Surfaces in terms of 2 by 2 matrices. Old and new integrable cases

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1 Introduction

Many of the equations which now are called integrable have been known in differential geometry for a long time. Probably the first was the famous sine-Gordon equation, which was derived to describe surfaces with constant negative Gaussian curvature. At that time many features of integrability of the sine-Gordon and other integrable equations were discovered [1], namely those which have clear geometrical interpretation (for example, the Bäcklund transform).

The theory of solitons appeared much later, in the 1960's. Though it was oriented basically to problems of mathematical physics, it deals in many cases with the same equations. Moreover, the starting point of this theory — the representation (which is called the Lax representation or Zakharov-Shabat representation) of the nonlinear equation in a form of compatibility condition

$$U_y(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)] = 0$$

of two linear equations

$$\Psi_x = U(\lambda)\Psi, \qquad \Psi_y = V(\lambda)\Psi$$

also has a transparent geometrical origin. In differential geometry it is the Gauss-Codazzi equation represented as a compatibility condition of linear equations for the moving frame (the Gauss-Weingarten equations). The spectral parameter λ in this representation describes deformations of surfaces preserving their properties:

$$\text{spectral parameter } \lambda \Longleftrightarrow \text{deformation parameter}, \tag{1.1}$$

[1]For the history of that period see the contribution of M.Melko and I.Sterling in this volume.

that is, integrable surfaces come in families! For the integrable equations of mathematical physics, which were found first, λ was interpreted as a spectral parameter in the corresponding linear problem. Therefore it was quite natural to investigate how the solution of this linear problem Ψ depends on λ. This free treatment of λ, independent of its geometrical interpretation, resulted in the construction of powerful analytical methods of solution for the nonlinear integrable equations. Some of these methods do not have a transparent geometrical interpretation and therefore are new to geometry. First of all here one should mention the finite gap integration method. Whereas the dressing procedure in the soliton theory and the Bäcklund transform essentially coincide and produce the same multisoliton solutions, only a few of the simplest solutions constructed by the finite gap integration method were known before. Although from the very beginning of the soliton theory a close relation with the differential geometry was clear, the first new essential results in classical differential geometry of surfaces were obtained in the late 1980's (see the survey [2] and [12]), when the methods of the finite gap theory were applied.

In the present paper we consider only the classical case of surfaces in a 3-dimensional Euclidean space. Our goal here is to reformulate the classical theory of surfaces in a form familiar to the soliton theory, which makes possible an application of the analytical methods of this theory to integrable cases. In Sect. 2 the moving frame for a general surface is described in terms of quaternions $\Psi \in \mathbf{H}_*$. Such a description is more convenient for the analytical treatment since it analytically characterizes the spin structure of the immersion and deals with 2×2 matrices. This analytic characterization of the spin structure of the minimal surfaces is given in Sect. 3 in terms of the Weierstrass representation. All the necessary results about the spinors on the Riemann surfaces are presented in the Appendix.

All the rest of the paper is devoted to a description of integrable cases and their deformation families (1.1). Some of these cases are well known, some are not well known and some are possibly new. The list below presents these integrable cases:

1) Minimal surfaces: $H = 0$;

2) Constant mean curvature surfaces: $H = \text{const}$;

3) Constant positive Gaussian curvature surfaces: $K = \text{const} > 0$;

4) Constant negative Gaussian curvature surfaces: $K = \text{const} < 0$;

5) Bonnet surfaces: surfaces, possessing nontrivial families of isometries preserving principal curvatures;

6) Surfaces with harmonic inverse mean curvature: $\partial_z \partial_{\bar{z}}(1/H) = 0$, z is a conformal variable of the first fundamental form;

7) Bianchi surfaces: $\partial_x \partial_y (1/\sqrt{-K}) = 0$, x, y are asymptotic coordinates;

8) Bianchi surfaces of positive curvature: $\partial_z \partial_{\bar{z}}(1/\sqrt{K}) = 0$, z is a conformal coordinate of the second fundamental form.

Only the cases 6 and 8 of this list can pretend to be new, and the case 8 is just another real form of the Bianchi surfaces (case 7). The surfaces with $\partial_z\partial_{\bar{z}}(1/H) = 0$ seems to be new, but upon seeing a wonderful 3-page long paper by Tzitzéica [16] on affine spheres, shown me by Sergey Tsarev, where the Bullough-Dodd-Jiber-Shabat equation $f_{uv} = e^f - e^{-2f}$ together with its representation (1.1) ("système complètement intégrable") are presented, one would have to have a lot of courage to claim that a new integrable case in differential geometry of surfaces is found.

Formulas for the moving frame of integrable surfaces can be integrated by an expression, first suggested by A. Sym for the constant negative curvature surfaces [15] (see also Sect. 8). Namely, the immersion function in many cases is given by

$$\Psi^{-1}\frac{\partial}{\partial\lambda}\Psi.$$

We prove that with slight modifications this immersion formula is valid for all the cases above except 1 and 5.

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2 Surfaces in a 3-dimensional Euclidean space

2.1 Differential equations of surfaces

Let $\mathcal{F}$ be a smooth surface in a 3-dimensional Euclidean space. The Euclidean metric induces a metric Ω on this surface, which in turn generates the complex structure of a Riemann surface. The surface is covered by domains $\mathcal{D}_i$ with $\cup_i\mathcal{D}_i = \mathcal{F}$, and in each of these there is defined a local coordinate $z_i : \mathcal{D}_i \to U_i \subset \mathbf{C}$. If the intersection $\mathcal{D}_i \cap \mathcal{D}_j \neq \emptyset$ is non-empty, the glueing functions $z_i \circ z_j^{-1}$ are holomorphic. Under such a parametrization, which is called *conformal*, the surface is given by a the vector-valued function:

$$F = (F_1, F_2, F_3) : \mathcal{R} \to \mathbf{R}^3,$$

and the metric is conformal: $\Omega = e^{u_i} dz_i d\bar{z}_i$. In the sequel we suppose that $\mathcal{F}$ is sufficiently smooth. We also omit the superscript i of the local coordinate z in case when it is not confusing.

The conformal parametrisation gives the following normalization of the function $F(z,\bar{z})$:

$$< F_z, F_z >=< F_{\bar{z}}, F_{\bar{z}} >= 0, \; < F_z, F_{\bar{z}} >= \frac{1}{2}e^u, \tag{2.1}$$

where the brackets mean the scalar product

$$< a, b >= a_1b_1 + a_2b_2 + a_3b_3,$$

and F_z and $F_{\bar z}$ are the partial derivatives $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \bar z}$ where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The vectors $F_z, F_{\bar z}$ as well as the normal N,

$$< F_z, N > = < F_{\bar z}, N > = 0, \quad < N, N > = 1, \tag{2.2}$$

define a moving frame on the surface, which due to (2.1, 2.2) satisfies the following Gauss-Weingarten (GW) equations:

$$\sigma_z = \mathcal{U}\sigma, \ \sigma_{\bar z} = \mathcal{V}\sigma, \ \sigma = (F_z, F_{\bar z}, N)^T, \tag{2.3}$$

$$\mathcal{U} = \begin{pmatrix} u_z & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2e^{-u}Q & 0 \end{pmatrix}, \ \mathcal{V} = \begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & u_{\bar z} & \bar Q \\ -2e^{-u}\bar Q & -H & 0 \end{pmatrix} \tag{2.4}$$

where

$$Q = < F_{zz}, N >, \quad < F_{z\bar z}, N > = \frac{1}{2}He^u. \tag{2.5}$$

The first and the second quadratic forms

$$\begin{aligned} < dF, dF > &= < I\begin{pmatrix} dx \\ dy \end{pmatrix}, \begin{pmatrix} dx \\ dy \end{pmatrix} >, \ z = x + iy, \\ - < dF, dN > &= < II\begin{pmatrix} dx \\ dy \end{pmatrix}, \begin{pmatrix} dx \\ dy \end{pmatrix} > \end{aligned}$$

are given by the matrices

$$I = e^u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ II = \begin{pmatrix} Q + \bar Q + He^u & i(Q - \bar Q) \\ i(Q - \bar Q) & -(Q + \bar Q) + He^u \end{pmatrix}.$$

The principal curvatures k_1 and k_2 are the eigenvalues of the matrix $II \cdot I^{-1}$, which gives the following expressions for the mean and the Gaussian curvatures:

$$\begin{aligned} H &= \tfrac{1}{2}\,(k_1 + k_2) = \tfrac{1}{2}\ \mathrm{tr}\ (II \cdot I^{-1}), \\ K &= k_1 k_2 = \det\ (II \cdot I^{-1}) = H^2 - 4Q\bar Q e^{-2u}. \end{aligned}$$

The Gauss-Codazzi (GC) equations, which are the compatibility conditions of equations (2.3, 2.4),

$$\mathcal{U}_{\bar z} - \mathcal{V}_z + [\mathcal{U}, \mathcal{V}] = 0,$$

have the following form (cf. [17]):

$$u_{z\bar z} + \frac{1}{2}H^2 e^u - 2Q\bar Q e^{-u} = 0, \tag{2.6}$$

$$Q_{\bar z} = \frac{1}{2}H_z e^u, \tag{2.7}$$

$$\bar Q_z = \frac{1}{2}H_{\bar z} e^u. \tag{2.8}$$

2.2 Quaternionic description

We construct and investigate surfaces in $\mathbf{R}^3$ by analytical methods. For this purpose it is more convenient to use 2×2 matrices instead of 3×3 matrices (2.4), therefore first we rewrite the equations (2.3) for the moving frame in terms of quaternions. This also allows us to control the spin structure of the immersion (see the next section) and makes the presentation familiar to the specialists in the theory of integrable equations.

Let us denote the algebra of quaternions by $\mathbf{H}$, the multiplicative quaternion group by $\mathbf{H}_* = \mathbf{H} \setminus \{0\}$ and their standard basis by $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$

$$\mathbf{ij} = \mathbf{k}, \ \mathbf{jk} = \mathbf{i}, \ \mathbf{ki} = \mathbf{j}. \tag{2.9}$$

The Pauli matrices σ_α are related with this basis as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\ \mathbf{i}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\ \mathbf{j},$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\ \mathbf{k}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.10}$$

with the multiplication in (2.9) being just the matrix multiplication. We identify a 3-dimensional Euclidean space with the space of imaginary quaternions Im $\mathbf{H}$

$$X = -i \sum_{\alpha=1}^{3} X_\alpha \sigma_\alpha \in \mathrm{Im}\, \mathbf{H} \ \longleftrightarrow \ X = (X_1, X_2, X_3) \in \mathbf{R}^3. \tag{2.11}$$

The scalar product of vectors in terms of matrices is then

$$< X, Y >= -\frac{1}{2}\mathrm{tr} XY. \tag{2.12}$$

We also denote by F and N the matrices obtained in this way from the vectors F and N.

Let us take $\Phi \in \mathbf{H}_*$

$$\Phi = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \ |a|^2 + |b|^2 \neq 0, \tag{2.13}$$

which transforms the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the basis F_x, F_y, N:

$$F_x = e^{u/2}\Phi^{-1}\mathbf{i}\Phi, \ F_y = e^{u/2}\Phi^{-1}\mathbf{j}\Phi, \ N = \Phi^{-1}\mathbf{k}\Phi. \tag{2.14}$$

Then

$$F_z = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi, \ F_{\bar{z}} = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi, \tag{2.15}$$

and all the conditions (2.1) are automatically satisfied.

The quaternion Φ satisfies linear differential equations. To derive them we introduce matrices

$$U = \Phi_z \Phi^{-1}, \quad V = \Phi_{\bar z} \Phi^{-1}. \tag{2.16}$$

The compatibility condition $F_{z\bar z} = F_{\bar z z}$ for (2.15) implies

$$-ie^{u/2}\frac{u_{\bar z}}{2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi - ie^{u/2}\Phi^{-1}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, V\right]\Phi =$$
$$= -ie^{u/2}\frac{u_z}{2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi - ie^{u/2}\Phi^{-1}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, U\right]\Phi,$$

or, eqivalently,

$$u_{\bar z} = 2(V_{22} - V_{11}), \quad u_z = 2(U_{11} - U_{22}), \quad U_{21} = -V_{12},$$

where U_{kl} and V_{kl} are the matrix elements of U and V. In the same way the equalities (2.15) imply

$$\begin{aligned} F_{z\bar z} = \frac{1}{2}He^u N \quad &\to \quad U_{21} = -V_{12} = \frac{1}{2}He^{u/2} \\ F_{zz} = u_z F_z + QN \quad &\to \quad U_{12} = -Qe^{-u/2} \\ F_{\bar z\bar z} = u_{\bar z} F_{\bar z} + \bar Q N \quad &\to \quad V_{21} = \bar Q e^{-u/2} \end{aligned}$$

Now only the coeffitients U_{22} and V_{11} are still not determined. To fix them, we recall that Φ was defined by (2.14) up to a multiplication by a scalar factor. We normalize this factor by the condition

$$\det \Phi = e^{u/2}, \tag{2.17}$$

the reason for which is clarified in the next section. For the traces of U and V this implies

$$U_{11} + U_{22} = u_z/2, \quad V_{11} + V_{22} = u_{\bar z}/2.$$

Finally we get the following

Theorem 1 *Using the isomorphism (2.11), the moving frame $F_z, F_{\bar z}, N$ of the conformally parametrised surface (z is a conformal coordinate) is described by formulas (2.14),(2.15), where $\Phi \in H_*$ satisfies the equations (2.16) with U, V of the form*

$$U = \begin{pmatrix} \dfrac{u_z}{2} & -Qe^{-u/2} \\ \dfrac{1}{2}He^{u/2} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -\dfrac{1}{2}He^{u/2} \\ \bar Q e^{-u/2} & \dfrac{u_{\bar z}}{2} \end{pmatrix}. \tag{2.18}$$

Corollary 1 *Φ satisfies the Dirac equation*

$$e^{-u/2}\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}\Phi = \frac{1}{2}H\Phi. \tag{2.19}$$

2.3 Spin structure

Let us now determine the dependence of Φ on the holomorphic coordinate z.

Lemma 1

$$\Phi_{sp} = \begin{pmatrix} \sqrt{dz} & 0 \\ 0 & \sqrt{d\bar{z}} \end{pmatrix} \Phi$$

is invariant under analytical changes of z.

