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The discrete quantum pendulum ☆

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We study a doubly discrete quantum sine-Gordon equation, which is derived from the quantum Volterra model solved by Faddeev and Volkov [Teor. Mat. Fiz. 92 (1992) 207]. The discrete quantum pendulum is obtained as the simplest special case. The eigenfunctions of its Hamiltonian, displayed as densities on the phase space, are very localized around the classical energy levels.

1. The classical system

Consider a classical particle moving on a circle (parametrized by a coordinate $q \in S^1/2\pi\mathbb{Z}$) subject to a potential V: $S^1 \to \mathbb{R}$. The equation of motion will look like

$$\ddot{q} = f(q) , \quad f = -V' . \tag{1}$$

Let us consider the mathematical pendulum, for which $V = -4k\cos q$, and $\ddot{q} = -4k\sin q$. The phase space of the system is a cylinder $M = S^1 \times \mathbb{R}$, parametrized by q and $p = \dot{q}$.

Here we will be concerned with time discrete systems. The most obvious way to discretize (1) is to look for a sequence $t \to q_t$, $t \in \mathbb{Z}$ satisfying

$$q_{t-1} - 2q_t + q_{t+1} = f(q_t) . \tag{2}$$

The use of a lattice constant h different from 1 for the time discretization could be easily realized by multiplying f with h^2.

The main feature of (2) is that, as (1), it is still a Hamiltonian system: The whole sequence ..., q_{-1}, q_0, q_1, ... is uniquely defined by giving (q_{t-1}, q_t) for any fixed $t \in \mathbb{Z}$. So, for any fixed $t \in \mathbb{Z}$, q_{t-1} and q_t should be considered as the coordinate functions on the phase space M, which is therefore diffeomorphic to $T^2 = S^1 \times S^1$. The time evolution is then given by

$$T: \quad M \to M$$
$$(q_{t-1}, q_t) \mapsto (q_t, q_{t+1}) = (q_t, 2q_t + f(q_t) - q_{t-1}) . \tag{3}$$

Observe, that for any function g: $S^1 \to \mathbb{R}$ the map $(q, \tilde{q}) \mapsto (\tilde{q}, g(\tilde{q}) - q)$ preserves the standard symplectic from $dq \wedge d\tilde{q}$ on M.

Figure 1 shows some orbits of the map (3) with $f(q) = -4k\sin q$, $k = 0.7$.

Pictures like those in fig. 1 are familiar from KAM

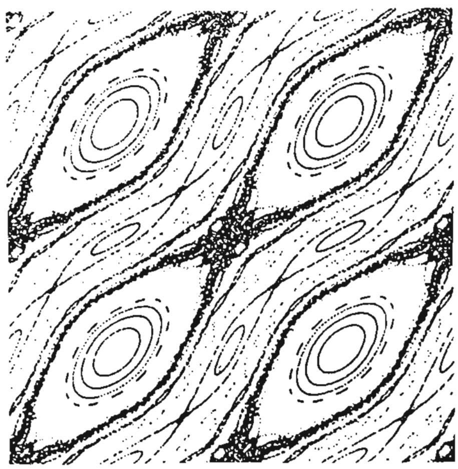

Fig. 1. Four copies of phase space.

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theory, and in fact the present T is nothing else but the so-called "standard map" which is a well-studied example in chaos theory. The crucial point is, that this T does not admit any continuous integral of the motion: the system is *not* integrable.

Inspired by the idea of looking at solutions of the pendulum equation as special solutions for the sine-Gordon equation $\ddot{q}-q''=k\sin q$, namely those for which $q(x,t)=q(t)$, we consider the discrete analog of the sine-Gordon equation in light-cone coordinates:

$$\begin{aligned} q_{m+1,n+1}-q_{m+1,n}-q_{m,n+1}+q_{m,n} &= 2\arg[1+k\exp(-\mathrm{i}q_{m+1,n})] \\ &\quad +2\arg[1+k\exp(-\mathrm{i}q_{m,n+1})]\,. \end{aligned} \tag{4}$$

This equation describes the angles of discrete surfaces with constant negative curvature and was derived and investigated in ref. [1].

The variables n, m are the cone variables. In order to reduce eq. (4) to the discrete pendulum equation we consider space independent solutions $q_{m+1,n-1}=q_{m,n}$.

