present a more complete picture of the (SG)-equation and its perturbations we end each part of the paper with a brief discussion of the corresponding properties of large-amplitude solutions, following [BiK].

Results similar to ours were known for the nonlinear string equation with a "typical" potential $V(x)$,

$$u_{tt} = u_{xx} - V(x)u + \varepsilon f(u). \tag{12}$$

It was proved [K1, K4] that if the potential $V(x)$ depends on an n-dimensional external parameter in "a nondegenerate way", than for most values of the parameter time-quasiperiodic solutions of the linear equation $(12)\,|_{\varepsilon=0}$ with $\le n$ frequencies persist in (12) (the equation should be supplemented by (D) or (N) boundary conditions). Similar result was obtained by Wayne [W] provided that the potential $V(x)$ is random and the funciton $f(u)$ satisfies (2). See in [CW] another approach to prove persistence of *time-periodic* solutions which is also applicable to the equation (12) under periodic boundary conditions.

Time-periodic solutions of $(1)+(\mathrm{N})$ and $(1)+(\mathrm{D})$ have been studied by many authors (see survey [Bre]). Still, results of the Corollary also are new: in the previous works under different restrictions on the nonlinear term $f(u)$ of the equation it was proved that the equation has a *countable family* of time-periodic solutions. We prove that the time-periodic solutions form infinitely many *one-dimensional* families.

Notations

We denote by D_ρ^{2n} and D_ρ^c the polydisc of radius ρ and its complexification:

$$D_\rho^{2n} = \{(p, q) \in \mathbb{R}^{2n} \mid p_j^2 + q_j^2 < 2\rho\}, \qquad D_\rho^c = \{p, q) \in \mathbb{C}^{2n} \mid |p_j|^2 + |q_j|^2 < 2\rho\};$$

by μ_j we denote the actions $\mu_j = \frac{1}{2}(p_j^2 + q_j^2)$ and by M_ρ^+ and M_ρ^c the polydisc in the action-representation and its complexification

$$M_\rho^+ = \{\mu \in \mathbb{R}_+^n \mid 0 \le \mu_j < \rho\}, \qquad M_\rho^c = \{\mu \in \mathbb{C}^n \mid |\mu_j| < \rho\}.$$

By C, C_1 etc., we denote different positive constants in estimates and denote by ρ, ρ' positive radii of manifolds $\mathscr{T}_\rho$, different in different parts of the text (so the manifold $\mathscr{T}_\rho$ in Part 1 is larger than in Part 6).

1. Small-amplitude finite-gap solutions of boundary-valued problems for the Sine–Gordon equation

We consider the Sine–Gordon equation

$$u_{tt} = u_{xx} - \sin u \tag{SG}$$

under Neumann

$$u'(t, 0) \equiv u'(t, \pi) \equiv 0 \tag{N}$$

or Dirichlet

$$u(t, 0) \equiv u(t, \pi) \equiv 0 \tag{D}$$

boundary conditions.

In a contrast with the tradition we treat (SG) as a system of first order (in time) equations *not* for pairs of functions $(u(t, x), u_t(t, x))$, but for the pairs $(u, A^{-1/2}u_t)$. Here A is the differential operator $-\partial^2/\partial x^2 + 1$, supplemented by the boundary conditions (N) or (D). The operator A is positive selfadjoint, so the square root $A^{1/2}$ and its inverse $A^{-1/2}$ are well defined. We write down (SG) + (N) (or +(D)) as

$$\dot{u} = -\sqrt{A}\, v, \qquad \dot{v} = \sqrt{A}\,(u + A^{-1}(\sin u - u)) \tag{1.1}$$

(the function v can be excluded from the equations; after this reduction we obtain for u exactly the (SG) equation). The linear part of equations (1.1) is symmetric with respect to u and v, which is convenient for our analytic tools.

We denote

$$U(t, x) = (u(t, x), v(t, x))$$

and observe that the first component $u(t, x)$ contains all the information about the solution, because $v = -A^{-1/2}\dot{u}$.

We start with some basic facts from the finite-gap theory of the (SG) equation (see [McK, EF, DN, BBEIM] for the proofs and details). Let $X = \{P = (\lambda, \mu)\}$ be the hyperelliptic Riemann surface of the polynomial

$$\mu^2 = \lambda \prod_{i=1}^{g} (\lambda - \lambda_i)(\lambda - \bar{\lambda}_i), \tag{1.2}$$

where $\lambda_1, \ldots, \lambda_g$ are pairwise different complex numbers from the upper half-plane $\mathbb{C}_+$ (we restrict ourself to the solutions with complex branching points because the small-amplitude finite-gap solutions we are interested in are of this type). We denote the hyperelliptic involution and the conjugation involution as follows:

$$\tau_1(\lambda, \mu) = (\lambda, -\mu), \qquad \tau_2(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu}).$$

Let us make on X the cut $\gamma_0 = [0, \infty)$ and the cuts γ_i, $i = 1, \ldots, g$, where γ_i is a path from $\bar{\lambda}_i$ to λ_i; let us choose the canonical basis of circles (a_i, b_i), $i = 1, \ldots, g$, on Γ in such a way that the circle a_j surrounds the cut γ_j (see Fig. 1), and fix a basis of holomorphic differentials $d\omega_1, \ldots, d\omega_g$ of X normalized by the conditions

$$\oint_{a_m} \omega_j = 2\pi i \delta_{mj}, \qquad j, m = 1, \ldots, g.$$

The Riemann matrix $B = (B_{mj})$,

$$B_{mj} = \oint_{b_m} \omega_j, \qquad j, m = 1, \ldots, g,$$

defines the theta-function θ,

$$\theta(z \mid B) = \sum \exp\left(\frac{1}{2}\langle Bm, m\rangle + \langle z, m\rangle\right).$$

This function has the matrix of periods $(2\pi i I, B)$.

The function $\sqrt{\lambda}$ is not single-valued on X. To correlate the local parameters $\sqrt{\lambda}$ at the points $\lambda = 0$ and $\lambda = \infty$ we should fix a branch of $\sqrt{\lambda}$ on X. This branch is fixed if a contour $\mathscr{L}$ on X is specified, where $\sqrt{\lambda}$ has a jump alternating its sign ($\sqrt{\lambda}$ is analytic on $X - \mathscr{L}$ and boundary values of $\sqrt{\lambda}$ at two edges of $\mathscr{L}$ differ by a sign, $\sqrt{\lambda}\,|_{\mathscr{L}_+} = -\sqrt{\lambda}\,|_{\mathscr{L}_-}$). We choose $\mathscr{L}$ to be a union (see Fig. 1) of the contours surrounding the cuts γ_i, which are mapped to γ_j's by the projection $(\lambda, \mu) \to \lambda$. Let us consider the Abelian differentials $d\Omega_\infty$, $d\Omega_0$ with zero a-periods

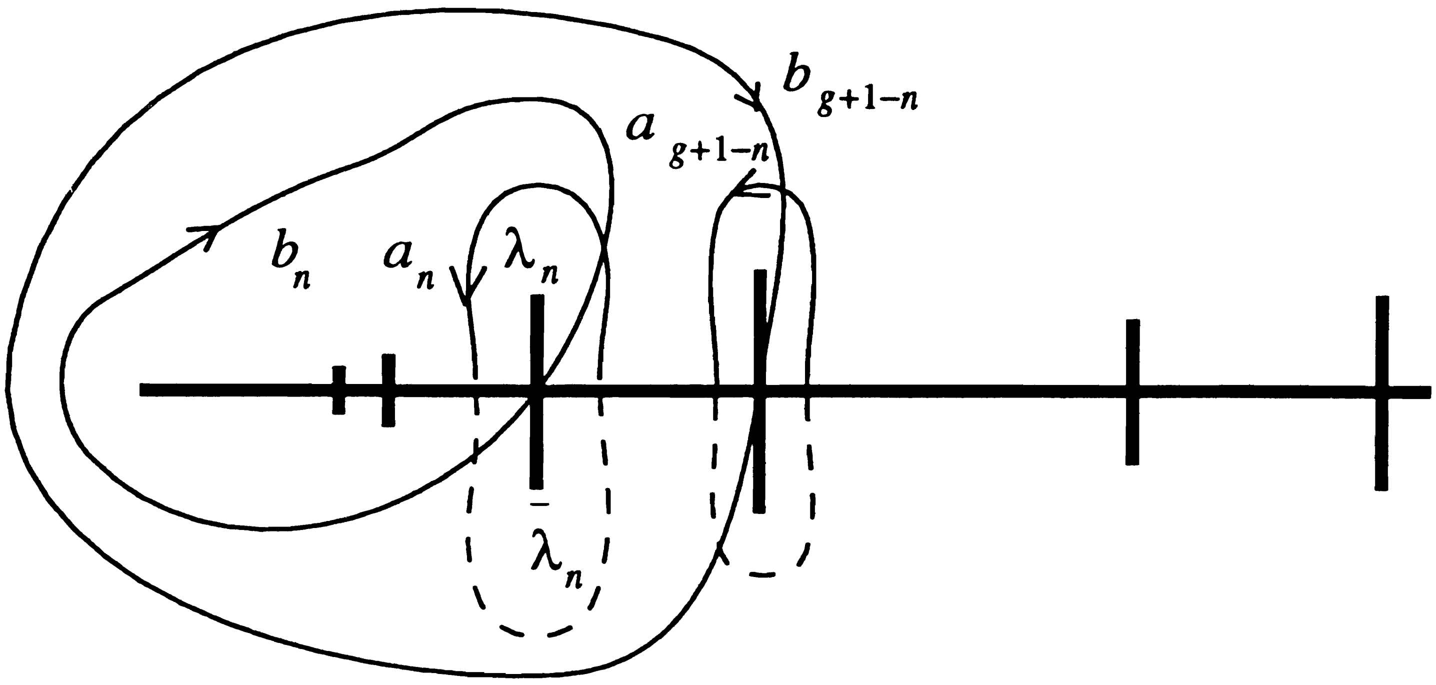

Figure 1. The spectral curve with the canonical basis.

and such that $d\Omega_\infty$ has the only pole in ∞ and $d\Omega_0$ has the only pole in zero:

$$d\Omega_\infty(P) = d(\sqrt{\lambda})\ (P \to \infty), \qquad d\Omega_0(P) = d\left(\frac{1}{\sqrt{\lambda}}\right)(P \longrightarrow 0). \tag{1.3}$$

We denote the b-periods of $d\Omega_\infty, d\Omega_0$ as B^∞, B^0:

$$B_n^{\infty,0} = \int_{b_n} d\Omega_{\infty,0},$$

and define the vectors

$$V = \frac{1}{4}(B^\infty - B^0), \qquad W = \frac{1}{4}(B^\infty + B^0).$$

The antiholomorphic involution τ_2 acts on the basis of the cycles and on the local parameters as follows: $\tau_2 a_\kappa = a_\kappa$, $\tau_2 b_\kappa = -b_\kappa + a_\kappa$, $\tau_2^* \sqrt{\lambda} = -\overline{\sqrt{\lambda}}$. These relations imply

$$\tau_2^*\, d\Omega^\infty = -\overline{d\Omega^\infty}, \qquad \tau_2^*\, d\Omega^0 = -\overline{d\Omega^0}$$

and prove the realvaluedness of the g-vectors V, W.

The finite-gap (theta functional) solutions of (1.1) are given by the formula

$$u(t, x; \boldsymbol{\lambda}, D) = 2i \log \frac{\theta(i(Vx + Wt + D + \Delta))}{\theta(i(Vx + Wt + D))}, \tag{1.4}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_g)$, $V = V(\boldsymbol{\lambda})$, $W = W(\boldsymbol{\lambda})$; $i\Delta = i(\pi, \ldots, \pi)$ is the vector of the half-periods and $D \in \mathbb{T}^g = \mathbb{R}^g/2\pi\mathbb{Z}^g$ is the phase of the solution.

The construction just described assigns to each vector[5] $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_g) \in \mathfrak{M}^g$, where

$$\mathfrak{M}^g = \{(\lambda_1, \ldots, \lambda_g) \mid \lambda_j \in \mathbb{C}_+, \lambda_j \neq \lambda_k\ \forall j \neq k\}, \tag{1.5}$$

the toroidal family of the finite-gap solutions (1.4), where the phase D varies in the g-torus.

[5] In fact, to each set $\{\lambda_1, \ldots, \lambda_g\}$. With some abuse of notations we do not distinguish a vector $(\lambda_1, \ldots, \lambda_g)$ from the set $\{\lambda_1, \ldots, \lambda_g\}$.

For $\boldsymbol{\lambda} \in \mathfrak{M}^g$ we denote by $\boldsymbol{\lambda}^{-1} \in \mathfrak{M}^g$ the g-vector with the inverse components, $(\boldsymbol{\lambda}^{-1})_j = (\lambda_j)^{-1}, j = 1, \ldots, g$, and denote by $\mathfrak{M}^g_{\mathrm{sym}}$ a set of all $\boldsymbol{\lambda} \in \mathfrak{M}^g$ such that

$$|\lambda_1| \leq 1, \ldots, |\lambda_n| \leq 1 \quad \text{and} \quad \boldsymbol{\lambda}^{-1} = \bar{\boldsymbol{\lambda}} \text{ as sets,}$$

where

$$n = n(g) = 1 + \left[\frac{g-1}{2}\right].$$

Since all λ_j's are different, then

$$|\lambda_n| = 1 \qquad \forall \boldsymbol{\lambda} \in \mathfrak{M}^g \quad \text{if } g \text{ is odd.}$$

We denote by T_3 the $g \times g$ matrix

$$T_3 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdots & & \\ 1 & & & \end{bmatrix}.$$

LEMMA 1.1. *Suppose that $\boldsymbol{\lambda} \in \mathfrak{M}^g_{\mathrm{sym}}$. Then the solution* (1.4) *is even in x if $D = T_3 D$ and is odd if $D = T_3 D + (\pi, \ldots, \pi)$. Besides, $T_3 W = W$, $T_3 V = -V$ and the vectors V, W are given by the formulas $V_k = \frac{1}{4}(B_k^\infty - B_{g+1-k}^\infty)$, $W_k = \frac{1}{4}(B_k^\infty + B_{g+1-k}^\infty)$.*

For a proof see [Bo, BiK, BoK2].

