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WILLMORE TORI WITH UMBILIC LINES AND MINIMAL SURFACES IN HYPERBOLIC SPACE

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1. Introduction. Let M be a torus in $\mathbf{R}^3$. The Willmore functional of M is defined as

$$W = \int_M H^2 \, dS, \tag{1}$$

where H is the mean curvature and dS is an area element. Critical points of W are called Willmore tori. Background information about Willmore surfaces can be found in [19, 23].

We add one more interesting family of Willmore tori to those already known [22, 12, 5, 6, 13]. The main result of this paper is a construction of the Willmore tori with umbilic lines. The tori we construct possess these properties:

(1) there is a plane $\mathscr{A}$ (infinity plane) intersecting M orthogonally and decomposing it into 3 parts $M = M_+ \cup M_0 \cup M_-$ (lying respectively above $\mathscr{A}$, on $\mathscr{A}$, and below $\mathscr{A}$);
(2) $M_\pm$ are minimal surfaces in hyperbolic spaces, realized as upper and lower (with respect to $\mathscr{A}$) half spaces with the Poincaré metric;
(3) M_0 is an umbilic set; i.e., for the points of M_0 two principal curvatures coincide.

To construct these tori we use the methods of the integrable equations. These methods were already successfully applied [17, 24, 6, 7] to differential geometric classification of some submanifolds. In particular the case of constant mean curvature (CMC) surfaces in 3-dimensional space forms $\mathbf{R}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$ has received a fairly complete treatment in [24, 5, 6].

Our starting points are the results on description of the tori with CMC H, $|H| > 1$ obtained in [6], based on the solution of the corresponding Gauss equation $\Delta u + \sinh u = 0$. In the present paper we are interested in the minimal surface case $H = 0$, the Gauss equation of which is the elliptic cosh-Gordon equation

$$\Delta u = \cosh u.$$

The corresponding modifications were made in [2] and are presented in Section 2.

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Because of the maximum principle there are no compact minimal surfaces in $\mathbf{H}^3$. But it turns out that some of the minimal surfaces we construct are analytic tori in two copies (!) of $\mathbf{H}^3$ suitably glued. How we get Willmore tori in this way is explained in Section 3.

To construct these tori we need doubly periodic solutions of the elliptic cosh-Gordon equation (which necessarily blow up; the lines where $u \to +\infty$ become later preimages of the umbilic lines). More precisely to construct surfaces we use the finite-gap solutions of the elliptic cosh-Gordon equation, which were found in [1]. These results as well as the formulas for the corresponding surfaces are presented in Sections 4, 5.

In general, the surfaces are not tori. To describe a torus the immersion function must be doubly periodic. The periodicity conditions (Section 7) are formulated in terms of the spectral curve. Although they are rather complicated, it is possible to investigate the simplest cases and to prove the nondegeneracy of these conditions. The simplest tori constructed are of rectangular conformal type with closed mean curvature lines (Section 9). The lowest possible genus of the spectral curve determining these tori is 3 (Section 10).

The famous Willmore conjecture is that for any torus $W \geqslant 2\pi^2$. This conjecture is proved for the canal tori [16] and for the tori with the conformal type close to square [20]. In Section 8 we derive a formula for the Willmore functional W in terms of the spectral curve, without hope, however, of finding a torus with $W < 2\pi^2$.

The present paper is parallel ideologically and technically to [6, 5]. To follow details of the theta function calculations, one can use these papers and the book [3].

We would like to mention also the recent paper [25], where minimal surfaces in $\mathbf{H}^3$ with one family of spherical curvature lines were constructed. These surfaces are close to those of ours considered in Section 9; they are also generated by the genus-3 solutions of the elliptic cosh-Gordon equation with the symmetric spectral curve. The relaton with Willmore tori and periodicity conditions we use in Section 10 are not discussed in [25].

In Figures 2, 3 with the help of Mathematica [27] we present the simplest examples of the surfaces constructed. They are described by Jacobi theta functions and elliptic integrals. Unfortunately we can not, in the same way, construct pictures of the tori since they are described by theta functions and abelian integrals of the Riemann surfaces of genus > 1, which are not implemented in Mathematica. At the present time this higher genus implementation is being developed in Berlin by Sonderforschungsbereich 288, and we hope to be able to look at these tori soon.

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2. Minimal surfaces in hyperbolic space. Formula for immersion. A hyperbolic space Q

$$\{F, F\} = -1$$

is embedded into the Lorentz space $\mathbf{R}^{3,1}$. The metric of $\mathbf{R}^{3,1}$

$$\{a, b\} = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_0 b_0$$

induces a positively definite metric on Q.

Let M be a smooth orientable surface in Q. The metric $\{\ ,\ \}$ induces a complex structure of a Riemann surface $\mathscr{R}$ on M. Let

$$F: \mathscr{R} \to M \subset Q \subset \mathbf{R}^{3,1}$$

be a conformal parametrization of M. This means that the immersion function is normalized as

$$\{F_z, F_z\} = \{F_{\bar{z}}, F_{\bar{z}}\} = 0,$$

where z is some local parameter on $\mathscr{R}$ and F_z, $F_{\bar{z}}$ are the partial derivatives. Vectors F, F_z, $F_{\bar{z}}$ may be supplemented with a normal N in such a way that

$$\{F_z, N\} = \{F_{\bar{z}}, N\} = \{F, N\} = 0, \qquad \{N, N\} = 1.$$

Let us also introduce the notation

$$\{F_z, F_{\bar{z}}\} = 2e^u, \{F_{z\bar{z}}, N\} = 2H^h e^u, \{F_{zz}, N\} = A^h,$$

where H^h and A^h are called respectively the mean curvature and the Hopf differential. The Gauss curvature is equal to

$$K^h = -1 - (H^h)^2 - \tfrac{1}{4} A^h \bar{A}^h e^{-2u}.$$

The variation of the basis F, F_z, $F_{\bar{z}}$, N with respect to motion along the surface is described by the equations

$$\sigma_z = U\sigma, \sigma_{\bar{z}} = V\sigma, \sigma = (F, F_z, F_{\bar{z}}, N) \tag{2}$$

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & u_z & 0 & A^h \\ 2e^u & 0 & 0 & 2H^h e^u \\ 0 & -H^h & -\frac{1}{2}A^h e^{-u} & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2e^u & 0 & 0 & 2H^h e^u \\ 0 & 0 & u_{\bar{z}} & \bar{A}^h \\ 0 & -\frac{1}{2}\bar{A}^h e^{-u} & -H^h & 0 \end{pmatrix}$$

The constant mean curvature (CMC) condition

$$H^h = const$$

leads to the Gauss-Peterson-Codazzi equation

$$u_{z\bar{z}} + 2((H^h)^2 - 1)e^u - A^h\bar{A}^h e^{-u}/2 = 0, \qquad A^h_{\bar{z}} = 0.$$

The case $(H^h)^2 > 1$ is described in [5, 6]. Here we consider the case $(H^h)^2 < 1$ (see also [2]). In new variables

$$z_H = \delta_H z,\ e^{2i\phi}A = \delta_H^{-1}A^h,\ \delta_H = \sqrt{1-(H^h)^2} \tag{3}$$

we obtain the equations

$$u_{z_H\bar{z}_H} - 2e^u - \tfrac{1}{2}A\bar{A}e^{-u} = 0 \tag{4}$$

$$A_{\bar{z}_H} = 0. \tag{5}$$

THEOREM 1. *A CMC surface in Q, conformally parametrized, generates a holomorphic quadratic differential $A^h(dz)^2$. The induced metric $u(z, \bar{z})$ satisfies (4).*

The system (4, 5) can be represented as the compatibility condition

$$U_{\bar{z}_H} - V_{z_H} + [U, V] = 0$$

of the system

$$\Phi_{z_H} = U\Phi,\ \Phi_{\bar{z}_H} = V\Phi \tag{6}$$

with the matrices

$$U = \frac{1}{2}\begin{pmatrix} 0 & 2\lambda e^{u/2} \\ Ae^{-u/2} & u_{z_H} \end{pmatrix},\ V = \frac{1}{2}\begin{bmatrix} u_{\bar{z}_H} & -\bar{A}e^{-u/2} \\ \dfrac{2}{\lambda}e^{u/2} & 0 \end{bmatrix} \tag{7}$$

depending on an extra parameter λ, which is called a spectral parameter in the theory of integrable equations.