Then in the discrete time variable $t=m+n$ the discrete pendulum equation reads as follows,

$$q_{t+1}-2q_t+q_{t-1}=4\arg[1+k\exp(-\mathrm{i}q_t)] \tag{5}$$

The corresponding new T: M→M, for which the standard map can be regarded as a perturbation, is integrable, i.e. there exists an integral of the motion

$$H=2(\cos q_n+\cos q_{n-1})+k\cos(q_n+q_{n-1}) +k^{-1}\cos(q_n-q_{n-1})\,.$$

Using $Q_n\overset{\text{def}}{=}\exp(-\mathrm{i}q_n)$ (5) can be written as

$$Q_{n+1}Q_{n-1}=\left(\frac{Q_n+k}{1+kQ_n}\right)^2\,. \tag{6}$$

In the following we find quantized analogs of (6) and (4).

2. The quantum discrete Volterra model

Let us consider a stairway with matrices L and M on its edges (L and M are associated to the different kinds of edges) of the following form,

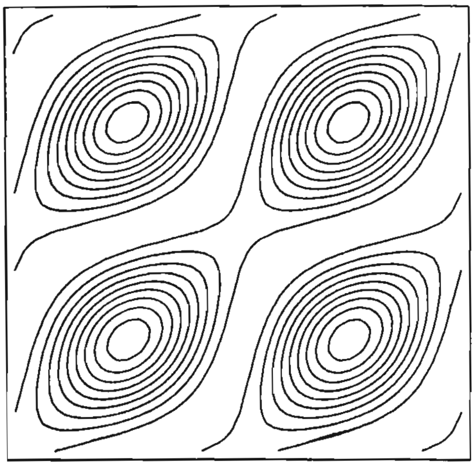

Fig. 2. Phase portrait of system (5).

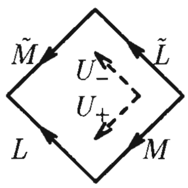

Fig. 3.

$$L=\begin{pmatrix} u^L & -\lambda k^{-1/2}(v^L)^{-1} \\ \lambda k^{-1/2}v^L & (u^L)^{-1} \end{pmatrix},$$

$$M=\begin{pmatrix} u^M & -\lambda k^{1/2}(v^M)^{-1} \\ \lambda k^{1/2}v^M & (u^M)^{-1} \end{pmatrix}. \tag{7}$$

Here the u and v are unitary operators, k is a parameter of the model, λ is a spectral parameter.

The operators u, v of different edges commute and on the same edge they satisfy the following commutation relation,

$$uv=\mathrm{e}^{-\mathrm{i}\gamma}vu\,. \tag{8}$$

In a recent paper [2] Faddeev and Volkov showed that this model is integrable by the quantum inverse transform method and described its discrete time evolution. They constructed operators $U_\pm$ (shift operators in cone directions) determining the evolution (see fig. 3)

$$\tilde{L}=U_+^{-1}LU_+\,,\quad \tilde{M}=U_-MU_-^{-1}\,.$$

This evolution has the following properties.

(1) The matrices $\tilde{L}$, $\tilde{M}$ have the same form as L, M and the commutation relations for the u and v are preserved (i.e. $\tilde{u}$, $\tilde{v}$ commute in the same way as u and v).

(2) The zero curvature condition is satisfied: $LM=\tilde{M}\tilde{L}$.

The last equation is equivalent to the following system,

$$u^L u^M = \tilde{u}^M \tilde{u}^L , \tag{9}$$

$$(v^L)^{-1} v^M = (\tilde{v}^M)^{-1} \tilde{v}^L , \tag{10}$$

$$k^{1/2}(u^L)^{-1} v^M + k^{-1/2} v^L u^M = k^{-1/2}(\tilde{u}^M)^{-1}\tilde{v}^L + k^{1/2}\tilde{v}^M \tilde{u}^L . \tag{11}$$

3. Quantum discrete sine-Gordon equation

Let us consider a bigger fragment of the lattice (see fig. 4). The initial stairway is marked by the thick line. We denote the four faces in fig. 4 by up, down, left, right (u, d, ℓ, r) and use these labels to distinguish fields on different edges and faces.

