

# DENSITY OF HEAT CURVES IN THE MODULI SPACE

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## 1. Introduction. Let

$$\Gamma = (L_1, 0)\mathbf{Z} \oplus (0, L_2)\mathbf{Z} \quad L_1, L_2 \neq 0.$$

For  $q$  in  $L^2(\mathbf{R}^2/\Gamma)$ , let

$$H + q = \frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2} + q(x_1, x_2)$$

be the corresponding heat operator. Set

$$\hat{\mathcal{H}}(q) = \{(k_1, k_2) \in \mathbf{C}^2 \mid \text{there is a nontrivial } \psi \in H_k^2(\mathbf{R}^2), \\ \text{such that } (H + q)\psi = 0\},$$

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where

$$H_k^2(\mathbf{R}^2) = \{\psi \in H_{loc}^2(\mathbf{R}^2) \mid \text{such that } \psi(x + \gamma) = e^{i\langle k, \gamma \rangle} \psi(x) \text{ for all } \gamma \in \Gamma\}.$$

The dual lattice

$$\Gamma^\sharp = \left(\frac{2\pi}{L_1}, 0\right) \mathbf{Z} \oplus \left(0, \frac{2\pi}{L_2}\right) \mathbf{Z}$$

acts by translation on  $\widehat{\mathcal{H}}$ . The heat curve associated to  $q$  is, by definition,

$$\mathcal{H}(q) = \widehat{\mathcal{H}}(q)/\Gamma^\sharp.$$

In [7] it is shown that  $\mathcal{H}$  is a complex analytic subvariety of  $\mathbf{C}^2$ .

In [10] it is shown that for generic  $q$  the heat curve  $\mathcal{H}(q)$  is smooth and of infinite genus. In [7] a normalized basis of  $L^2$ -holomorphic one forms is constructed, the Riemann operator and associated theta function are analyzed, a vanishing theorem is proved and a Torelli theorem is obtained. Using these results we expect to solve the periodic Kadomtsev-Petviashvili (KP) equation with initial data  $q$  and to show that the solution is almost periodic in the time whenever  $\mathcal{H}(q)$  is smooth. In [10] it is shown that for any  $T$  the solution of KP can be arbitrarily well approximated on  $0 \leq t \leq T$  by a finite (but large) gap solution of KP. This is done by approximating  $\mathcal{H}(q)$  with curves of finite genus.

The results mentioned above suggest that heat curves are a very natural class of transcendental curves. In this paper we further substantiate this suggestion, by proving that every compact Riemann surface can be arbitrarily well approximated by the normalisations of heat curves.

Recall that the normalisation  $\widetilde{\mathcal{H}}(q)$  of  $\mathcal{H}(q)$  is obtained by constructing the analytic configuration [12] (in the sense of Weyl) associated to any smooth germ of  $\mathcal{H}(q)$ . It is a Riemann surface. One can imagine that  $\widetilde{\mathcal{H}}(q)$  is the affine part of a finite genus curve when there are enough singularities.

**THEOREM 1.** *For each  $g \geq 1$ , the set of Riemann surfaces of genus  $g$  that are normalisations  $\widetilde{\mathcal{H}}(q)$  of heat curves  $\mathcal{H}(q)$ , where  $q \in L^2(\mathbf{R}^2/\Gamma)$  for some rectangular lattice  $\Gamma$ , is dense in the moduli space  $\mathcal{M}_g$  of all Riemann surfaces of genus  $g$ .*

Let  $C$  be any compact Riemann surface,  $p$  a point on  $C$  and  $\zeta$  a local coordinate centered at  $p$ . Also, let

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \Theta(xU + yV + tW + D) + 2c \quad (1)$$

be the solution of the Kadomtsev-Petviashvili equation

$$u_{yy} = \frac{4}{3} \left( u_t - \frac{1}{4} (uu_x + u_{xxx}) \right)_x$$

generated by  $C$  (see [9]). If this solution is periodic in  $x$  and  $y$ , then  $C - \{p\}$  is isomorphic to  $\tilde{\mathcal{H}}(u(x_1, x_2, t))$  for any  $t, D$ .