We note that $U - V$ pair (7) satisfies the reduction

$$\overline{U(-\bar{\lambda}^{-1})} = \sigma_2 V(\lambda)\sigma_2.$$

For the solution of the system (6) it gives

$$\Phi(\lambda) = \sigma_2\overline{\Phi(-\bar{\lambda}^{-1})}M(\lambda) \tag{8}$$

with some matrix $M(\lambda)$ independent of z, $\bar{z}$. Here and below σ_α denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us identify the Lorentz space with the space of (2×2) hermitian matrices $\overline{X^T} = X$

$$\mathbf{X} = (X_0, X_1, X_2, X_3) \in \mathbf{R}^{3,1} \leftrightarrow X = X_0 I + \sum_{k=1}^{3} X_k \sigma_k \tag{9}$$

with the scalar product

$$\{X, Y\} = -\tfrac{1}{2}\operatorname{tr}(X\sigma_2 Y^T \sigma_2).$$

THEOREM 2. *Let $A^h(z_H)(dz_H)^2$ be a holomorphic quadratic differential on $\mathcal{R}$, let $u(z_H, \bar{z}_H)$ be a solution of (4), and let*

$$\Phi_0(z_H, \overline{z_H}, \lambda = e^{-q}e^{2i\phi})$$

be a solution of (6). Then

$$F = \Phi_0^* \Phi_0 \frac{1}{\sqrt{\det \Phi_0 \det \Phi_0^*}}, N = \Phi_0^* \sigma_3 \Phi_0 \frac{1}{\sqrt{\det \Phi_0 \det \Phi_0^*}}, \tag{10}$$

where $\Phi_0^* = \overline{\Phi_0^T}$ in the variables (3) satisfy equations (2) for the moving frame and describe a surface in Q with the mean curvature

$$H = \tanh q.$$

Proof. Since $F = F^*$ and $\det F = 1$, we see that (10) describes a surface in Q. Multiplying Φ_0 by a complex number gives us $\det \Phi_0 \in \mathbf{R}$. Further we suppose the determinant to be real. Let us denote

$$\Phi_1 = \Phi_0, \Phi_2 = \sigma_2 \overline{\Phi_0} \sigma_2.$$

Using the identity

$$\sigma_2 X^T \sigma_2 = X^{-1} \det X$$

for invertible matrices, we have (compare with [6])

$$F = \Phi_2^{-1}\Phi_1, N = \Phi_2^{-1}\sigma_3\Phi_1.$$

The functions Φ_1 and Φ_2 satisfy (6) with $\lambda = \lambda_1 = e^{-q}e^{2i\phi}$ and $\lambda = \lambda_2 = -e^{q}e^{2i\phi}$ respectively. To see the last fact we should take (8) into account. Direct calculations prove all the identities (2). In particular, we have

$$\begin{aligned}
\{F_{z_H}, F_{\overline{z_H}}\} &= -\tfrac{1}{2}\,\mathrm{tr}((U_1 - U_2)\sigma_2(V_1 - V_2)^T\sigma_2) \\
&= 2e^u \cosh^2 q, \\
\{F_{z_H z_H}, N\} &= \tfrac{1}{2}\,\mathrm{tr}(((U_1 - U_2)U_1 - U_2(U_1 - U_2) + (U_1 - U_2)_z)\sigma_3) \\
&= e^{2i\phi} A \cosh q, \\
F_{z_H \overline{z_H}} &= \Phi_2^{-1}((U_1 - U_2)V_1 - V_2(U_1 - U_2) + (U_1 - U_2)_{\bar z})\Phi_1 \\
&= 2e^u F \cosh^2 q + 2e^u N \cosh q \sinh q,
\end{aligned}$$

where $U_i = U(\lambda_i)$, $V_i = V(\lambda_i)$. The change of variables (3) completes the proof.

Remark. The sign of $\sqrt{\det \Phi_0 \det \Phi_0^*}$ in (10) determines a sheet of Q. For the surfaces lying on both sheets of Q, which we will consider, the sign must be chosen correctly (see Section 5).

Remark. We restrict ourselves to the most interesting case of the minimal surfaces $H = 0$. In this case $q = 0$, and λ has to be taken on the unit circle. Further, we do not distinguish

$$z_H = z.$$

We will also use the Poincaré model (H-model) of the hyperbolic space, namely the half-space model

$$H_\pm = \{(G_1, G_2, G_3) \in \mathbf{R}^3 \mid \pm G_3 > 0\}.$$

It is related to that already described by the conformal map

$$S: (F_0, F_1, F_2, F_3) \to \left(\frac{F_1}{F_0 - F_3}, \frac{F_2}{F_0 - F_3}, \frac{1}{F_0 - F_3}\right).$$

More precisely, if we denote by $Q_\pm$ the upper ($F_0 \geqslant 1$) and the lower ($F_0 \leqslant 1$) sheets of Q, then

$$Q_\pm \xrightarrow{S} H_\pm.$$

H_+ and H_- are separated by the infinity plane $G_3 = 0$. For the induced metric of the H-model we keep the same notation

$$\{dA, dB\} = \frac{1}{|G_3|^2} \sum_{i=1}^{3} dA_i \, dB_i, \tag{11}$$

where $|G_3|$ is a euclidean distance from the point to the infinity plane $\mathscr{A}$.

THEOREM 3. *Let*

$$\Phi(z, \bar{z}, \lambda = e^{2i\phi})$$

be a solution of (6) with some A and u. Then in the H-model the corresponding minimal surface is described by the formulas

$$G_1 + iG_2 = \frac{a\bar{b} + \bar{d}c}{b\bar{b} + d\bar{d}}, \quad G_3 = \frac{\Delta}{b\bar{b} + d\bar{d}}, \tag{12}$$

where

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Delta = \sqrt{\det \Phi \det \Phi^*}. \tag{13}$$

3. Minimal surfaces in hyperbolic space and Willmore surfaces with umbilics. We will consider surfaces M in $\mathbf{R}^3$, which are sufficiently smooth (with immersion function $G = (G_1, G_2, G_3)\colon \mathscr{R} \to \mathbf{R}^3$, $G \in \mathscr{C}^4$).

Decompose M into three parts $M = M_+ \cup M_0 \cup M_-$:

$$M_\pm = \{P \in M \mid \pm G_3(P) > 0\}$$

$$M_0 = \{P \in M \mid G_3(P) = 0\}.$$

THEOREM 4. *Let M be a smooth ($\mathscr{C}^4$) surface in $\mathbf{R}^3$ such that both $M_\pm$ are minimal surfaces in $H_\pm$ respectively (i.e., with respect to the metric (11)). Then*

(i) *M is a Willmore surface;*
(ii) *M_0 is an umbilic line.*

Proof. We will use two metrics

$$\langle\ ,\ \rangle \text{ the euclidean metric in } \mathbf{R}^3,$$

$$\{\ ,\ \} = G_3^{-2}\langle\ ,\ \rangle \text{ the Poincaré metric on } H_\pm.$$

Let us consider the conformal parametrization $G(z, \bar{z})$ of M in $\mathbf{R}^3$. This parametrization is also conformal for the hyperbolic metric $\{\ ,\ \}$. The normal vectors N^e and N^h differ by a factor

$$N^h = G_3 N^e,$$

since $\langle N^e, N^e \rangle = \{N^h, N^h\} = 1$.