Proof. Since $e^u dz d\bar{z}$ is invariant under analytical changes of $z \to w(z)$, u is transformed as follows

$$e^{u/2}(z) = e^{u/2}(w)\sqrt{w'\bar{w}'}, \quad w' = dw/dz. \tag{2.20}$$

The most general transformation of Φ compatible with the formulas for the moving frame (2.14, 2.15) is

$$\Phi(z) = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix} \Phi(w),$$

where

$$\frac{c}{\bar{c}} = \sqrt{\frac{w'}{\bar{w}'}}. \tag{2.21}$$

To get the last formula we have to take into account the transformation law of the metric (2.20). The normalization condition (2.17) implies for c

$$c\bar{c} = \sqrt{w'\bar{w}'},$$

which, combined with (2.21), completes the proof

$$c = \sqrt{w'}.$$

The local variation of $\Phi(z, \bar{z})$ is described by system (2.16, 2.18). These equations are compatible, therefore there is no local monodromy of Φ. We see that

$$\Phi_{sp} = \begin{pmatrix} \sqrt{dz} & 0 \\ 0 & \sqrt{dz} \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

consists of two spinors $a\sqrt{dz}$ and $b\sqrt{dz}$. All the necessary information about spinors is presented in the Appendix. The spinors $a\sqrt{dz}$, $b\sqrt{dz}$ may change sign around cycles γ of $\mathcal{R}$.

Lemma 2 *The spin structures of the spinors $a\sqrt{dz}$ and $b\sqrt{dz}$ are the same. The flip numbers of these spinors coincide: $p^a(\gamma) = p^b(\gamma)$ for any contour $\gamma \subset \mathcal{R}$.*

Proof. This is straightforward. Since $\mathcal{F}$ is oriented, $F_z, F_{\bar{z}}, N$ are uniquely defined on $\mathcal{R}$, and formulas (2.14, 2.15) show that the only possible transformation of Φ_{sp} on passing around γ is

$$\Phi_{sp} \to (-1)^p \Phi_{sp}.$$

This spin structure, and generally the numbers $p(\gamma) \in \mathbf{Z}_2$, have a simple geometrical interpretation. Let us consider a closed contour $\Gamma \subset \mathcal{F}$. Γ together with the normal field N of the surface at this contour defines a closed orientable strip in space. Let $N_\Gamma \in \mathbf{Z}$ be the number of twists of this strip, or equivalently, the winding number of the contours Γ and $(\Gamma + \epsilon N)_\Gamma$, where ϵ is small. Isotopies of the band preserve N_Γ, whereas regular homotopies (self-intersections are allowed) preserve N_Γ (*mod* 2), which we from now on denote by

$$P(\Gamma) \in \mathbf{Z}_2$$

and call it the *parity of twists.*

Let us consider a little bit more general class of surfaces: surfaces with translational periods. For these surfaces the frame $F_z, F_{\bar{z}}, N$ is uniquely defined on $\mathcal{R}$, whereas the immersion function F can have periods around cycles on $\mathcal{R}$. All the previous considerations of spin structures are valid also for surfaces with periods, since only the formulas for the frames were used. The numbers $P(\Gamma) \in \mathbf{Z}_2$ can also be defined in this case. To do this we introduce a notion of a translation-holonomy strip.

Definition 1 *A translation-holonomy strip is a smooth curve in* $\mathbf{R}^3$ *equipped with a smooth normal field* N*, such that the orthonormal frames, consisting of the normal vector* N *and the tangential vector, coincide at the ends of the curve.*

If $F(s), N(s)$,

$$F : [0,1] \longrightarrow R^3, \quad N : [0,1] \longrightarrow S^2$$

is some parametrisation of the translation-holonomy strip, then

$$\frac{F_s}{|F_s|}(0) = \frac{F_s}{|F_s|}(1), \quad N(0) = N(1).$$

We consider smooth homotopies of translation-holonomy strips and define the parity of twists of a translation-holonomy strip S to be equal to a parity of twists of a closed strip S_0 smoothly homotopic to S

$$P(S) \stackrel{def}{=} P(S_0).$$

This is well-defined, i.e. $P(S)$ is independent of the choice of closed strip S_0 homotopic to S, which also shows that $P(S)$ is an invariant of the homotopy class.

Remark. The parity of twists of the straight strip $\{F(s) = \vec{f}s,\ N(s) = \vec{n},\ (\vec{f}, \vec{n}) = 0; \vec{f}, \vec{n} = \text{const}\}$ is equal to 1!

Theorem 2 *If $F:\mathcal{R}\longrightarrow \mathbf{R}^3$ is a conformal parametrisation of a surface $\mathcal{F}$ with translational periods and $\gamma\subset\mathcal{R}$ is a closed contour, then the spinor flip number $p(\gamma)$ is equal to the parity of twists $P(\Gamma)$:*

$$p(\gamma)=P(\Gamma),$$

where $\Gamma\subset\mathcal{F}$ is the image of γ.

Proof. Let $\gamma(s)$, $s\in[0,1]$, $\gamma(0)=\gamma(1)$ be a parametrisation of the contour and $z=e^{2\pi is}$ be a complex annular coordinate $|z(\gamma)|=1$ on γ. Formulas (2.14) imply for the frame F_s,N along γ:

$$\begin{aligned} F_s &= 2\pi e^{u/2}\Phi^{-1}(s)\begin{pmatrix} 0 & -e^{-2\pi is} \\ e^{2\pi is} & 0 \end{pmatrix}\Phi(s), \\ N &= -i\Phi^{-1}(s)\sigma_3\Phi(s). \end{aligned}$$

By general rotation of the translation-holonomy strip in $\mathbf{R}^3$ we normalize $\Phi(0)=I$. Since the parity of twists $P(\Gamma)$ is preserved by the smooth homotopies of translation-holonomy strips, we can replace $e^{u/2}$ by 1 when calculating $P(\Gamma)$.

The curves $\Phi(s)$ have different topology for different flip numbers $p(\gamma)$:

a) $p(\gamma)=0 \Rightarrow \Phi(1)=I$,

b) $p(\gamma)=1 \Rightarrow \Phi(1)=-I$.

The variety H_* is simply connected, therefore by smooth homotopies of the translation-holonomy strips the curves $\Phi(s)$ can be transformed in the cases a),b) above respectively to

a) $\Phi(s)=I$,

b) $\Phi(s)=\begin{pmatrix} e^{-\pi is} & 0 \\ 0 & e^{\pi is} \end{pmatrix}$.

For the immersion of the translation-holonomy strip it yields

a) $F=-i\begin{pmatrix} 0 & e^{-2\pi is} \\ e^{2\pi is} & 0 \end{pmatrix}$, $\quad N=\mathbf{k} \Rightarrow P(\Gamma)=0$,

b) $F=2\pi s\mathbf{j}$, $\quad N=\mathbf{k} \Rightarrow P(\Gamma)=1$,

which proves $p(\gamma)=P(\Gamma)$.

Definition 2 *The spin structure of Φ_{sp} is called the spin structure of the immersion.*

Usually, if a surface is given by its immersion function it is difficult to answer the important geometric question of whether it is an embedding or not. Sometimes existence of the self-intersection can be proved by purely topological arguments analyzing the spin structure of the immersion. If $\mathcal{R}$ is a Riemann surface with G handles and K punctures or holes, then this spin structure is characterized by the numbers

$$[\alpha, \beta, \delta] \in \mathbf{Z}_2^{2G+K}$$

(see Appendix). The numbers α_i, β_i describe flips along the handles, whereas the δ_k describe flips around the holes or the punctures. As it is mentioned in the Appendix the parity of the spin structure $< \alpha, \beta > \in \mathbf{Z}_2$ is invariant with respect to the choice of the basis in $H_1(\mathcal{R}, \mathbf{Z})$. As it was proved in [13], it completely classifies regular homotopies of compact orientable immersions.

Theorem 3 *[13] The parity of the spin structure completely classifies compact orientable immersions with respect to regular homotopies, i.e. there is a smooth homotopy F_t, $t \in [0,1]$ of two immersions, F_0 and F_1, of a surface of genus G which at each moment remains an immersion if and only if the parities of the spin structures corresponding to F_0 and F_1 coincide. For embeddings, $< \alpha, \beta >= 0$.*

Corollary 2 *If F is an embedding with G handles and K holes or punctures, then*

$$< \alpha, \beta >= 0, \quad \delta_k = 0, \quad k = 1, \ldots, K.$$

To prove this statement we consider a sphere big enough to contain all the handles inside. Replacing the outside parts of the surface by the corresponding pieces of the sphere (smooth glueing) and applying Theorem 3 to this compact surface we get $< \alpha, \beta >= 0$. The embeddedness of the ends imply the vanishing of the $\delta' s$.

3 Minimal surfaces

In the case of minimal surfaces ($H = 0$) the system (2.16, 2.18) can be solved. The elements $a(z), b(z)$ of Φ in (2.13) are holomorphic. The metric and the Hopf differential are expressed in the terms of the spinors $a(z), b(z)$ as follows:

$$e^{u/2} = |a|^2 + |b|^2, \quad Q = a_z b - b_z a.$$

Formulas (2.14, 2.15) for the frame yield

$$\begin{aligned} F_z &= -i \begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}, \qquad F_{\bar z} = -i \begin{pmatrix} -\bar a \bar b & \bar a^2 \\ -\bar b^2 & \bar a \bar b \end{pmatrix}, \\ N &= -i \frac{1}{|a|^2 + |b|^2} \begin{pmatrix} |a|^2 - |b|^2 & 2\bar a b \\ 2a\bar b & |b|^2 - |a|^2 \end{pmatrix}. \end{aligned}$$

Finally, for the coordinates of the immersion and the Gauss map we obtain the Weierstrass representation:

$$\begin{aligned} F_1 &= \mathrm{Re} \int^z (g^2 - 1)\eta, \qquad N_1 = \frac{2\,\mathrm{Re} g}{|g|^2 + 1}, \\ F_2 &= \mathrm{Im} \int^z (g^2 + 1)\eta, \qquad N_2 = \frac{2\,\mathrm{Im} g}{|g|^2 + 1}, \\ F_3 &= -2\mathrm{Re} \int^z g\eta, \qquad N_3 = \frac{|g|^2 - 1}{|g|^2 + 1}, \end{aligned} \tag{3.1}$$

where $g = a/b$ is an analytic function and $\eta = b^2 dz$ is a holomorphic differential on $\mathcal{R}$.

Proposition 1 *The spin structure of the minimal immersion is given by the spinor* $\sqrt{\eta}$, *where* η *is the holomorphic differential in the Weierstrass representation (3.1). For any closed contour* $\gamma \subset \mathcal{R}$ *the flip number* $p(\gamma)$ *of* $\sqrt{\eta}$ *is equal to the parity of twists* $P(\Gamma)$ *of the corresponding normal strip* $\Gamma = F(\gamma)$.

4 Dual surfaces

Let us consider the special case when the Hopf differential is real

$$Q \in \mathbf{R}. \tag{4.1}$$

The second fundamental form is diagonal and the preimages of the curvature lines are the lines $x = \text{const}$ and $y = \text{const}$ on the parameter domain. A conformal parametrisation with this property is called *isothermal.*

Definition 3 *Surfaces which admit isothermal coordinates are called isothermal.*

In terms of arbitrary conformal coordinate Property (4.1) can be reformulated as follows:

$$Q(z, \bar{z}) = \frac{1}{2} q(z, \bar{z}) f(z), \tag{4.2}$$

where $f(z)$ is holomorphic and $q(z, \bar{z})$ is real.

Definition 4 *Let* $F(z, \bar{z})$ *be a conformal immersion of an isothermal surface* $\mathcal{F}$ *with Hopf differential of the form (4.2). Then a surface* $\mathcal{F}^*$, *defined via the immersion function* $F^* : \mathcal{R} \to \mathbf{R}^3$ *with the following formulas for the moving frame*

$$F^*_z = e^{-u} f F_{\bar{z}}, \quad F^*_{\bar{z}} = e^{-u} \bar{f} F_z, \quad N^* = N, \tag{4.3}$$

is called a dual surface

Proposition 2 *The immersion* $F^* : \mathcal{R} \to \mathbf{R}^3$ *defined above is a conformal parametrisation of an isothermal surface. The metric* e^{u^*}, *the mean curvature* H^* *and the Hopf differential* Q^* *of this surface are given by the formulas:*

$$e^{u^*} = e^{-u} f \bar{f}, \quad H^* = q, \quad Q^* = \frac{1}{2} H f. \tag{4.4}$$

Proof. The definition (4.3) of F^* is self-consistent since the equality $F^*_{z\bar{z}} = F^*_{\bar{z}z}$ is equivalent to $(e^{-u} f F_{\bar{z}})_{\bar{z}} = \frac{1}{2} e^{-u} f \bar{f} q N = (e^{-u} \bar{f} F_z)_z$. Here we use (4.2) and the Gauss-Weingarten equations for $F_{zz}, F_{\bar{z}\bar{z}}$. The conformality of F^* is evident. The expressions (4.4) are obtained by straightforward calculation, for example

$$Q^* = -< F^*_z, N^*_z > = -e^{-u} f < F_{\bar{z}}, (-H F_z - 2e^{-u} Q F_{\bar{z}}) > = \frac{1}{2} H f,$$

which shows that $\mathcal{F}^*$ is also isothermal.

Remark. $\mathcal{F}^{**} = \mathcal{F}$ up to a scaling in $\mathbf{R}^3$.

5 Constant mean curvature (CMC) surfaces.

5.1 Formula for immersion

If the mean curvature of $\mathcal{F}$ is constant, then the Gauss-Codazzi equations

$$u_{z\bar{z}} + \frac{1}{2}H^2 e^u - 2Q\bar{Q}e^{-u} = 0, \qquad Q_{\bar{z}} = 0, \tag{5.1}$$

are invariant with respect to the transformation

$$Q \to Q^t = \lambda Q, \quad |\lambda| = 1, \tag{5.2}$$

where $\lambda = e^{2it}$ is a complex number of unit modulus which is the same for all points of $\mathcal{F}$. Integrating the equations for the moving frame with the coefficient Q replaced by $Q^t = \lambda Q$ we obtain a one-parameter family $\mathcal{F}^t$ of surfaces. The transformation (5.2) does not effect the metric and the mean curvature, therefore all the surfaces $\mathcal{F}^t$ are isometric and have the same constant mean curvature. Treating t as a deformation parameter we obtain a classical theorem of Bonnet.

Theorem 4 (Bonnet) *Every constant mean curvature surface has a one-parameter family of isometric deformations preserving both principal curvatures. The deformation is described by the transformations (5.2).*

The invariance of the principal curvatures follows from the fact that K is an isometric invariant $K = -2u_{z\bar{z}}e^{-u}$.

The quaternion Φ solving the system (2.16, 2.18) describes the moving frame $F_z, F_{\bar{z}}, N$ (2.14, 2.15) on the surface. In [2] it was shown that knowing the family $\Phi(z, \bar{z}, \lambda)$ for all $\lambda = e^{2it}$ allows us to integrate the formulas for the moving frame explicitly replacing the integration with respect to $z, \bar{z}$ by a differentiation with respect to t.