Starting with the discrete Volterra model we construct an algebra of operators on faces. First, we associate to each face the pair of commuting operators

$$A = u^L u^M = \tilde{u}^M \tilde{u}^L , \quad B = (v^L)^{-1} v^M = (\tilde{v}^M)^{-1} \tilde{v}^L .$$

Equation (11) can be written as follows

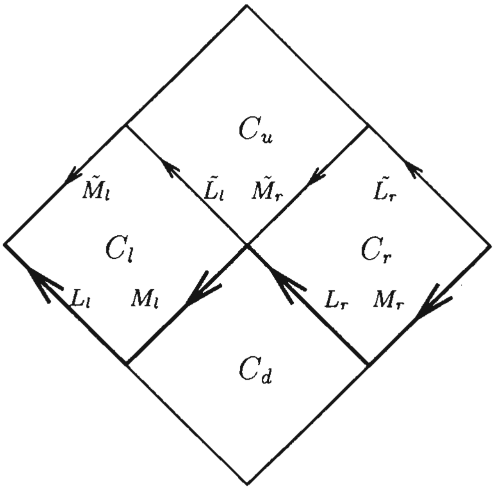

Fig. 4.

$$(u^L)^{-1} v^L (e^{-i\gamma} A + kB) = (\tilde{u}^M)^{-1} (\tilde{v}^M)(e^{-i\gamma} kA + B) ,$$

or equivalently

$$(v^L)^{-1} u^L (\tilde{u}^M)^{-1} \tilde{v}^M = \frac{A + kB e^{i\gamma}}{kA + B e^{i\gamma}} ,$$

where we use the fraction notation for commuting operators. Combining two of these equations written for the faces ℓ and r with the trivial observation following from (9) and (10),

$$(v_\ell^L)^{-1} u_\ell^L (\tilde{u}_\ell^M)^{-1} \tilde{v}_\ell^M = (u_\ell^M)^{-1} (v_\ell^M)^{-1} \tilde{v}_\ell^L \tilde{u}_\ell^L = A_d^{-1} B_d (v_r^L)^{-1} u_r^L (\tilde{u}_r^M)^{-1} \tilde{v}_r^M B_u^{-1} A_u ,$$

we finally get for $C = BA^{-1} = (v^L)^{-1} v^M (u^L)^{-1} \times (u^M)^{-1}$

$$\frac{1 + k e^{i\gamma} C_\ell}{k + e^{i\gamma} C_\ell} = C_d \frac{1 + k e^{i\gamma} C_r}{k + e^{i\gamma} C_r} C_u^{-1} .$$

Only for the operators on neighbouring faces the commutation relations are nontrivial: $C_u C_\ell = e^{2i\gamma} C_\ell C_u$, $C_d C_\ell = e^{2i\gamma} C_\ell C_d$.

We prefer to change the sign of our "quantum angle" along the diagonals. For the combinations of four faces shown in fig. 2 we get consequently the following redifinitions,

$$Q_u = C_u , \quad Q_\ell = C_\ell , \quad Q_r = C_r^{-1} , \quad Q_d = C_d^{-1} ;$$

$$Q_u = C_u^{-1} , \quad Q_\ell = C_\ell^{-1} , \quad Q_r = C_r , \quad Q_d = C_d .$$

In both cases the equation and the commutation relations for Q look as follows,

$$Q_u Q_d = \frac{k + e^{i\gamma} Q_\ell}{1 + e^{i\gamma} k Q_\ell} \frac{k + e^{i\gamma} Q_r}{1 + e^{i\gamma} k Q_r} , \tag{12}$$

$$Q_d Q_r = e^{-2i\gamma} Q_r Q_d , \quad Q_d Q_\ell = e^{-2i\gamma} Q_\ell Q_d \tag{13}$$

(all other Q along the stairway commute with Q_d). We call this equation the quantum discrete sine-Gordon equation [#1].

4. The discrete quantum pendulum

The simplest reduction, which it is possible to im-

[#1] Independently this equation was obtained by L.D. Faddeev.

pose on (12) and (13), is $Q_\ell = Q_r$. In this case we get a sequence of unitary operators Q_n satisfying

$$Q_{n+1}Q_{n-1} = \left(\frac{k+e^{i\gamma}Q_n}{1+e^{i\gamma}kQ_n}\right)^2 ,$$

$$Q_{n+1}Q_n = e^{2i\gamma}Q_nQ_{n+1} . \tag{14}$$

For obvious reasons we call this system a (discrete) quantum pendulum.