The first fundamental forms of the hyperbolic and euclidean models are simply related

$$I^h = \{dG, dG\} = G_3^{-2}\langle dG, dG \rangle = G_3^{-2} I^e.$$

The expression for the second fundamental form is a little bit more complicated

$$\begin{aligned} II^h &= -\frac{1}{2}\frac{d}{dt}\{d(G + tN^h), d(G + tN^h)\}_{G+tN^h}|_{t=0} \\ &= -\{dG, dN^h\}_G - \frac{1}{2}\frac{d}{dt}\{dG, dG\}_{G+tN^h}|_{t=0} \\ &= -G_3^{-1} II^e + G_3^{-2} N_3^e I^e. \end{aligned} \tag{14}$$

Here the upper indexes h an e denote hyperelliptic and euclidean models, and the lower index of $\{\ ,\ \}$ denotes a point where it is calculated. Using the definitions of the Gaussian and mean curvatures

$$K^e = \det(II^e(I^e)^{-1}),\ K^h + 1 = \det(II^h(I^h)^{-1}),$$

$$H^e = \tfrac{1}{2}\operatorname{tr}(II^e(I^e)^{-1}),\ H^h = \tfrac{1}{2}\operatorname{tr}(II^h(I^h)^{-1}),$$

one can easily derive the relation

$$(H^e)^2\, dS^e = ((H^h)^2 - 1)\, dS^h + K^e\, dS^e - K^h\, dS^h, \tag{15}$$

where $dS^{e,h}$ are the area forms in e and h-models: $dS^h = G_3^{-2}\, dS^e$. Let us integrate (15) over some domain $D \subset M$, $D \cap M_0 = \varnothing$

$$\int_D (H^e)^2\, dS^e = \int_D (H^h)^2\, dS^h - S^h(D) + \int_D K^e\, dS^e - \int_D K^h\, dS^h. \tag{16}$$

Due to the Gauss-Bonnet theorem, the last two integrals are reduced to the integrals over the boundary ∂D. Let us consider variations of D vanishing along with their derivatives on the boundary ∂D. With respect to these variations the last two integrals are constant. The minimal surfaces are critical for both integrals $\int_D (H^h)^2\, dS^h$ and $\int_D dS^h$; therefore D is critical for the Willmore functional (1). This proves that $M_\pm$ satisfy the Euler-Lagrange equation corresponding to (1). This is an elliptic equation

$$\Delta H + 2H(H^2 - K) = 0. \tag{17}$$

The simple behaviour of M near the infinity plane $\mathscr{A}$ (orthogonal intersection) allows us to represent the surface in the neighbourhood of $\mathscr{A}$ as a graph. The equation (17) shows that the graph function satisfies some elliptic equation of the fourth order. Combined with general $\mathscr{C}^4$ differentiability of G this fact allows us by standard methods[1] ([4], p. 467) to prove that the same equation is valid for all points of M. This finishes the proof of (i).

The equality (15) in particular gives

$$G_3\{G_{zz}, N^h\} = G_3 A^h = A^e = \langle G_{zz}, N^e \rangle. \tag{18}$$

Zeros of A^e are the umbilic points of M

$$K - H^2 = -A^e \bar{A}^e \langle G_z, G_{\bar{z}} \rangle^{-2}.$$

Let us denote $\mathscr{R}_+, \mathscr{R}_0, \mathscr{R}_- \subset \mathscr{R}$ preimages of M_+, M_0, M_- respectively. Vanishing of the derivative $\partial A^h/\partial \bar{z}$ on $\mathscr{R}_\pm$ gives

$$\frac{A^e}{G_3} = \frac{\partial A^e/\partial \bar{z}}{\partial G_3/\partial \bar{z}}. \tag{19}$$

Since M is orthogonal to the infinity plane $\mathscr{A}$, the derivative $\partial G_3/\partial \bar{z}$ does not vanish on $\mathscr{R}_0$. It determines a continous continuation of A^h to $\mathscr{R}_0$ by the formula (19)

$$A^h = \frac{\partial A^e/\partial \bar{z}}{\partial G_3/\partial \bar{z}} \qquad \text{near } \mathscr{R}_0. \tag{20}$$

Rewriting (19)

$$A^e = G_3 \frac{\partial A^e/\partial \bar{z}}{\partial G_3/\partial \bar{z}}, \tag{21}$$

we see that A^e vanishes on M_0. The theorem is proved.

Holomorphicity of A^h on $\mathscr{R}_\pm$ and continuity on $\mathscr{R}_0$ gives the following corollary.

COROLLARY 1. *Let M be as in Theorem 4. Then $A^h = \{F_{zz}, N^h\}$ is holomorphic everywhere on $\mathscr{R}$.*

[1] The authors are thankful to Bennett Palmer for explanation of this.

The aim of this paper is to construct Willmore tori with umbilic lines. If $\mathscr{R}$ is a torus, it can be represented as a quotient $\mathbf{C}/\Lambda$ with respect to some lattice. There is only one holomorphic quadratic differential on $\mathbf{C}/\Lambda$. Therefore we can normalize

$$A = 2, \tag{22}$$

which reduces equation (4) and its Lax representaton to the form

$$u_{z\bar{z}} = 4\cosh u, \tag{23}$$

$$\Phi_z = U\Phi, \Phi_{\bar{z}} = V\Phi, U = \begin{pmatrix} 0 & \lambda e^{u/2} \\ e^{-u/2} & u_z/2 \end{pmatrix}, V = \begin{bmatrix} u_{\bar{z}}/2 & -e^{-u/2} \\ \frac{1}{\lambda}e^{u/2} & 0 \end{bmatrix}. \tag{24}$$

Finally, to construct a Willmore torus with umbilic lines, we should find a doubly periodic solution of (23) such that the immersion function given by (12, 13, 24) is also doubly periodic and describes a smooth torus in $\mathbf{R}^3$. It is possible to construct these tori using the finite-gap doubly periodic solutions of (23).

4. Complex-valued finite-gap solutions of the cosh-Gordon equation. The equation (23) is one of the real versions of the sine-Gordon equation

$$u_{\xi\eta} = 4\sin u,$$

finite-gap solutions of which are well known [18, 21, 8, 14, 3, 9]. Generally these solutions are analytic functions of ξ and η, given by explicit formulas in terms of theta functions. In this section we present these formulas as well as formulas for the corresponding Baker-Akhiezer function in a modified form. This modification will be used in the next section to get real-valued solutions of (23).

We start with introducing some standard ingredients of the theory. Consider the Riemann surface X of the hyperelliptic curve

$$\mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - E_i). \tag{25}$$

It is a double cover of λ-plane with E_i, 0, ∞ being ramification points. We denote the hyperelliptic involution as

$$\pi(\lambda, \mu) = (\lambda, -\mu). \tag{26}$$

Let a_n, b_n, $n = 1, \ldots, g$ be the canonical basis of cycles and du_n be the normalized holomorphic differentials on X

$$\int_{a_m} du_n = 2\pi i \delta_{nm}.$$

The associated period matrix

$$B_{nm} = \int_{b_m} du_n$$

defines the Riemann theta function with characteristics $\alpha, \beta \in \mathbf{R}^g$

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) = \sum_{m \in \mathbf{Z}^g} \exp\left\{\frac{1}{2}\langle B(m+\alpha), m+\alpha\rangle + \langle z + 2\pi i\beta, m+\alpha\rangle\right\}, \tag{27}$$

$z \in \mathbf{C}^g$. For zero characteristics we will use the notation

$$\theta(z) = \theta\begin{bmatrix}0\\ 0\end{bmatrix}(z).$$

The function $v = \sqrt{\lambda}$, multivalued on X, defines a Riemann surface $\hat{X}$, which is an unramified covering of X. This covering can be defined by a contour $\mathscr{L}$ on X, which fixes a branch of v on $X \backslash \mathscr{L}$. The function v is multiplied by a factor $(-1)^{\langle\gamma,\mathscr{L}\rangle}$, when a circuit of γ is tranversed. Here $\langle\gamma, \mathscr{L}\rangle$ is the intersection number of γ and $\mathscr{L}$.

We need also the abelian integrals of the second kind

$$\Omega_{0,\infty} = \int_{\lambda=\infty} d\Omega_{0,\infty},$$

normalized by the condition

$$\int_{a_n} d\Omega_{0,\infty} = 0, \qquad n = 1, \ldots, g$$

and the asymptotic behaviour at the singularities:

$$d\Omega_\infty \to dv,\ v \to \infty, \qquad d\Omega_0 \to -\frac{dv}{v^2},\ v \to 0.$$

Recall that we have fixed a certain branch of v on X. Periods of $d\Omega_\infty$ and $d\Omega_0$ over the b-cycles we denote as

$$U_n = \int_{b_n} d\Omega_\infty, \qquad V_n = \int_{b_n} d\Omega_0. \tag{28}$$

Now we consider z and $\bar{z}$ as independent complex variables. Let

$$\mathscr{L} = \sum_{i=1}^{g} \Delta_2^i a_i + \sum_{i=1}^{g} \Delta_1^i b_i = \langle\Delta_2, a\rangle + \langle\Delta_1, b\rangle$$

be a decomposition of the cycle $\mathscr{L}$ with respect to the chosen basis of cycles.