A solution $U = (u, v)$ of (1.1) satisfies Neumann boundary conditions (N) if it satisfies “even periodic” boundary conditions with the doubled period:

$$U(t, x) \equiv U(t, x + 2\pi), \qquad U(t, x) \equiv U(t, -x). \tag{EP}$$

Similarly $U(t, x)$ satisfies Dirichlet boundary conditions (D) if it satisfies the “odd periodic” boundary conditions:

$$U(t, x) \equiv U(t, x + 2\pi), \qquad U(t, x) \equiv -U(t, -x). \tag{OP}$$

By Lemma 1.1, to extract from the set of even (odd) solutions (1.4) the solutions of (SG) + (OP) ((SG) + (EP)) we should solve the equation

$$(V_1, \ldots, V_g)(\boldsymbol{\lambda}) \in \mathbb{Z}^g \tag{1.6}$$

for $\boldsymbol{\lambda} \in \mathfrak{M}^g_{\text{sym}}$. We start an analysis of this equation with simple small-gap limits for V and W vectors when $\boldsymbol{\lambda} \in \mathfrak{M}^g_{\text{sym}}$ tends to a real vector $\mathbf{l}$ with positive components:

$$V(\boldsymbol{\lambda}) \longrightarrow V^0(\mathbf{l}), \qquad W(\boldsymbol{\lambda}) \longrightarrow W^0(\mathbf{l}) \qquad \text{as } \boldsymbol{\lambda} \longrightarrow \mathbf{l} \in \mathbb{R}^g_+,$$

where

$$V^0_j(\mathbf{l}) = V^0_j(l_j) = \frac{1}{2}\left(\sqrt{l_j} - \frac{1}{\sqrt{l_j}}\right), \qquad W^0_j(\mathbf{l}) = W^0_j(l_j) = \frac{1}{2}\left(\sqrt{l_j} + \frac{1}{\sqrt{l_j}}\right) \tag{1.7}$$

(see [McK, EFM] and Theorem 1.2 below). As $\boldsymbol{\lambda} \in \mathfrak{M}^g_{\text{sym}}$, then for the limiting vector $\mathbf{l}$ we have: $0 < l_1, \dots, l_n \le 1 < l_{n+1}, \dots, l_g$.

We suppose that all components of the vector $\mathbf{l}$ are different. Then, after unessential reordering of the first and the last n of them, we have:

$$0 < l_n < \cdots < l_1 \le 1 < l_g \cdots < l_{n+1}, \qquad l_j \cdot l_{g+1-j} = 1 \qquad \forall j.$$

After this reordering the components of the vector V_0 are increasing:

$$V^0_n < \cdots < V^0_1 \le 0 < V^0_g < \cdots < V^0_{n+1}, \; V^0_j = -V^0_{g+1-j}.$$

As a suitable parameter for the families of solutions we choose the integer n-vector $\mathbf{V} = -(V^0_1, V^0_2, \dots, V^0_n)$, varying in the set $\mathscr{V}^g$, where $n = n(g)$ and

$$\mathscr{V}^g = \{\mathbf{V} = (V_1, \dots, V_n) \in \mathbb{Z}^n \mid V_n > \cdots > V_1 \ge 0, \; V_1 = 0 \text{ iff } g \text{ is odd}\}.$$

For $\mathbf{V} \in \mathscr{V}^g$ fixed we denote

$$\mathbb{N}_n = \mathbb{N}_n(\mathbf{V}) = (\mathbb{N} \cup \{0\}) \setminus \{-V^0_1, \dots, -V^0_n\}.$$

We treat $\mathbf{V} = \{-V^0_1, \dots, -V^0_n\}$ and $\mathbb{N}_n$ as the lists of open and closed gaps of the solution (1.4).

By (1.7) components W^0_j of the limiting vector W^0 have the form

$$W^0_j = (V^0_j)^*, \qquad 1 \le j \le g,$$

where for real l we denote $l^* = \sqrt{l^2 + 1}$.

Small-amplitudes solutions we are discussing now correspond to the situation when all the cuts in Fig. 1 are small. They are studied in our work [BoK2]. Below in Theorem 1.2 we give the final results of this analysis.

THEOREM 1.2. *For every* $\mathbf{V} \in \mathscr{V}^g$ *there exists* $\rho > 0$ *and real-analytic map*

$$\boldsymbol{\lambda} : M_\rho^C = \{\mu \in \mathbb{C}^n \mid |\mu_j| < \rho \ \forall j\} \to \mathbb{C}^g, \qquad \mu \mapsto \boldsymbol{\lambda}(\mu),$$

such that

(a) *for* $\mu \in M_\rho^+ = M_\rho^C \cap \mathbb{R}_+^n$ *the vector* $\boldsymbol{\lambda}(\mu)$ *lies in* $\mathfrak{M}_{\text{sym}}^g \subset \mathbb{C}_+^g$ *and the Riemann surface* (1.2) *with* $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mu)$ *satisfies* (1.6);

(b) *the maps*

$$\mu \mapsto U(t, x; \boldsymbol{\lambda}(\mu), D), \qquad \mu \mapsto W(\boldsymbol{\lambda}(\mu))$$

are analytic in M_ρ^C *and* $U(t, x; \boldsymbol{\lambda}(0), D) \equiv 0$, $W_j(0) = W_j^0$;

(c) *the vector* $V(\boldsymbol{\lambda}(\mu))$ *equals to* V^0 *for all* μ;

(d) *the matrix* $\partial W / \partial \mu$ *at the point* $\mu = 0$ *equals to*

$$\partial W_j / \partial \mu_k \,|_{\mu = 0} = \begin{cases} -16/W_j^0, & j \neq k, \\ -12/W_j^0, & j = k; \end{cases} \tag{1.8}$$

(e) *for* $\mu = (0, \ldots, \mu_j, \ldots, 0)$, *where* $\mu_j \geq 0$,

$$U(0, x; \boldsymbol{\lambda}(\mu), D) = 16\sqrt{\mu_j}(\cos V_j^0 x \cos D_j, \cos V_j^0 x \sin D_j) + O(\mu). \tag{1.9}$$

COROLLARY 1.3. *The map* $M_\rho^C \to \mathbb{C}^n$, $\mu \mapsto (W_1, \ldots, W_n)(\mu)$, *is an analytic diffeomorphism on its image, provided* ρ *is sufficiently small.*

Proof. We should check that $\det \partial W_j / \partial \mu_k \neq 0$ at $\mu = 0$. This determinant differs by a nonzero factor from the determinant of the matrix $m = (m_{jk})$, where $m_{jj} = 3$ and $m_{jk} = 4$ if $j \neq k$. The matrix m clearly defines an invertible linear map, so $\det m \neq 0$. □

Thus, g-gap solutions $U(t, x; \mu, D) = U(t, x; \boldsymbol{\lambda}(\mu), D)$ of (SG) + (N) analytically depend on μ, D and are parameterized by the discrete parameter $\mathbf{V} \in \mathscr{V}^g$. Below in parts 2–5 the vector $\mathbf{V}$ is fixed.

Due to the symmetry relations, the vectors V, W and D are uniquely defined by their first n components (belonging to $\mathbb{R}^n$ and $\mathbb{T}^n$). With some abuse of notations we denote these n-vectors by the same symbols V, W and D.

The coordinate system (μ, D) is singular in the points, where some μ_j vanishes, because for $\mu_j = 0$ the zone $[\lambda_j, \bar{\lambda}_j]$ shrinks to a point and the solution U does not depend on the phase D_j. This observation hints that the functions $\{(\sqrt{2\mu_j}, D_j) \mid j = 1, \ldots, n\}$ form a "good" polar coordinate system and the solution

u analytically depends on the corresponding Cartesian coordinates (p, q),

$$p_j = \sqrt{2\mu_j}\cos D_j, \qquad q_j = \sqrt{2\mu_j}\sin D_j. \tag{1.10}$$

Direct calculations, given in [BoK2], prove this conjecture:

LEMMA 1.4. *The map*

$$\Phi_0 : D_\rho^{2n} := \{(p, q) \mid p_j^2 + q_j^2 < 2\rho \ \forall j\} \to H_s, \qquad \Phi_0(p, q)(x) = U(0, x; p, q),$$

is real-analytic for every $s \in \mathbb{N}$, *and*

$$\frac{\partial}{\partial p_j}\Phi_0(0) = 8\sqrt{2}(\cos V_j^0 x, 0), \qquad \frac{\partial}{\partial q_j}\Phi_0(0) = 8\sqrt{2}(0, \cos V_j^0 x). \tag{1.11}$$

Moreover, the map Φ_0 *is odd*: $\Phi_0(p, q)(x) \equiv -\Phi_0(-p, -q)(x)$.

In the lemma we denote by H_s the Sobolev space of vector-valued even periodic functions $U(x) = (u(x), v(x))$. That is,

$$H_s = \left\{ U(x) \mid U(x) \equiv U(-x) \equiv U(x + 2\pi), \int_0^{2\pi} |\partial_x^l U(x)|^2 \, dx < \infty \ \forall l \leq s \right\}.$$

The formula (1.11) results from (1.9). The last statement of the lemma follows directly from the formula (1.4), since the transformation $D \mapsto D + \Delta$ interchanges the numerator and the denominator of the logarithm's argument in (1.4).

The following statement (with ρ sufficiently small) is an immediate consequence of the lemma:

COROLLARY 1.5. *The set* $\mathcal{T}_\rho = \Phi_0(D_\rho^{2n})$ *is a* $2n$*-dimensional analytic submanifold of* H_s. *This manifold passes through zero* $0 \in H_s$ *with the tangent space*

$$T_0\mathcal{T}_\rho = E^{2n} := \operatorname{span}\{(\cos V_j^0 x, 0), (0, \cos V_j^0 x) \mid j = 1, \ldots, n\}.$$

The manifold is invariant under the flow of (SG) + (N) *and is foliated by the invariant analytic tori of the form*

$$\Phi_0(T^n(\mu)), \qquad T^n(\mu) = \{p_j^2 + q_j^2 = 2\mu_j \geq 0 \mid j = 1, \ldots, n\}. \tag{1.12}$$

The dimension of the torus $T^n(\mu)$ *equals* n *in general case and drops by one if some* μ_j *vanishes.*

Thus, equation (1.6) defines an n-dimensional analytic subvariety of the g-dimensional domain $\mathfrak{M}^g_{\mathrm{sym}}$. Due to Theorem 1.2, this subvariety has nonempty components $\mathfrak{M}^g_V$, parameterized by the vectors $\mathbf{V}$ from $\mathscr{V}^g$. The g-gap solutions of (SG) + (N), corresponding to vectors from $\mathfrak{M}^g_{\mathbf{V}}$, form in H_s a $2n$-dimensional variety $\mathscr{T}^{2n} = \mathscr{T}^{2n}(\mathbf{V})$, diffeomorphic to $\mathfrak{M}^g_{\mathbf{V}} \times \mathbb{T}^n$. The intersection of $\mathscr{T}^{2n}$ with a small enough neighborhood of zero in the phase-space forms smooth analytic manifold; its closure is a $2n$-dimensional smooth analytic manifold $\mathscr{T}_\rho = \mathscr{T}_\rho(\mathbf{V})$, diffeomorphic to the $2n$-dimensional polydisk D^{2n}_ρ.

Due to Corollary 1.5, manifold $\mathscr{T}_\rho$ is stratified as follows:

$$\mathscr{T}_\rho = \mathscr{T}^0_\rho \cup \left(\bigcup_{g' < g} \mathscr{T}_{\rho, g'} \right),$$

where $\mathscr{T}^0_\rho = \mathscr{T}^{2n} \cap \mathscr{T}_\rho$ is an open part of $\mathscr{T}_\rho$, filled with g-gap solutions, and nonconnected analytic submanifolds $\mathscr{T}_{\rho,g'}$, are filled with $(g' < g)$-gap solutions of (SG) + (N).

The object of this paper is to study behavior of solutions of (SG) and perturbed (SG) equation near manifold $\mathscr{T}_\rho$, including its lower-dimensional submanifolds $\mathscr{T}_{\rho,g'}$, $g' < g$.

In [BiK] the whole variety $\mathscr{T}^{2n}$ *without* lower-dimensional subvarieties $\mathscr{T}_{\rho,g'}$ was considered[6]. The variety $\mathscr{T}$ is formed by the components of $\mathscr{T}(\mathbf{V})$, containing small-amplitude solutions. It does not exhaust all finite-gap solutions; in particular, because the solutions in $\mathscr{T}^{2n}$ have trivial topological charge. So the theory, developed in [BiK] can be called *half-global*. The local situation, which is being considered in this paper, *can not* be covered by the half-global theory from [BiK], because small-amplitude solutions were excluded there from the consideration.

2. Solutions of the linearized equation

We consider equation (1.1) linearized about the g-gap solution $U = (u, v)$:

$$\delta\dot{u} = -\sqrt{A}\,\delta v, \qquad \delta\dot{v} = \sqrt{A}\,(\delta u + A^{-1}(\cos u(t,x)\delta u - \delta u)). \tag{2.1}$$

Clearly, we can exclude δv from this system and obtain for $\delta u(t, x)$ the linearized

---

[6] It was stated in Lemma 2 of [BiK] that the variety $\mathscr{T}^{2n}$ is smooth. At this moment both the authors of [BiK] can not prove this more general statement. However, the information about $\mathscr{T}$ we possess (an analytic variety, smooth near zero) is quite sufficient to carry out the proofs of [BiK].

(SG) equation:

$$\delta\ddot{u} = \delta u_{xx} - (\cos u(t, x))\delta u, \tag{LSG}$$

supplemented by (N) (or (D)) boundary conditions (because the functions δu and δv belong to the domain of definition of the operator A).

There is a natural way to construct solutions $\delta U = (\delta u, \delta v)(t, x)$ of (2.1):

(1) to write $U(t, x; \mu, D) \equiv U(t, x; \lambda(\mu), D)$ as a degenerate $(g + 2)$-zone solution

$$U(t, x; \mu, D) = U^{n+1}(t, x; \mu, \mu_{n+1}; D, D_{n+1})|_{\mu_{n+1}=0},$$

where U^{n+1} is a $(g + 2)$-gap solution of (SG) + (N), corresponding to a vector $\mathbf{V}^{n+1} = (\mathbf{V}, V^0_{n+1}) \in \mathscr{V}^{g+2}(\mathbf{V} \in \mathscr{V}^g$ corresponds to the solution U and $V^0_{n+1} \in \mathbb{N}_n)$;

(2) to obtain a solution of (LSG) as

$$\lim_{\mu_{n+1} \longrightarrow 0} \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial U^{n+1}}{\partial D_{n+1}}, \tag{2.2}$$

(the factor $\mu_{n+1}^{-1/2}$ appears in the formula because not (D_{n+1}, μ_{n+1}) but (p_{n+1}, q_{n+1}) forms a smooth coordinate system near $\mu_{n+1} = 0$).

The solution (2.2) depends on the choice of the phase D_{n+1}. Different solutions are parameterized by elements of the set $\mathbb{N}_n$ which enumerates the closed gaps of the solution U.

We recall that by D^c_ρ we denote the set $\{(p, q) \in \mathbb{C}^{2n} \mid |p_j|^2 + |q_j|^2 < 2\rho \ \forall j\}$.