Theorem 5 *Let $\Phi(z, \bar{z}, \lambda = e^{2it})$ be a solution of the system*

$$\Phi_z = U(\lambda)\Phi, \qquad \Phi_{\bar{z}} = V(\lambda)\Phi,$$

$$\begin{aligned} U(\lambda) &= \begin{pmatrix} \frac{u_z}{2} & -\lambda Q e^{-u/2} \\ \frac{1}{2}He^{u/2} & 0 \end{pmatrix}, \\ V(\lambda) &= \begin{pmatrix} 0 & -\frac{1}{2}He^{u/2} \\ \frac{1}{\lambda}\bar{Q}e^{-u/2} & \frac{u_{\bar{z}}}{2} \end{pmatrix} \end{aligned} \tag{5.3}$$

belonging to the quaternion group $\Phi(z, \bar{z}, \lambda = e^{2it}) \in H_*$ with the norm $\det\Phi = e^{u/2}$. *Then F and N, defined by the formulas*

$$F = -\frac{1}{H}(\Phi^{-1}\frac{\partial}{\partial t}\Phi - i\Phi^{-1}\sigma_3\Phi), \qquad N = -i\Phi^{-1}\sigma_3\Phi, \tag{5.4}$$

describe a CMC surface with metric e^u the mean curvature H and Hopf differential $Q^t = e^{2it}Q$.

Conversely, let F be a conformal parametrisation of a CMC surface with metric e^u, mean curvature H and Hopf differential Q^t. Then F is given by Formula (5.4) where Φ is a solution of (5.3) as above.

Proof. First we note that both F and N take values in the imaginary quaternions Im**H** and therefore can be identified (2.11) with vectors in $\mathbf{R}^3$. The system (5.3) coincides with the quaternionic representation (2.18) for the equations for the moving frame with the Hopf differential λQ. Differentiating (5.4), we get

$$\begin{aligned} F_z &= -\frac{1}{H}(\Phi^{-1}\frac{\partial U(\lambda)}{\partial t}\Phi - i\Phi^{-1}[\sigma_3, U(\lambda)]\Phi) = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi, \\ F_{\bar z} &= -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi, \end{aligned}$$

which coincides with (2.15).

For a given e^u, H and Q^t the surface is determined up to an Euclidean motion of $\mathbf{R}^3$. The solution $\Phi(z, \bar z, \lambda = e^{2it}) \in H_*, \det\Phi = e^{u/2}$ is defined up to multiplication on the right by a factor $R(\lambda) \in SU(2)$. This right multiplication $\Phi \to \Phi R(\lambda)$ describes all Euclidean motions of the surface

$$F \to R^{-1}FR + R^{-1}\frac{\partial}{\partial t}R, \qquad R \in SU(2), \quad R^{-1}\frac{\partial}{\partial t}R \in su(2). \tag{5.5}$$

The immersion function $F(z, \bar z, \lambda = e^{2it_0})$ with a fixed $\lambda = e^{2it_0}$ (one CMC surface from the family) determines $\Phi(z, \bar z, \lambda)$ uniquely up to the transformation

$$\Phi(z, \bar z, \lambda) \to \Phi(z, \bar z, \lambda)R(\lambda), \quad R(\lambda) = \pm(I + O(t - t_0)^2), \quad t \sim t_0. \tag{5.6}$$

System (5.3) represents itself the Lax representation for the nonlinear equations (5.1). The Lax representation is a starting point of the integration procedure of the soliton theory, which allows us to construct explicit solutions of the corresponding nonlinear integrable equations. The main tool of this procedure is the study of the analytic properties of $\Phi(\lambda)$ with respect to the spectral parameter λ. Moreover, as a by-product of the integration procedure, $\Phi(\lambda)$ is also constructed. This explains why formula (5.4) for the CMC immersion, which seems not to have been known classically, is very useful for analytic treatment of the surfaces. It allows us to eliminate the double integration of the GW equations and supplies us with the final formula for the immersion. For the CMC tori case this helps to control the periodicity of the immersion and, finally, to describe all the CMC tori explicitly [2].

We also mention here a well known fact, which can be easily checked cf. [17].

Proposition 3 *The Gauss map $N : \mathcal{R} \to S^2$ of the CMC surface is harmonic, i.e.*

$$N_{z\bar z} = qN, \qquad q : \mathcal{R} \to \mathbf{R}.$$

Remark. In the neighborhood of a non-umbilic point $Q \neq 0$ by a conformal change of coordinate $z \to \tilde{z}(z)$ one can always normalize $Q = H/2$. In this parametrisation the Gauss equation and the system (5.3) become the elliptic sinh-Gordon equation

$$u_{z\bar{z}} + H \sinh u = 0$$

and its Lax representation.

5.2 Monodromy of Φ and balanced diagrams

Let $\Phi(z, \bar{z}, \lambda)$ be a solution of the system (5.3) as in Theorem 5, $\Phi(z, \bar{z}, \lambda = e^{2it}) \in H_*$, $\det \Phi = e^{u/2}$. It is defined on the universal covering of $\mathcal{R}$. Under passage around a closed contour γ on $\mathcal{R}$ this solution gets a monodromy

$$\Phi(z, \bar{z}, \lambda) \xrightarrow{\gamma} {}^{\gamma}\Phi(z, \bar{z}, \lambda) = \Phi(z, \bar{z}, \lambda)\ {}^{\gamma}M(\lambda).$$

Since the norm of Φ is preserved, M is unitary

$${}^{\gamma}M(\lambda = e^{2it}) \in SU(2).$$

Monodromy of the solution depends not on a particular cycle γ but on its homotopy class $[\gamma] \in \pi_1(\mathcal{R})$.

Lemma 3 *Let $F : \mathcal{R} \to \mathbf{R}^3$ be a CMC immersion defined by Formula (5.4) with $\Phi(z, \bar{z}, \lambda = e^{2it_0})$ and suppose that the image of the contour $\gamma \subset \mathcal{R}$ is a closed contour $\Gamma = F(\gamma)$ in $\mathbf{R}^3$. Then*

$${}^{\gamma}M(\lambda = e^{2it}) = \pm(I + A[\gamma](t - t_0)^2 + B[\gamma](t - t_0)^3 + O(t - t_0)^4), t \sim t_0, \quad (5.7)$$

where the sign is determined by the spin structure of the immersion (see Section 2.3)

Proof. Formula (5.4) yields the following transformation law for the immersion function under the passage around γ

$$F \longrightarrow {}^{\gamma}F = M^{-1}FM + M\frac{\partial}{\partial t}M_{|_{t=t_0}}, \quad (5.8)$$

which implies ${}^{\gamma}M(t_0) = \pm I, \ \ \partial\, {}^{\gamma}M/\partial t_{|_{t=t_0}} = 0$.

Since $M(\lambda = e^{2it}) \in SU(2)$, both A and B lie in the Lie algebra

$$A, B \in su(2)$$

and can be identified (2.11) with the vectors in $\mathbf{R}^3$.

Proposition 4 *Let $F : \mathcal{R} \to \mathbf{R}^3$ be a CMC immersion. Then to any homology class $[\Gamma] \in H_1(\mathcal{F}, \mathbf{Z})$ there can be associated two vectors $A[\Gamma], B[\Gamma]$. The maps $A : H_1(\mathcal{F}, \mathbf{Z}) \to \mathbf{R}^3$ and $B : H_1(\mathcal{F}, \mathbf{Z}) \to \mathbf{R}^3$ are homomorphisms. The group $E(3)$ of Euclidean motions*

$$F \longrightarrow F^R = R^{-1}FR + r, \quad R \in SU(2), \quad r \in su(2) \tag{5.9}$$

acts on A, B as follows ($A, B \in su(2)$):

$$A \longrightarrow A^R = R^{-1}AR, \quad R \longrightarrow B^R = R^{-1}BR + [A^R, r]. \tag{5.10}$$

Proof. Proving Theorem 5 we have shown that the immersion function $F(z, \bar{z}, \lambda = e^{2it_0})$ determines the solution $\Phi(z, \bar{z}, \lambda)$ uniquely up to the transformation (5.6). This transformation does not affect the terms $O((t-t_0)^2)$ and $O((t-t_0)^3)$ of the monodromy (5.7), therefore the vectors $A[\Gamma], B[\Gamma]$ characterize the immersion function F and not a special Φ. The Euclidean motion (5.9) transforms Φ as follows:

$$\Phi \longrightarrow \Phi^R = \Phi R(I + r(t - t_0)),$$

which implies

$${}^{\gamma}M \longrightarrow {}^{\gamma}M^R = (I - r(t-t_0) + O((t-t_0)^2)R^{-1}\ {}^{\gamma}MR(I + r(t-t_0))$$

for the monodromy and (5.10) for A and B.

The multiplication law for the monodromy ${}^{\gamma_1+\gamma_2}M = {}^{\gamma_1}M \cdot {}^{\gamma_2}M$ yields

$$A[\Gamma_1 + \Gamma_2] = A[\Gamma_1] + A[\Gamma_2], \quad B[\Gamma_1 + \Gamma_2] = B[\Gamma_1] + B[\Gamma_2], \quad \Gamma_i = F(\gamma_i).$$

This shows that A and B are the homomorphisms $H_1(\mathcal{F}, \mathbf{Z}) \to \mathbf{R}^3$.

The maps $A, B : H_1(\mathcal{F}, \mathbf{Z}) \to \mathbf{R}^3$ were constructed in [9] by the variational principle, which provides for these maps a more transparent geometrical and physical interpretations ($A[\Gamma]$ and $B[\Gamma]$ were called force and torque respectively). Namely, $A[\Gamma]$ and $B[\Gamma]$ transform exactly as linear and angular momentum for a moving body. If the translational force $A[\Gamma]$ is non-zero, then a natural "balancing" line in $\mathbf{R}^3$ can be associated to $[\Gamma]$ - the line along which the center of mass travels. For surfaces of revolution the balancing line is exactly the axis. For many cases it is proved that balancing line and homology class representative are "close" to each other. Moreover, sometimes one can proceed further and build a balanced diagram of the surface — a graph in $\mathbf{R}^3$ consisting of the segments of the balancing lines. Using this approach one can characterize the structure of CMC embeddings [10], in particular the localization of the surface with respect to its balancing diagram.

Remark. One more interpretation of the vector $A[\Gamma]$ can be given. For $t \sim t_0, t \neq t_0$ the immersion is not periodic. Formula (5.4) implies

$${}^{\gamma}F = F + 2(t - t_0)A[\Gamma] + O((t-t_0)^2),$$

which allows us to suggest the following interpretation for $A[\Gamma]$:

$$A[\Gamma] = \frac{1}{2}\frac{\partial}{\partial t}({}^{\gamma}F - F)_{|t=t_0}.$$

Remark. There is one more map, which can be defined on homologies and is standard in the theory of solitons. The right multiplication $\Phi \to \Phi R(\lambda)$ transforms ${}^{\gamma}M$ as follows:

$$^{\gamma}M \to R^{-1}(\lambda)^{\gamma}MR(\lambda).$$

The eigenvalues of ${}^{\gamma}M(\lambda)$, which we denote by ${}^{[\gamma]}m^{\pm 1}(t), |m| = 1$, are preserved by this transformation. To each element $[\gamma] \in H_1(\mathcal{R}, \mathbf{Z})$ there corresponds a function ${}^{[\gamma]}m : S^1 \to S^1$, which depends only on geometry of the surface and not on its disposition.

5.3 Three parallel surfaces

It is a classical result that surfaces parallel to a CMC surface and lying in the normal direction at distances $1/2H$ and $1/H$ are of constant Gaussian and of constant mean curvature respectively. To see how this fact comes in our description of surfaces we note that the formula (5.4) for the immersion is a sum of two vectors wherethe second one is a normal vector. It is thus natural to consider the surface described by the first term.

Proposition 5 *Let F a CMC surface, described by F, N, Φ as in Theorem 5. Then*

$$F^* = -\frac{1}{H}\Phi^{-1}\frac{\partial}{\partial t}\Phi = F + \frac{1}{H}N, \quad N^* = i\Phi^{-1}\sigma_3\Phi = -N \tag{5.11}$$

is a conformal parametrisation of another surface $\mathcal{F}^*$ and its Gauss map. The metric e^{u^*}, the mean curvature H^* and the Hopf differential Q^{t*} of this surface are given by

$$e^{u^*} = 4\frac{Q\bar{Q}}{H^2}e^{-u}, \quad H^* = H, \quad Q^{t*} = Q^t. \tag{5.12}$$

The surface $\mathcal{F}^*$ is dual to $\mathcal{F}$.

Proof. Differentiating (5.11) we get

$$\begin{aligned} F_z^* &= -\frac{1}{H}\Phi^{-1}\frac{\partial U}{\partial t}\Phi = \frac{2iQ}{H}e^{2it}e^{-u/2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi, \\ F_{\bar{z}}^* &= \frac{2i\bar{Q}}{H}e^{-2it}e^{-u/2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi, \end{aligned} \tag{5.13}$$

which shows the conformality of the parametrisation and orthogonality of $F_z^*, F_{\bar{z}}^*$ to N^*. Calculating scalar products via traces (2.12) we easily get (5.12)

$$\begin{aligned} e^{u^*} &= 2 < F_z^*, F_{\bar{z}}^* >= \frac{4Q\bar{Q}}{H^2}e^{-u}\mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \frac{4Q\bar{Q}}{H^2}e^{-u}, \\ H^* &= -2e^{-u^*} < F_z^*, N_{\bar{z}}^* >= \\ &= -\frac{2Qe^{-u^*}}{H}e^{2it}e^{-u/2}\mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, [\sigma_3, V]\right) = H, \\ Q^{t*} &= < F_z^*, N_{\bar{z}}^* >= -\frac{Q}{H}e^{2it}e^{-u/2}\mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, [\sigma_3, U]\right) = Q^t. \end{aligned}$$

Equating Q^t and (4.2)

$$q = H, \quad f = 2Q^t H^{-1}$$

shows that (5.12) coincides with (4.4) and that (5.13) and (4.3) differ only by a sign. This proves the duality of $\mathcal{F}$ and $\mathcal{F}^*$.