The Baker-Akhiezer function $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ is a vector-function on X, given by the formula

$$\psi_1(P) = \frac{\theta(u+\Omega)\theta(D)}{\theta(u+D)\theta(\Omega)}\exp(z\Omega_\infty + i\bar{z}\Omega_0),$$

$$\psi_2(P) = \frac{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(u+\Omega)\theta(D)}{\theta(u+D)\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega)}\exp(z\Omega_\infty + i\bar{z}\Omega_0). \tag{29}$$

Here $\Omega = zU + i\bar{z}V + D$, u is an Abel map $u = \int_\infty^P du$, $P = (\lambda, u) \in X$, $\Omega_{0,\infty} = \int_\infty^P d\Omega_{0,\infty}$, U and V are the vectors of b-periods

$$U = (U_1, \ldots, U_g),\ V = (V_1, \ldots, V_g),$$

and $D \in \mathbf{C}^g$ is arbitrary. The paths of integration in u and $\Omega_{0,\infty}$ are identical.

The functions ψ_1 and $v\psi_2$ are single valued on X. The divisor of poles $\mathscr{D}$ of ψ_1 as well as ψ_2 is of degree g and is determined by the theta function in the denominators of ψ.

In a standard way [3, 9, 8, 21] it is proved that ψ satisfies the system

$$\psi_z = U\psi, \qquad \psi_{\bar{z}} = V\psi,$$

$$U = \begin{pmatrix}-u_z/2 & v\\ v & u_z/2\end{pmatrix}, \qquad V = \frac{1}{v}\begin{pmatrix}0 & -e^{-u}\\ e^u & 0\end{pmatrix}. \tag{30}$$

Let l be a path on X from ∞ to $P \in X$ and v_l be an analytic continuation of v along this path. (We denote by f_l the analytic continuation of the function f along the path l.) Let l^* be a path from ∞ to πP such that $\langle \mathscr{L}, l\rangle - \langle \mathscr{L}, l^*\rangle$ is odd. Then $v_l = v_{l^*}$. Let us denote $\psi_i = \psi_{il}$, $\psi_i^* = \psi_{il^*}$. They comprise a matrix-valued function on $\hat{X}$

$$\Psi = \begin{pmatrix}\psi_1 & \psi_1^*\\ \psi_2 & \psi_2^*\end{pmatrix},$$

which also satisfies the system (30). Considering this function at $v = 0$ we get the formula for u

$$u = 2\log\frac{\theta(\Omega)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega)} - \frac{\pi i}{2}\langle\Delta_1, \Delta_2\rangle + \frac{\pi i}{2} + \pi i k.$$

Here k is the parity of the intersection index $\langle \mathscr{L}, l_{[0,\infty]} \rangle$, where $l_{[0,\infty]}$ is a path from $\lambda = \infty$ to $\lambda = 0$ on X such that $\mathscr{L} = l_{[0,\infty]} - \pi l_{[0,\infty]}$ (π is the hyperelliptic involution of X).

From now on we set

$$\langle \Delta_1, \Delta_2 \rangle = 1.$$

It turns out that real finite-gap solutions of (23) can be described by this specialization (see Section 5).

Remark. If $\mathscr{L}_0$ and $\mathscr{L}_1$ are two $\mathscr{L}$-cycles on X with $k = 0$ and $k = 1$ respectively, then the corresponding solutions are related in a simple way. Replacing $\mathscr{L}_0$ by $\mathscr{L}_1$, we add to $\mathscr{L}_0$ a small contour around the point $\lambda = 0$ and in this way change a sign of the local parameter at $\lambda = 0$. Therefore, if $u(z, \bar{z})$ is a solution generated by $\mathscr{L}_0$, then $\mathscr{L}_1$ determines the solution $u(z, -\bar{z}) + \pi i$.

From now on we consider only the case $k = 0$ as determining two families of solutions

$$u_+ = 2 \log \frac{\theta(\Omega_+)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_+)} \qquad (k = 0), \tag{31}$$

$$u_- = 2 \log \frac{\theta(\Omega_-)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_-)} + \pi i \qquad (k = 1), \tag{32}$$

$$\Omega_\pm = zU \pm i\bar{z}V + D_\pm .$$

Beside the symbols $\pm$ for these two families we keep the symbol k, which is equal to 0 for the first and to 1 for the second families respectively.

We can summarize the above arguments in the following theorem.

THEOREM 5. *The function*

$$\Psi = \begin{pmatrix} \dfrac{\dfrac{\theta(\Omega_\pm + u)}{\theta(\Omega_\pm)}}{\dfrac{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm + u)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm)}} & -\dfrac{\dfrac{\theta(\Omega_\pm - u)}{\theta(\Omega_\pm)}}{\dfrac{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm - u)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm)}} \end{pmatrix} \begin{pmatrix} \dfrac{e^{\omega_\pm}\theta(D_\pm)}{\theta(D_\pm + u)} & 0 \\ 0 & \dfrac{e^{-\omega_\pm}\theta(D_\pm)}{\theta(D_\pm - u)} \end{pmatrix}, \tag{33}$$

$$\omega_\pm = z\Omega_\infty \pm i\bar{z}\Omega_0$$

is a solution of (30) with $u(z, \bar{z})$ given by (31, 32). In (30) $v = v_l$ and $\langle l, \mathscr{L} \rangle$ is even.

The systems (30) and (24) are gauge equivalent

$$\Phi = \begin{pmatrix} ve^{u/2} & 0 \\ 0 & 1 \end{pmatrix} \Psi. \tag{34}$$

We obtain also a useful formula for

$$d = \psi_1 \psi_2^* - \psi_2 \psi_1^*,$$

analysing its analytical properties. It is a meromorphic function on X with divisor of poles $\mathscr{D} + \pi\mathscr{D}$ and divisor of zeros $E_1 + \cdots + E_{2g}$ at the branch points of (25). In addition $vd(P)$ is a single-valued function on X and its asymptotics at $\lambda = \infty$ give $d(\infty) = -2$. Finally we get the expression in terms of theta functions:

$$d = -2 \frac{\theta^2(d)}{\theta(D - u)\theta(D + u)} \frac{\theta\begin{bmatrix} \Delta_1/2 \\ 0 \end{bmatrix}(u)\theta\begin{bmatrix} 0 \\ \Delta_2/2 \end{bmatrix}(u)}{\theta\begin{bmatrix} \Delta_1/2 \\ 0 \end{bmatrix}(0)\theta\begin{bmatrix} 0 \\ \Delta_2/2 \end{bmatrix}(0)}. \tag{35}$$

Now we are in a position to present a final formula for Φ.

THEOREM 6. *The matrix*

$$\Phi = \begin{bmatrix} ve^{\pi ik/2}\theta(\Omega_\pm + u) & ve^{\pi ik/2}\theta(\Omega_\pm - u) \\ \theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm + u) & -\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm - u) \end{bmatrix} \frac{\exp(\omega_\pm \sigma_3)}{\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm)} \tag{36}$$

solves (24) with $u(z, \bar{z})$ given by (31, 32). The paths of analytic continuation of v and integration in u, Ω are identical. The determinant of Φ is

$$\det \Phi = -2ve^{\pi ik/2} \frac{\theta(\Omega_\pm)}{\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm)} \frac{\theta\begin{bmatrix} \Delta_1/2 \\ 0 \end{bmatrix}(u)\theta\begin{bmatrix} 0 \\ \Delta_2/2 \end{bmatrix}(u)}{\theta\begin{bmatrix} \Delta_1/2 \\ 0 \end{bmatrix}(0)\theta\begin{bmatrix} 0 \\ \Delta_2/2 \end{bmatrix}(0)}. \tag{37}$$

For the proof we use (31–35) and cancel Φ by a nonessential constant right factor.