THEOREM 2.1. *For each* $j = V^0_{n+1} \in \mathbb{N}$ *there exists a linear combination* $\mathfrak{z}_j$ *of two solutions* (2.2) *with different phases* D_{n+1}, *having the form*

$$\mathfrak{z}_j(D, t; \mu)(x) = e^{iw_j(\mu)t}\Psi^j(W(\mu)t + D, \mu)(x), \tag{2.3}$$

where w_j *and* Ψ^j *are analytic functions. The frequency* $w_j(\mu)$ *equals to the* $(n + 1)$'*th component of the* W-*vector of the solution* U^{n+1} *with* $\mu_{n+1} = 0$. *It can be analytically extended to some complex polydisc* $M^c_\rho = \{|\mu_j| < \rho\}$, *where*

$$|w_j(\mu) - j^*| \le C \min(|\mu|, (1 + j)^{-1}). \tag{2.4}$$

The function Ψ^j *is even in* (p, q). *It can be analytically extended to some domain*

$$\mathcal{O}_\rho = \{(p, q) \in D^c_\rho\} \times \{x \in \mathbb{C} \mid |\operatorname{Im} x| < \rho\},$$

where it is close to $(\cos jx, i \cos jx)$:

$$\Psi^i = (\cos jx, i\cos jx) + \Psi^{j0}(W(\mu)t + D, \mu)(x)$$

and

$$\Psi^{j0}(D, \mu) = \frac{1}{2}(e^{ijx}\Psi^{j1}(D, \mu)(x) + e^{-ijx}\Psi^{j1}(D, \mu)(-x)). \tag{2.5}$$

The function Ψ^{j1} *is analytic in* x *and* (p, q)*-variables and everywhere in* $\mathcal{O}_\rho$

$$|\Psi^{j1}| \le C|\mu|(1 + j)^{-1}. \tag{2.6}$$

Proof. In [BoK2] we construct a linear combination of solutions (2.2) with the u-component equal to

$$\Psi^j_u = e^{iw_j t}(\cos jx + \Psi^{j0}_u),$$

where $w_j(\mu)$ satisfies (2.4), the function Ψ^{j0}_u is analytic in $\mathcal{O}_{2\rho}$ with some $\rho > 0$ and has the form (2.5) with Ψ^{j1} replaced by Ψ^{j1}_u. The function Ψ^{j1}_u does not exceed $C|\mu|(1 + j)^{-1}$.

Since $v = A^{-1/2}\dot{u}(t, x)$ and

$$A^{-1/2}\sin(\cos)(kx) = k^{*-1}\sin(\cos)(kx),$$

then the v-component of the solution equals

$$v(t, x; D, \mu) = ie^{iw_j t}(\cos jx + \Psi^{j0}_v),$$

where the function Ψ^{j0}_v has the form (2.5) and the analytic function Ψ^{j1}_v is bounded in $\mathcal{O}_\rho$ by $C'|\mu|(1 + j)^{-1}$. To obtain this estimate one should use the direct and inverse estimates for the norm of an analytic function in a complex strip via its Fourier coefficients (see [A2] and [K1], appendix B to Part 3).

The v-component of the solution is analytic and even in (p, q)-variables as well as the u-component. □

It occurs that the frequencies w_j satisfy nonresonance relations, important for subsequent constructions.

PROPOSITION 2.2. *For all $s \in \mathbb{Z}^n$ and all $l > r$ in $\mathbb{N}_n$ we have*

$$\sum_{j=1}^{n} W_j(\mu)s_j + 2w_r(\mu) \not\equiv 0, \tag{2.7}$$

$$\sum_{j=1}^{n} W_j(\mu)s_j + w_r(\mu) \pm w_l(\mu) \not\equiv 0. \tag{2.8}$$

Moreover, for each function as in the l.h.s. of (2.7) *or* (2.8) *either the function itself, or its gradient does not vanish at* $\mu = 0$.

Proof. We proove more complicated relation (2.8) only. Denote the l.h.s. in (2.8) by $\chi(\mu)$ and suppose that

$$\chi(0) = 0, \qquad \frac{\partial}{\partial \mu_j}\chi(0) = 0 \qquad j = 1, \ldots, n. \tag{2.9}$$

Abbreviating $\sum_{j \in \mathbf{V}}$ to $\sum_j$ we can rewrite the first relation in (2.9) as

$$0 = \chi(0) = \sum_j j^* s_j + r^* \pm l^*.$$

Using (1.8) we can rewrite the second one as

$$-4\left(\sum_k \frac{4}{k^*} s_k - \frac{s_j}{j^*} + \frac{4}{r^*} \pm \frac{4}{l^*}\right) = 0, \qquad j = 1, \ldots, n; \tag{2.10}$$

in particular, $s_j / j^* = C$ for all j in $\mathbf{V}$ with some real C. Hence,

$$C \sum_k k^{*2} + r^* \pm l^* = 0$$

and

$$C(4|\mathbf{V}| - 1) + \frac{4}{r^*} \pm \frac{4}{l^*} = 0. \tag{2.11}$$

We can eliminate C from these equations and find that

$$(r^2 + 1)(l^2 + 1) = (r^* l^*)^2 = \left(\frac{4 \sum_j (1 + j^2)}{4|\mathbf{V}| - 1}\right)^2.$$

Thus, $(r^2+1)(l^2+1)=16N^2$ with some integer N. We have obtained a contradiction because a number m^2+1 with integer m never can be divided by four. $\square$

We have proved Proposition 2.2 for 2π-periodic solutions. If the period equals $2\pi/L$ with some $L>0$, then the numbers $W_j^0=j^*$ in the statements (b), (c) of Theorem 1.2 should be replaced by $\sqrt{j^2L^2+1}$ and it becomes more complicated to prove that the system of $(n+1)$ equations (2.9) has no integer solution $(s_1,\dots,s_n)$. We do not prove the statement in this general setting, but observe the following:

AMPLIFICATION 2.3. (1) *The set of all $L>0$ for which the statement of Proposition* 2.2 *fails has no more than finitely many points in each finite segment* $[a,b]$, $0<a<b<\infty$.

(2) *The statement holds for all L if* $\mathbf{V}=\{0,1,\dots,n-1\}$ (*i.e., if all the first gaps of the finite-gap solution* (1.4) *are open*).

Proof of the first statement see in [BiK].

To prove the second one we observe that all the formulas from the above proof of Proposition 2.2 till (2.11) remain true for an arbitrary $L>0$ if we define r^* as $r^*=\sqrt{r^2L^2+1}$. In particular, the numbers $s_1,\dots,s_n$ have the same sign (and are nonzero). We rewrite (2.10) with $j=n-1$ as follows:

$$4\sum_{k=0}^{n-1}\frac{s_k}{k^*}-\frac{s_{n-1}}{(n-1)^*}=\pm\frac{4}{l^*}-\frac{4}{r^*}. \tag{2.12}$$

As $|s_k|\geq 1$ for all k, then the modulus of the l.h.s. is larger than

$$4\sum_{k=0}^{n-1}\frac{1}{\sqrt{L^2k^2+1}}-\frac{1}{\sqrt{L^2(n-1)^2+1}}>\frac{4n-1}{\sqrt{L^2(n-1)^2+1}},$$

and the modulus of the r.h.s. is less than $8/\sqrt{L^2(n-1)^2+1}$. So (2.12) is impossible if $n\geq 3$.

If $n=2$ the equality is also impossible because $|s_0|+|s_1|\geq 3$ (the choice $|s_1|=|s_2|=1$ contradicts the equality $s_0/0^*=s_1/1^*$). For $n=1$ the equality is impossible for similar arguments. $\square$

As we explained in Part 1, g-gap solutions (1.4) of the equation (SG) + (N) form $2n$-dimensional analytic varieties embedded into the phase space Z. The connected components of these varieties, containing $0\in Z$ in their closures, were denoted as $\mathscr{T}^{2n}=\mathscr{T}^{2n}(\mathbf{V})$, $\mathbf{V}\in\mathscr{V}^g$. Their closures are smooth near zero and contain the small-amplitude manifolds $\mathscr{T}_\rho$ we are studying. The Bloch-like solutions (2.3) can be also constructed for the equation (SG) + (N), linearized about a solution $U = (u, v) \subset \mathcal{T}^{2n}$. For large μ (corresponding to a large-amplitude solution U) the functions $w_j(\mu)$ can have nontrivial branching points. After crossing these points the functions w_j become complex [EFM, BiK] and the solutions (2.3) become exponentially growing as $t \to \infty$. The branching points for the functions w_j can occur outside the singularities of $\mathcal{T}^{2n}$ (and only outside the manifold $\mathcal{T}_\rho$).

The statements of Theorem 2.1 remain essentially the same when $\mathcal{T}_\rho$ is replaced by $\mathcal{T}^{2n}$. Besides, due to uniqueness of the analytic extension the claims of Proposition 2.2 hold for the Bloch-like solutions corresponding to $U \subset \mathcal{T}^{2n}$.

3. Symplectic structure of the phase space and manifold $\mathcal{T}_\rho$. Action-angle variables on $\mathcal{T}_\rho$

We start with defining some functional spaces we need in what follows.

Let $\mathfrak{Z}_k$ be the Sobolev space $H_e^{k+1}(S^1)$ of even 2π-periodic scalar functions (i.e., the space of even 2π-periodic functions with square summable derivatives up to the order $k+1$). We provide $\mathfrak{Z}_0$ with the scalar product

$$\langle u, v \rangle = \int_0^{2\pi} (u_x w_x + uw)\, dx$$

and provide $\mathfrak{Z}_s$, $s \geq 0$, with the scalar product

$$\langle u, v \rangle_s = \langle A^{s/2} u, A^{s/2} w \rangle,$$

where, as above, $A^{s/2}$ is a power of the positive selfadjoint in $\mathfrak{Z}_0$ operator A, $A(u) = -u_{xx} + u$. By the definition of the spaces $\mathfrak{Z}_s$, the operator A isomorphically maps $\mathfrak{Z}_s$ to $\mathfrak{Z}_{s-2}$ (i.e., A is an isomorphism of the scale $\{\mathfrak{Z}_s\}$ of order two).

Let us define the Hilbert spaces Z_s of vector-valued functions,

$$Z_s = \mathfrak{Z}_s \times \mathfrak{Z}_s, \qquad s \geq 0.$$

The scalar product, inherited by Z_s from $\mathfrak{Z}_s$, will be also denoted $\langle \cdot, \cdot \rangle_s$. We abbreviate $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0$.

The operator $J(u, v) = (-\sqrt{A}v, \sqrt{A}u)$ defines unbounded skew-symmetric operators in the spaces Z_s and defines an isomorphism of the scale $\{Z_s\}$ of order one. The operator J^{-1} is bounded skew-symmetric in Z_s, $s \geq 0$, and defines there the

2-form

$$\omega_2 = -\langle J^{-1}\, dz, dz\rangle.^7$$

Let us set $r(u) = -\cos u - \frac{1}{2}u^2$. The functional

$$H(u(x), v(x)) = \int_0^{2\pi} r(u(x))\, dx$$

is analytic in the spaces Z_s, $s \geq 0$. Its gradient with respect to the scalar product $\langle \cdot, \cdot \rangle$ is

$$\nabla H(u, v) = (A^{-1} r'(u(x)), 0).^8 \tag{3.1}$$

Under the symplectic structure given by the two-form ω_2, the Hamiltonian equation corresponding to the hamiltonian

$$\mathscr{H}(z) = \frac{1}{2}\langle z, z\rangle + H(z), \qquad z \in Z,$$

has the form

$$\dot{z} = J\nabla\mathscr{H}(z), \qquad z = (u(x), v(x)) \in Z \tag{3.2}$$

(see [K1]). By (3.1), the last equation may be written as follows:

$$\dot{u} = -\sqrt{A}\, v, \qquad \dot{v} = \sqrt{A}\,(u + A^{-1}(r'(u)).$$

I.e., the Hamiltonian equation with the hamiltonian $\mathscr{H}$ is exactly the (SG) equation, written in the form (1.1).

Now we turn to the manifold $\mathscr{T}_\rho = \Phi_0(D_\rho^{2n})$ and denote by α_2 the form in D_ρ^{2n}, equal to the pull-back of ω_2:

$$\alpha_2 = \Phi_0^* \omega_2.$$

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[7] By definition, $-\langle J^{-1}\, dz, dz\rangle(\mathfrak{z}_1, \mathfrak{z}_2) = -\langle J^{-1}\mathfrak{z}_1, \mathfrak{z}_2\rangle$.

[8] To prove the formula one should observe that

$$\langle \nabla H(u, v), (u_1, v_1)\rangle = dH(u, v)(u_1, v_1) = \int r'(u(x))u_1(x)\, dx = \langle (A^{-1}r'(u(x)), 0), (u_1(x), v_1(x))\rangle.$$

By (1.11), $\alpha_2(0) = \sum B_j^2\, dp_j \wedge dq_j$, where $B_j^2 = 128\pi j^*$. In the dilated variables

$$\tilde{p}_j = B_j p_j, \qquad \tilde{q}_j = B_j q_j, \qquad \tilde{\mu}_j = B_j^2 \mu_j$$

the form $\alpha_2(0)$ is just $d\tilde{p} \wedge d\tilde{q}$. We pass to the tilde-variables and (as usual) omit the tildes in what follows. So

$$\alpha_2 = dp \wedge dq + O(|p, q|),$$

and the form α_2 is nondegenerate on $\mathscr{T}_\rho$ provided that ρ is sufficiently small. Thus, $\mathscr{T}_\rho$ carries the natural symplectic structure.

The restriction of equation (1.1) to $\mathscr{T}_\rho$ is a Hamiltonian vector field V_h with the hamiltonian h equal to the restriction of $\mathscr{H}$ to $\mathscr{T}_\rho$. The open dense subdomain $\mathscr{T}_\rho^0$,

$$\mathscr{T}_\rho^0 = \{(\mu, D) \in \mathscr{T}_\rho \mid \mu_j \neq 0\ \forall j\},$$

if filled with the invariant n-tori $T^n(\mu)$ as in (1.12):

$$\mathscr{T}_\rho^0 = \bigcup \{T^n(\mu) \mid \mu_j > 0\ \forall j\}, \tag{3.3}$$

and restriction of V_h to the torus $T^n(\mu)$ is the Kronecker vector-field,

$$V_h\,|_{T^n(\mu)} = W_j(\mu)\frac{\partial}{\partial D_j}. \tag{3.4}$$

Due to Corollary 1.3,

$$\det \partial W_j / \partial \mu_k \not\equiv 0, \tag{3.5}$$

and for almost all μ trajectories of (3.4) are dense in the torus $T^n(\mu)$. It occurs that the decomposition (3.3) and the nondegeneracy relation (3.5) jointly imply the Liouville–Arnold integrability of V_h (see appendix 1 below). So locally near each torus $T^n(\mu)$ we can construct analytic action-angle variables (I, φ), where the actions I vary in some n-dimensional domain, angles $\varphi \in \mathbb{T}^n$ and

$$\omega_2 = dI \wedge d\varphi, \qquad h = h(I). \tag{3.6}$$

Fortunately, the variables (I, φ) may be analytically extended to the whole domain $\mathscr{T}_\rho$:

THEOREM 3.1. *If ρ is sufficiently small, then there exists an odd analytic transformation*

$$(p, q) \mapsto (\tilde{p}, \tilde{q}), \tag{3.7}$$

such that $(\tilde{p}, \tilde{q}) = (p, q) + O(|p, q|^2)$, $\omega_2 = d\tilde{p} \wedge d\tilde{q}$ and the hamiltonian h, written in the $(\tilde{p}, \tilde{q})$-variables, depends on the actions $I_j = \frac{1}{2}(\tilde{p}_j^2 + \tilde{q}_j^2)$, $j = 1, \ldots, n$ and does not depend on the angles $\varphi_j = \arctan \tilde{q}_j/\tilde{p}_j$. In the variables (μ, D) and (I, φ) the transformation (3.7) has the form

$$(\mu, D) \mapsto (I = I(\mu), \varphi = D + \varphi^0(\mu)),$$

with some analytic map φ^0.