Denote by  $M$  the space of triples  $(C, p, \zeta)$ . Here,  $C$  is a compact Riemann surface,  $p$  a point on  $C$  and  $\zeta$  the 3-jet of a local coordinate centered at  $p$ . Theorem 1 is a direct consequence of

**THEOREM 2.** *Let  $M_{per}$  be the set of points  $(C, p, \zeta)$  for which there are  $L_1, L_2 \neq 0$  such that the solution of KP given above satisfies*

$$u(x + L_1, y, t) = u(x, y + L_2, t) = u(x, y, t).$$

*Then,  $M_{per}$  is dense in  $M_g$ .*

The technique of Schottky uniformisation used to prove Theorem 2 also yields similar results for curves that generate real solutions of the KP-equation and the KdV-equation. To formulate them, let  $M_{\mathbf{R}}$  be the set of all quadruples  $(C, p, \zeta, \sigma)$  with  $(C, p, \zeta) \in M$  and  $\sigma$  an antiholomorphic involution on  $C$  whose fixed point set consists of  $g + 1$  ovals, contains  $p$  and satisfies  $\sigma^*(\zeta) = \bar{\zeta}$ . For these quadruples, the solution (1) is real whenever  $D$  is real. We have

**THEOREM 3.** *Let  $M_{\mathbf{R},per}$  be the set of points  $(C, p, \zeta, \sigma)$  in  $M_{\mathbf{R}}$  for which there are  $L_1, L_2, T \neq 0$  such that the corresponding solution (1) satisfies*

$$u(x + L_1, y, t) = u(x, Y + L_2, t) = u(x, y, t), \quad u(x, y, t + T) = u(x, y, t).$$

*Then,  $M_{\mathbf{R},per}$  is dense in  $M_{\mathbf{R}}$ .*

Theorem 3 has been proved in [8] by a different method. A stronger version of Theorem 2 has been announced by I. Krichever in [11].

To formulate the last result, let  $N$  be the set of points  $(C, p, \zeta, \sigma)$  in  $M_{\mathbf{R}}$  for which  $C$  is hyperelliptic and  $p$  is Weierstrass point on  $C$ . In this case, (1) is independent of  $y$  and is a solution of the Korteweg-de Vries equation

$$4u_t = 6uu_x + u_{xxx}.$$

**THEOREM 4.** *Let  $N_{per}$  be the set of points in  $N$  for which there are  $L, T \neq 0$  such that the corresponding solution of the Korteweg-de Vries equation satisfies*

$$u(x + L, t) = u(x, t + T) = u(x, t).$$

*Then,  $N_{per}$  is dense in  $N$ .*

Again, a stronger version of Theorem 4, namely that the differential of the map (7) below has a maximal rank everywhere, has been proven by I. Krichever (see [3]).

**2. Proofs.** To begin the proofs we first specify the constants in (1). Let  $(C, p, \zeta) \in M$  and  $a_1, \dots, a_g, b_1, \dots, b_g$  be a normalized basis of  $H_1(C, \mathbf{Z})$  and  $w = (w_1, \dots, w_g)$  the vector of differentials on  $C$  normalized so that

$$\int_{a_n} w_m = 2\pi i \delta_{mn}, \quad m, n = 1, \dots, g.$$

Furthermore, let

$$R(C)_{nm} = \int_{b_m} w_n$$

be the Riemann period matrix of  $C$ . Observe that the theta function of  $C$  depends on the choice of the basis of homology, but the second logarithmic derivative in (1) does not and that  $u(x, y, t)$  only depends on the point  $xU + yV + tW + D$  in  $\text{Jac}(C) = \mathbf{C}^{2g} / \{2\pi\mathbf{Z}^g \oplus R(C)\mathbf{Z}^g\}$ .

The constants  $U, V, W$  of (1) are characterized by

$$w = Ud\zeta + V\zeta d\zeta + W\zeta^2 d\zeta$$

near  $p$ . We first prove Theorem 2. Write

$$\begin{aligned} U &= 2\pi i \Delta_1 + R(C) \Delta_2 \\ V &= 2\pi i \Delta_3 + R(C) \Delta_4 \end{aligned} \tag{2}$$