In a standard way the Hamiltonian of this quantum pendulum can be calculated as

$$H = \frac{d}{d\lambda}\,\mathrm{tr}(L_\ell M_\ell L_r M_r)\,|_{\lambda=0} .$$

It turns out that H depends only on the Q,

$$H = k(e^{i\gamma}Q_{n-1}Q_n + e^{-i\gamma}Q_n^*Q_{n-1}^*) + k^{-1}(e^{i\gamma}Q_nQ_{n-1}^* + e^{-i\gamma}Q_{n-1}Q_n^*) + 2(Q_n + Q_n^* + Q_{n-1} + Q_{n-1}^*) . \tag{15}$$

The evolution operator U is a unitary operator transforming Q_n into Q_{n+1},

$$Q_{n+1} = UQ_nU^* , \quad \forall n . \tag{16}$$

In case of rational γ there is a finite dimensional representation of the algebra of the Q_n, and U can be calculated explicitly. As a matter of fact, let $h = e^{2i\gamma}$ be such that $h^N = 1$. Then, taking into account the commutation relation $Q_mQ_{m-1} = hQ_{m-1}Q_m$, the operators Q_m and Q_{m-1} can be represented in the following form,

$$Q_{m-1} = \begin{pmatrix} 0 & & & 0 & 1 \\ 1 & 0 & & & 0 \\ & 1 & \ddots & & \\ & & & & \\ 0 & & & 1 & 0 \end{pmatrix} ,$$

$$Q_m = \begin{pmatrix} 1 & & & 0 \\ & h & & \\ & & \ddots & \\ 0 & & & h^{N-1} \end{pmatrix} .$$

Let F be a Fourier transformation matrix,

$$F_{kl} = \frac{1}{\sqrt{N}} h^{(k-1)(l-1)} , \quad h = \exp(2\pi i/N) .$$

Then

$$FQ_{m-1}F^* = Q_m , \quad FQ_mF^* = Q_{m-1}^* .$$

Comparing the first relation with (16), we have

$$U = DF ,$$

where D is some diagonal matrix

$$D_{kl} = \delta_{kl}\exp(id_k) .$$

To determine D we use the evolution equation

$$Q_{m+1} = UQ_mU^* = DFQ_mF^*D^* = DQ_{m-1}^*D^*$$

or, equivalently,

$$\left(\frac{\sqrt{h}\,Q_m + k}{1+\sqrt{h}\,kQ_m}\right)^2 = DQ_{m-1}^*D^*Q_{m-1} . \tag{17}$$

Both matrices in (17) are diagonal. For the elements of D this implies

$$\exp(id_{j+1}) = \exp(id_j)\left(\frac{1+kh^{j-1/2}}{k+h^{j-1/2}}\right)^2$$

which completes the calculation of U.

Figure 5 presents eigenstates of H (15) for $k=0.5$, $h=\exp(i\pi/16)$. Let us explain how these pictures were made. First we construct a coherent state Ψ_{00}, which is characterized by the fact that the probability distributions of both observables Q_n and Q_{n-1} are maximally localized around $Q=1$. The state Ψ_{00} can be obtained as a solution of the following variational problem. First one should consider a set S_a of states with a fixed value of $\mathrm{Re}\langle\Phi|Q_n|\Phi\rangle$,

$$S_a\{\Phi\in\mathscr{H}\,|\,\langle\Phi,\Phi\rangle = 1,\ \mathrm{Re}\langle\Phi|Q_n|\Phi\rangle = a\} .$$

Here $\mathscr{H}$ is the Hilbert space, which is $\mathscr{H} = \mathbb{C}^N$ in our case. The maximum

$$b = \max_{\Phi\in S_a}\mathrm{Re}\langle\Phi|Q_{n-1}|\Phi\rangle$$

is achieved in some state Φ^a, which is equal to $\Phi_{00} := \Phi^a$, when $b=a$. In fact, Ψ_{00} is the ground state of $-(Q_n + Q_n^* + Q_{n-1} + Q_{n-1}^*)$.

Then we obtain states $\Psi_{kl} = Q_n^kQ_{n-1}^l\Psi_{00}$ which are localized around the point $(\exp(2\pi ik/N), \exp(2\pi il/N))$ in the classical phase space $S^1\times S^1$. To make a picture of an eigenstate $\Psi\in\mathbb{C}^N$ we then display the "Husimi function" $(k,l)\mapsto|\langle\Psi,\Psi_{kl}\rangle|^2$ by grey levels. Figure 5 shows characteristic eigenstates out of 32, including the top and the bottom energy levels.