This statement is a version of the Liouville–Arnold theorem for a hamiltonian vector-field with a singularity. For rather sophisticated results of this type see [Ito] and references therein. We give a simple proof of the theorem in appendix 1 (our situation is much simplified by *a priori* knowledge that the tori (1.12) are invariant for the equation).

We finish with a brief discussion of the half-global analytic variety $\mathscr{T}^{2n}$. The restriction of the symplectic form ω_2 to $\mathscr{T}^{2n}$ is nondegenerate almost everywhere (because it is analytic in $\mathscr{T}^{2n}$ and nondegenerate in $\mathscr{T}_\rho$) and the restriction of (SG) + (N) to $\mathscr{T}^{2n}$ is an integrable equation outside some subvariety $\mathscr{T}_{cr}$ of a positive codimension. So $\mathscr{T}^{2n} \backslash \mathscr{T}_{cr}$ is a smooth analytic symplectic manifold with the integrable system on it. Locally (near each invariant n-torus) the action-angle variables can be introduced.

4. Symplectic structure of the infinitesimal vicinity of manifold $\mathscr{T}_\rho$

In Part 2 we constructed "Bloch-like" solutions (2.3) of the linearized Sine–Gordon equation (2.1) and proved nonresonance relations (2.7), (2.8). In this part we show that the corresponding vectors Ψ^j, $\bar{\Psi}^j$, $j \in \mathbb{N}_n$, form a symplectic basis of the skew-orthogonal complement to the tangent space to the manifold $\mathscr{T}_\rho$. It is remarkable that this important property is a rather simple consequence of the nonresonance relations and the asymptotics (2.4) (cf. direct proofs of similar statements in [EFM], [Kri]).

THEOREM 4.1. *If ρ is sufficiently small, then for each (μ, D) the vectors $\{\bar{\Psi}^j(\mu, D), \Psi^j(\mu, D) \mid j \in \mathbb{N}_n\}$ lie in the complexification of the skew-orthogonal com-*

plement to the tangent space $T_{(\mu, D)}\mathcal{T}_\rho$ in Z_s and form a complex basis of this space such that

$$\omega_2(\bar{\Psi}^j, \bar{\Psi}^l) \equiv \omega_2(\Psi^j, \Psi^l) \equiv 0, \qquad \omega_2(\bar{\Psi}^j, \Psi^l) = \delta_{jl} 2i\pi j^* \varkappa_j(\mu), \tag{4.1}$$

where $\varkappa_j$ is real and

$$|\varkappa_j(\mu) - 1| \leq C \min(|\mu|, (1+j)^{-1}). \tag{4.2}$$

The basis from this theorem analytically depends on (μ, D).To state the corresponding result we observe that by (2.5), (2.6)

$$\Psi_0^j := \Psi^j(0, 0; x) = (\cos jx, i \cos jx), \qquad j \in \mathbb{N}_n;$$

and by Corollary 1.5 the tangent space $T_0\mathcal{T}_\rho$ equals to E^{2n}. (In particular, for $(\mu, D) = 0$ the statement of the last theorem is trivial).

Let us denote by Y_s the skew-orthogonal complement to E^{2n} in Z_s,

$$Y_s = \overline{\operatorname{span}}\{\operatorname{Re} \Psi_0^j, \operatorname{Im} \Psi_0^j \mid j \in \mathbb{N}_n\},^9$$

and denote by Φ_1^0 the natural embedding of Y_s to Z_s. The system of the complex vectors $\{\Psi_0^j, \bar{\Psi}_0^j \mid j \in \mathbb{N}_n\}$ forms a symplectic basis of the complexification Y_s^c of the space Y_s:

$$\omega^2(\bar{\Psi}_0^j, \bar{\Psi}_0^l) \equiv \omega^2(\Psi_0^j, \Psi_0^l) \equiv 0, \qquad \omega^2(\bar{\Psi}_0^j, \Psi_0^l) = \delta_{jl} 2i\pi j^*. \tag{4.3}$$

Let us define the map

$$\Phi_1 : D_\rho^{2n} \times Y_s \to Z_s, \qquad (\tilde{p}, \tilde{q}, y) \mapsto \Phi_1(\tilde{p}, \tilde{q})y,$$

which is linear in the third variable, for fixed $(\tilde{p}, \tilde{q})$ sends a vector Ψ_0^j to $\Psi^j(\tilde{p}, \tilde{q})\varkappa_j^{-1/2}(\mu)$ and is extended to all of Y_s by linearity ($(\tilde{p}, \tilde{q})$-variables are the Cartesian coordinates in $\mathcal{T}_\rho$, corresponding to the action-angle variables (I, φ), see Theorem 3.1). By (4.1) and (4.3) for each $(\tilde{p}, \tilde{q})$ the map $\Phi_1(\tilde{p}, \tilde{q}) : Y \to Z$ is symplectic.

The following regularity properties of the map Φ_1 mostly result from the estimate (2.5):

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[9] Here and below bar above a set means its closure.

THEOREM 4.2. *For* $s \geq 0$ *the map* Φ_1 *is Fréchet-analytic jointly in both arguments. The following estimate for the linear map* $\Phi_1(\tilde{p}, \tilde{q})$ *holds after an analytic extension to* D^c_ρ:

$$\|\Phi_1(\tilde{p}, \tilde{q}) - \Phi^0_1\|_{s,s+1} \leq C_s |(\tilde{p}, \tilde{q})|, \tag{4.4}$$

provided that ρ *is small enough. The map* Φ_1 *is even in* $(\tilde{p}, \tilde{q})$. *For fixed* $(\tilde{p}, \tilde{q})$ *it defines a symplectic isomorphism of* Y_s *and the skew-orthogonal complement to* $T_{(\tilde{p}, \tilde{q})}\mathscr{T}_\rho$ *in* Z_s.

Theorems 4.1, 4.2 are proved in Part 4 of [BiK]. Below for the reader's convenience we sketch the proofs:

Proof of Theorem 4.1. To prove that

$$F(D, \mu) := \omega_2(\Psi^j, \Psi^l)(D, \mu) \equiv 0$$

we shall check that the function

$$\begin{aligned}\varphi(D, t; \mu) &:= e^{i(w_j + w_l)t}\omega_2[\Psi^j(Wt + D, \mu), \Psi^l(Wt + D, \mu)]\\ &\equiv \omega_2[\mathfrak{z}_j(D, t; \mu), \mathfrak{z}_l(D, t; \mu)]\end{aligned}$$

vanishes identically. As the skew-product of any two solutions of the linear equation (2.1) is time-independent, then $d/dt\ \varphi \equiv 0$. Thus,

$$0 = \frac{d}{dt}\bigg|_{t=0} \varphi = i(w_j + w_l)F + \frac{\partial F}{\partial q} W.$$

Write F as Fourier series:

$$F(D, \mu) = \sum e^{is \cdot D}\hat{F}(s, \mu).$$

From the last identity we have

$$\hat{F}(s, \mu)((w_j + w_l) + s \cdot W)(\mu) = 0$$

for all s and μ. By (2.8) the second factor is nonzero for almost all μ, so $\hat{F}(s, \mu) \equiv 0$ and $F(q, \mu) \equiv 0$.

In a similar way one proves that $\omega_2(\bar{\Psi}^j, \bar{\Psi}^k) \equiv 0$ and $\omega_2(\Psi^j, \Psi^k) \equiv 0$ if $j \neq k$.

The skew-product $\omega_2(\bar{\Psi}^j, \Psi^j)$ is D-independent because the corresponding function φ as above is time-independent. The estimate (4.2) results from (2.5) and (4.3).

To prove that each vector Ψ^j and $\bar{\Psi}^j$ is skew-orthogonal to the tangent space to $\mathscr{T}_\rho$ one should consider the skew-product of the solution $\mathfrak{z}_j$ with any trajectory of (2.1), starting from a tengent vector to $\mathscr{T}_\rho$, and use the relation (2.7).

By (4.2) we have in (4.1) $\varkappa_j(\mu) \neq 0$. So the vectors $\{\Psi^j, \bar{\Psi}^j \mid j \in \mathbb{N}\}$ are linearly independent. By (2.5), (2.6) and Fredholm theorem

$$\operatorname{codim} \overline{\operatorname{span}}\{\Psi^j, \bar{\Psi}^j \mid j \in \mathbb{N}_n\} = \operatorname{codim} Y_s^c = 2n.$$

As the vectors Ψ^j, $\bar{\Psi}^j$ lie in the skew-orthogonal complement to the $2n$-dimensional space $T_{(\mu,D)}\mathscr{T}_\rho$, and are linearly independent, then they form its basis. □

Proof of Theorem 4.2. The estimate

$$\|\Phi_1(\tilde{p}, \tilde{q}) - \Phi_1^0\|_{s,s+1} \leq C_s$$

results from (2.4), (2.6) because the norm of an operator in a Hilbert space can be estimated by supremum of the l^1-norms of the rows and columns of its matrix. This estimate implies analyticity of the map $\Phi_1 - \Phi_1^0$, because each matrix element of the latter is analytic in $(\tilde{p}, \tilde{q})$ by Theorems 2.1 and 3.1. Now (4.4) results from the Cauchy estimate. □

The vectors Ψ^j and the map Φ_1 are well-defined on the half-global variety $\mathscr{T}^{2n} \subset Z$ outside its singularities, zeros of the functions $\varkappa_j$ (see (4.1)) and branching points of the exponents w_j. Proposition 2.2 (the nonresonance relations) and the asymptotics (2.4), (2.6) also hold there. So the statements of Theorems 4.1, 4.2 remain true for $\mathscr{T}_\rho$ replaced by $\mathscr{T}^{2n}$, after we cut of from the latter a "bad" analytic subvariety $\mathscr{T}_{\text{bad}}$ of a positive codimension.

5. Normal form of the SG equation near manifold $\mathscr{T}_\rho$

By $\mathcal{O}_s(\rho, D_\rho^{2n})$ where $s \geq 0$, $\rho > 0$, we denote the set

$$\mathcal{O}_s(\rho, D_\rho^{2n}) = D_\rho^{2n} \times \{y \in Y_s \mid \|y\|_s < \rho\},$$

endowed the symplectic structure by means of the 2-form $\Omega_2 = d\tilde{p} \wedge d\tilde{q} \oplus \omega_2|_{Y_s}$. In what follows we omit the tildes and write (p, q) instead of $(\tilde{p}, \tilde{q})$. We consider the map

$$\Phi : \mathcal{O}_s(\rho, D_\rho^{2n}) \to Z_s, \qquad (p, q, y) \mapsto \Phi_0(p, q) + \Phi_1(p, q)y.$$

Clearly,

$$\Phi(p, q, 0)_*(\delta p, \delta q, \delta y) = \Phi_0(p, q)_*(\delta p, \delta q) + \Phi_1(p, q)\delta y.$$

By Theorems 3.1, 4.2 the map $\Phi(p, q, 0)_*$ sends the form Ω_2 to ω_2. Thus, if ρ is sufficiently small, then Φ is an analytic diffeomorphism (onto its image) and

$$\Phi^*\omega_2 = \Omega_2 + O(\|y\|_s).$$

The map Φ is odd because Φ_0 is odd (Theorem 1.2) and the map $(p, q) \mapsto \Phi_1(p, q)$ is even (Theorem 4.2).

Now we can apply the Moser–Weinstein theorem [Wei] to get an analytic diffeomorphism

$$\Delta : \mathcal{O}_s(\rho', D_\rho^{2n}) \to \mathcal{O}_s(\rho, D_\rho^{2n})$$

(ρ' is some positive number) such that

$$\Delta_* |_{D_\rho^{2n} \times \{0\}} = id$$

and $\Delta^*(\Phi^*\omega_2) = \Omega_2$. Then

$$\mathfrak{F}^*\omega_2 = \Omega_2 \qquad \text{for } \mathfrak{F} = \Phi \circ \Delta.$$

The map Δ, and so also the map $\mathfrak{F}$, is odd.

The pull-back of the vector-field of the equation (1.1) is a hamiltonian vector-field in $\mathcal{O}_s(\rho', D_\rho^{2n})$ with the hamiltonian $K = \mathscr{H} \circ \mathfrak{F}$ and has the form

$$\dot{q} = \nabla_p K, \qquad \dot{p} = -\nabla_q K, \qquad \dot{y} = J\nabla_y K.$$

Let us write K as

$$K = h^0(p, q) + \langle h^1(p, q), y\rangle + \frac{1}{2}\langle h^2(p, q)y, y\rangle + h^3(p, q, y), \qquad h^3 = O(\|y\|_s^3), \tag{5.1}$$

where h^1 is a vector in Y and h^2 is a selfadjoint operator.

As the set $\{y = 0\}$ is invariant for the equations, then $h^1 \equiv 0$ and $h^0(p, q) = h(I)$, see (3.6) and Theorem 3.1.

In the (I, φ, y)-variables the finite gap solutions $U(t, x)$ take the form

$$I(t) = \text{const}, \qquad \varphi(t) = \varphi_0 + tW(I), \qquad y \equiv 0. \tag{5.2}$$

So the equation, linearized about these solutions, (i.e., the equations (2.1) in the (q, p, y)-variables) has the form

$$\delta \dot{I} = 0, \qquad \delta\dot{\varphi} = W(I)_* \delta I, \qquad \delta \dot{y} = Jh^2(I, \varphi(t))\delta y. \tag{5.3}$$

The map $\mathfrak{F}_*$ transforms solution of (5.3) to solutions of (2.1). As $\mathfrak{F}_*(I, \varphi(t), 0)\delta y = \Phi_1(I, \varphi(t))\delta y$, then by the construction of the map Φ_1 the map $\mathfrak{F}_*$ sends the curves

$$e^{iw_j(I)t}\Psi_0^j, \qquad j \in \mathbb{N}_n,$$

to solutions (2.3) of (2.1). Thus, these curves are solutions of (5.3) and so

$$h^2(I, \varphi)\Psi_0^j = \lambda_j^A(I)\Psi_0^j, \qquad \text{where } \lambda_j^A = w_j(I)/j^*,$$

because $J\Psi_0^j = ij^*\Psi_0^j$. So the operator

$$h^2(I, \varphi) = A(I)$$

is a φ-independent linear operator with the double spectrum $\{\lambda_j^A(I) \mid j \in \mathbb{N}_n\}$, diagonal in the basis $\{\operatorname{Re}\Psi_0^j, \operatorname{Im}\Psi_0^j \mid j \in \mathbb{N}_n\}$ of the space Y.