The immersion function of the original CMC surface can also be written in a form of a logarithmic derivative with respect to t.

$$F = -\frac{1}{H}\Psi_{[2]}^{-1}\frac{\partial}{\partial t}\Psi_{[2]},$$

where $\Psi_{[2]}$ is a gauge transformed quaternion

$$\Psi_{[2]} = \begin{pmatrix} 1/\sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \Phi, \qquad \sqrt{\lambda} = e^{it}.$$

The coefficients in the linear equations for $\Psi_{[2]}$ are the same as for Φ, but λ is inserted in a different way

$$U_{[2]} = \begin{pmatrix} \frac{u_z}{2} & -Qe^{-u/2} \\ \frac{1}{2}\lambda H e^{u/2} & 0 \end{pmatrix}, V_{[2]} = \begin{pmatrix} 0 & -\frac{1}{2\lambda}He^{u/2} \\ \bar{Q}e^{-u/2} & \frac{u_{\bar{z}}}{2} \end{pmatrix}. \tag{5.14}$$

It is also natural to consider the intermediate case

$$\Psi_{[3]} = \begin{pmatrix} 1/\sqrt[4]{\lambda} & 0 \\ 0 & \sqrt[4]{\lambda} \end{pmatrix} \Phi, \qquad \sqrt[4]{\lambda} = e^{it/2},$$

where the parameter λ enters symmetrically into the $U - V$ pair

$$\begin{aligned} U_{[3]} &= \begin{pmatrix} \frac{u_z}{2} & -\sqrt{\lambda}Qe^{-u/2} \\ \frac{1}{2}\sqrt{\lambda}He^{u/2} & 0 \end{pmatrix}, \\ V_{[3]} &= \begin{pmatrix} 0 & -\frac{1}{2}\frac{1}{\sqrt{\lambda}}He^{u/2} \\ \frac{1}{\sqrt{\lambda}}\bar{Q}e^{-u/2} & \frac{u_{\bar{z}}}{2} \end{pmatrix}. \end{aligned} \tag{5.15}$$

Renaming F, Φ, U, V by

$$F = F_{[1]}, \quad \Phi = \Psi_{[1]}, \quad U = U_{[1]}, \quad V = V_{[1]}$$

we can formulate the following already partially proved

Theorem 6 *The formulas*

$$F_{[i]} = -\frac{1}{H}\Psi_{[i]}^{-1}\frac{\partial}{\partial t}\Psi_{[i]}, \qquad i = 1, 2, 3 \tag{5.16}$$

describe 3 parallel surfaces $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$,

$$F_{[2]} = F_{[1]} + \frac{1}{H}N, \qquad F_{[3]} = F_{[1]} + \frac{1}{2H}N, \tag{5.17}$$

where

$$N = \Psi_{[i]}^{-1}\mathbf{k}\Psi_{[i]} \tag{5.18}$$

is their Gauss map. Surfaces $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are of constant mean curvature* H *and dual* $\mathcal{F}_1^* = \mathcal{F}_2$. *The surface* $\mathcal{F}_3$ *is of constant Gaussian curvature* $K = 4H^2$. *Variation of* t *preserves both principal curvatures of* $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$*; for* $\mathcal{F}_1, \mathcal{F}_2$ *it is an isometry, whereas for* $\mathcal{F}_3$ *the second fundamental form is preserved.*

Proof. All statements about $\mathcal{F}_1$ and $\mathcal{F}_2$ are already proved. The calculation of the fundamental forms of $\mathcal{F}_3$ is quite similar to the calculations in Proposition 5. For the first fundamental form we get

$$A =< F_{[3]z}, F_{[3]z} >= -\frac{Q}{2H}e^{2it}, \quad B =< F_{[3]z}, F_{[3]\bar{z}} >= \frac{1}{2}\frac{Q\bar{Q}}{H^2}e^{-u} + \frac{1}{8}e^{u}.$$

For the determinant of the first fundamental form

$$< dF_{[3],}, dF_{[3],} >= A(dz)^2 + \bar{A}(d\bar{z})^2 + 2Bdzd\bar{z},$$

$$I_{[3]} = \begin{pmatrix} 2B + A + \bar{A} & i(A - \bar{A}) \\ i(A - \bar{A}) & 2B - (A + \bar{A}) \end{pmatrix},$$

this implies

$$\det I_{[3]} = \left(\frac{Q\bar{Q}}{H^2}e^{-u} - \frac{1}{4}e^{u}\right)^2. \tag{5.19}$$

Since

$$< F_{[3]z}, N_z >=< F_{[3]\bar{z}}, N_{\bar{z}} >= 0,$$

the parametrisation of $\mathcal{F}_3$ is conformal with respect to the second fundamental form

$$- < dF_{[3]}, dN >= -2 < F_{[3]z}, N_{\bar{z}} > dzd\bar{z}.$$

Calculating

$$< F_{[3]z}, N_{\bar{z}} >= \frac{Q\bar{Q}}{H}e^{-u} - \frac{1}{4}He^{u},$$

we see that the second fundamental form $II_{[3]}$ does not depend on t and the Gaussian curvature equals

$$K_{[3]} = \det II_{[3]} / \det I_{[3]} = 4H^2.$$

For the mean curvature of $\mathcal{F}_3$ we have

$$H_{[3]} = \frac{1}{2}\mathrm{tr}(II_{[3]} \cdot I_{[3]}^{-1}) = \left(\frac{1}{4}He^u - \frac{Q\bar{Q}}{H}e^{-u}\right)\frac{4B}{\det I_{[3]}} = 2H\frac{\frac{1}{4}H^2e^u + Q\bar{Q}e^{-u}}{\frac{1}{4}H^2e^u - Q\bar{Q}e^{-u}}.$$

Both principal curvatures of $\mathcal{F}_3$ are independent of t.

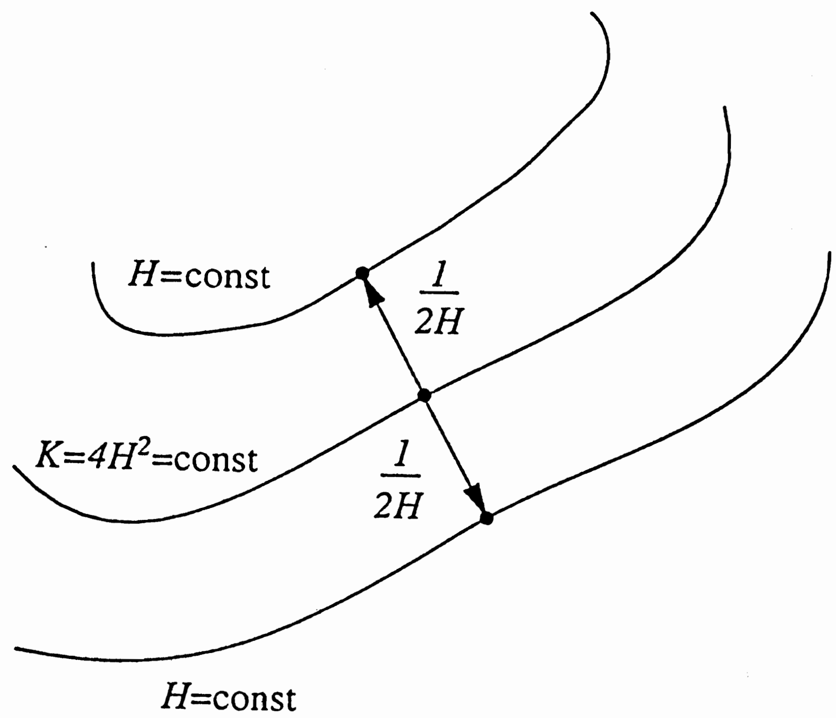


Fig.1 Three parallel surfaces

Remark. Transformations $\mathcal{F}_1 \to \mathcal{F}_2$ and $\mathcal{F}_1 \to \mathcal{F}_3$ can be singular. If $\mathcal{F}_1$ is smooth, then $\mathcal{F}_2 = \mathcal{F}_1^*$ is degenerate ($e^{u^*} = 0$) at images of umbilic points ($Q = 0$) of $\mathcal{F}_1$. The surface $\mathcal{F}_3$ is degenerate at images of flat points of $\mathcal{F}_1$; $K_{[1]} = 0$ implies $u_{z\bar{z}} = 0$ and, finally, $\det I_{[3]} = 0$ in (5.19).

6 Bonnet surfaces

In the present section we study the following

Problem. *To characterize non-trivial families of isometric surfaces having the same principal curvatures.* By a non-trivial family of surfaces we mean surfaces which do not differ by rigid motions. We suppose also that they do not contain umbilics and are sufficiently smooth. This problem was first studied by Bonnet, therefore we call these surfaces Bonnet surfaces. The most detailed results concerning these surfaces are presented in papers by E. Cartan [3] and S. Chern [4], where they were classified and, in particular, it was proved that they are Weingarten surfaces (the principal curvatures are algebraically related).

As we have shown already in Sect. 5.1, the CMC surfaces possess non-trivial isometries. In this section we exclude the CMC case and suppose that H is a non-trivial function on $\mathcal{F}$. To characterize other Bonnet surfaces let us note that since e^u and H are both preserved, only the Hopf differential Q can be varied by the deformations. Since the Gauss equation (2.6) guarantees that $Q\bar{Q}$ is also preserved, the deformation parameter is inserted, in the GW equations via the transformation

$$Q \to \lambda Q, \qquad \bar{Q} \to \frac{1}{\lambda}\bar{Q}. \tag{6.1}$$

In this way we get the $U - V$ system (5.3) (or $U_{[1]}, V_{[1]}$ in notations of Sect. 5.3), where now λ is allowed to depend on z and $\bar{z}$.

The transformation (6.1) does not effect the right hand sides of the Codazzi equations (2.7, 2.8), therefore the left hand sides should also be preserved:

$$(\lambda Q - Q)_{\bar{z}} = (\frac{\bar{Q}}{\lambda} - \bar{Q})_z = 0.$$

These equations can be easily solved:

$$Q = \frac{\bar{a} + \bar{b}}{b\bar{b} - a\bar{a}}, \qquad \lambda = \frac{1 + b/a}{1 + \bar{a}/\bar{b}}, \tag{6.2}$$

where both a and b are holomorphic. The transformation

$$\begin{aligned} a &\to \frac{1}{1-k}(a + kb), \qquad \bar{a} \to \frac{1}{1-k}(\bar{a} + k\bar{b}), \\ b &\to \frac{1}{1-k}(b + ka), \qquad \bar{b} \to \frac{1}{1-k}(\bar{b} + k\bar{a}) \end{aligned}$$

preserves Q and transforms λ

$$\lambda \to \lambda = \left(\frac{a+b}{\bar{a}+\bar{b}}\right)\left(\frac{\bar{b} + k\bar{a}}{a + kb}\right).$$

Now k is an independent parameter (k does not depend on z and $\bar{z}$). Deformations correspond to the case

$$|\Lambda| = 1 \quad \Longleftrightarrow \quad |k| = 1.$$

One can see that Q in (6.2) is of the form (4.2). In terms of a new conformal variable $\tilde{z}$ (we avoid umbilic points):

$$\left(\frac{dz}{d\tilde{z}}\right)^2 = a + b$$

Q becomes real-valued

$$Q(\tilde{z}) = Q(z)\left(\frac{dz}{d\tilde{z}}\right)^2 = \frac{|a+b|^2}{|b|^2 - |a|^2}.$$

Further we omit the tilde and use the old notation z for this new isothermal coordinate. Finally Q and λ are parametrised by one holomorphic function

$$h(z) = \frac{1}{2}\,\frac{b-a}{b+a}$$

of an isothermal coordinate z:

$$\frac{1}{Q} = h + \bar{h}, \tag{6.3}$$

$$\lambda = \frac{1 - 2it\bar{h}}{1 + 2ith}, \qquad it = \frac{k-1}{k+1}, \tag{6.4}$$

where t (or equivalently k) is a deformation parameter. For unimodular $k = e^{2i\alpha}$ we have real valued $t = \tan\alpha$.

Proposition 6 *The Bonnet surfaces are isothermal and $1/Q$ is a harmonic function of an isothermal coordinate.*

The harmonicity of $1/Q$ is necessary but not sufficient. Substitution of (6.3) into the Codazzi equations (2.7, 2.8) yields

$$h_z H_z = \bar{h}_{\bar{z}} H_{\bar{z}}. \tag{6.5}$$

Since $h(z)$ is holomorphic, formula (6.5) shows that H depends on one variable s only:

$$H_s = H_w = H_{\bar{w}}, \tag{6.6}$$

where

$$s = w + \bar{w}, \qquad \frac{dw}{dz} = \frac{1}{h_z}. \tag{6.7}$$

For the metric this implies

$$e^u = 2\frac{Q_{\bar{z}}}{H_z} = -\frac{2(1/Q)_{\bar{z}}}{(1/Q)^2 H_s \frac{dw}{dz}} = -\frac{2|h_z|^2}{(h+\bar{h})^2 H_s}. \tag{6.8}$$

Differentiating the metric twice:

$$u_{z\bar{z}} = 2\frac{|h_z|^2}{(h+\bar{h})^2} - \frac{1}{|h_z|^2}\left(\frac{H_{ss}}{H_s}\right)_s,$$

and substituting all this into the Gauss equation (2.6), we get

$$\left(\left(\frac{H_{ss}}{H_s}\right)_s - H_s\right) R^2 = 2 - \frac{H^2}{H_s}, \tag{6.9}$$

$$R = \frac{h+\bar{h}}{|h_z|^2} \tag{6.10}$$

The vanishing of $2H_s - H^2$ implies the vanishing of $H_s^2 - H_{sss} + H_{ss}^2/H_s$, therefore two cases are possible in principle:

i) $2H_s - H^2 = 0$,

ii) R depends on s only

$$R_w = R_{\bar{w}} \qquad \Longleftrightarrow \qquad h_z R_z = \bar{h}_{\bar{z}} R_{\bar{z}}. \tag{6.11}$$

The case (i) has no geometrical meaning, since formula (6.8) shows that H_s must be negative. Finally we end up with the following equation for $h(z)$, which is equivalent to (6.11)

$$\frac{\partial^2}{\partial z^2} \log(h + \bar{h}) = \frac{\partial^2}{\partial \bar{z}^2} \log(h + \bar{h}).$$

This equation shows that $(z = x + iy)$

$$\frac{\partial^2}{\partial x \partial y} \log(1/Q) = 0,$$

therefore $1/Q$ must be a product of two functions, depending on x and y respectively

$$\frac{1}{Q} = p(x)q(y).$$

The harmonicity condition allows us to find p and q explicitly, since the variables separate

$$\frac{p_{xx}}{p} = -\frac{q_{yy}}{q} = l, \tag{6.12}$$

where l does not depend on x and y. To normalize the solutions of (6.12) we have a reparametrization $z \to az + b + ic \quad (a, b, c \in \mathbf{R})$ and a general scaling of $\mathbf{R}^3$ at our disposal. Also, reconstructing $h(z)$ from p and q one can always add an imaginary constant to $h(z)$. This constant is not important for our calculations, since only the sum $h + \bar{h}$ and the derivative h_z are used, therefore we put this constant equal to zero. After suitable normalization one can easily prove the following

Lemma 4 *All non-constant normalized functions $h(z)$, generated by solutions of (6.12), are listed in the table below:*

h	w	s	$R(s)$
$-iz/2$	$2iz$	$-4y$	$-s$
e^z	$-e^{-z}$	$-(e^{-z} + e^{-\bar{z}})$	$-s$
iz^2	$\frac{1}{2i} \log z$	$\arg z$	$-\frac{1}{2} \sin 2s$
$2 \sinh z$	$\frac{1}{2i} \log \frac{e^z - i}{e^z + i}$	$\frac{1}{2i} \log \left(\left(\frac{e^z - i}{e^z + i} \right) \left(\frac{e^{\bar{z}} - i}{e^{\bar{z}} + i} \right) \right)$	$-\frac{1}{2} \sin 2s$
$2 \cosh z$	$\frac{1}{2} \log \frac{e^z - 1}{e^z + 1}$	$\frac{1}{2} \log \left(\left(\frac{e^z - 1}{e^z + 1} \right) \left(\frac{e^{\bar{z}} - 1}{e^{\bar{z}} + 1} \right) \right)$	$-\frac{1}{2} \sinh 2s$

where the corresponding coordinates w, s and the function $R(s)$ are also indicated.