5. Reality conditions. From now on we again consider z and $\bar{z}$ as complex conjugate variables

$$z = x + iy, \qquad \bar{z} = x - iy.$$

To construct surfaces we need real-valued $u(z, \bar{z})$. Real finite-gap solutions of the equation (23) were constructed in [1]. Here we present the main results of this paper.[2]

For the described complex finite-gap solutions to be real valued, X necessarily has an antiholomorphic involution (see symmetry (8))

$$\tau: \lambda \to -\bar{\lambda}^{-1}.$$

In particular, it implies that the genus of X is odd $g = 2n + 1$. More exactly, let us consider the spectral curve

$$\mu^2 = \lambda(\lambda - E_{2N+1})(\lambda + \bar{E}_{2N+1}^{-1}) \prod_{k=1}^{N} (\lambda - E_k)(\lambda + \bar{E}_k^{-1})(\lambda - E_{k+N})(\lambda + \bar{E}_{k+N}^{-1}). \quad (38)$$

A canonical basis of cycles can be chosen in such a way that τ acts on it as (see Figure 1):

$$\tau a_i = -a_{i+N},\ \tau a_{i+N} = -a_i,\ \tau a_{2N+1} = -a_{2N+1},$$
$$\tau b_i = b_{i+N},\ \tau b_{i+N} = b_i,\ \tau b_{2N+1} = b_{2N+1}, \qquad i = 1, \ldots, N. \quad (39)$$

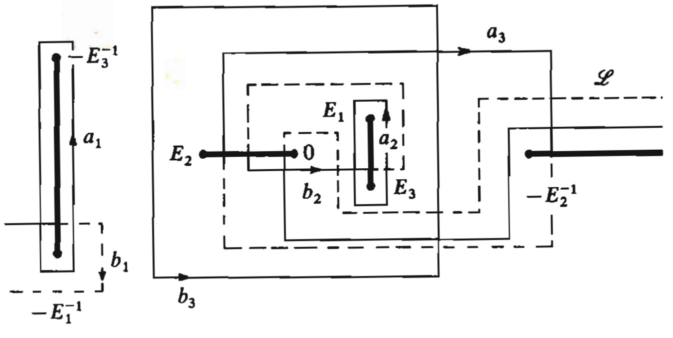

FIGURE 1

[2] There are some misprints in [1]. The correct version of (5a, b) and z_4 in the final table is

$$z_1 = (iV_0 + V_\infty)t + (V_0 + iV_\infty)x,$$
$$z_2 = (iV_0 - V_\infty)t + (V_0 - iV_\infty)x,$$
$$z_4 = i(iV_0 - V_\infty)t + i(V_0 - iV_\infty)x + z_4^0.$$

In the English translation of [1] the table with the final results is split into two parts, which makes it completely nonunderstandable.

We use the matrix notation

$$\tau a = Ta,\ \tau b = -Tb,\ T = T^{-1} = T^T,$$

$$T = -\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where I is the $N \times N$ identity matrix. The cycle $\mathscr{L}$ is

$$\mathscr{L} = \sum_{n=1}^{2N+1} a_n + b_{2N+1},\ \Delta_1 = (0, 0, 1),\ \Delta_2 = (1, 1, 1). \tag{40}$$

THEOREM 7. *Real finite-gap solutions of the equation (23) are given by the formulas (31, 32), where X is a real curve (38) with the basis (39, 40) and* $D_\pm$ *are*

$$D_+ = i(d_+, \bar{d}_+, d_+^0),\ D_- = (d_-, \bar{d}_-, d_-^0), \qquad d_\pm \in \mathbf{C}^N,\ d_\pm^0 \in \mathbf{R}. \tag{41}$$

Proof. A reciprocity law [11] for the abelian integrals of the second kind allows us to express their periods in terms of the normalized holomorphic differentials

$$du_n = -U_n d(1/\sqrt{\lambda}),\ \lambda \sim \infty, \qquad du_n = -V_n d(\sqrt{\lambda}),\ \lambda \sim 0.$$

For the branch of $\sqrt{\lambda}$ fixed by the contour $\mathscr{L}$ shown in Figure 1, we have

$$\sqrt{\lambda(\tau P)}\,\overline{\sqrt{\lambda(P)}} = i, \qquad P \sim 0, \tag{42}$$

when P is in a neighbourhood of $\lambda = 0$. To prove (42) we mention the connection of $\sqrt{\lambda}$ with μ

$$\begin{aligned} \sqrt{\lambda}(P) &= (-1)^N(-i)\exp\left(-i\sum \arg E_i\right)\mu(P)(1 + o(1)), \qquad P \sim 0, \\ 1/\sqrt{\lambda}(P) &= \mu(P)\lambda^{-g-1}(P)(1 + o(1)), \qquad P \sim \infty \end{aligned} \tag{43}$$

and the transformation law of μ

$$\mu(\tau P) = (-1)^N \exp\left(i\sum \arg E_i\right)\overline{\mu(P)\lambda^{-g-1}(P)}.$$

The differentials du transform as

$$\tau^*\,\overline{du} = -T\,du.$$

The straightforward calculation

$$-T\overline{Vd\sqrt{\lambda(P)}} = -T\,\overline{du(P)} = du(\tau P) = -Ud(1/\sqrt{\lambda(\tau P)}) = iU\overline{d(\sqrt{\lambda(P)})},$$

where $\lambda(P) \sim 0$, proves the conjugation law

$$iT\overline{U} = -V.$$

For $\Omega_\pm$ we have

$$T\overline{\Omega}_\pm = \pm\Omega_\pm .$$

The vectors $D_\pm$ given by (41) have the same symmetry as $\Omega_\pm$.

The theta function also satisfies a simple conjugation law

$$\overline{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)} = \theta\begin{bmatrix}-T\alpha\\ T\beta\end{bmatrix}(-T\bar{z}),$$

which is proved by the change of the summation index $m = -Tn$, taking into account the symmetry of the period matrix $\overline{B} = TBT$. Finally, we see that both theta functions in (31) as well as the one in the nominator of (32) are real. On the other hand, the theta function in the denominator of (32) is imaginary, since it is odd and $T\Delta_{1,2} = -\Delta_{1,2}$,

$$\begin{aligned}\overline{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_-)} &= \theta\begin{bmatrix}\Delta_1/2\\ -\Delta_2/2\end{bmatrix}(-T\overline{\Omega}_-)\\ &= \theta\begin{bmatrix}\Delta_1/2\\ -\Delta_2/2\end{bmatrix}(\Omega_-)\\ &= -\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_-).\end{aligned}$$

Since

$$e^{\pi ik/2}\frac{\theta(\Omega_\pm)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm)}$$

is real, using (37) we may choose the square root in (13) as

$$\Delta = 2e^{\pi ik/2}\frac{\theta(\Omega_\pm)}{\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm)}\left|\frac{\theta\begin{bmatrix}\Delta_1/2\\ 0\end{bmatrix}(u)\theta\begin{bmatrix}0\\ \Delta_2/2\end{bmatrix}(u)}{\theta\begin{bmatrix}\Delta_1/2\\ 0\end{bmatrix}(0)\theta\begin{bmatrix}0\\ \Delta_2/2\end{bmatrix}(0)}\right|. \qquad (44)$$

This shows that in this case formulas (12, 13) describe an analytic surface in $\mathbf{R}^3$. Due to Theorem 4 it is a Willmore surface. Taking into account the last factor in (36) we have

$$G_3 = \theta(\Omega_\pm)\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm)f(z,\bar{z}),$$

where $f(z,\bar{z})$ does not vanish. This means that the umbilic line $G_3 = 0$ on the z-plane is given by the equation

$$\theta(\Omega_\pm)\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm) = 0.$$

6. Simplest surfaces. The simplest possible case under consideration is the case of elliptic spectral curve

$$\mu^2 = \lambda(\lambda - E)(\lambda + E^{-1}), \qquad E \in \mathbf{R}.$$

The integral

$$\Omega_\infty + i\Omega_0 = \mu/\lambda \tag{45}$$

is a function, its periods vanish, and arguments of the theta function depend only on y for $k = 0$ and only on x for $k = 1$. We denote

$$\Omega_1 = \Omega_\infty - i\Omega_0 = 2\Omega_\infty - \mu/\lambda$$

an additional integral to (45). For $\omega_\pm$ in (36) we have

$$\omega_+ = \mu/\lambda x + i\Omega_1 y, \qquad \omega_- = \Omega_1 x + iy\mu/\lambda.$$