Now we discuss the last term $h^3(p, q, y)$ in (5.1). As the map $\mathfrak{F}$ is odd and the hamiltonian $\mathscr{H}$ is even, then K is also even. So h^3 contains no cubic terms and

$$h^3 = O(\|y\|_s^3) \cdot O(\|p\| + \|q\| + \|y\|_s). \tag{5.4}$$

An additional nontrivial and essential property of h^3 is its smoothness. This function turns out to be as smooth as the hamiltonian H (see (3.1)):

LEMMA 5.1 (see [K2, K3]). *For $s \geq 0$ the map $\nabla_y h^3$ may be analytically extended to a bounded analytic map*

$$\nabla_y h^3 : D_\rho^c \times \{y \in Y_s^c \mid \|y\|_s < \rho\} \to Y_{s+2}^c, \tag{5.5}$$

where Y_s^c is the complexification of the space Y_s.[10]

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[10] Here and in similar statements below $\rho > 0$ is sufficiently small and depends on s.

We have obtained

THEOREM 5.2. *The odd map* $\mathfrak{F}^{-1}$ *transforms solutions of equation* (1.1) *into solutions of hamiltonian equation on the domain* $\mathcal{O}_s(\rho, D_\rho^{2n})$ *with hamiltonian* K *of the form*

$$K(p, q, y) = h(I) + \frac{1}{2}\langle A(I)y, y\rangle + h^3(p, q, y). \tag{5.6}$$

The function h^3 *satisfies* (5.4), *the gradient map* (5.5) *is analytic and bounded.*

In the half-global situation the normal form (5.6) is available in a neighborhood of $\mathcal{T}^{2n}\backslash\mathcal{T}_{\text{bad}}$ (see the end of the previous part). As some frequencies w_j, corresponding to solutions in $\mathcal{T}^{2n}\backslash\mathcal{T}_{\text{bad}}$ with large norms, can be complex, then the spectrum of the operator $JA(I)$ can contain a finite number of points with nontrivial real parts (these points are not real and form quandruples $\pm\lambda$, $\pm\bar{\lambda}$). Now the operator $A(I)$ has some more complicated form: it is diagonal in the basis $\{\text{Re (Im) } \Psi_0^j\}$ only "up to a finite subsystem" of these vectors. See [BiK] and Part 2.7 in [K1].

6. Perturbed Sine–Gordon equation

Now we start to study perturbations of solutions (1.4), which fill some finite-gap manifold $\mathcal{T}_\rho \subset Z_s$. The number $s \geq 0$ and the set $\mathbf{V} \subset \mathscr{V}^g$ of open gaps are fixed and we abbreviate

$$\|\cdot\| = \|\cdot\|_s.$$

We recall that $\mathcal{T}_\rho$ is an image of the map Φ_0,

$$\Phi_0 : D_\rho^{2n} \to Z_s, \qquad \Phi_0(0) = 0.$$

In D_ρ^{2n} we use the coordinates $(\tilde{p}, \tilde{q})$ constructed in Theorem 3.1 (and omit the tildes), or the corresponding action-angle variables (I, φ). So

$$\{(p, q)\} = D_\rho^{2n} \simeq M_\rho^+ \times \mathbb{T}^n, \qquad M_\rho^+ = \{I\}, \qquad \mathbb{T}^n = \{\varphi\}.$$

The solutions $U = (u, v)$ of (SG) + (N) on the manifold $\mathcal{T}_\rho$ have the form

$$U(t, x) = \Phi_0(I, \varphi + W(I)t)(x)$$

and fill the invariant tori $T^n(I)$,

$$T^n(I) = \Phi_0(\{I\} \times \mathbb{T}^n), \qquad I \in M_\rho^+ .$$

The tangent space at zero $T_0\mathcal{T}_\rho$ equals the image of the tangent map $\Phi_{0*}(0)$ and equals the space E^{2n} (see Corollary 1.5).

We are going to attack the following problem: how do the solutions $U(t, x)$ and the invariant tori $T^n(I)$ they fill behave under higher-order perturbations, in the equation

$$u_{tt} = u_{xx} - \sin u + F_u(u, x), \tag{PSG}$$

$$u_x(t, 0) \equiv u_x(t, \pi) \equiv 0, \tag{N}$$

where F is an analytic in u, C^{s+1}-smooth in x, u function such that

$$|F(u, x)| \leq C|u|^6; \qquad F(u, x) \equiv F(u, x + 2\pi) \equiv F(u, -x). \tag{6.1}$$

Observe that $\sin u = u - \frac{1}{6}u^3 + O(|u|^5)$. So the equation (PSG) may be rewritten as

$$u_{tt} = u_{xx} - u + \frac{1}{6} u^3 + \tilde{F}_u(u, x), \tag{6.2}$$

where $\tilde{F}$ also satisfies (6.1).

The boundary-valued problem (PSG) + (N) may be written down as the Hamiltonian system (3.2) with the hamiltonian $\mathcal{H} = \mathcal{H}_{\text{pert}}$,

$$\dot{U} = J\nabla\mathcal{H}_{\text{pert}}(U), \qquad U = (u(x), v(x)) \in Z,$$

where

$$\mathcal{H}_{\text{pert}}(U) = \frac{1}{2} \langle U, U\rangle + H(U) + H_\Delta(U), \qquad H_\Delta(U) = \int_0^{2\pi} F(u(x), x)\, dx.$$

The functional H_Δ is analytic in Z_s and its gradient map ∇H_Δ is two-smoothing (it sends Z_s to Z_{s+2}).

We can perform the change of variables $\mathfrak{F}$ from Theorem 5.2 and rewrite (PSG) + (N) as the system

$$\dot{q} = \nabla_p K_1, \qquad \dot{p} = -\nabla_q K_1, \qquad \dot{y} = J\nabla_y K_1 \tag{6.3}$$

in $\mathcal{O}_s(\rho, D_\rho^{2n}) = D_\rho^{2n} \times \{\|y\| < \rho\}$ where $K_1 = K + K_\Delta$, $K_\Delta = H_\Delta \circ \mathfrak{F}$ and the hamiltonian K is as in (5.1). For the perturbation K_Δ the gradient map

$$\nabla_y K_\Delta : D_\rho^C \times \{\|y\|_s < \rho\} \to Y_{s+2}^C$$

is analytic. This follows from analyticity of the map

$$H_e^{s+1}(S^1) \longrightarrow H_e^{s+1}(S^1), \qquad u(x) \mapsto f(u(x); x) \equiv F_u(u(x), x),$$

("e" stands for "even", $s \geq 0$), which in turn results from analyticity of the map

$$H^{s+1}(S^1) \longrightarrow H^{s+1}(S^1), \qquad u \mapsto f(u, x), \tag{*}$$

since (*) preserves the closes subspace $H_e^{s+1}(S^1) \subset H^{s+1}(S^1)$.

Remark. Analyticity of the maps $\nabla_y K_\Delta$ and ∇H_Δ is less obvious in the odd periodic case which corresponds to the Neumann problem $(1) + (N)$. Now the maps clearly are analytic (with the same proof) if $f(u, x) \equiv f(u, x + 2\pi) \equiv -f(-u, -x)$ (this holds if F is 2π-periodic in x and even in (x, u)). Consider f which is not odd and for the sake of simplicity suppose that it is x-independent: $f = f(u)$. We pass from the space $H_o^{s+1}(S^1)$ ("o" for odd) to the space $H_{tr}^{s+1}(0, \pi)$ of the traces on the segment $[0, \pi]$ and accordingly modify the phase space Z_k. This change is inessential since the trace-map defines an isomorphism of H_o^{s+1} and H_{tr}^{s+1}. For $s = 0$ (this choice agrees with the restrictions of our theorems) we have $H_{tr}^1(0, \pi) = \mathring{H}^1(0, \pi)$ and the map clearly is analytic. We omit discussion of the higher-smoothness case ($s > 0$) but just mention that under the restriction (2) the map (*) is analytic in H_{tr}^{s+1} if $s \leq 5$.

We study perturbations of solutions (1.4) with a norm of order $\zeta \ll 1$. This is equivalent to suppose that the corresponding actions I's vary in the domain $\mathcal{I}$ of the form

$$\mathcal{I} = \mathcal{I}(\zeta) = \{I \in \mathbb{R}^n \mid 0 < I_j < \zeta^2 \ \forall j\}.$$

We cut away solutions with one of the actions too small and consider the solutions with $I \in \mathcal{I}_r$, where

$$\mathcal{I}_r = \mathcal{I}_r(\zeta) = \{I \in \mathbb{R}^n \mid r\zeta^2 < I_j < \zeta^2 \ \forall j\}$$

and $r < 1$ is fixed for a moment. In the new variables the invariant tori $T^n(I)$ have the form $\{I = \text{const}, y = 0\}$. To study the perturbed equations near some n-torus

$T^n(I)$ with $I \in \mathscr{I}_r$ we stretch the variables by means of the substitution

$$I := I + \zeta^2 \tilde{I}, \qquad \varphi := \tilde{\varphi}, \qquad y := \zeta \tilde{y}. \tag{6.4}$$

In the tilde-variables the perturbed equation has the form (6.3) with the hamiltonian K_2,

$$K_2 = \text{const} + \nabla h(I) \cdot \tilde{I} + \frac{1}{2} \langle A(I)\tilde{y}, \tilde{y} \rangle + \tilde{h},$$

where

$$\tilde{h} = \zeta^{-2}((h(I + \zeta^2 \tilde{I}) - h(I) - \zeta^2 \nabla h(I) \cdot \tilde{I})$$
$$+ \zeta^2 \langle (A(I + \zeta^2 \tilde{I}) - A(I))\tilde{y}, \tilde{y} \rangle + h^3(I + \zeta^2 \tilde{I}, \tilde{\varphi}, \zeta \tilde{y}) + K_\Delta(I + \zeta^2 \tilde{I}, \tilde{\varphi}, \zeta \tilde{y})).$$

The functions h, h^3, K_Δ and the operator A are analytic in $\{|\tilde{I}| < r/2\} \times \mathbb{T}^n \times \{\|\tilde{y}\| < 1\}$, and h^3 satisfies (5.4). So the hamiltonian $\tilde{h}$ is analytic, the gradient map $\nabla_{\tilde{y}} \tilde{h}$ is 2-smoothing as in (5.5) and

$$\tilde{h} = O(\zeta^2(|\tilde{I}|^2 + |\tilde{I}| \|\tilde{y}\|^2 + \|\tilde{y}\|^3) + \zeta^4).$$

Now we treat I as a parameter of the equation, which we shall study for small $\tilde{I}, \tilde{y}$. The parameter I varies in the domain $\mathscr{I}_r$ of the "effective radius" $\delta_a = \zeta^2$:

$$\operatorname{diam} \mathscr{I}_r \le C\delta_a, \qquad \operatorname{mes} \mathscr{I}_r \ge C^{-1}\delta_a^n,$$

with some ζ-independent C. We denote $\varepsilon = \zeta^4$ and treat ε as a magnitude of the perturbation. Then $\varepsilon = \zeta^4 = \delta_a^2$.

The function h and the operator A are analytic in I from the complex polydisc M_ρ^c, so their gradients in $I \in \mathscr{I}$ can be estimated via the Cauchy inequality. The functions h^3 and K_Δ can be analytically extended to a complex neighborhood of $\mathscr{I}_r$ of the radius $\delta_a r/C$. So their I-gradients for I in $\mathscr{I}_r$ are majorized by $C(\delta_a r)^{-1}|h^3|$ and $C(\delta_a r)^{-1}|K_\Delta|$.

We summarize our knowledge about the hamiltonian K_2 as follows:

(i) the map

$$\omega : \bar{\mathscr{I}} \to \mathbb{R}^n, \qquad I \mapsto \omega = \nabla h(I) \tag{6.5}$$

is an analytic diffeomorphism (so we can pass from the parameter $I \in \mathscr{I}_r$ to $\omega \in \nabla h(\mathscr{I}_r)$);

(ii) $|\tilde{h}| + r\delta_a |\nabla_I \tilde{h}| = O(\delta_a(|\tilde{I}|^2 + |\tilde{I}| \|\tilde{y}\|^2 + \|\tilde{y}\|^3) + \varepsilon)$ and the gradient map $\nabla_y \tilde{h}$ is two-smoothing as in (5.5);

(iii) the operator $JA(I)$ is diagonal in the complex basis $\{\Psi_0^j, \bar{\Psi}_0^j\}$ with analytic in $I \in \bar{\mathscr{I}}$ eigenvalues $\{\pm i w_j(I)\}$, obeying (2.4);

(iv) for each finite system of resonance relations

$$W(I) \cdot s \pm 2w_j(I), \qquad W(I) \cdot s \pm w_j(I) \pm w_k(I),$$

$$|s| \leq M_1, \qquad j < k \leq j_1$$

there exists ζ-independent $C_* > 0$ such that each function as above or its I-gradient is $\geq C_*^{-1}$ everywhere in $\mathscr{I}$, provided that ζ is small enough.[11]

By the properties (i)–(iv) the abstract theorem on perturbations of finite-dimensional invariant tori in parameter-depending linear hamiltonian systems [K1, K4] can be applied to prove persistence most of the tori $T^n(I)$, $I \in \mathscr{I}_r$, in the perturbed equation.

An application of Theorem 3.12 from [K1, p. 53] with ω as a parameter, $\omega \in \Omega = \nabla h(\mathscr{I}_r)$, implies (see Appendix 2 for a correction), that

THEOREM 6.1. *For each given* $0 < r, \gamma < 1$ *and for* $0 < \zeta < \zeta(r, \gamma)$ *there exists a Borel subset* $\tilde{\mathscr{I}}_r \subset \mathscr{I}_r$, $\operatorname{mes}(\mathscr{I}_r \setminus \tilde{\mathscr{I}}_r) \leq \gamma \operatorname{mes} \mathscr{I}_r$,[12] *and for* $I \in \tilde{I}_r$ *there exists an analytic map*

$$\tilde{\Sigma}_I : \mathbb{T}^n \mapsto \mathbb{R}^n \times \mathbb{T}^n \times Y_s = \{\tilde{I}, \tilde{\varphi}, \tilde{y}\}$$

and an n-vector $\tilde{W}_r(I)$ *such that the curves*

$$t \mapsto \tilde{\Sigma}_I(\varphi + \tilde{W}_r(I)t) \tag{6.6}$$

are time-quasiperiodic solutions of the system with hamiltonian K_2. *All Lyapunov exponents of these solutions equal zero. The vector* $\tilde{W}_r$ *is close to* W *and the map*

$$\tilde{\Sigma} : \tilde{\mathscr{I}}_r \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{R}^n \times Y_s, \qquad (I, \varphi) \mapsto \tilde{\Sigma}_I(\varphi)$$

[11] This statement is a reformulation of Proposition 2.2.