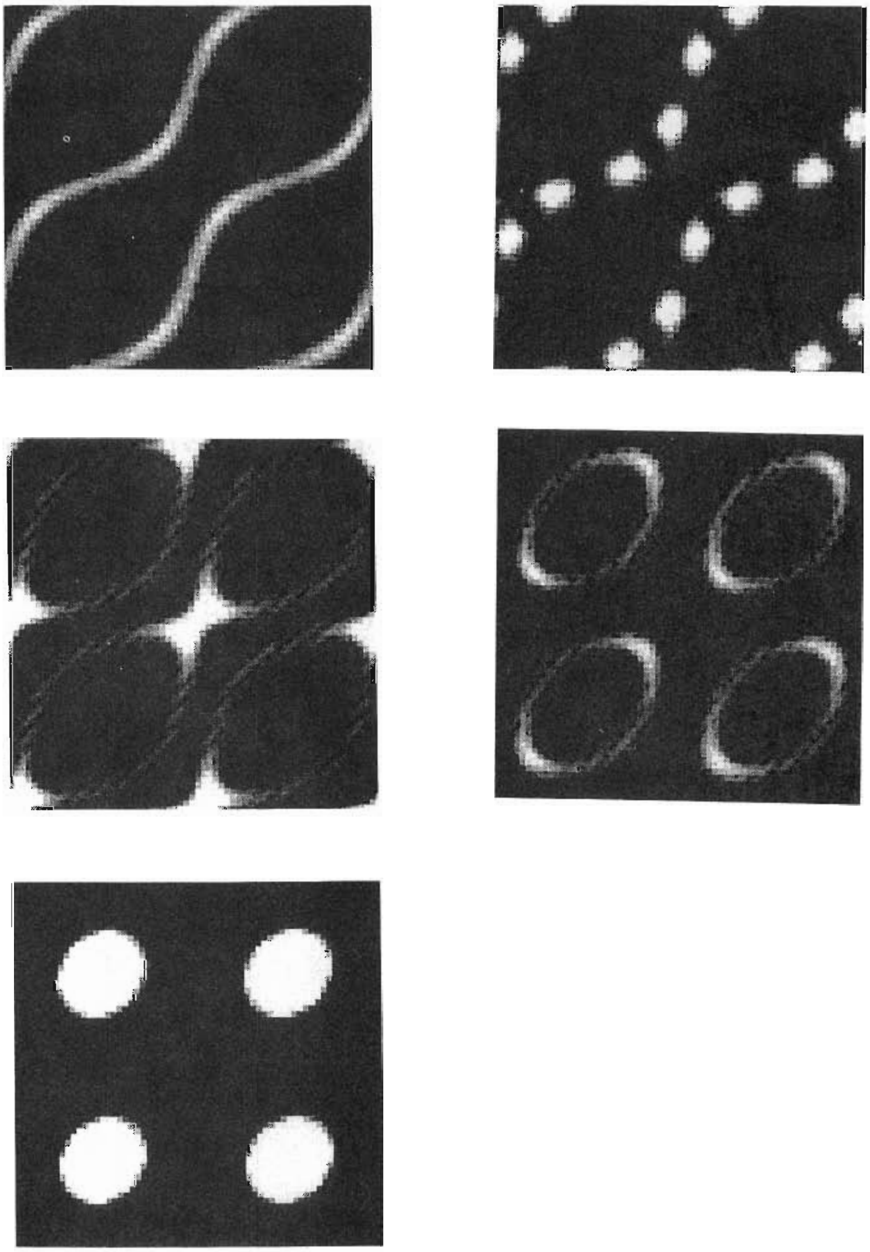

Fig. 5.

The results presented in fig. 5 were obtained numerically. Using the formulas of this section the operator H was represented as an $N\times N$ dimensional matrix, the eigenvector system of which was calculated using Mathematica [3]. The problem of analytical calculation of the spectrum and the eigenfunctions of H (QISM sometimes implies a possibility of this calculation) is now under consideration.

5. Other related results

At the end we would like to mention some important papers closely related to the present one.

First of all we should mention refs. [4,5], where the first examples of integrable quantum mappings appeared and where their quantum Yang–Baxter structure was investigated. These were examples of quantum mappings associated with the lattice analogues of the KdV and the MKdV equations. A first discrete version of the quantum sine-Gordon equation (discrete space, continuous time) was obtained and investigated in ref. [6]. The Lax operator of that paper has a similar form to the one used by us.

We would also like to refer to two good surveys [7,8] on integrable classical symplectic mappings.

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