Actually, not all the cases listed above are geometrically different. Only the cases corresponding to different $R(s)$ differ.

Lemma 5 *Surfaces corresponding to the same $R(s)$ and the same solution $H(s)$ of equation (6.9) belong to the same deformation family, i.e. the metrics and the mean curvatures of these surfaces coincide and the Hopf differentials differ by the transformation (6.1, 6.4).*

Proof. Rewriting characteristics of surfaces in terms of variables w we see, that the statement concerning the mean curvature is trivial since $s = w + \bar{w}$. The metric also depends on s only

$$e^{u(z,\bar{z})} dz d\bar{z} = -\frac{2|h_z|^2}{(h+\bar{h})^2 H_s} dz d\bar{z} = -\frac{2}{R^2 H_s} dw d\bar{w}. \tag{6.13}$$

The calculation for the Hopf differentials is more complicated:

$$\lambda Q(dz)^2 = \frac{1}{R(s)} r(w,\bar{w},t)(dw)^2, \quad r(w,\bar{w},t) = \frac{h_z}{\bar{h}_{\bar{z}}} \frac{(1-2it\bar{h})}{(1+2ith)}.$$

Direct calculation for the cases 1-4 of the table above yields

$$r_1 = \frac{-\frac{i}{2}(1+t_1\bar{z})}{\frac{i}{2}(1+t_1 z)} = -\frac{(1+it_1\bar{w}/2)}{(1-it_1 w/2)}, \tag{6.14}$$

$$r_2 = \frac{e^z(1-2it_2 e^{\bar{z}})}{e^{\bar{z}}(1+2it_2 e^z)} = -\frac{(1-\frac{i}{2t_2}\bar{w})}{(1+\frac{i}{2t_2}w)},$$

$$r_3 = \frac{2iz(1-2t_3\bar{z}^2)}{-2i\bar{z}(1-2t_3 z^2)} = -\frac{(e^{2i\bar{w}} - 2t_3 e^{-2i\bar{w}})}{(e^{-2iw} - 2t_3 e^{2iw})}, \tag{6.15}$$

$$r_4 = \frac{2\cosh z(1-4it_4\sinh\bar{z})}{2\cosh\bar{z}(1+4it_4\sinh z)} = -\frac{\left(e^{2i\bar{w}} - \left(\frac{1+4t_4}{1-4t_4}\right)e^{-2i\bar{w}}\right)}{\left(e^{-2iw} - \left(\frac{1+4t_4}{1-4t_4}\right)e^{2iw}\right)},$$

where t_i denotes the deformation parameter t corresponding to the i-th case of the table. The identification

$$t_1 = -\frac{1}{t_2}, \quad 2t_3 = \frac{1+4t_4}{1-4t_4}$$

proves that the cases 1 and 2 as well as 3 and 4 are isomorphic.

For completeness let us write down the expression for the Hopf differential in case 5 in terms of the variable w:

$$\lambda Q_5(dz)^2 = \frac{1}{R(s)} r_5(w,\bar{w},t_5)(dw)^2,$$

$$r_5 = \frac{2\sinh z(1-4it_5\cosh\bar z)}{2\sinh\bar z(1+4it_5\cosh z)} = -\frac{e^{2\bar w}-e^{-2\bar w}\dfrac{1-4it_5}{1+4it_5}}{e^{2w}-e^{-2w}\dfrac{1+4it_5}{1-4it_5}}. \tag{6.16}$$

So there are 3 cases to consider, which we denote by A, B and C following E. Cartan [3]:

$$\begin{aligned} \mathrm{A}: &\quad R_A(s) = -\frac{1}{2}\sin 2s, \\ \mathrm{B}: &\quad R_B(s) = -\frac{1}{2}\sinh 2s, \\ \mathrm{C}: &\quad R_C(s) = -s. \end{aligned} \tag{6.17}$$

Till now the consideration was a local one and dealt with pieces of surfaces. It turns out that it is possible to combine all these pieces into global surfaces in such a way that the isometries preserving the principal curvatures are described by translations along the surface. For this purpose the coordinate w rather then the starting isothermal coordinate z is more convenient. Both the metric (6.13) and the mean curvature depend on the real part of w only, whereas a change of the imaginary part of w corresponds to the transformation (6.1, 6.4) of Q.

Theorem 7 *The non-trivial families of isometric surfaces having the same principal curvatures are the CMC surfaces and families A, B, C, which can be described as follows. The mean curvature $H(s) = H(w+\bar w)$ is a solution of (6.9) with a negative derivative $H_s < 0$ (here and below the coefficient R should be replaced by R_A, R_B, R_C (6.17) for the cases A, B, C respectively), the metric equals*

$$e^{u(w,\bar w)}dwd\bar w = -\frac{2}{R^2 H_s}dwd\bar w, \tag{6.18}$$

and the Hopf differentials for the families A, B, C are as follows:

$$\begin{aligned} Q_A(w,\bar w)(dw)^2 &= -\frac{2\sin 2\bar w}{\sin 2(w+\bar w)\sin 2w}(dw)^2, \\ Q_B(w,\bar w)(dw)^2 &= \frac{2\sinh 2\bar w}{\sinh 2(w+\bar w)\sinh 2w}(dw)^2, \\ Q_C(w,\bar w)(dw)^2 &= -\frac{\bar w}{(w+\bar w)w}(dw)^2. \end{aligned} \tag{6.19}$$

The isometries preserving the principal curvatures are given by the shift transformations of these surfaces

$$w \longrightarrow w + iT, \qquad T \in \mathbf{R}. \tag{6.20}$$

Proof. Only formulas (6.19) for Q's and the transformation (6.20) need clarification. To get (6.19) we put $t_3 = 1/2,\ t_5 = 0,\ t_1 = \infty$ in (6.15, 6.16, 6.17). Clearly the parameters t_i can be fixed. The deformations described by them are given by the shift (6.20) as well. To identify t_i and T we substitute (6.20) in (6.19):

$$2t_3 = e^{-4T}, \quad \frac{1+4it_5}{1-4it_5} = e^{-4iT}, \quad t_1 = \frac{2}{T}.$$

The curvature lines on the w-domain for the cases A, B, C are presented in Fig.2.

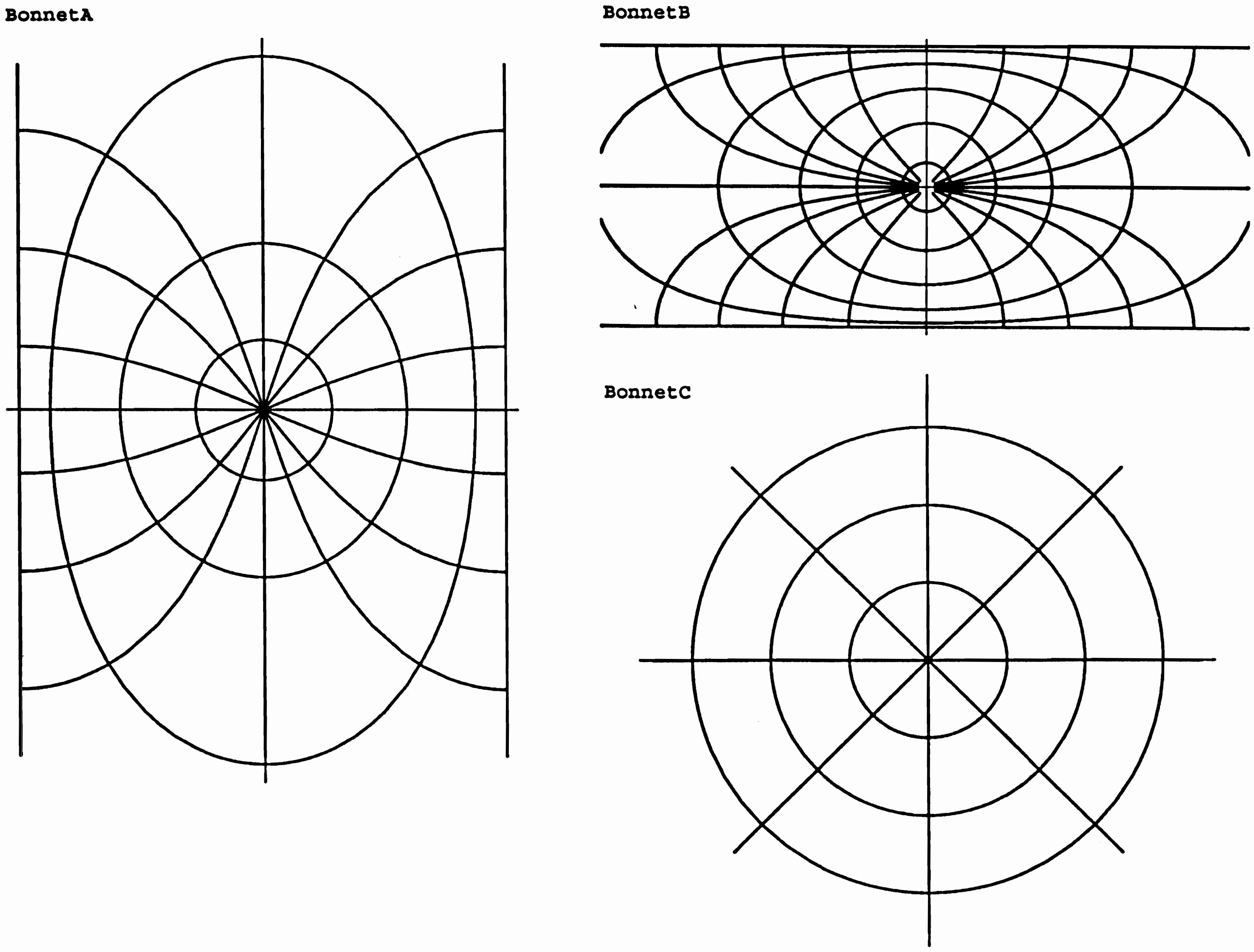


Fig. 2 Curvature lines in the w-domain for the cases A, B, C

A crucial point for the analytical description of the Bonnet surfaces is a solution of equation (6.9). The coefficient $R(s)$ in this equation is of the form

$$R(s) = -\frac{\sinh \alpha s}{\alpha} \tag{6.21}$$

for all the cases A ($\alpha = 2i$), B ($\alpha = 2$) and C ($\alpha \to 0$). Hazzidakis [6] was able to integrate equation (6.9) once[2] with $R(s)$ of the form (6.21):

$$\left[\left(\frac{H_{ss}}{H_s}\right) - 2\sigma_1\right]^2 + 4H\sigma_1 - 2\sigma\frac{H^2}{H_s} - 2H_s = C, \tag{6.22}$$

$$\sigma(s) = \left(\frac{\alpha}{\sinh \alpha s}\right)^2, \qquad \sigma_1(s) = -\alpha \coth \alpha s,$$

where C =const. One should mention that equation (6.22) has extra solutions compared with (6.9).

[2] I am grateful to Prof. Voss, who pointed out to me this result of Hazzidakis

Equation (6.9) arises as a compatibility condition of the system (5.3), starting with which one can derive a Lax representation [3] for equation (6.9) similar to those, used for integration of the Painleve equations [5]. Following the usual terminology of the soliton theory equation (6.9) can be called an *integrable* ordinary differential equation. Nevertheless, a general integration of equation (6.9) seems not to be a simple problem. One can find a solution $H = r/s$, where r is an arbitrary constant, for the case C. The surface described by this solution is a cone.

Surfaces dual to the Bonnet surfaces can be defined, since the latter are isothermal. Propositions 2 and 6 show that the inverse mean curvature $1/H$ of these dual surfaces is harmonic. In the next section we show that the class of surfaces with harmonic $1/H$ is much bigger then the surfaces dual to the Bonnet families and this class can be put into the framework of soliton theory.

7 Surfaces with harmonic inverse mean curvature.

In Sect. 6 we generalized the Lax representation (5.3) (or $U_{[1]}, V_{[1]}$ in the notations of Sect. 5.3) to the case of λ depending on $z, \bar{z}$. Surfaces defined in this way retain the property of the CMC surfaces of possessing isometries preserving the principal curvatures and are described by solutions of an integrable ordinary differential equation. In the present section we suggest another generalization of the CMC surfaces, namely the property $H =$ const itself is generalized. Here we deal with the Lax representation $U_{[2]}, V_{[2]}$ (5.14) with λ depending on $z, \bar{z}$, which gives rise to integrable partial differential equations — the more usual situation for the soliton theory than the one of Sect. 6.

The compatibility condition

$$U_{[2]\bar{z}} - V_{[2]z} + [U_{[2]}, V_{[2]}] = 0 \tag{7.1}$$

for the system (5.14) yields equation (2.6) and

$$Q_{\bar{z}} = \frac{1}{2}\left(\frac{H}{\lambda}\right)_z e^u, \qquad \bar{Q}_z = \frac{1}{2}(\lambda H)_{\bar{z}} e^u. \tag{7.2}$$

We suppose that e^u, H and Q correspond to some surface and therefore the GC equations (2.6-2.8) are also satisfied. Subtraction of (7.2) from (2.7, 2.8) gives the equations

$$\left(H\left(\frac{1}{\lambda} - 1\right)\right)_z = 0, \qquad (H(\lambda - 1))_{\bar{z}} = 0,$$

which can be easily solved:

$$H = \frac{1}{h + \bar{h}}, \qquad \lambda = -\frac{\bar{h}}{h}, \tag{7.3}$$

[3]One can reproduce the integration step (6.22) using this Lax representation

where $h(z)$ is holomorphic.

The transformation

$$h \to h + \frac{1}{2it}, \qquad t \in \mathbf{R}$$

preserves H and transforms λ

$$\lambda \to \lambda = \frac{1 - 2i\bar{h}t}{1 + 2iht}, \tag{7.4}$$

where t is an independent of $z, \bar{z}$ parameter.

The form (7.3) of the mean curvature is equivalent to $1/H$ being a harmonic function:

$$\partial_z \partial_{\bar{z}} \left(\frac{1}{H}\right) = 0.$$

Here the complex structure is determined by the immersion. The parameter t in (7.4) can be considered as a deformation parameter.