[12] *mes* = Lebesgue measure.

is close to the map $\Sigma(I, \varphi) = (0, \varphi, 0)$:

$$|W - \tilde{W}_r| \le C\zeta^4, \qquad \mathrm{Lip}_I |W - \tilde{W}_r| \le C\zeta^2, \tag{6.7}$$

$$\|\tilde{\Sigma} - \Sigma\| \le C\zeta^2, \qquad \mathrm{Lip}_\varphi \|\tilde{\Sigma} - \Sigma\| \le C\zeta^2, \qquad \mathrm{Lip}_I \|\tilde{\Sigma} - \Sigma\| \le C \tag{6.8}$$

with some $C = C(r, \gamma)$.[13]

Now we use the formulas (6.4) to go back to the variables (I, φ, y) in the domain $O_s(\varphi, D^{2n}_\rho)$. After this we pass in D^{2n}_ρ from the action-angle variables (I, φ) to the Cartesian variables which we denote (p, q) in the preimage and (p_r, q_r) in the image. We use the map $\mathfrak{F}$ to go to the "usual" variable in a neighborhood of T_ρ in Z_s and denote the resulting map by $\tilde{\Phi}_r$:

$$\begin{tikzcd}
D^{2n}_\rho \ni (p, q) \arrow[dddr, "\tilde{\Phi}_r"'] \arrow[r, mapsto] & (I, \varphi) \mapsto \tilde{\Sigma}(I, \varphi) = (\Sigma^{\tilde{I}}, \Sigma^{\tilde{\varphi}}, \Sigma^{\tilde{y}}) \arrow[d, mapsto] \\
& (I + \zeta^2 \Sigma^{\tilde{I}}, \Sigma^{\tilde{\varphi}}, \zeta \Sigma^{\tilde{y}}) \arrow[d, mapsto] \\
& (p_r, q_r, y_r) \in O_s(\rho, D^{2n}_\rho) \arrow[d, mapsto] \\
& \mathfrak{F}(p_r, q_r, y_r) \in Z_s
\end{tikzcd}$$

As $\sqrt{r}\,\zeta < |(p_j, q_j)| = \sqrt{2|I_j|} \le \sqrt{2}\,\zeta$ for $j = 1, \dots, n$, then as a trivial consequence of (6.8) we get the estimate

$$\|\tilde{\Phi}_r - \Phi_0\| \le C_1 \zeta^3.$$

More cumbersome but as elementary as above arguments show that

$$\mathrm{Lip}\|\tilde{\Phi}_r - \Phi_0\| \le C_1 \zeta^2.$$

In particular, the map $\tilde{\Phi}_r$ is an embedding because it is Lipschitz-close to the embedding Φ_0.

The constant C_1 in the last inequalities (as well as C in (6.7), (6.8)) depends on r and γ. To avoid this dependence we observe that for each $\tilde{\varkappa} > 0$ the inequalities

[13] In (6.8) by $\mathrm{Lip}_\varphi \|\tilde{\Sigma} - \Sigma\|$ is denoted the Lipschitz constant in φ of the map $(\tilde{\Sigma} - \Sigma) : \tilde{\mathscr{I}}_r \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n \times Y_s$, etc.

imply existence of $\tilde{\zeta}(r, \gamma)$, $0 < \tilde{\zeta} < \zeta(r, \gamma)$, such that

$$\|\tilde{\Phi}_r - \Phi_0\| \le \zeta^{3-\tilde{\varkappa}}, \qquad \mathrm{Lip}\|\tilde{\Phi}_r - \Phi_0\| \le \zeta^{2-\tilde{\varkappa}}, \tag{6.9}$$

and

$$|\tilde{W}_r - W| \le \zeta^{4-\tilde{\varkappa}}, \qquad \mathrm{Lip}|\tilde{W}_r - W| \le \zeta^{2-\tilde{\varkappa}}, \tag{6.10}$$

provided that $\zeta \le \tilde{\zeta}(r, \gamma)$. We can suppose that the positive function $\tilde{\zeta}$ is monotonic:

$$\tilde{\zeta}(r, \gamma) \ge \tilde{\zeta}(r_1, \gamma_1) \qquad \text{if } r \ge r_1, \gamma \ge \gamma_1$$

(otherwise we replace $\tilde{\zeta}$ by the function which sends (r, γ) to $\sup_{r_1 \le r, \gamma_1 \le \gamma} \tilde{\zeta}(r_1, \gamma_1)$).

Now we shall iterate the application of Theorem 6.1 to construct perturbations of arbitrarily small finite-gap solutions (i.e., without the restriction $I_j \ge \zeta^2 r$). We remind that a Borel subset $\tilde{M}$ of a Borel set M, $M \subset \mathbb{R}^n$, has a density $\tilde{\varkappa}(0 \le \tilde{\varkappa} \le 1)$ at a point $m_* \subset M$, if

$$\frac{\mathrm{mes}\{m \in \tilde{M} \mid |m - m_*| < v\}}{\mathrm{mes}\{m \in M \mid |m - m_*| < v\}} \longrightarrow \varkappa \qquad \text{as } v \longrightarrow 0$$

(we suppose that the denominator does not vanish for positive v). Clearly, a subset $\tilde{M}$ has density $\tilde{\varkappa}$ at m_* if and only if $M \backslash \tilde{M}$ has there density $1 - \tilde{\varkappa}$.

THEOREM 6.2. *For each $\varkappa > 0$ there exists a Borel subset $\tilde{D} \simeq \tilde{M} \times \mathbb{T}^n$ of $D^{2n}_\rho \simeq M^n_\rho \times \mathbb{T}^n$, having density one at zero and Lipschitz maps $\tilde{\Phi} : \tilde{D} \times \mathbb{T}^n \to Z_s$, $\tilde{W} : \tilde{M} \to \mathbb{R}^n$ such that the curves*

$$t \mapsto \tilde{\Phi}(I, \varphi + \tilde{W}(I)t) \tag{6.11}$$

are time-quasiperiodic solutions of (PSG) + (N) *with zero Lyapunov exponents. The map $\tilde{\Phi}$ is close to Φ_0 and the vector $\tilde{W}$ is close to W for small (p, q)*:

$$\|\tilde{\Phi}(p, q) - \Phi_0(p, q)\| \le C|(p, q)|^{3-\varkappa}, \qquad \mathrm{Lip}\|\tilde{\Phi} - \Phi_0\| \le C\rho^{2-\varkappa}; \tag{6.12}$$

$$|\tilde{W}(p, q) - W(p, q)| \le C|(p, q)^{4-\varkappa}, \qquad \mathrm{Lip}|\tilde{W} - W| \le C\rho^{2-\varkappa}. \tag{6.13}$$

COROLLARY 6.3. *The set $\tilde{\mathcal{T}}_\rho = \tilde{\Phi}(\tilde{D})$ has the tangent space at zero, equal to E^{2n}. This set is of positive Hausdorff measure $\mathcal{H}^{2n}$ and $\mathcal{H}^{2n}(\tilde{\mathcal{T}}_\rho)/\mathcal{H}^{2n}(\mathcal{T}_\rho) \to 1$ as $\rho \to 0$.*

Proof. The first statement results from the first estimate in (6.12). The second one follows from the basic properties of the Hausdorff measure and the second estimate in (6.12), because a map, which is Lipschitz-close to the identity, changes $\mathscr{H}^{2n}$ only a little [Fe]. □

Proof of Theorem. For $j = 0, 1, 2, \ldots$ let us set

$$\zeta_j = 2^{-j}\zeta, \qquad r_j = \Gamma_j r, \qquad \gamma_j = \Gamma_j \gamma,$$

where $\Gamma_0 = 1$, $\Gamma_j \searrow 0$ $(j \to \infty)$ and $\zeta_j \le \tilde{\zeta}(r_j, \gamma_j)$. The sequence $\{\Gamma_j\}$ exists because the function

$$(0, 1] \longrightarrow \mathbb{R}, \qquad \Gamma \mapsto \tilde{\zeta}(\Gamma r, \Gamma \gamma),$$

is positive and increasing.

For $j = 0, 1, 2, \ldots$ we can apply Theorem 6.1 to the sets $\mathscr{I}^j = \mathscr{I}_{r_j}(\zeta_j)$ (first two of them are represented on Fig. 2 below) and construct the subjects $\tilde{\mathscr{I}}^j \subset \mathscr{I}^j$, the maps $\tilde{\Phi}^j : \tilde{\mathscr{I}}^j \times \mathbb{T}^n \to Z_s$ and the n-vectors $\tilde{W}^j(I)$, $I \in \tilde{\mathscr{I}}^j$, satisfying the estimates (6.9), (6.10) with $\zeta = \zeta_j$ and defining solutions of (PSG) + (N) of the form (6.11) with $\tilde{\Phi} = \tilde{\Phi}^j$, $\tilde{W} = \tilde{W}^j$.

For $v > 0$ we denote by $K(v)$ the cube

$$K(v) = \{I \mid 0 \le I_l \le v \ \forall l\}$$

(so $\mathscr{I}^j \subset K(\zeta_j^2)$) and construct the subset $\mathscr{I}_1 \subset M_\rho^+$ as the disjoint union

$$\mathscr{I}_1 = \bigcup_{j=0}^{\infty} (\tilde{\mathscr{I}}^j \backslash K(\zeta_{j+1}^2)).$$

LEMMA 6.4. *The subset $\mathscr{I}_1 \subset M_\rho^+$ has density one at zero.*

We omit an elementary proof which follows from the convergences $\gamma_j \searrow 0$, $r_j \searrow 0$.

Choose in (6.9), (6.10) $\tilde{\varkappa} = \varkappa/2$ and define the maps $\tilde{\Phi} : \mathscr{I}_1 \times \mathbb{T}^n \to Z_s$ and $\tilde{W} : \mathscr{I}_1 \to \mathbb{R}^n$ be equal to $\tilde{\Phi}^j$ and $\tilde{W}^j$ in $\tilde{\mathscr{I}}^j \times \mathbb{T}^n$, $j = 0, 1, \ldots$ It results from (6.9), (6.10) that the map $\tilde{\Phi}$ meets the first estimates in (6.12), (6.13) everywhere in $\mathscr{I}_1 \times \mathbb{T}^n$. The map $\tilde{\Phi}$ is analytic in q; both maps $\tilde{\Phi}$ and $\tilde{W}$ are Lipschitz in each component $(\tilde{\mathscr{I}}^j \backslash K(\zeta_{j+1}^2)) \times \mathbb{T}^n$, but they may be discontinuous in I at boundary points of the cubes $K(\zeta_j^2)$. To improve this imperfection we cut off from the set $\mathscr{I}_1$

small neighborhoods of the boundaries of the cubes and denote

$$\tilde{M} = \mathscr{I}_1 \backslash \mathscr{I}_\zeta, \qquad \mathscr{I}_\zeta = \bigcup_{j=1}^{\infty} (K(\zeta_j^2 + \zeta_j^{2+\nu}) \backslash K(\zeta_j^2 - \zeta_j^{2+\nu}))$$

with $\nu = \varkappa/2$ (see Fig. 2).

Now we can estimate the increments of the map $\tilde{\Phi} - \Phi_0$, corresponding to points in different components of $\tilde{D} = \tilde{M} \times \mathbb{T}^n$, by the first estimate in (6.9) and the increments, corresponding to points in the same component of $\tilde{D}$ by the second one. Thus we obtain the estimate (6.12) for $\mathrm{Lip}\|\tilde{\Phi} - \Phi_0\|_s$ and the estimate (6.13) for $\mathrm{Lip}|\tilde{W} - W_0|$.

The set $\mathscr{I}_\zeta$ has zero density at zero. So Lemma 6.4 implies that $\tilde{M}$ has unit density at zero. As $dp\, dq = dI\, d\varphi$, then the set $\tilde{D} = \tilde{M} \times \mathbb{T}^n$ has unit density at zero as well, and the theorem is proved. □

Theorem 6.2 deals with small-amplitude solutions of the (PSG) equation (equivalent to (6.2)) under even 2π-periodic boundary conditions (equivalent to (N)). The only part of the proof where we have used the exact value of the period is Proposition 2.2. So Theorem 6.2 remains true for even T-periodic solutions if for this value of the period we can prove Proposition 2.2. In particular, Amplification 2.3 implies the following result.

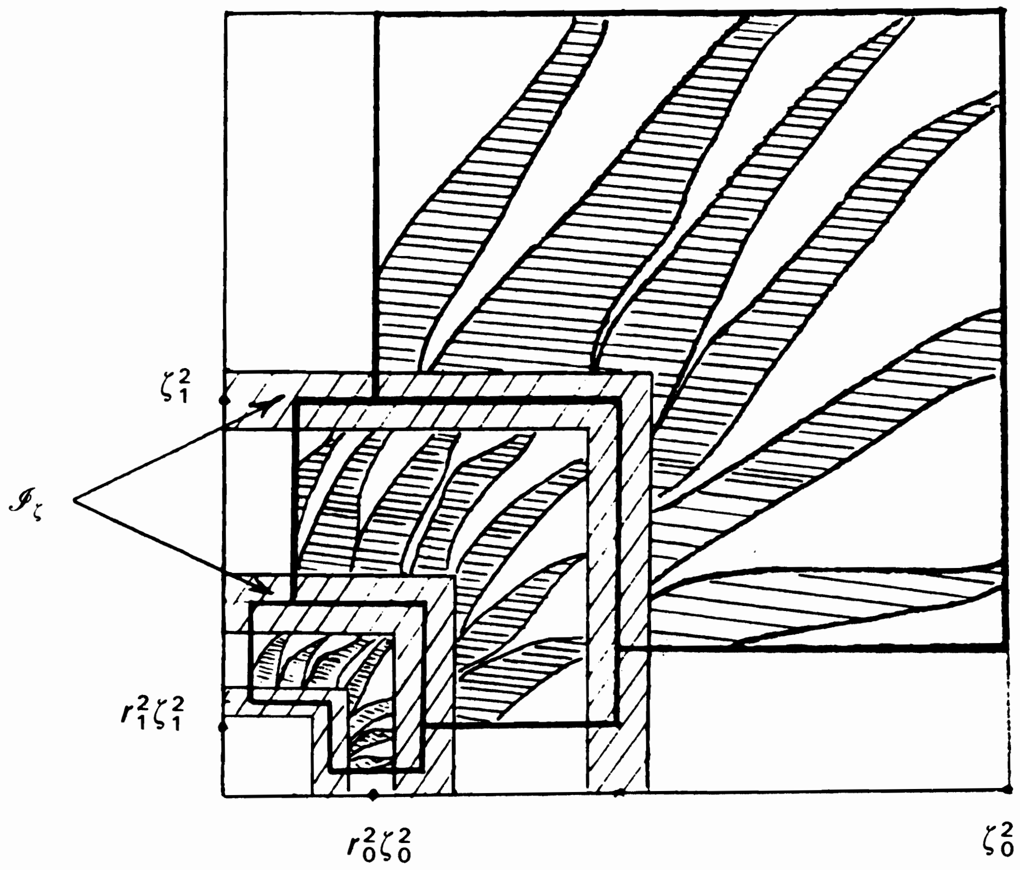

Figure 2. The set $\tilde{M}$.