with vectors  $\Delta_i \in \mathbf{Q}^g$ . Clearly (1) has the required periodicity if  $\Delta_i \in \mathbf{Q}^g$ . Observe that this property is independent of the choice of the basis of homology. Let  $\widetilde{M}$  be the covering of  $M$  consisting of points  $(C, p, \zeta)$  of  $M$  together with a choice of a basis of homology. Associating to each element of  $\widetilde{M}$  the vector  $(\Delta_1, \dots, \Delta_4) \in \mathbf{R}^{4g}$  as above we get a real analytic mapping

$$\Delta : \widetilde{M} \rightarrow \mathbf{R}^{4g}.$$

To prove Theorem 2 it suffices to show that  $\Delta$  has maximal rank almost everywhere. As this map is analytic it is enough to show that  $\Delta$  has maximal rank at one point of  $\widetilde{M}$ . For this purpose and also for the proofs of Theorems 3, 4 we use the technique of Schottky uniformization [2], [4], [5], which we now briefly review.

For  $(A, B, \mu) \in \mathbf{C}^g \times \mathbf{C}^g \times \mathbf{C}^{*g}$  let  $\sigma_n$  be the linear transformations defined by

$$\frac{\sigma_n z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n} \quad (z \in \mathbf{C}), \quad n = 1, \dots, g.$$

The map  $\sigma_n$  maps the outside of the circle of radius

$$\frac{|A_n - B_n|}{\left| \sqrt{\mu_n} - \frac{1}{\sqrt{\mu_n}} \right|}$$

and centered at

$$\frac{B_n \sqrt{\mu_n} - A_n \frac{1}{\sqrt{\mu_n}}}{\sqrt{\mu_n} - \frac{1}{\sqrt{\mu_n}}}$$

to the inside of the circle of the same radius and center at

$$\frac{A_n \sqrt{\mu_n} - B_n \frac{1}{\sqrt{\mu_n}}}{\sqrt{\mu_n} - \frac{1}{\sqrt{\mu_n}}}.$$

Let  $S$  be the set of all  $(A, B, \mu) \in \mathbf{C}^g \times \mathbf{C}^g \times \mathbf{C}^{*g}$  for which all the discs bounded by these circles are disjoint. Clearly  $S$  is an open subset of  $\mathbf{C}^g \times \mathbf{C}^g \times \mathbf{C}^{*g}$  and contains the full-dimensional subset of all points for which all  $A$ 's,  $B$ 's are mutually different and  $\mu$ 's are sufficiently small. For  $(A, B, \mu) \in S$  the complement of the discs mentioned above is a fundamental domain for the Schottky group  $G$  generated by  $\sigma_1, \dots, \sigma_g$ . Let  $\Omega$  be the region of discontinuity for  $G$ . Then  $C = \Omega/G$  is a compact Riemann surface of genus  $g$ . It has a distinguished local coordinate  $\zeta = z^{-1}$ . Further the images of the circles above define  $a$ -cycles on  $C$ , so that  $C$  is extended with a marking of homology. Thus we get a map  $Y : S \rightarrow \widetilde{M}$  whose image is an open subset of  $\widetilde{M}$ .

In order to describe the map  $\Delta \circ Y$  we give formulas for the relevant data in terms of  $A, B, \mu$ . Denote by  $G_n$  the cyclic subgroup of  $G$  generated by  $\sigma_n$ . The series

$$w_n = \sum_{\sigma \in G/G_n} \left( \frac{1}{z - \sigma B_n} - \frac{1}{z - \sigma A_n} \right) dz, \quad n = 1, \dots, g,$$

(where  $G/G_n$  is the right coset space) define the normalized holomorphic differentials on  $G$ . The period matrix is given by

$$\begin{aligned} R(C)_{nm} &= \sum_{\sigma \in G_m \setminus G/G_n} \log\{B_m, A_m, \sigma B_n, \sigma A_n\}, \quad m \neq n, \\ R(C)_{nn} &= \log \mu_n + \sum_{\sigma \in G_n \setminus G/G_n, \sigma \neq I} \log\{B_n, A_n, \sigma B_n, \sigma A_n\}, \end{aligned} \quad (3)$$

where the curly brackets indicate the cross-ratio

$$\{z_1, z_2, z_3, z_4\} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

These are  $(-2)$ -dimensional Poincaré theta-series and they converge absolutely, if  $\mu$  is sufficiently small [1], [6]. Furthermore

$$U_n = \sum_{\sigma \in G/G_n} (\sigma A_n - \sigma B_n),$$

$$\begin{aligned}
V_n &= \sum_{\sigma \in G/G_n} \left( (\sigma A_n)^2 - (\sigma B_n)^2 \right), \\
W_n &= \sum_{\sigma \in G/G_n} \left( (\sigma A_n)^3 - (\sigma B_n)^3 \right).
\end{aligned} \tag{4}$$