Theorem 8 *Let $\mathcal{F}$ be a conformally parametrised $F : \mathcal{R} \to \mathbf{R}^3$ surface with metric e^u, Hopf differential Q and harmonic inverse mean curvature*

$$\frac{1}{H} = h(z) + \overline{h(z)}, \tag{7.5}$$

where $h(z)$ is holomorphic. Then the compatibility condition (7.1, 5.14) with λ of the form (7.4) is satisfied for all t. There is a one-parametric deformation family of surfaces $\mathcal{F}^t$, $t \in \mathbf{R}$ such that:

i) $\mathcal{F} = \mathcal{F}^{t=0}$;

ii) The metrics $e^{u^t} dz d\bar{z}$ of $\mathcal{F}^t$ are conformally equivalent

$$e^{u^t} dz d\bar{z} = \frac{e^u}{(1 + 2iht)^2 (1 - 2i\bar{h}t)^2} dz d\bar{z};$$

iii) The inverse mean curvature $1/H^t$ of $\mathcal{F}^t$ is harmonic

$$\frac{1}{H^t} = h^t + \overline{h^t}, \tag{7.6}$$

where h^t is given by

$$h^t = \frac{h}{1 + 2iht}; \tag{7.7}$$

iv) The ratio of the principal curvatures k_1/k_2 is preserved by the deformation;

v) Let $\Psi(z,\bar z,t) \in H_*$ be a quaternion, normalized so that $\det \Psi$ *is independent of t, which solves the system*

$$\Psi_z = U_{[2]}\Psi, \quad \Psi_{\bar z} = V_{[2]}\Psi,$$

where λ is of the form (7.4). Then the immersion function

$$F = \Psi^{-1}\frac{\partial}{\partial t}\Psi \tag{7.8}$$

describes a conformal parametrisation $F : \mathcal{R} \to \mathbf{R}^3$ of $\mathcal{F}^t$. Its Gauss map is given by

$$N = \Psi^{-1}\mathbf{k}\Psi, \qquad \mathbf{k} = -i\sigma_3. \tag{7.9}$$

Proof. The calculations are quite similar to those of Sect. 5. F and N given by formulas (7.8, 7.9) lie in Im$\mathbf{H}$. Differentiating F

$$F_z = \Psi^{-1}\frac{\partial U_{[2]}}{\partial t}\Psi, \qquad F_{\bar z} = \Psi^{-1}\frac{\partial V_{[2]}}{\partial t}\Psi,$$

we see, that the parametrisation is conformal and (2.1, 2.2) are satisfied. For the metric, the mean curvature and the Hopf differential of $\mathcal{F}^t$ determined by (7.8, 7.9) we get

$$\begin{aligned}
e^{u^t} &= 2 < F_z, F_{\bar z} >= -\mathrm{tr}\left(\frac{\partial U_{[2]}}{\partial t}\frac{\partial V_{[2]}}{\partial t}\right) = \\
&= \frac{1}{4}H^2 e^u \frac{\partial\lambda}{\partial t}\frac{\partial\lambda^{-1}}{\partial t}\mathrm{tr}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \\
&= \frac{e^u}{(1-2i\bar h t)^2(1+2iht)^2}, \\
-\frac{1}{2}e^{u^t}H^t &= < F_z, N_{\bar z} >= \frac{i}{2}\mathrm{tr}\left(\frac{\partial U_{[2]}}{\partial t}[\sigma_3, V_{[2]}]\right) = \\
&= -\frac{1}{2}e^u H\frac{1}{(1-2i\bar h t)(1+2iht)}, \\
Q^t &= -< F_z, N_z >= -\frac{i}{2}\mathrm{tr}\left(\frac{\partial U_{[2]}}{\partial t}[\sigma_3, V_{[2]}]\right) = \frac{Q}{(1+2iht)^2}.
\end{aligned}$$

The transformation of the mean curvature

$$H^t = H(1+2iht)(1-2i\bar h t) \tag{7.10}$$

can be rewritten in the form (7.6, 7.7). The Gaussian curvature $K = -2u_{z\bar z}e^{-u}$ transforms as follows:

$$K^t = K(1+2iht)^2(1-2i\bar h t)^2,$$

which, combined with (7.10), shows the conservation of $(H^t)^2/K^t$, or equivalently, proves (iv).

Remark. For a given H the holomorphic function $h(z)$ is determined uniquely up to an imaginary constant

$$h(z) \to h(z) + ic, \qquad c \in \mathbf{R}. \tag{7.11}$$

The family $\mathcal{F}^t$ is preserved by the transformation (7.10), which is equivalent to a redefinition of the parameter t only. Combining (7.7) with the transformations (7.11) for h and h^t we get a general Möbius transformation SL(2, $\mathbf{R}$) for ia

$$ih^t = \frac{\alpha(ih) + \beta}{\gamma(ih) + \delta}, \qquad \gamma = 2t, \qquad \alpha, \beta, \gamma, \delta \in \mathbf{R}, \qquad \alpha\delta - \beta\gamma = 1.$$

Remark. The Lax representation $U_{[2]}, V_{[2]}$ with λ of the form (7.4) allows us to apply the soliton theory to construct explicit examples of surfaces with the harmonic inverse mean curvature. In particular the Bäcklund transformation can be derived.

8 Surfaces with negative Gaussian curvature

8.1 Quaternionic description

The calculations of this section are similar to those of Sect. 2.2.

Let us consider a surface $\mathcal{F}$ with negative Gaussian curvature. For each regular point of $\mathcal{F}$ there are two directions in which the curvature vanishes. They are called the *asymptotic directions*. We use the asymptotic line parametrisation of $\mathcal{F}$

$$F : (x, y) \in \mathbf{R}^2 \to \mathbf{R}^3.$$

For this parametrisation the vectors F_x, F_y, F_{xx}, F_{yy} are orthogonal to the normal vector N

$$F_x, F_y, F_{xx}, F_{yy} \perp N.$$

The fundamental forms are as follows:

$$\begin{aligned} I &= \; < dF, dF >= A^2(dx)^2 + 2AB\cos\phi\, dxdy + B^2(dy)^2 \\ II &= \; - < dF, dN >= 2 < F_{xy}, N > dxdy, \end{aligned}$$

where ϕ is the angle between the asymptotic lines and

$$A = |F_x|, \; B = |F_y|. \tag{8.1}$$

We consider weakly regular surfaces, i.e. suppose that A, B do not vanish. Let us suppose also that the Gaussian curvature is strictly negative

$$K = -\frac{1}{\rho^2}, \qquad \rho > 0. \tag{8.2}$$

For the second fundamental form this implies

$$-\frac{1}{\rho^2} = \frac{\det II}{\det I} = -\frac{< F_{xy}, N >^2}{A^2 B^2 \sin^2\phi}$$

and finally, choosing a suitable direction of the normal vector N, we get

$$II = 2AB\frac{\sin\phi}{\rho}\, dx\, dy.$$

Let $\Phi \in SU(2)$ be a unitary quaternion, which transforms the basis

$$A(\mathbf{i}\cos\frac{\phi}{2} + \mathbf{j}\sin\frac{\phi}{2}), \qquad B(\mathbf{i}\cos\frac{\phi}{2} - \mathbf{j}\sin\frac{\phi}{2}), \qquad \mathbf{k}$$

to the basis F_x, F_y, N:

$$\begin{aligned} F_x &= -iA\Phi^{-1}\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}\Phi, \\ F_y &= -iB\Phi^{-1}\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}\Phi, \\ N &= -i\Phi^{-1}\sigma_3\Phi. \end{aligned} \tag{8.3}$$

Then the first fundamental form is as above.

To derive linear differential equations for Φ, as in Sect. 2.2 let us introduce the matrices

$$U = \Phi_x\Phi^{-1}, \qquad V = \Phi_y\Phi^{-1} \tag{8.4}$$

lying in the imaginary quaternions $su(2)$. Orthogonality: $F_{xx} \perp N, F_{yy} \perp N$, i.e.

$$\begin{aligned} 0 &= < F_x, N_x >= \frac{A}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}[\sigma_3, U]\right), \\ 0 &= < F_y, N_y >= \frac{B}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}[\sigma_3, V]\right) \end{aligned}$$

shows that the off-diagonal parts of U and V are proportional to $\Phi F_x\Phi^{-1}$ and $\Phi F_y\Phi^{-1}$ respectively. One can calculate the coefficients of proportionality using the identities

$$< F_x, N_y >=< F_y, N_x >= -\frac{AB\sin\phi}{\rho}.$$

For U and V this implies

$$\begin{aligned} U &= -iu_3\sigma_3 - \frac{ia}{2}\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}, \\ V &= -iv_3\sigma_3 + \frac{ib}{2}\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}, \end{aligned}$$

where

$$a = \frac{A}{\rho}, \qquad b = \frac{B}{\rho}, \tag{8.5}$$

and u_3, v_3 are some real coefficients. To calculate u_3, v_3 one should use the compatibility condition $F_{xy} = F_{yx}$ of (8.3):

$$\begin{aligned}
- \quad & iA_y \begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix} + \frac{A}{2}\phi_y \begin{pmatrix} 0 & -e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix} - \\
- \quad & iA \left[\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}, V \right] = \\
= \quad & -iB_x \begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix} + \frac{B}{2}\phi_x \begin{pmatrix} 0 & e^{i\phi/2} \\ -e^{-i\phi/2} & 0 \end{pmatrix} - \\
- \quad & iB \left[\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}, U \right],
\end{aligned}$$

which yields

$$\begin{aligned}
u_3 &= u - \frac{\phi_x}{4}, \qquad u = \frac{B_x \cos\phi - A_y}{2B \sin\phi}, \\
v_3 &= v + \frac{\phi_y}{4}, \qquad v = \frac{B_x - A_y \cos\phi}{2A \sin\phi}.
\end{aligned} \tag{8.6}$$

The diagonal part of the compatibility condition of (8.4)

$$U_y - V_x + [U, V] = 0 \tag{8.7}$$

yields the Gauss equation

$$\phi_{xy} + 2v_x - 2u_y - ab \sin\phi = 0, \tag{8.8}$$

whereas the off-diagonal part gives rise to other formulas for u and v:

$$u = \frac{a_y + b_x \cos\phi}{2b \sin\phi}, \qquad v = -\frac{b_x + a_y \cos\phi}{2a \sin\phi} \tag{8.9}$$

Comparing (8.6) and (8.9) we get the Codazzi equations

$$\begin{aligned}
a_y + \frac{\rho_y}{2\rho} a - \frac{\rho_x}{2\rho} b \cos\phi &= 0, \\
b_x + \frac{\rho_x}{2\rho} b - \frac{\rho_y}{2\rho} a \cos\phi &= 0
\end{aligned} \tag{8.10}$$

and one more representation for u, v

$$u = -\frac{\rho_y a}{4\rho b} \sin\phi, \qquad v = \frac{\rho_x b}{4\rho a} \sin\phi. \tag{8.11}$$

The following theorem is proved:

Theorem 9 *Using the isomorphism (2.11), the moving frame F_x, F_y, N of the surface with negative Gaussian curvature $K = -1/\rho^2$ in the asymptotic line parametrisation is described by formulas (8.3), where $\Phi \in SU(2)$ satisfies equations (8.4) with U, V of the form*

$$\begin{aligned} U &= -i(u - \frac{\phi_x}{4})\sigma_3 - \frac{ia}{2}\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}, \\ V &= -i(v + \frac{\phi_y}{4})\sigma_3 + \frac{ib}{2}\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}. \end{aligned} \tag{8.12}$$

The coefficients u, v here can be written in one of the equivalent forms (8.6, 8.9, 8.11). Equations (8.8, 8.10) are the Gauss-Codazzi equations.

8.2 Surfaces with constant negative Gaussian curvature

For the constant curvature case $\rho = \text{const}$ the GC equations (8.8, 8.10) simplify a lot

$$\phi_{xy} - ab\sin\phi = 0, \qquad a_y = b_x = 0$$

(which implies also $u = v = 0$ in (8.2)). These equations are invariant with respect to the Lorentz transformation

$$a \to \lambda a, \qquad b \to b/\lambda, \qquad \lambda \in \mathbf{R}, \tag{8.13}$$

which plays the role of the transformation (5.2) in the CMC case.

Treating

$$\lambda = e^t, \qquad t \in \mathbf{R}$$

as a deformation parameter we get the following

Theorem 10 *Every surface with constant negative Gaussian curvature has a one-parameter family of deformations preserving the second fundamental form*

$$II = 2\rho ab\sin\phi dxdy,$$

the Gaussian curvature and the angle ϕ between the asymptotic lines. The deformation is described by the transformation (8.13).

The quaternion Φ solving the system (8.4) with U, V of the form (8.12), where one should put $u = v = 0$, describes the moving frame of the surface $\mathcal{F}^t$ with $|F_x| = \rho a e^t, |F_y| = \rho b e^{-t}$. As it was shown by A. Sym [15], knowing the family $\Phi(x, y, \lambda)$ for all $\lambda = e^t$ allows us to integrate the formulas for the moving frame explicitly replacing the integration with respect to x, y by a differentiation with respect to t.

Theorem 11 *Let $\Phi(x, y, \lambda = e^t) \in SU(2)$ be a solution of the system (8.4) with*

$$U = \begin{pmatrix} \frac{i\phi_x}{4} & -\frac{ia}{2}\lambda e^{-i\phi/2} \\ -\frac{ia}{2}\lambda e^{i\phi/2} & -\frac{i\phi_x}{4} \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{i\phi_y}{4} & \frac{ib}{2\lambda}e^{i\phi/2} \\ \frac{ib}{2\lambda}e^{-i\phi/2} & \frac{i\phi_y}{4} \end{pmatrix}. \tag{8.14}$$

Then F and N defined by the formulas

$$F = 2\rho\Phi^{-1}\frac{\partial}{\partial t}\Phi, \qquad N = -i\Phi^{-1}\sigma_3\Phi \tag{8.15}$$

describe a constant negative Gaussian curvature surface with the fundamental forms

$$\begin{aligned} I &= \rho^2(\lambda^2 a^2 dx^2 + 2ab\cos\phi dxdy + \lambda^{-2}b^2(dy)^2), \\ II &= 2\rho ab\sin\phi\, dx\, dy. \end{aligned} \tag{8.16}$$

A surface with constant negative Gaussian curvature in asymptotic line parametrisation with the fundamental forms (8.16) is described by formula (8.15), where Φ is as above.

Proof. Both F and N lie in Im**H**. The system (8.14) coincides with (8.12) for $\mathcal{F}^t$. Differentiating (8.15), we get

$$F_x = 2\rho\Phi^{-1}\frac{\partial U}{\partial t}\Phi, \qquad F_y = 2\rho\Phi^{-1}\frac{\partial V}{\partial t}\Phi,$$

which coincides with (8.3) for $\mathcal{F}^t$. The proof of the second part of the theorem is identical to the proof of the corresponding part of Theorem 5.

For the weakly regular surfaces, i.e. the surfaces with $A \neq 0, B \neq 0$ for all x, y, the conformal change of coordinates $x \to \tilde{x}(x), y \to \tilde{y}(y)$ reparametrises the surface so, that the asymptotic lines are parametrised by arc-lengths (generally different for x and y directions)

$$A = |F_x| = \text{const}, \quad B = |F_y| = \text{const}.$$

In this parametrisation (which is called a *Chebyshev net* if $A = B$) the Gauss equation and the system (8.14) become the sine-Gordon equation with the standard Lax representation. Lastly we mention also a well known fact, which can easily be checked.