Let us consider the $k = 0$ case and introduce

$$v = u/(2i).$$

Combining formulas (12, 13) and (36, 37), we get

$$G_1 + iG_2 = \frac{\theta_3(Uy+v)\overline{\theta_3}(Uy-v) - \theta_1(Uy+v)\overline{\theta_1}(Uy-v)}{\theta_3(Uy-v)\overline{\theta_3}(Uy-v) + \theta_1(Uy-v)\overline{\theta_1}(Uy-v)}e^{2\omega_+},$$

$$G_3 = h\frac{\theta_3(Uy)\theta_1(Uy)}{\theta_3(Uy-v)\overline{\theta_3}(Uy-v) - \theta_1(Uy-v)\overline{\theta_1}(Uy-v)}e^{\omega_+ + \overline{\omega_+}}, \tag{46}$$

$$h = 2\left|\frac{\theta_2(v)\theta_4(v)}{\theta_2(0)\theta_4(0)}\right|,$$

where we used Jacobi theta functions [26] related with the ones we used before as

$$\theta_1(v) = -\theta\begin{bmatrix}1\\1\end{bmatrix}(2iv), \qquad \theta_2(v) = \theta\begin{bmatrix}1\\0\end{bmatrix}(2iv),$$

$$\theta_3(v) = \theta\begin{bmatrix}0\\0\end{bmatrix}(2iv), \qquad \theta_4(v) = \theta\begin{bmatrix}0\\1\end{bmatrix}(2iv),$$

$$\theta(v) = \theta(v, q), \qquad q = e^{B/2}.$$

In (46) U is as above the b-period (28) of Ω_∞.

Theta functional factors in (46) are periodic in y. Due to the exponential factors, formulas (46) generally describe surfaces of spiral shape. There are cases when these surfaces are especially symmetric. Let us take a curvature line parametrization: $\lambda = -1$. Due to (3, 22), the Hopf differential $A = \langle F_{zz}, N\rangle = -2G_3$ is real, and the parameter curves $x = const$, $y = const$ are curvature lines on the surface. There are two different cases:

(a) $\lambda = -1$, $E > 1$, both μ/λ and Ω_1 are imaginary,
(b) $\lambda = -1$, $E < 1$, both μ/λ and Ω_1 are real.

We see that (a) gives surfaces of revolution. An example[3] of this surface is in Figures 2a and 2b.

Family (b) consists of cones. If the ratio of the periods of the exponent and of the theta functions in (46) is rational, i.e.,

$$\frac{\Omega_1(\lambda = -1)}{U} \in \overline{\mathbf{Q}},$$

then the cone closes up. The case with minimal possible number of folds equal to 2 is shown in Figure 3.

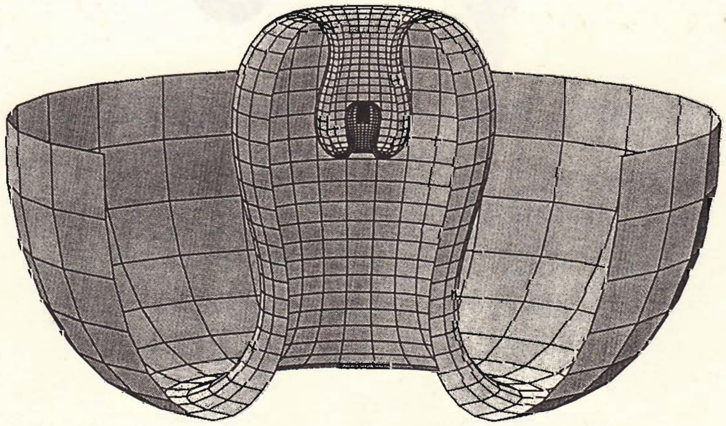

FIGURE 2a. Willmore surface of revolution, $E = 2$

[3] The examples shown in Figures 2, 3 were calculated using Mathematica [27].

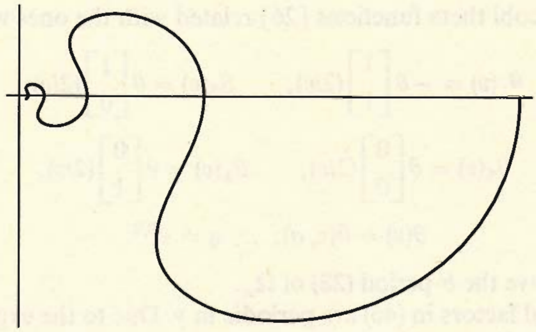

FIGURE 2b. Meridian curve

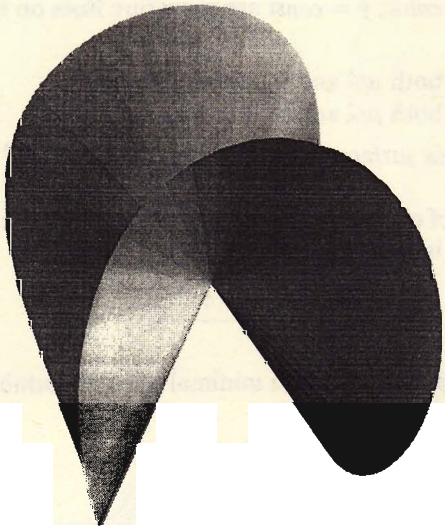

FIGURE 3. Willmore cone with two folds, $E = 0.53$

7. Periodicity conditions. In general the immersion $G(z, \bar{z})$ (12) determined by a real finite-gap solution is not periodic. We derive periodicity conditions simultaneously for both cases $k = 0$ and $k = 1$ using, as above, the notation $\pm$ to distinguish them.

Let Z be a period of the immersion

$$G(z + Z, \bar{z} + \bar{Z}) = G(z, \bar{z}). \tag{47}$$

It is a period of the metric $u(z, \bar{z})$. Due to the periodicity properties of theta functions, the vector $UZ \pm iV\bar{Z}$ must be a lattice vector

$$UZ \pm iV\bar{Z} = 2\pi iM + BN, \qquad M, N \in \mathbf{Z}^g.$$

Let us consider an abelian differential

$$d\hat{\omega}_\pm = d\Omega_\infty Z \pm i d\Omega_0 \bar{Z} - \langle du, N \rangle.$$

All periods of this differential are imaginary and proportional to $2\pi i$

$$\int_a d\hat{\omega}_\pm = -2\pi i N, \int_b d\hat{\omega}_\pm = 2\pi i M.$$

LEMMA 1. *The immersion $G(z, \bar{z})$ given by (12, 13, 36, 44) is a periodic function (47) with a period Z if and only if there exists a differential $d\hat{\omega}_\pm$ of the second kind such that*

(i) *the only singularities of $d\omega$ are at the points $\lambda = 0, \infty$ and they are of the form*

$$d\hat{\omega}_\pm = Zd(\sqrt{\lambda}), \qquad \lambda \sim \infty,$$

$$d\hat{\omega}_\pm = \pm i\bar{Z}d(1/\sqrt{\lambda}), \qquad \lambda \sim 0; \tag{48}$$

(ii) *for any closed cycle γ on X*

$$\frac{1}{2\pi i}\int_\gamma d\hat{\omega}_\pm \in \mathbf{Z}; \tag{49}$$

(iii)

$$\frac{1}{\pi i}\int_\infty^{\lambda = e^{2i\phi}} d\hat{\omega}_\pm \in \mathbf{Z}. \tag{50}$$