LEMMA 1. *The asymptotics of the series (3), (4) as  $\mu \rightarrow 0$  are*

$$\begin{aligned}
U_n &= A_n - B_n + O(|\sqrt{\mu}|), & V_n &= A_n^2 - B_n^2 + O(|\sqrt{\mu}|), \\
R_{nm} &= O(1), & n \neq m, & R_{nn} = \log \mu_n + O(1).
\end{aligned}$$

*These  $O$ -estimates are uniform in derivatives in  $A$  and  $B$ .*

We postpone the proof of this Lemma and continue with the proof of Theorem 2. From (2) one deduces that

$$\begin{pmatrix} \Delta_1 & \Delta_3 \\ \Delta_2 & \Delta_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi}(\operatorname{Im} R)(\operatorname{Re} R)^{-1} \\ 0 & (\operatorname{Re} R)^{-1} \end{pmatrix} \begin{pmatrix} \operatorname{Im} U & \operatorname{Im} V \\ \operatorname{Re} U & \operatorname{Re} V \end{pmatrix}.$$

Using the Lemma this gives

$$\begin{aligned}
(\Delta_1)_n &= \frac{1}{2\pi} \operatorname{Im}(A_n - B_n) + O\left(\frac{1}{\log |\mu|}\right), \\
(\Delta_2)_n &= \frac{1}{\log |\mu_n|} \operatorname{Re}(A_n - B_n) + O\left(\frac{1}{(\log |\mu|)^2}\right), \\
(\Delta_3)_n &= \frac{1}{2\pi} \operatorname{Im}(A_n^2 - B_n^2) + O\left(\frac{1}{\log |\mu|}\right), \\
(\Delta_4)_n &= \frac{1}{\log |\mu_n|} \operatorname{Re}(A_n^2 - B_n^2) + O\left(\frac{1}{(\log |\mu|)^2}\right).
\end{aligned}$$

So if we choose  $(\hat{A}, \hat{B}, \hat{\mu}) \in S$  with  $\hat{A}_i, \hat{B}_i$  all different and  $\hat{\mu}$  sufficiently small then the derivative of  $\Delta \circ Y$  with respect to  $A$  and  $B$  is invertible at this point, since the map  $\mathbf{C}^{2g} \rightarrow \mathbf{R}^{4g}$ ,

$$(A, B) \mapsto \left( \operatorname{Re}(A_n - B_n); \operatorname{Im}(A_n - B_n); \operatorname{Re}(A_n^2 - B_n^2); \operatorname{Im}(A_n^2 - B_n^2) \right)$$

is invertible at  $\hat{A}, \hat{B}$ .

For the proof of Theorem 3 we need a more detailed expansion of  $(U, V)$  than in Lemma 1.

LEMMA 2.

$$\begin{aligned}
U_n &= A_n - B_n + \sum_{k \neq n} \mu_k u_{nk} + O(|\mu|^2), \\
V_n &= A_n^2 - B_n^2 + \sum_{k \neq n} \mu_k v_{nk} + O(|\mu|^2), \\
W_n &= A_n^3 - B_n^3 + \sum_{k \neq n} \mu_k w_{nk} + O(|\mu|^2),
\end{aligned}$$

where

$$\begin{aligned} u_{nk} &= (A_n - B_n)(A_k - B_k)^2 \left[ \frac{1}{(A_k - A_n)(A_k - B_k)} + \frac{1}{(B_k - A_n)(B_k - B_n)} \right], \\ v_{nk} &= (A_n - B_n)(A_k - B_k)^2 \left[ \frac{2B_k}{(A_k - A_n)(A_k - B_k)} + \frac{2A_k}{(B_k - A_n)(B_k - B_n)} \right], \\ w_{nk} &= (A_n - B_n)(A_k - B_k)^2 \left[ \frac{3B_k^2}{(A_k - A_n)(A_k - B_k)} + \frac{3A_k^2}{(B_k - A_n)(B_k - B_n)} \right], \end{aligned} \quad (5)$$