Proposition 7 *The Gauss map $N : \mathbf{R^2} \to S^2$ of the surface with $K = -1/\rho^2 =$ const < 0 is Lorentz-harmonic, i.e.*

$$N_{xy} = qN, \quad q : \mathbf{R}^2 \to \mathbf{R}.$$

It forms in S^2 the same kind of Chebyshev net as the immersion function does in $\mathbf{R}^3$

$$|N_x| = a, \quad |N_y| = b.$$

8.3 Bianchi surfaces

To generalize the Lax representation (8.14) to the case of λ depending on x, y let us first rewrite the GC equations of a surface in asymptotic line parametrisation in the form

$$\begin{aligned} \phi_{xy} + 2v_x - 2u_y - ab\sin\phi &= 0, \\ a_y \sin\phi - 2av\cos\phi - 2bu &= 0, \\ b_x \sin\phi + 2bu\cos\phi + 2av &= 0, \end{aligned} \tag{8.17}$$

as they come in the compatibility equation $U_y - V_x + [U, V] = 0$ with the $U - V$ pair (8.12).

If λ is inserted into the $U - V$ pair (8.12) in the same way as in Sect. 8.2

$$\begin{aligned} U^{Bi}(\lambda) &= -i(u - \frac{\phi_x}{4})\sigma_3 - \frac{ia\lambda}{2}\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}, \\ V^{Bi}(\lambda) &= -i(v + \frac{\phi_y}{4})\sigma_3 + \frac{ib}{2\lambda}\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}, \end{aligned} \tag{8.18}$$

the Gauss equation is preserved, whereas the two last equations in (8.17) transform as follows:

$$\begin{aligned} \lambda a_y \sin\phi + \lambda_y a \sin\phi - 2\lambda av\cos\phi - 2b\lambda^{-1}u &= 0, \\ \lambda^{-1}b_x \sin\phi + (\lambda^{-1})_x b\sin\phi + 2(\lambda^{-1})bu\cos\phi + 2\lambda av &= 0. \end{aligned} \tag{8.19}$$

We suppose that ρ, ϕ, a, b correspond to the some surface, therefore the GC equations (8.17) are also satisfied. Subtracting (8.17) from (8.19) and using formulas (8.11) for u, v, we get the equations

$$\lambda_y = (\lambda - \frac{1}{\lambda})\frac{\rho_y}{2\rho}, \qquad (\frac{1}{\lambda})_x = (\frac{1}{\lambda} - \lambda)\frac{\rho_x}{2\rho},$$

which can be easily solved:

$$\rho = f(x) + g(y), \qquad \lambda = \sqrt{-\frac{g(y)}{f(x)}}, \tag{8.20}$$

where $f(x)$ and $g(y)$ are two arbitrary functions.

The transformation

$$f \to f - \frac{1}{2t}, \qquad g \to g + \frac{1}{2t}, \qquad t \in \mathbf{R}$$

preserves ρ and transforms λ

$$\lambda \longrightarrow \lambda = \sqrt{\frac{1 + 2tg}{1 - 2tf}}, \tag{8.21}$$

where t is an independent of x, y parameters.

The form (8.20) of the Gaussian curvature $K = -1/\rho^2$ is equivalent to $1/\sqrt{-K}$ being Lorentz-harmonic

$$\partial_y \partial_x \left(\frac{1}{\sqrt{-K}} \right) = 0. \tag{8.22}$$

Surfaces with a curvature of this form were investigated by Bianchi [1], therefore they are called Bianchi surfaces, although may be it would be fairer to call them *Peterson surfaces* after Peterson, who probably studied this problem first (see a comment of Stäckel in [14] [4]) .

Theorem 12 *Let $\mathcal{F}$ be a surface with negative Gaussian curvature $K = -1/\rho^2$, $F(x,t)$ its asymptotic line parametrisation and*

$$\begin{aligned} I &= \langle dF, dF \rangle = \rho^2(a^2(dx)^2 + 2ab\cos\phi\, dx\, dy + b^2(dy)^2), \\ II &= -\langle dF, dN \rangle = 2\rho ab\sin\phi\, dx\, dy \end{aligned}$$

its fundamental forms. If $\rho = 1/\sqrt{-K}$ is Lorentz-harmonic (8.22)

$$\rho(x,y) = f(x) + g(y),$$

then the compatibility condition $U_y^{Bi} - V_x^{Bi} + [U^{Bi}, V^{Bi}] = 0$ with λ of the form (8.21) is satisfied for all t. There is a one-parametric deformation family of surfaces $\mathcal{F}^t$, $t \in \mathbf{R}$ such that:

i) $\mathcal{F} = \mathcal{F}^{t=0}$;

ii) the fundamental forms of $\mathcal{F}^t$ are as follows:

$$\begin{aligned} I^t &= \langle dF^t, dF^t \rangle = \\ &= (\rho^t)^2((a^t)^2(dx)^2 + 2a^t b^t\cos\phi\, dx\, dy + (b^t)^2(dy)^2), \\ II^t &= -\langle dF^t, dN^t \rangle = 2\rho^t a^t b^t \sin\phi\, dx\, dy, \\ a^t &= \lambda a, \qquad b^t = \frac{1}{\lambda} b, \qquad \rho^t = \frac{\rho}{(1+2tg)(1-2tf)}. \end{aligned}$$

In particular, the angle ϕ between the asymptotic lines is preserved, $\rho^t = 1/\sqrt{-K^t}$ is Lorentz-harmonic $\rho^t(x,y) = f^t(x) + g^t(y)$, where f^t and g^t are given by the formulas

$$f^t = \frac{f}{1-2tf}, \qquad g^t = \frac{g}{1+2tg};$$

iii) let $\Psi(x,y,t) \in SU(2)$ be a solution of the system

$$\Psi_x = U^{Bi}\Psi, \qquad \Psi_y = V^{Bi}\Psi,$$

[4] I am grateful to Prof. Voss for showing me this paper

where λ is of the form (8.21). Then the immersion function

$$F = 2\Psi^{-1}\frac{\partial}{\partial t}\Psi \tag{8.23}$$

describes an asymptotic line parametrisation of $\mathcal{F}^t$. Its Gauss map is given by

$$N = \Psi^{-1}\mathbf{k}\Psi, \qquad \mathbf{k} = -i\sigma_3.$$

We omit the proof of this theorem, which is given by a direct calculation analogous to ones used to prove Theorems 8 and 11.

The Lax representation (8.18, 8.21) for the Bianchi surfaces was obtained in [11]. Using this representation the Bäcklund-Darboux transformation [11] and finite gap solutions [8] were constructed.

9 Surfaces with positive Gaussian curvature

At the end of the Sect. 5 the parametrisation conformal with respect to the second fundamental form appeared, when the surface $\mathcal{F}_3$ of constant positive Gaussian curvature was described. This parametrisation can be used to describe general surfaces with $K > 0$ since the second fundamental form of these surfaces is positive. All calculations in this case are quite parallel to the calculations of Sect. 8. Here we present the final results, which can be obtained by a simple replacement of the symbols of Sect. 8:

$$\begin{array}{lll} \rho \longrightarrow -i\sigma, & x \longrightarrow z, & y \longrightarrow \bar{z}, \\ -ia \longrightarrow c, & -ib \longrightarrow \bar{c}, & i\phi \longrightarrow \psi. \end{array}$$

The Gaussian curvature is positive

$$K = \frac{1}{\sigma^2} \qquad \sigma > 0,$$

and the fundamental forms are follows:

$$\begin{aligned} I &= \; < dF, dF >= \sigma^2(c^2(dz)^2 + 2c\bar{c}\cosh\psi\, dz\, d\bar{z} + \bar{c}^2(d\bar{z})^2), \\ II &= \; - < dF, dN >= 2\sigma c\bar{c}\sinh\psi\, dz\, d\bar{z}. \end{aligned} \tag{9.1}$$

The mean curvature in this parametrisation is

$$H = \frac{1}{\sigma}\coth\psi.$$

Theorem 13 *The moving frame $F_z, F_{\bar z}, N$ of the surface with positive Gaussian curvature $K = 1/\sigma^2$ in the parametrisation conformal with respect to the second fundamental form is described by the following formulas*

$$
\begin{aligned}
F_z &= -i\sigma c\Phi^{-1}\begin{pmatrix} 0 & e^{-\psi/2} \\ e^{\psi/2} & 0 \end{pmatrix}\Phi, \\
F_{\bar z} &= -i\sigma \bar c\Phi^{-1}\begin{pmatrix} 0 & e^{\psi/2} \\ e^{-\psi/2} & 0 \end{pmatrix}\Phi, \\
N &= -i\Phi^{-1}\sigma_3\Phi,
\end{aligned}
$$

where $\Phi \in SU(2)$ satisfies the GW equations

$$\Phi_z = U\Phi, \qquad \Phi_{\bar z} = V\Phi \tag{9.2}$$

with U, V of the form

$$
\begin{aligned}
U &= (\frac{\psi_z}{4} + \mathsf{u})\sigma_3 + \frac{c}{2}\begin{pmatrix} 0 & e^{-i\psi/2} \\ e^{i\psi/2} & 0 \end{pmatrix}, \\
V &= (-\frac{\psi_{\bar z}}{4} + \mathsf{v})\sigma_3 - \frac{\bar c}{2}\begin{pmatrix} 0 & e^{\psi/2} \\ e^{-\psi/2} & 0 \end{pmatrix}.
\end{aligned}
$$

The coefficients u, v here are given by

$$\mathsf{u} = \frac{\sigma_z c}{4\sigma\bar c}\sinh\psi, \qquad \mathsf{v} = -\frac{\sigma_z \bar c}{4\sigma c}\sinh\psi.$$

The Gauss-Codazzi equations are as follows:

$$
\begin{aligned}
\psi_{z\bar z} + 2\mathsf{u}_{\bar z} - 2\mathsf{v}_z + c\bar c\sinh\psi &= 0, \\
c_{\bar z} + \frac{\sigma_{\bar z}}{2\sigma}c - \frac{\sigma_z}{2\sigma}\bar c\cosh\psi &= 0, \\
\bar c_z + \frac{\sigma_z}{2\sigma}\bar c - \frac{\sigma_{\bar z}}{2\sigma}c\cosh\psi &= 0.
\end{aligned}
$$

The following theorem is just a reformulation of the statements of Theorem 6 concerning the surface $\mathcal{F}_3$

Theorem 14 *Every surface with constant positive Gaussian curvature $K = 1/\sigma^2$ has a one-parameter deformation family preserving the Gaussian and the mean curvatures and the second fundamental form. This deformation is described by the transformation*

$$c \longrightarrow \lambda c, \qquad \bar c \longrightarrow \frac{1}{\lambda}\bar c, \qquad \lambda = e^{it}, \qquad t \in \mathbf{R}.$$

Let $\Phi(z, \bar z, \lambda = e^{it}) \in SU(2)$ be a solution of the system (9.2) with

$$U = \begin{pmatrix} \frac{\psi_z}{4} & \lambda\frac{c}{2}e^{-\psi/2} \\ \lambda\frac{c}{2}e^{\psi/2} & -\frac{\psi_z}{4} \end{pmatrix}, \qquad V = \begin{pmatrix} -\frac{\psi_{\bar z}}{4} & -\frac{\bar c}{2\lambda}e^{\psi/2} \\ -\frac{\bar c}{2\lambda}e^{-\psi/2} & \frac{\psi_{\bar z}}{4} \end{pmatrix}.$$

Then F and N, defined by the formulas

$$F = -2\sigma\Phi^{-1}\frac{\partial}{\partial t}\Phi, \qquad N = -i\Phi^{-1}\sigma_3\Phi,$$

describe a surface with constant positive Gaussian curvature and the fundamental forms

$$\begin{aligned} I &= \sigma^2(\lambda^2 c^2 (dz)^2 + 2c\bar{c}\cosh\psi\, dz\, d\bar{z} + \lambda^{-2}c^{-2}(d\bar{z})^2), \\ II &= 2\sigma c\bar{c}\sinh\psi\, dz\, d\bar{z}. \end{aligned}$$

The surfaces of positive curvature analogous to the Bianchi surfaces are described by the following

Theorem 15 *Let $\mathcal{F}$ be a surface with positive Gaussian curvature $K = 1/\sigma^2$ and (9.1) be its fundamental forms. If $\sigma = 1/\sqrt{K}$ is harmonic*

$$\sigma(z,\bar{z}) = h(z) + h(\bar{z}),$$

where $h(z)$ is holomorphic, then the compatibility condition

$$U^{\mathrm{Bi}}_{\bar{z}} - V^{\mathrm{Bi}}_{z} + [U^{\mathrm{Bi}}, V^{\mathrm{Bi}}] = 0$$

with the matrices

$$\begin{aligned} U^{\mathrm{Bi}} &= (\frac{\psi_z}{4} + \mathsf{u})\sigma_3 + \frac{c\lambda}{2}\begin{pmatrix} 0 & e^{-\psi/2} \\ e^{\psi/2} & 0 \end{pmatrix}, \\ V^{\mathrm{Bi}} &= (\frac{-\psi_{\bar{z}}}{4} + \mathsf{v})\sigma_3 - \frac{\bar{c}}{2\lambda}\begin{pmatrix} 0 & e^{\psi/2} \\ e^{-\psi/2} & 0 \end{pmatrix}, \end{aligned}$$

and λ of the form

$$\lambda = \sqrt{\frac{1-2it\bar{h}}{1+2ith}}$$

is satisfied for all t. There is a one-parameter deformation family of surfaces $\mathcal{F}^t, t \in$ **R**, *such that:*

i) $\mathcal{F} = \mathcal{F}^{t=0}$;

ii) the fundamental forms of $\mathcal{F}^t$ are (9.1), where one should replace $c, \bar{c}, \sigma$ by

$$c \to c^t = \lambda c, \quad \bar{c} \to \bar{c}^t = \lambda^{-1}c, \quad \sigma \to \sigma^t = \frac{\sigma}{(1+2iht)(1-2i\bar{h}t)}.$$

In particular, $\sigma^t = 1/\sqrt{K^t}$ remains harmonic $\sigma^t(z,\bar{z}) = h^t(z) + \bar{h^t}(z)$, where $h^t(z)$ is given by

$$h^t = \frac{h}{1+2iht}.$$

The ratio of the principal curvatures is preserved by the deformation;

iii) let $\Psi(z,\bar{z},t) \in SU(2)$ be a solution of the system

$$\Psi_z = U^{\mathrm{Bi}}(t)\Psi, \qquad \Psi_{\bar{z}} = U^{\mathrm{Bi}}(t)\Psi.$$

Then the immersion function

$$F = 2\Psi^{-1}\frac{\partial}{\partial t}\Psi$$

describes the parametrisation of $\mathcal{F}^t$ conformal with respect to the second fundamental form. Its Gauss map is given by

$$N = \Psi^{-1}\mathbf{k}\Psi.$$

The formulas for the Bianchi surfaces of positive curvature are similar to the formulas of Sect. 7 for the surfaces with harmonic inverse mean curvature. It would be interesting to find a geometrical relation between these surfaces.