Proof. We already proved (i) and (ii). To prove (iii) we use the identities

$$\frac{\theta\left(u+\Omega_\pm+\begin{bmatrix}N\\M\end{bmatrix}\right)}{\theta\begin{bmatrix}\Delta_1/2\\\Delta_2/2\end{bmatrix}\left(\Omega_\pm+\begin{bmatrix}N\\M\end{bmatrix}\right)}$$

$$=\exp\{-\langle u,N\rangle+\pi i(\langle N,\Delta_2\rangle-\langle M,\Delta_1\rangle)\}\frac{\theta(u+\Omega_\pm)}{\theta\begin{bmatrix}\Delta_1/2\\\Delta_2/2\end{bmatrix}(\Omega_\pm)},$$

$$\frac{\theta\begin{bmatrix}\Delta_1/2\\\Delta_2/2\end{bmatrix}\left(u+\Omega_\pm+\begin{bmatrix}N\\M\end{bmatrix}\right)}{\theta\begin{bmatrix}\Delta_1/2\\\Delta_2/2\end{bmatrix}\left(\Omega_\pm+\begin{bmatrix}N\\M\end{bmatrix}\right)}=\exp\{-\langle u,N\rangle\}\frac{\theta\begin{bmatrix}\Delta_1/2\\\Delta_2/2\end{bmatrix}(u+\Omega_\pm)}{\theta\begin{bmatrix}\Delta_1/2\\\Delta_2/2\end{bmatrix}(\Omega_\pm)}, \tag{51}$$

$$\begin{bmatrix}N\\M\end{bmatrix}=2\pi iM+BN.$$

Under the shift $z\to z+Z$ the exponent in (36) acquires a factor

$$\exp\left\{\int_\infty^{\lambda=e^{2i\phi}}(Z\,d\Omega_\infty\pm i\bar{Z}\,d\Omega_0)\sigma_3\right\},$$

which jointly with (51) gives the transformation law for Φ

$$\Phi\to\begin{pmatrix}\pm1&0\\0&1\end{pmatrix}\Phi\exp\left\{\sigma_3\int_\infty^{\lambda=e^{2i\phi}}d\hat{\omega}_\pm\right\},$$

where $d\hat{\omega}_\pm$ are defined by (48).

THEOREM 8. *The immersion function $G(z,\bar{z})$ determined by (12, 13, 36, 44) describes an analytic Willmore surface M.*

(i) *M is a Willmore torus if and only if $G(z,\bar{z})$ is doubly periodic*

$$G(z+Z_1,\bar{z}+\bar{Z}_1)=G(z+Z_2,\bar{z}+\bar{Z}_2)=G(z,\bar{z}).$$

(ii) *$G(z,\bar{z})$ is doubly periodic if and only if there exist on X two abelian differentials of the second kind $d\hat{\omega}_\pm^1$ and $d\hat{\omega}_\pm^2$ (the same sign for both) satisfying the conditions (i)–(iii) (with Z_1 and Z_2 respectively) of Lemma 1.*

(iii) *All these Willmore tori have umbilic lines with preimages on the z-plane given by*

$$\theta\begin{bmatrix}\Delta_1/2\\ \Delta_2/2\end{bmatrix}(\Omega_\pm) = 0.$$

Proof. The properties (i), (ii) follow from Theorem 4 and Lemma 1. By the maximum principle there are no compact minimal surfaces in $\mathbf{H}^3$; therefore the tori constructed must intersect the infinity plane $G_3 = 0$. The metric $u(z, \bar{z})$ is singular on the umbilic line (see Section 3)

$$e^u \sim \frac{1}{|G_3|} \to +\infty .$$

Combined with (31, 32) this proves (iii).

There is a unique differential $d\hat{\omega}_\pm$ with the asymptotics (48), normalized by the condition that all its periods are imaginary. The number of periods is $4N + 2$, but due to

$$\overline{\tau^* \, d\hat{\omega}_\pm} = \mp d\hat{\omega}_\pm \tag{52}$$

only $2N + 1$ of them are independent. To prove (52) we compare the singularities of the both sides using (42).

Finally for a doubly periodic immersion we get $2(2N + 1)$ intrinsic periodicity conditions (49) ($2N + 1$ for $d\hat{\omega}^1_\pm$ and $2N + 1$ for $d\hat{\omega}^2_\pm$) and 4 (real) extrinsic periodicity conditions (50). All of them are conditions on the branch points of the spectral curve ($4N + 2$ real parameters) and on the periods Z_1, Z_2 (4 real parameters). The situation is quite similar to the case of constant mean curvature tori [5], [6]. The numbers of parameters and conditions coincide. It is not difficult to see that for any $N \geqslant 1$ a countable number of spectral curves exists defining distinct Willmore tori with umbilic lines. The case $N = 1$ is considered in Section 10. For general N this statement can be rigorously proved by the methods of [10].

There is no restriction on the vector $D_\pm$. The change of this vector in the directions transversal to the plane $Uz \pm iV\bar{z}$ changes the torus. This means that the Willmore tori constructed have $2N - 1$ commuting flows of deformations.

8. Willmore functional. Let Π be a fundamental region of the lattice Λ on the z-plane. It is a parallelogram determined by the vectors Z_1, Z_2.

The relation (15) allows us to express the Willmore functional for tori as

$$W = \int_{T^2} (H^e)^2 \, dS^e = -\int_{T^2} (K^h + 1) \, dS^h .$$

Here we used the minimality $H^h = 0$ and the Gauss-Bonnet theorem for the tori

$\int_{T^2} K^e\, dS^e = 0$. On the other hand, the normalization (22) gives

$$K^h + 1 = -e^{-2u},$$

which together with $dS^h = 4e^u\, dx\, dy$, $z = x + iy$ yields

$$W = 4\int_{\Pi} e^{-u}\, dx\, dy. \tag{53}$$

Fortunately, it is possible to perform integration in (53) explicitly and hence to obtain a more effective formula for W. The substitution of the asymptotics

$$\psi_{1,2} = (1 + b_{1,2}\nu^{-1} + \cdots)e^{z\nu}, \qquad \nu \to \infty$$

in the equation $\psi_{\bar z} = V\psi$ gives

$$e^{-u} = -b_{1\bar z}. \tag{54}$$

On the other hand, a direct calculation, using (29), leads to the formula for $b_{1\bar z}$:

$$\begin{aligned}\frac{\partial}{\partial \bar z} b_1 &= \frac{\partial}{\partial \bar z}\frac{\partial}{\partial p}\log\left\{\frac{\theta(\Omega + u)\theta(D)}{\theta(\Omega + D)\theta(\Omega)} e^{\pm i\bar z k p}\right\}\Bigg|_{u=0}\\ &= \pm ik - \frac{\partial}{\partial \bar z}\frac{\partial}{\partial p}\log\theta(\Omega + u)\Bigg|_{u=0}\\ &= \pm ik + \frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}\log\theta(\Omega),\end{aligned} \tag{55}$$

where $p = \nu^{-1}$ is a local parameter at $\lambda = \infty$, and k is defined by the condition that $d\Omega_0 = k\, dp$ at $\nu = \infty$. To derive (55) we used the form of Abel's map near $\lambda = \infty$

$$u = -Up, \tag{56}$$

which is proved in a standard way using a reciprocity law [11] for $d\Omega_\infty$ and du. Because of (56) the derivative $\partial/\partial p$ in (55) may be replaced by

$$\partial/\partial p \to -\partial/\partial z.$$

A reciprocity law also allows us to define k in a more convenient way

$$d\Omega_\infty = k\, d\nu, \qquad \nu \to 0.$$

Applying the Stokes formula to (53, 54, 55) we obtain the following theorem.

THEOREM 9. *For the constructed Willmore tori the Willmore functional is equal to*

$$W = \int (H^e)^2\, dS^e = \mp 4ikS(\Pi),$$

where $S(\Pi)$ *is the area of fundamental parallelogram* Π.

9. Tori of rectangular conformal type. From now on we consider an asymptotic line parametrization of minimal surfaces in Q

$$\lambda = e^{2i\phi} = i.$$

Due to (3, 22), $A = \langle F_{zz}, N\rangle = 2iG_3$ is imaginary, and curvature of the parameter lines $x = const$, $y = const$ on the surface is equal to the mean curvature. This is the mean curvature line parametrization of surfaces in $\mathbf{R}^3$.