Proof of Lemma 2 and Lemma 1: We only discuss the formula for  $U$ , the other cases being similar. Every element  $\sigma$  of Schottky group is a loxodromic linear transformation and is represented by a matrix in  $PSL(2, \mathbf{C})$  of the form

$$\frac{1}{F_1 - F_2} \begin{pmatrix} rF_1 - r^{-1}F_2 & F_1F_2(r^{-1} - r) \\ r - r^{-1} & r^{-1}F_1 - rF_2 \end{pmatrix},$$

where  $F_1, F_2$  are the fixed points of  $\sigma$ . If  $\ell$  is the length of  $\sigma$  represented as a word in  $\sigma_1, \dots, \sigma_g$  (i.e.  $\ell = |m_1| + \dots + |m_k|$  if  $\sigma = \sigma_{n_1}^{m_1} \dots \sigma_{n_k}^{m_k}$  is a reduced representation), then  $r = O(|\mu|^\ell)$ . In particular

$$\sigma z_1 - \sigma z_2 = r \frac{(z_1 - z_2)(F_1 - F_2)}{(-z_1 + F_1)(-z_2 + F_1)} + O(|\mu|^{\ell+1}). \quad (6)$$

If we take  $\sigma = I$  in the series (4) for  $U_n$  we get the term  $A_n - B_n$ . If we take  $\sigma = \sigma_k$  and  $\sigma = \sigma_k^{-1}$  we get the terms  $\mu_k u_{nk} + O(|\mu|^2)$ . Because of (6) all other terms in this series are of order  $|\mu|^2$ .

Proof of Theorem 3: Let

$$S_{\mathbf{R}} := \{(A, B, \mu) \in S \mid B = \bar{A}, \mu \in \mathbf{R}^g\}.$$

The image of  $S_{\mathbf{R}}$  under the map from  $S$  to  $M$  described above is  $M_{\mathbf{R}}$  [2], [5]. In this case the vectors  $U, V, W$  are purely imaginary. The associated solution of the  $KP$  equation is triply periodic if  $(\frac{1}{2\pi i}U, \frac{1}{2\pi i}V, \frac{1}{2\pi i}W) \in \mathbf{Q}^{3g}$ . Therefore it is enough to show that the map given by

$$(U, V, W) : S_{\mathbf{R}} \rightarrow i\mathbf{R}^{3g}$$

has maximal rank at one point. The determinant of the Jacobian of this map with respect to  $A, B, \mu$  at  $\mu = 0$  is

$$\begin{vmatrix} \frac{\partial U_n}{\partial A_k} & \frac{\partial U_n}{\partial B_k} & \frac{\partial U_n}{\partial \mu_k} \\ \frac{\partial V_n}{\partial A_k} & \frac{\partial V_n}{\partial B_k} & \frac{\partial V_n}{\partial \mu_k} \\ \frac{\partial W_n}{\partial A_k} & \frac{\partial W_n}{\partial B_k} & \frac{\partial W_n}{\partial \mu_k} \end{vmatrix}_{\mu=0} =$$

where in  $u_{nk}$  and  $w_{nk}$  given by (5) we should substitute  $B = -A$ . After this substitution the determinant of the Jacobian of the map (7) with respect to  $A$  and  $\mu$  at  $\mu = 0$  is

$$\left| \begin{array}{cc} \frac{\partial U_n}{\partial A_k} & \frac{\partial U_n}{\partial \mu_k} \\ \frac{\partial W_n}{\partial A_k} & \frac{\partial W_n}{\partial \mu_k} \end{array} \right|_{\mu=0} = \left| \begin{array}{cc} 2I & u_{nk} \\ 6 \operatorname{diag}(A_1^2, \dots, A_g^2) & w_{nk} \end{array} \right| = 2^g \det \tilde{F},$$

$$\tilde{F}_{nk} = W_{nk} - 3A_n^2 U_{nk},$$

$$\det \tilde{F} = (24)^g \prod_{k=1}^g A_k^3 \det F,$$

$$F_{nk} = 2, \quad k \neq n, \quad F_{nn} = 0.$$

Again,  $\det F \neq 0$  proves the nondegeneracy.

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