10 Appendix. Spinors and spin structures on Riemann surfaces

Let $\mathcal{R}$ be a Riemann surface (compact or not) and z a local coordinate on $\mathcal{R}$.

Definition 5 *Differentials $f(z,\bar{z})\sqrt{dz}$ and $f(z,\bar{z})\sqrt{d\bar{z}}$ of order $1/2$ are called spinors on $\mathcal{R}$ if $f(z,\bar{z})$ have no local monodromy (no branch points) on $\mathcal{R}$.*

These two types of spinors can be considered in exactly the same way. ¿From now on we concentrate on the type $f\sqrt{dz}$, which means that under conformal changes $z \to w(z)$ of the local parameter $f(z,\bar{z})$ transforms as follows:

$$f(z,\bar{z}) = f(w,\bar{w})\sqrt{\frac{dw}{dz}}. \tag{10.1}$$

Let us fix some spinor and investigate structures induced by it. Locally $f(z,\bar{z})$ is defined up to a sign (but it has no branch points !). We show that it supplies closed contours on $\mathcal{R}$ with nontrivial $\mathbf{Z}_2$ numbers, allowing us to define a spin structure — a quadratic form $H_1(\mathcal{R},\mathbf{Z}_2) \to \mathbf{Z}_2$

Let $\gamma \in \mathcal{R}$ be an embedding of S^1 in $\mathcal{R}$, i.e. a smooth closed contour on $\mathcal{R}$ without self-intersections (we shall also call such contours *simple* contours). In a small neighborhood $A(\gamma)$ of γ, which is topologically an annulus, we introduce a complex coordinate z. Then it is possible to control the global behaviour of the spinor along γ. The function $f(z,\bar{z})$ is not necessarily single-valued on $A(\gamma)$: it can have monodromy +1 or -1 around γ. The transformation law (10.1) shows that if z

and w are two different coordinates on $A(\gamma)$ then the monodromies of $f(z,\bar{z})$ and $f(w,\bar{w})$ along γ are different if $\sqrt{dw/dz}$ changes sign on passing around γ. To avoid this ambiguity we ask for z to be a map from $A(\gamma)$ to an annular domain on a complex plane. Then the number $p(\gamma) \in \mathbf{Z}_2$ is defined by the monodromy of the spinor, written in terms of this annular coordinate

$$f(z,\bar{z}) \xrightarrow{\gamma} (-1)^{p(\gamma)} f(z,\bar{z}).$$

We say that spinor changes or does not change sign along γ if $p(\gamma)$ equals -1 or +1 respectively. It is not difficult to show that:

1) $p(\gamma)$ is independent of the annular map z,

2) $p(\gamma)$ depends only on γ, but not on $A(\gamma)$,

3) $p(\gamma)$ depends only on the isotopy class of γ, i.e. if γ_1 is isotopic to γ_2 (there is a smooth deformation of γ_1 to γ_2, which is an embedding at each stage), then $p(\gamma_1) = p(\gamma_2)$.

Let us consider some characteristic examples.

Example 1. Let γ be an embedding of S^1 in the complex z-plane. Then $\sqrt{dz}$ does not change sign along γ

$$\sqrt{dz} \xrightarrow{\gamma} \sqrt{dz}.$$

Example 2. Let $C = \mathbf{C}/\{z \to z + 2\pi i\}$ be a cylinder represented as the complex z-plane factored by a shift. To prove that $\sqrt{dz}$ changes a sign around the cycle γ of the cylinder

$$\sqrt{dz} \xrightarrow{z \to z+2\pi i} -\sqrt{dz}$$

we note that z is not the annular coordinate. In terms of coordinate $w = e^z$ the spinor $\sqrt{dz}$ looks as follows

$$\sqrt{dz} = \frac{\sqrt{dw}}{\sqrt{w}}.$$

On the w-plane γ is a loop $|w| = const$ around the origin, so w-coordinate maps γ to an annulus. As we have seen in the first example, $\sqrt{dw}$ does not change sign around γ, whereas $\sqrt{w}$ does.

Example 3. Let $T = \mathbf{C}/\{z \to z + a,\ z \to z + b\}$, Im $a/b \neq 0$, be a torus. Example 2 shows that $\sqrt{dz}$ flips under passage around the cycles of the torus

$$\begin{aligned} \sqrt{dz} &\xrightarrow{z \to z+a} -\sqrt{dz}, \\ \sqrt{dz} &\xrightarrow{z \to z+b} -\sqrt{dz}, \\ \sqrt{dz} &\xrightarrow{z \to z+a+b} -\sqrt{dz}, \end{aligned} \tag{10.2}$$

since the annular coordinate w for any of the 3 contours above is constructed as in Example 2.

Flip numbers associated with immersed contours in $\mathcal{R}$ can be introduced in exactly the same way. Let γ be an immersion of S^1 in $\mathcal{R}$ and z a conformal map of γ to a simple contour in $\mathbf{C}$. Then the flip number $p(\gamma)$ is defined by the monodromy of the spinor, written in the z-coordinate as

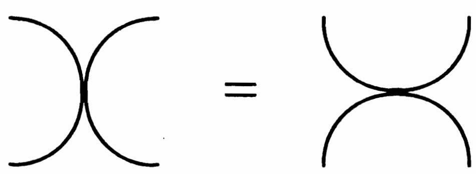
$$f(z,\bar{z}) \xrightarrow{\gamma} (-1)^{p(\gamma)} f(z,\bar{z}). \tag{10.3}$$

For example, if γ_k is a closed contour in the z-plane having k points of self-intersection, then the flip number of $\sqrt{dz}$ is equal to $p(\gamma_k) = (-1)^k$.

This number $p(\gamma)$ is a characteristic not of the particular immersion γ but of the regular homotopy class $[\gamma]$ of γ (two immersions γ_1 and γ_2 are called *regularly homotopic* if there is a deformation of γ_1 and γ_2 which is an immersion at each stage).

Definition 6 *A flip number $p([\gamma]) \in \mathbf{Z}_2$ of a regular homotopy class $[\gamma]$ is the monodromy (10.3) of the spinor under passage around γ, parametrised by annular coordinate.*

Now we define another structure associated with the spinor - a map s from $\mathbf{Z}_2$-homologies of $\mathcal{R}$ to $\mathbf{Z}_2$. The group $H_1(\mathcal{R}, \mathbf{Z}_2)$ coincides with the cobordism group of embedded sets of simple contours in $\mathcal{R}$. The elements of this group are cobordant classes of embedded sets of simple contours in $\mathcal{R}$. Two embedded sets of simple contours are called *cobordant* if one can be transformed into another by isotopies of contours and "touching transformations"



when the contours touch.

Proposition 8 *There is a map $s : H_1(\mathcal{R}, \mathbf{Z}_2) \to \mathbf{Z}_2$ defined by the rule*

$$s(\alpha) = \sum p(\alpha_k) \tag{10.4}$$

where $\alpha = \sum \alpha_k$ is a representation of an element $\alpha \in H_1(\mathcal{R}, \mathbf{Z}_2)$ by a sum of simple non-intersecting contours α_k.

Proof. We have seen already the invariance of $p(\gamma)$ with respect to isotopies. To prove that s is well-defined by the above we must prove the invariance of s with respect to the touching transformation. To show this we use the following

Lemma 6 *Let $\mathcal{P}$ be a pair of pants, i.e. a Riemann surface, which is topologically a sphere with 3 discs removed. The boundary $\partial\mathcal{P} = \alpha + \beta + \gamma$ consists of 3 simple non-intersecting contours. Then*

$$p(\alpha) + p(\beta) + p(\gamma) = 0 \tag{10.5}$$

(all the equalities here and below are in $\mathbf{Z}_2$).

Proof. Let z be a conformal map of $\mathcal{P}$ to the complex plane. The image of it is a plane domain $P \subset \mathbf{C}$, which is topologically a disc with 2 holes. The coordinate z is an annular coordinate for all simple cycles on $\mathcal{P}$. Identity (10.5) is the identity for monodromy of $f(z,\bar{z})$ in the plane domain P.

It is easy to see that the touching transformation always transforms two simple contours into one simple contour, which is cobordant to their sum, or vice versa. These three contours form a pair of pants, the equality (10.5) for which completes the proof of the proposition.

Lemma 7 *Let T be a torus or a torus with a hole and a, b be a canonical basis of cycles ($a \circ b = 1$) of T. Then the form s calculated at the element $na+mb \in H_1(T, \mathbf{Z}_2)$ is given by*

$$s(na+mb) = ns(a) + ms(b) + nm. \tag{10.6}$$

Proof. Since we consider $H_1(T, \mathbf{Z}_2)$, the only equality to be proved is

$$s(a+b) = s(a) + s(b) + 1.$$

Let z be a flat coordinate of the torus. As we have seen in Example 3, $\sqrt{dz}$ changes sign along any simple contour on T. This means that $f(z,\bar{z})$ acquires the factors $(-1)^{s(a)+1}$ and $(-1)^{s(b)+1}$ along the cycles a and b respectively. The group of $f(z,\bar{z})$ is multiplicative, therefore the factor along $a+b$ is equal to $(-1)^{s(a)+s(b)}$. To calculate s at $a+b$ we should multiply the monodromy of $f(z,\bar{z})$ by (-1) since $\sqrt{dz}$ changes sign along $a+b$:

$$f(z,\bar{z})\sqrt{dz} \xrightarrow{a+b} (-1)^{s(a)+s(b)+1} f(z,\bar{z})\sqrt{dz},$$

which completes the proof.

Corollary 3 *For any two cycles γ_1, γ_2 on a torus or on a torus with a hole,*

$$s(\gamma_1+\gamma_2) = s(\gamma_1) + s(\gamma_2) + \gamma_1 \circ \gamma_2, \tag{10.7}$$

where $\gamma_1 \circ \gamma_2$ is the intersection number.

Proof. Let $\gamma_1 = n_1 a + m_1 b$, $\gamma_2 = n_2 a + m_2 b$. The following calculation proves (10.7):

$$\begin{aligned} s(\gamma_1+\gamma_2) &= (n_1+n_2)s(a) + (m_1+m_2)s(b) + (n_1+n_2)(m_1+m_2) = \\ &= (n_1 s(a) + m_1 s(b) + n_1 m_1) + (n_2 s(a) + m_2 s(b) + n_2 m_2) + \\ &+ (n_1 m_2 + n_2 m_1) = s(\gamma_1) + s(\gamma_2) + \gamma_1 \circ \gamma_2. \end{aligned}$$

Proposition 9 *$s : H_1(\mathcal{R}, \mathbf{Z}_2) \to \mathbf{Z}_2$ is a quadratic form, i.e. for any two elements $\alpha, \beta \in H_1(\mathcal{R}, \mathbf{Z}_2)$,*

$$s(\alpha+\beta) = s(\alpha) + s(\beta) + \alpha \circ \beta, \tag{10.8}$$

where $\alpha \circ \beta$ is the intersection number.

Proof. Let us fix some standard basis of cycles on a Riemann surface $\mathcal{R}$ of genus G with K holes or punctures and let

$$\begin{aligned}\alpha &= \alpha_1 + \ldots + \alpha_G + \gamma,\\ \beta &= \beta_1 + \ldots + \beta_G + \delta\end{aligned}$$

be a decomposition of α and β in $H_1(\mathcal{R}, \mathbf{Z}_2)$, where α_i, β_i are the simple contours corresponding to the i-th handle of $\mathcal{R}$, whereas γ and δ correspond to homology of holes and punctures. These elements have the following intersection numbers:

$$\alpha_i \circ \beta_i = 1 \quad \alpha_i \circ \alpha_j = \beta_i \circ \beta_j = \alpha_i \circ \beta_j = 0, \quad i \neq j,$$
$$\alpha_i \circ \gamma = \alpha_i \circ \delta = \beta_i \circ \gamma = \beta_i \circ \delta = \gamma \circ \delta = 0.$$

Let us take a part of $\mathcal{R}$ corresponding to the i-th handle. This part can be chosen to be topologically a torus with a hole. Corollary 3, applied to such a part, gives

$$s(\alpha_i + \beta_i) = s(\alpha_i) + s(\beta_i) + \alpha_i \circ \beta_i.$$

Using this we easily get (10.8):

$$s(\alpha + \beta) = \sum_i s(\alpha_i + \beta_i) + s(\gamma + \delta) =$$
$$= \sum_i (s(\alpha_i) + s(\beta_i)) + \sum_i \alpha_i \circ \beta_i + s(\gamma) + s(\delta) =$$
$$= s(\sum_i \alpha_i + \gamma) + s(\sum_i \beta_i + \delta) + \alpha \circ \beta.$$

Definition 7 *The quadratic form $s : H_1(\mathcal{R}, \mathbf{Z}_2) \to \mathbf{Z}_2$ is called a spin structure of the spinor $f(z, \bar{z})\sqrt{dz}$*

Let $\mathcal{R}$ be a Riemann surface with G handles and K punctures and d a contour a homologous to zero surrounding all the punctures. Let us chose a basis a_n, b_n, d_k of $H_1(\mathcal{R}, \mathbf{Z})$ in such a way, that the cycles $d_k, k = 1, \ldots, K$ surround each its own puncture, the cycles $a_n, b_n, n = 1, \ldots, G$ form a canonical basis of the compact part of $\mathcal{R}$ and do not intersect d. Then to the spinor there correspond characteristics $[\alpha, \beta, \delta] \in \mathbf{Z}_2^{2G+K}$

$$\alpha = (\alpha_1, \ldots, \alpha_G), \quad \beta = (\beta_1, \ldots, \beta_G), \quad \delta = (\delta_1, \ldots, \delta_K),$$

where $\alpha_n = s(a_n), \quad \beta_n = s(b_n), \quad \delta_k = s(d_k).$

Since d is homologous to zero the sum of $\delta' s$ in $\mathbf{Z}_2$ always vanishes

$$\delta_1 + \ldots + \delta_K = 0.$$

All the other characteristics are independent, which shows that there are 2^{2G+K-1} if $K \neq 0$ and 2^{2K} if $K = 0$ different spin structures.

The characteristics $[\alpha, \beta, \delta]$ depend on the choice of the basis of the compact part of $\mathcal{R}$ and on the arrangement of the punctures.

Definition 8 *The scalar product*

$$< \alpha, \beta >= \sum_{i=1}^{G} \alpha_i \beta_i \in \mathbf{Z}_2$$

is called the parity of the spin structure.

Proposition 10 *The parity of the spin structure and the number of δ_k's, which are equal to zero are independent of the choice of basis in $H_1(\mathcal{R}, \mathbf{Z})$.*

The independence of the number of vanishing δ_k's is evident. The invariance of the parity of the spin structure needs some calculation, which can be found, for example, in [7]. One should consider a symplectic transformation, which relates canonical bases a, b and $a^{'}, b^{'}$ and then prove $< \alpha, \beta >=< \alpha^{'}, \beta^{'} >$ using (10.8).

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