AMPLIFICATION 6.5. (1) *If* $\mathbf{V}=\{V_1^0,\ldots,V_n^0\}=\{0,\ldots,n-1\}$[14], *then the statements of Theorem* 6.2 *remain true for all periods* T. (2) *The statements are true for all* $\mathbf{V}$ *and all periods* $T\in\mathbb{R}_+\backslash\mathfrak{T}$, *where* $\mathfrak{T}$ *is a discrete set which has no more than finitely many points in each finite segment* $[a,b]$, $0<a<b<\infty$.

Remark. Due to the complete analogy between Dirichlet and Neumann boundary conditions (see Part 1) all the results proven above remain true for the (PSG) equation under the boundary conditions

$$u(t,0)\equiv u(t,\pi)\equiv 0, \tag{D}$$

if we replace $\mathscr{T}_\rho$ by a $2n$-dimensional submanifold of the phase-space, filled with finite-gap solutions of (SG) + (D) (and accordingly replace cos's by sin's in the definition of the spaces E^{2n}). □

In the half-global situation one deals with finite-gap solutions filling the manifold $\mathscr{T}^{2n}=\mathscr{T}^{2n}(\mathbf{V})\simeq\mathfrak{M}_{\mathbf{V}}^g\times\mathbb{T}^n$ (see the end of Part 1). Now the equation (SG) should be perturbed by a small function (rather than by a higher-order term as in (PSG)):

$$u_{tt}=u_{xx}-\sin u+\varepsilon F_u(u,x), \tag{6.14}$$

where the function F is analytic in u, C^{s+1}-smooth in x and

$$F(u,x)\equiv F(u,x+2\pi)\equiv F(u,-x).$$

The half-global analogy of Theorem 6.2, proven in [BiK], states existence of a Borel subset $\mathfrak{M}_\varepsilon\subset\mathfrak{M}_{\mathbf{V}}^g$ such that $\operatorname{mes}(\mathfrak{M}_{\mathbf{V}}^g\backslash\mathfrak{M}_\varepsilon)\to 0$ as $\varepsilon\to 0$ and the solutions (1.4) with μ in $\mathfrak{M}_\varepsilon$ persist in the perturbed equation (6.14) + (N).

7. Application to the φ^4-equation

The φ^4-equation with positive mass has the form

$$u_{tt}=u_{xx}-mu+Cu^3, \tag{φ^4}$$

where $m>0$ and $C\neq 0$. Suppose $C>0$ (as we explained in the introduction, the

[14] i.e., if the first n gaps of the solutions (1.4) forming the mainfold $\mathscr{T}_\rho$ are open.

case $C < 0$ can be treated similar with the Sine–Gordon equation replaced by the Sinh–Gordon). We start with the unit-mass case: $m = 1$. Then by means of a trivial dilation of the u-variable the equation can be normalized as follows:

$$u_{tt} = u_{xx} - u + \frac{1}{6}u^3. \tag{Nφ^4}$$

This is exactly equation (6.2) with $\tilde{F} = 0$, and the results of the last part are applicable to study its small-amplitude solutions under Neumann boundary conditions (N).

We denote by i the natural embedding of the space E^{2n} to Z_s and formulate assertions of Theorem 6.2 as follows:

THEOREM 7.1. *There is a Borel subset* $\tilde{E} \subset E^{2n} \simeq \mathbb{R}^m_+ \times \mathbb{T}^n$ *of unit density at zero and of the form* $\tilde{E} \simeq \tilde{M} \times \mathbb{T}^n$ *and a Lipschitz embedding* $\tilde{\Phi} : \tilde{E} \to Z_s$, *analytic in* $\varphi \in \mathbb{T}^n$, *such that*

(i) *the tori* $\tilde{\Phi}(\{\mu\} \times \mathbb{T}^n) \subset Z_s$, $\mu \in \tilde{M}$, *are invariant for* (Nφ^4) + (N) *and are filled with time-quasiperiodic solutions with zero Lyapunov exponents*;

(ii) *the Lipschitz constant* $\mathrm{Lip}\|\Phi - i\| \le \frac{1}{2}$ *and for* ξ *in* $\tilde{E}$, $\Phi(\xi) = i(\xi) + O(|\xi|^2)$. *Moreover, the sets* $\tilde{\mathscr{T}} = \tilde{\Phi}(\tilde{E})$ *and the manifold* $\mathscr{T}_\rho$ *have second-order tangency at zero.*

In the general case $(m, C > 0)$ we rescale x-, t- and u-variables to rewrite $(\varphi^4) + (\mathrm{N})$ as the normalized equation (Nφ^4) under the boundary conditions

$$u_x(t, 0) \equiv u_x(t, \sqrt{m}\,\pi) \equiv 0. \tag{7.1}$$

Now the linearized at zero equation has the form $u_{tt} = u_{xx} - mu$, so the wave-numbers in the definition of the invariant spaces E^{2n} should be replaced accordingly:

$$E^{2n} = \mathrm{span}\{(\cos(V_j^0 x/\sqrt{m}), 0), (0, \cos(V_j^0 x/\sqrt{m})) \mid j = 1, \ldots, n\}. \tag{7.2}$$

We can apply Amplification 6.5 to get

AMPLIFICATION 7.1. (1) *Statements of Theorem* 7.1 *remain true for the equation* $(\varphi^4) + (\mathrm{N})$ *with an arbitrary* m, $C > 0$ *and the space* E^{2n} *defined as in* (7.2) *provided that*

$$\{V_1^0, \ldots, V_n^0\} = \{0, 1, \ldots, n-1\}. \tag{7.3}$$

(2) *If the wave-numbers* $\{V_j^0\}$ *are just any n numbers, then the statements hold provided that* $m \notin \{m_1, m_2, \ldots\}$, *where the only possible limiting points for the set* $\{m_1, m_2, \ldots\} \subset \mathbb{R}_+$ *are* 0 *and* ∞.

So the equation $(\varphi^4) + (\mathrm{N})$ has many small-amplitude time-quasiperiodic solutions. To make this statement quantitative we rescale $u = \varepsilon \tilde{u}$, $\varepsilon \ll 1$, and obtain for $\tilde{u}$ the equation

$$\tilde{u}_{tt} = \tilde{u}_{xx} - m\tilde{u} + C\varepsilon^2\tilde{u}^3. \tag{7.4}$$

Denote by $QP_\varepsilon \subset Z_s$ the "quasiperiodic set of the equation", equal to the union in the phase-space Z_s all the curves corresponding to time-quasiperiodic solutions of $(7.4) + (\mathrm{N})$ with zero Lyapunov exponents.

PROPOSITION 7.3. *For any* $\mathfrak{z}(x) \in Z_s$

$$\mathrm{dist}_{Z_s}(\mathfrak{z}, QP_\varepsilon) \longrightarrow 0 \qquad \text{as } \varepsilon \to 0. \tag{7.5}$$

Proof. Fix any $\delta > 0$. For n large enough one can find a point $\mathfrak{z}_1$ in the space E^{2n} as in (7.2), such that $\|\mathfrak{z} - \mathfrak{z}_1\| < \delta/3$. This point lies in some ball $B = \{\mathfrak{z} \in E^{2n} \mid \|\mathfrak{z}\| < R\}$. Under the rescaling $u = \varepsilon\tilde{u}$ this ball corresponds to the ball εB in the linear subspace E^{2n} of the phase-space of $(\varphi^4) + (\mathrm{N})$. Consider the subset $\tilde{E} \subset \tilde{E}^{2n}$, constructed in Theorem 7.1[15]. As $\tilde{E}$ has the unit density at zero, then for ε sufficiently small $\tilde{E}$ has nonempty intersection with the $\varepsilon\delta/3$-neighborhood of the point $\varepsilon\mathfrak{z}_1 \in \varepsilon B$. Fix any point $\varepsilon\mathfrak{z}_2$ in this intersection. By the statement (ii) of Theorem 7.1 we have $\|\varepsilon\mathfrak{z}_2 - \tilde{\Phi}(\varepsilon\mathfrak{z}_2)\| \le C\varepsilon^2$. The point $\varepsilon^{-1}\tilde{\Phi}(\varepsilon\mathfrak{z}_2)$ lies in QP_ε. So $\mathrm{dist}(\mathfrak{z}_2, PQ_\varepsilon) \le C\varepsilon \le \delta/3$ if ε is small enough. Thus,

$$\mathrm{dist}(\mathfrak{z}, QP_\varepsilon) \le \|\mathfrak{z} - \mathfrak{z}_1\| + \|\mathfrak{z}_1 - \mathfrak{z}_2\| + \mathrm{dist}(\mathfrak{z}_2, QP_\varepsilon) \le \delta,$$

if ε is sufficiently small. The statement is proved. $\square$

Remark. Results similar to Proposition 7.3 hold for nonlinear wave equation with random potential $V_\omega(x)$ with "good randomness properties",

$$u_{tt} = u_{xx} - V_\omega(x)u + \varepsilon\varphi(u), \tag{7.6}$$

if we replace in (7.5) the usual convergence by the convergence in probability. This

[15] We use amplification 7.2.

statement is proved in [K1, Part 2.4] for nonlinear Schrödinger equation with random potential; the same proof holds for (7.6).

Appendix 1. Liouville–Arnold theorem near singularity

By $D_\rho = D_\rho^{2n}$ we denote the polydisk

$$\{(p, q) \in \mathbb{R}^{2n} \mid \mu_j = \frac{1}{2}(p_j^2 + q_j^2) < \rho \ \forall j\};^{16}$$

by $M = M_\rho^+$ the open n-cube $\{\mu \in \mathbb{R}^n \mid 0 < \mu_j < \rho\}$ and by M_0 the half-closed cube $\{\mu \mid 0 \leq \mu_j < \rho\}$. The polydisk D_ρ is given the symplectic structure by an analytic 2-form ω_2 such that

$$\omega_2 = dp \wedge dq + O(|p, q|). \tag{A.1}$$

In D_ρ we consider hamiltonian vector field V_h with analytic hamiltonian h such that $V_h(0) = 0$ and for all $\mu \in M$, $D_0 \in \mathbb{T}^n$ the curves

$$\mu = \text{const}, \qquad D = D_0 + W(\mu)t \tag{A.2}$$

are trajectories of V_h, where $W: M \to \mathbb{R}^n$ is an analytic map.

THEOREM. *If* $\det \partial W/\partial\mu \not\equiv 0$, *then after decrease* ρ, *in* D_ρ *analytic coordinates* $(\tilde{p}, \tilde{q})$ *may be constructed such that*

(i) $(\tilde{p}, \tilde{q}) = (p, q) + O(|p, q|^2)$,

(ii) $d\tilde{p} \wedge d\tilde{q} = \omega_2$,

(iii) *the actions* $I_j = \frac{1}{2}(\tilde{p}_j^2 + q_j^2)$ *and the angles* $\varphi_j = \arctan\, \tilde{q}_j/\tilde{p}_j$ *forms action-angle variables for the vectorfield* V_h:

$$V_h = \sum_{j=1}^{n} (\partial/\partial I_j \tilde{h}(I)) \frac{\partial}{\partial\varphi_j},$$

where the hamiltonian $\tilde{h}$ *is analytic in* M_0;

(iv) *the transformation* $(\mu, D) \mapsto (I, \varphi)$ *has the form*

$$(\mu, D) \mapsto (I = I(\mu), \varphi = D + \Psi(\mu)),$$

where the maps $I(\mu)$ *and* $\Psi(\mu)$ *are analytic in* M_0.

---

[16] The angles, correspnding to μ_j's, are denoted D_j's. See (1.10).

Proof. Denote

$$D_{-} = \{(p, q) \in D_\rho \mid \mu_j > 0 \; \forall j\},$$

and for $\mu \in M$ denote by $T^n(\mu) \subset D$ the n-torus $\{(\mu, D) \mid \mu = \text{fixed}\}$.

LEMMA. *Near each torus $T^n(\mu)$ the vectorfield V_h is Liouville–Arnold integrable.*

Proof. The vectorfield V_h restricted to $T^n(\mu)$ equals $\Sigma W_j(\mu)\partial/\partial D_j$, and by the theorem's assumption the flow of V_h on $T^n(\mu)$ is ergodic for almost all μ. The tori with ergodic flow of the form (A.2) are Lagrangian [Her][17]. So all the tori $T^n(\mu)$ are Lagrangian.

Consider the functions

$$f_j : (p, q) \mapsto \mu_j(p, q), \qquad j = 1, \ldots, n.$$

As f_j's are constant on each torus $T^n(\mu)$, then for $q \in T^n(\mu)$ and $\xi \in \Pi := T_q T^n(\mu)$ we have

$$0 = \langle df_j(q), \xi \rangle = \omega_2(V_{f_j}(q), \xi).$$

Thus, the vectors $V_{f_j}(q)$ lie in the skew-orthogonal complement to Π, equal Π because the torus $T^n(\mu)$ is Lagrangian. Hence, the functions f_j are in involution:

$$[f_j, f_k](q) = \omega_2(V_{f_j}(q), V_{f_k}(q)) = 0.$$

Similarly $[f_j, H] = 0$, and the lemma is proved. □

For $(p, q) = (\mu, D) \in D_{-}$ and $j = 1, \ldots, n$ we define

$$C_j(p, q) = \{(\mu', D') \mid \mu' = \mu, D'_l = D_l \text{ for } l \neq j, D'_j \in \mathbb{T}^1\}.$$

We use (A.1) to construct an analytic Liouvillean form ω_1, $d\omega_1 = \omega_2$, such that

$$\omega_1 = pdq + O(|p, q|^2).$$

[17] We sketch the proof. Denote by Ω_2 the form ω_2 restricted to some ergodic torus. As the flow of V_h preserves the form ω_2, then the flow of the ergodic vectorfield $\Sigma W_j \partial/\partial D_j$ on the torus preserves Ω_2. Thus $\Omega_2 = \sum_{i<j} a_{ij}\, dD_i \wedge dD_j$ with some constant coefficients a_{ij}. The coefficient a_{ij} equals averaging Ω_2 along the two-torus $\{q \mid q_l = 0 \text{ if } l \neq i, j\}$. So it vanishes because the form Ω_2 is exact as well as the form ω_2.

Fix $\mu_* \in M$. Due to the lemma and Liouville–Arnold theorem in the vicinity of $T^n(\mu_*)$ there exist analytic action-angle variables (I, Φ) such that

$$I_j(\mu, D) = \oint_{C_j(\mu, D)} \omega_1, \qquad j = 1, \ldots, n.$$

The actions depend only on the n-torus. So $I_j = I_j(\mu)$.

LEMMA. *The functions I_j are analytic in M_0 and*

$$I_j(\mu) = \mu_j(1 + O(|\mu|).$$

Proof. By the formulas for $C_j(p, q)$ and ω_1, the functions I_j are analytic in D_ρ and $I_j = \mu_j + O(|p, q|^3)$. Denote

$$z_j = p_j + iq_j = \sqrt{2\mu_j}\, e^{iD_j}, \qquad j = 1, \ldots, n.$$

As the functions $I_j(p, q)$ are analytic, then

$$I_j = \sum_{\alpha,\beta} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta = \sum_{\alpha,\beta} a^j_{\alpha,\beta} \prod_l (2\mu_l)^{1/2(\alpha_l + \beta_l)} e^{iD_l(\alpha_l - \beta_l)}.$$

As I_j is D-independent, then $a^j_{\alpha,\beta} = 0$ if $\alpha \neq \beta$. So

$$I_j = \sum a^j_{\alpha,\alpha} |z|^{2\alpha} = \sum a^j_{\alpha,\alpha} (2\mu)^\alpha$$

is an analytic function of μ such that $I_j = \mu_j + o(|\mu|)$. As I_j vanishes with μ_j, then $o(|\mu|) = \mu_j O(|\mu|)$. □

LEMMA. *Near the fixed torus $T^n(\mu_*)$ we have $\Phi = D + \Psi(\mu)$ with some map Ψ which is defined and analytic near μ_*.*

Proof. On each torus $T^n(\mu)$ with μ near μ_* the vectorfield V_h equals $\Sigma W_j(\mu)\, \partial/\partial D_j$ (by (A.2)) and equals $\Sigma \tilde{W}_j(\mu) \partial/\partial \Phi_j$, because (I, Φ) are the action-angle variables. As the trajectories (A.2) are dense in $T^n(\mu)$ for most μ, then $\Phi = LD + \Psi(\mu)$ with some unimodular matrix L. By the formulas for the actions I_j the cycles $\tilde{C}_j$ on the tori $T^n(\mu)$,

$$\tilde{C}_j = \{\Phi \mid \Phi_l \text{ is fixed for } l \neq j,\ \Phi_j \in \mathbb{T}^1\},$$

are homologous to C_j. So $L = Id$. □

As $dI \wedge d\Phi = \omega_2$, then the last lemma implies that

$$\omega_2 = dI \wedge dD + dI \wedge d\Psi^0 \equiv \gamma_1 + \gamma_2. \tag{A.3}$$

Observe that the form γ_1 is analytic in D_ρ[18]. As $\gamma_2 = \omega_2 - \gamma_1$, then the form γ_2, originally defined in the vicinity of $T^n(\mu_*)$ can be analytically extended to D_ρ.

LEMMA. *There exists a 2-form $\tilde{\gamma}_2$, defined and analytic in M_0, such that $\gamma_2 = \Pi^* \tilde{\gamma}_2$ where*

$$\Pi : D_{\rho^-} \to M, \qquad (p, q) \mapsto I.$$

Proof. For $j = 1, \dots, n$ denote

$$z_j = x_j + iy_j, \qquad z_j^+ = z_j, \qquad z_j^- = \bar{z}_j, \qquad \bar{z}_j^+ = \bar{z}_j, \qquad \bar{z}_j^- = z_j.$$

As the form γ_2 is analytic in D_ρ, then

$$\gamma_2 = \sum_{i,j=1}^{n} \sum_{\mu,\nu=\pm} a_{ij}^{\mu\nu}(z, \bar{z})\, dz_i^\mu \wedge dz_j^\nu,$$

where the functions $a_{ij}^{\mu\nu}$ are analytic in D_ρ. Near the torus $T^n(\mu_*)$

$$\gamma_2 = \sum_{i,j} A_{ij}(I)\, dI_i \wedge dI_j = \frac{1}{4} \sum_{i,j} A_{ij}\left(\frac{1}{2}|z|^2\right) \sum_{\mu,\nu} \bar{z}_i^\mu \bar{z}_j^\nu\, dz_i^\mu \wedge dz_j^\nu.$$

These two representations for the analytic form γ_2 jointly imply that the functions A_{ij} are analytic in M_0, and the lemma's assertion follows. □

Observe that $\gamma_2 = \sum dI_j \wedge d\Psi_j = d(\Psi\, dI)$. So the form γ_2 is exact and closed and the form $\tilde{\gamma}_2$ is closed. By the Poincaré lemma there exists an analytic in M_0 1-form $\varphi^0(I)\, dI$, $\varphi^0(0) = 0$, such that $d(\varphi^0(I)\, dI) = \gamma_2$. By (A.3),

$$\omega_2 = dI \wedge d(D + \varphi^0(I)). \tag{A.4}$$

So $(I, \varphi = D + \varphi^0(I))$ are action-angle variables.

---

[18] Because $dI_j = d\mu_j + d(\mu_j J^j(\mu))$ with some analytic in M_0 funcitons J^j (by the first lemma) and the form $d\mu \wedge dD$ is analytic in D_ρ.

Define the Cartesian variables

$$\tilde{p}_j = \sqrt{2I_j}\cos\varphi_j, \qquad \tilde{q}_j = \sqrt{2I_j}\sin\varphi_j.$$

By the first lemma,

$$\tilde{p}_j = \sqrt{1+O(\mu)}\sqrt{2\mu_j}(\cos D_j \cos\varphi_j^0(\mu) - \sin D_j \sin\varphi_j^0(\mu)) = p_j P_j(\mu) - q_j Q_j(\mu),$$

where P_j, Q_j are analytic in M_0 and $P_j(0) = 1$, $Q_j(0) = 0$. Similar with $\tilde{q}_j$. So the analytic map $(p, q) \mapsto (\tilde{p}, \tilde{q})$ has the form given in the statement (i) of the theorem.

Statement (ii) results from (A.4).

In the coordinates $(\tilde{p}, \tilde{q})$ the vectorfield V_h is hamiltonian with the analytic hamiltonian $h(\tilde{p}, \tilde{q})$, depending on the actions I only. By the same arguments as in the proof of second lemma, $h = \tilde{h}(I)$, where the function $\tilde{h}$ is analytic in M_0. So the statement (iii) follows.

The last statement results from the definition of (I, φ)-variables.

Appendix 2. Correction

In Part 6 above we essentially use Theorem 3.1.2 from [K1]. The secon author (S.K.) admits that the proof of Theorem 3.1.2 (more exactly, its reduction to the main theorem of [K1]) contains a gap which was drawn to his attention by J. Pöschel. The gap affects the theorem exactly in the specific case we use above. Below we give the corrected statement.[19] We use notations of [K1].

CORRECTION (to Theorem 3.1.2 in [K1]). *If $d_1 = 1$ (i.e., if the frequencies $\lambda_j(\theta)$ of the unperturbed system have linear growth),*[20] *then the spectral asymptotics* (1.12) ([K1], p. 50) *should be strengthened as*

$$|\lambda_j(\theta) - K_2^0 j - K_2^1| \le K_1 j^{-1}. \tag{1.12'}$$

Besides, the radius δ_a should be larger than $C^{-1}\varepsilon^{1-\mu}$, where $\mu > (2-\Delta)/(4-\Delta)$ with $\Delta = min(1, -d_H)$ and d_H is the (negative) order of the nonlinear part of the perturbation.

[19] It is somewhat weaker than the one given in [K1] but is sufficient for the purposes of the current paper.

[20] As the frequencies $\{w_j\}$ in (2.4).

For the perturbed (SG)-equation (1.12′) is fulfilled, $d_H = -1$, $(2-\Delta)/(4-\Delta) = 1/3$ and $\mu = 1/2$. So the theorem can be applied to (6.3).

The mistake is contained in the estimate (4.11), p. 77 (which is needed for the case $d_1 = 1$): the correct version of the estimate has no factor δ_a in the r.h.s.

Therefore under an appropriate choice of the small "bad set" Θ^2, for "good parameters" $\theta \notin \Theta^2$ one has

$$|D| \geq \delta_a^2 \frac{\langle j-k \rangle}{C(m)\langle s \rangle^c}$$

(not $|D| \geq \delta_a \ldots$ as in the book). So

(1) the proof given in the book works without additional corrections if $\delta_a = \delta > C^{-1}\varepsilon^{1/2-\mu'}$ with $\mu' > 0$ (see (8.11), p. 88, where δ^{-1} should be replaced by δ^{-2}).

This restriction is too hard since it is not fulfilled for the (PSG)-equation. To obtain a better result we make one more observation.

(2) For $b \in [0, 1)$ one can construct a small "bad set" Θ^2 in such a way that

$$|D| \geq \delta_a^{2-b} \frac{\langle j-k \rangle}{\langle k \rangle^b \tilde{C}(m)\langle s \rangle^{c_2}}$$

for $\theta \notin \Theta^2$. With denominators like that the nonlinear part of the transformed vectorfield will loose b "units of smoothness". So if we take $b \in [0, \Delta = \min(1, -d_H))$, then the transformed nonlinear vectorfield will be still of the negative order $d_H^n = d_H + b < 0$ – i.e. still smoothing. With this choice of the bad set after the first step of the normalizing procedure we get as a new magnitude of the perturbation $\varepsilon_{(1)} = \varepsilon^2\delta^{2-b}$. As $\mu > (2-\Delta)/(4-\Delta)$, then one can find $b \in [0, \Delta)$ such that

$$\delta^2 > C^{-1}\varepsilon_{(1)}^{1-\mu'}, \qquad \mu' > 0.$$

After this we can proceed as in 1).

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REFERENCES

[A1] ARNOLD, V. I., *Mathematical methods in classical mechanics*, Nauka, Moscow, 1974; English transl., 2nd ed. Springer-Verlag, Berlin, 1989.

[A2] ARNOLD V. I., *Proof of a theorem of A. N. Kolmogorov on the conservation of quasiperiodic motions under a small change of the Hamiltonian function*, Uspekhi Mat. Nauk *18:5* (1963), 13–40; English transl. in Russ. Math. Surv. *18:5* (1963), 9–36.

[BBEIM] BELOKOLOS, E. D., BOBENKO, A. I., ENOLSKII, V. Z., ITS, A. R., MATVEEV, V. B., *Algebro-geometric approach to nonlinear integrable equations*, Springer-Verlag, Berlin, 1994.

[BiK] BIKBAEV, R. F., KUKSIN S. B., *A periodic boundary value problem for the Sine–Gordon equation, small Hamiltonian perturbations of it, and KAM-deformations of finite-gap tori*, Algebra i Analiz *4:3* (1992), 42–78; English transl. in St.-Petersburg Math. J. *4* (1993), 439–468.

[Bo] BOBENKO, A. I., *Eigenfunctions of the Dirichlet and Neumann boundary value problems for the elliptic sine–Gordon equation*, Zap. Nauchn. Sem. LOMI *179* (1989), 32–36: English transl. in J. Soviet Math. *57:3* (1991).

[BoK1] BOBENKO, A. I., KUKSIN, S. B., *Finite-gap periodic solutions of the KdV equation are nondegenerate*, Phis. Lett. A *161:3* (1991), 274–276.

[BoK2] BOBENKO, A. I., KUKSIN, S. B., *Small-amplitude solutions of the Sine–Gordon equaiton on an interval under Dirichlet or Neumann boundary conditions*, ~~to appear in~~ Journal of Nonlinear Science. 5, 207–232 (1995)

[Bre] BREZIS, H., *Periodic solutions of nonlinear vibrating string and duality principles*, Bull. Amer. Math. Soc. *8* (1983), 409–426.

[CW] CRAIG, W., WAYNE, C. E., *Newton's method and periodic solutions of nonlinear wave equation*, Commun. Pure Appl. Math. *46* (1992), 1409–1498.

[DN] DUBROVIN, B. A., NATANZON, S. M., *Real two-gap solutions of the sine–Gordon equation*, Funkts. Anal. i Prilozhen. *16:1*, (1982), 27–43; English transl. in Funct. Anal. Appl. *16* (1982).

[EF] ERCOLANI, N., FOREST, M. G., *The geometry of real sine–Gordon wavetrains*, Comm. Math. Phys. *99* (1985), 1–49.

[EFM] ERCOLANI, N., FOREST, M. G., MCLAUGHLIN, D., *Geometry of the modulational instability.* Part 3, Physica *43 D* (1990), 349–384.

[Fe] FEDERER, H., *Geometric measure theory*, Springer-Verlag, Berlin, 1969.

[Her] HERMAN, M., *Inégalités a priori pour des tores Lagrangiens invariants par des difféomorphismes symplectiques*, Publ. Math. IHES *70* (1989).

[Ito] ITO, H., *Action-angle coordinates at singularities for analytic integrable systems*, Math. Z *206* (1991), 363–407.

[K1] KUKSIN, S. B., *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes Math. 1556 (1993), Springer-Verlag, Berlin.

[K2] KUKSIN, S. B., *Perturbation theory for quasiperiodic solutions of infinite-dimensional Hamiltonian systems, and its applications to the Korteweg-de Vries equation*, Matem. Sbornik *136(178):3* (1988); English transl in Math. USSR Sbornik *64* (1989), 397–413.

[K3] KUKSIN, S. B., *Reducible variational equations and the perturbation of invariant tori of Hamiltonian systems*, Matem. Zametki *45:5* (1989), 38–49; English transl. in Mathematical Notes *45:5* (1989), 373–381.

[K4] KUKSIN, S. B., *KAM-theory for partial differential equations*, in Proceedings of the First European Congress of Mathematics (Paris, 6–10 July 1992) Birkhäuser, 1994.

[KK] KOZEL, V. A., KOTLYAROV, V. P., *Quasiperiodic solutions of the equation* $U_{tt} - U_{xx} + \sin U = 0$, Dokl. Akad. Nauk Ukr. SSR *10A* (1976), 878–881. (Russian).

[Kri] KRICHEVER, I. M., *Perturbation theory in periodic problems for two-dimensional integrable systems*, Sov. Sci. Rev. C. Math. Phys. *9* (1991), 1–101.

[McK] MCKEAN, H. P., *The sine–Gordon and singh–Gordon equation on the circle*, Comm. Pure Appl. Math. *34* (1981), 197–257.

[M] MOSER, J., *Stable and random motions in dynamical systems*, Princeton Univ. Press, Princeton, 1973.

[Mat] MATEEV, V., *Abelian functions and solitions*, Preprint 373 Univ. of Wroclaw (1976).

[P] PÖSCHEL, J., *On elliptic lower dimensional tori in Hamiltonian systems*, Math. Z. *202* (1989), 559–608.

[W] WAYNE, C. E., *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Commun. Math. Phys. *127* (1990), 479–528.

[Wei] WEINSTEIN A., *Lectures on symplectic manifolds*, A.M.S. Conf. Board, Reg. Conf. in Math. *29* (1977).

[ZIS] ZAKHAROV, V. E., IVANOV, M. F., SHUR, L. N., *On the abnormally slow stochastisation in some two-dimensional field theory models*, Pis'ma Zh. Eksp. Teor. Fiz. *30:1* (1979), 39–44; English transl. in JETP Letters *30:1* (1979).

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