The case of tori of rectangular conformal type can be formulated in terms of the spectral curve. Namely, in this case X possesses a holomorphic involution

$$\lambda \to -1/\lambda.$$

The equation of X is

$$\mu^2 = \lambda^{2N+2} \prod_{k=1}^{2N+1} (\lambda - \lambda^{-1} - e_k), \tag{57}$$

where the set of the points e_k is invariant with respect to complex conjugation. The coverings $X \to X_1$ and $X \to X_2$ are given by

$$X_1 \equiv \left\{(s, t_1) | t_1^2 = \prod_{k=1}^{2N+1} (s - e_k)\right\},$$

$$X_2 \equiv \left\{(s, t_2) | t_2^2 = (s^2 + 4) \prod_{k=1}^{2N+1} (s - e_k)\right\}, \tag{58}$$

$$s = \lambda - \lambda^{-1},\ t_1 = \mu\lambda^{-N-1},\ t_2 = \mu\lambda^{-N-1}(\lambda + \lambda^{-1}).$$

Now the two integrals $d\hat{\omega}_\pm^{1,2}$ we are looking for can be considered as integrals on X_1 and X_2 with singularities at $s = \infty$. Analysis of the singularities with the help of (43, 58) shows that the following theorem holds.

THEOREM 10. *The following specialization of the spectral data determines Willmore tori, as above, of rectangular conformal type* ($Z_{1+} = X_+$, $Z_{2+} = iY_+$; $Z_{1-} = X_-$, $Z_{2-} = iY_-$, *where* $\pm$ *denote two families of solutions as above*):

(1) *The spectral curve is of the form (57),*
(2) $d\hat{\omega}^{1,2}_{\pm}$ *are differentials on* $X_{1,2}$ *respectively with all periods integer multiples of* $2\pi i$ *and the asymptotics at* $s = \infty$:

$$d\hat{\omega}^1_+ = X_+ d\left(\frac{s^{N+1}}{t_1}\right), \quad d\hat{\omega}^1_- = iY_- d\left(\frac{s^{N+1}}{t_1}\right), \tag{59}$$

$$d\hat{\omega}^2_+ = iY_+ d\left(\frac{s^{N+2}}{t_2}\right), \quad d\hat{\omega}^2_- = X_- d\left(\frac{s^{N+2}}{t_2}\right), \tag{60}$$

(3)

$$\frac{1}{\pi i}\int_{\infty}^{s=2i} d\hat{\omega}^1_{\pm} \in \mathbf{Z}. \tag{61}$$

The mean curvature lines of these tori are closed.

To finish the proof, we mention that the extrinsic periodicity conditions (50) for $d\hat{\omega}^2_{\pm}$ are automatically satisfied since $s = 2i$ is a branch point of X_2.

Again we have equal numbers $2N + 3$ of parameters (e_k, X, Y) and conditions ($2N + 1$ intrinsic and 2 extrinsic (61)), which isolate a discrete set of the spectral curves.

10. Simplest Willmore tori with umbilic lines. We consider the symmetric case $N = 1$ in more detail. We suppose that not all branch points of X are real

$$\mu^2 = \lambda^4(\lambda - \lambda^{-1} - e_1)(\lambda - \lambda^{-1} - e_2)(\lambda - \lambda^{-1} - \overline{e_2}), \qquad e_1 \in \mathbf{R}. \tag{62}$$

The genera of X_1 and X_2

$$t_1^2 = (s - e_1)(s - e_2)(s - \overline{e_2}),\; t_2^2 = (s^2 + 4)(s - e_1)(s - e_2)(s - \overline{e_2}) \tag{63}$$

are equal to 1 and 2 respectively.

10.1. Intrinsic periodicity. Since $Jac(X_1)$ is one dimensional, $u_+(z, \bar{z})$ is always periodic in the x-direction, and $u_-(z, \bar{z})$ is periodic in the y-direction. The periodicity in another orthogonal direction is described by the following theorem.

THEOREM 11. *The solutions* $u_{\pm}(z, \bar{z})$ *of equation (23), generated by the curve (62), are doubly periodic if and only if* q *is rational*

$$q = \frac{\int_{l^1_{\pm}} ds/t_2}{\int_{l^2_{\pm}} ds/t_2} \in \overline{\mathbf{Q}} \equiv \mathbf{Q} \cup \{\infty\}, \tag{64}$$

where $l^{1,2}_{\pm}$ *are indicated in Figure 4.*

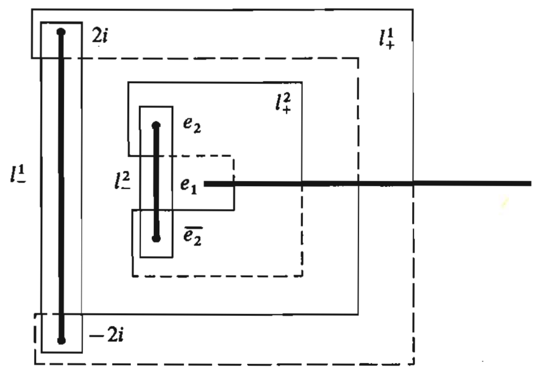

FIGURE 4

Proof. In Section 7 we saw that existence of the differential of the second kind with all periods integer multiples of $2\pi i$ is equivalent to existence of the normalized differential with the same singularity, b-period of which belong to the lattice of the Jacobian. Let us denote by $d\tilde{\omega}_\pm$ the corresponding normalized differentials on X_2 with the singularities (60). Then the condition

$$\int_{b_n} d\tilde{\omega}_\pm = \int_{\gamma_\pm} d\tilde{u}_n \tag{65}$$

must be satisfied, where $d\tilde{u}_n$ are the normalized holomorphic differentials of X_2, and $\gamma_\pm$ are some cycles on X_2. On the other hand a reciprocity law [11] allows us to express b-periods of $d\tilde{\omega}_\pm$ in terms of $d\tilde{u}_n$

$$\int_{b_n} d\tilde{\omega}_+ = -iY_+ \frac{d\tilde{u}_n}{d(s^{-3}t_2)}\bigg|_{s=\infty}, \int_{b_n} d\tilde{\omega}_- = -X_- \frac{d\tilde{u}_n}{d(s^{-3}t_2)}\bigg|_{s=\infty}. \tag{66}$$

To use reality arguments, we choose the a-cycles coinciding with $l^{1,2}_-$ shown in Figure 4. The normalized differentials change sign $\tau^* \overline{d\tilde{u}} = -d\tilde{u}$ under the action of

$$\tau\colon (s, t_2) \to (\bar{s}, -\bar{t}_2).$$

For the periods of $d\tilde{\omega}_\pm$ we have

$$\int_{b_n} d\tilde{\omega}_+, i\int_{b_n} d\tilde{\omega}_- \in \mathbf{R},$$

which implies that $\gamma_\pm$ must be decomposable with respect to $l^{1,2}_\pm$ respectively $(\tau l^{1,2}_\pm = \mp l^{1,2}_\pm)$. Combining (65, 66) with the decomposition argument above, we get

$$\begin{pmatrix} -iY_+ \\ -X_- \end{pmatrix} \frac{d\tilde{u}_n}{d(s^{-3}t_2)}\bigg|_{s=\infty} = q_\pm^1 \int_{t_\pm^1} d\tilde{u}_n + q_\pm^2 \int_{t_\pm^2} d\tilde{u}_n \tag{67}$$

with $q_\pm^{1,2}$ an integer. The relation (67) is independent of the choice of basis of holomorphic differentials. Written down in the basis ds/t_2, $s\,ds/t_2$ it gives (64) and determines the periods Y_+, X_-.

10.2. Extrinsic periodicity. To get a torus it is now enough to satisfy the extrinsic periodicity condition for $d\hat{\omega}_\pm^1$ only, since for $d\hat{\omega}_\pm^2$ they are automatically satisfied (see Section 8). Let us consider the elliptic curve

$$\check{t}_1^2 = \check{s}(\check{s} - e^{i\delta})(\check{s} - e^{-i\delta}) \tag{68}$$

and the differential $d\check{\omega}_\pm$ on it with the asymptotics

$$d\check{\omega}_+ = \check{X}_+ d(\check{s}^2\check{t}^{-1}),\ d\check{\omega}_- = i\check{Y}_- d(\check{s}^2\check{t}^{-1}), \qquad \check{s} \sim \infty$$

and all periods imaginary. The real part of the integral $\int_\infty^P d\check{\omega}_\pm$ is a well-defined function. It is not difficult to prove the following lemma.

LEMMA 2. *The zero sets of* $Re \int_\infty^P d\check{\omega}_\pm$

$$S_\pm = \left\{P \,\middle|\, Re \int_\infty^P d\check{\omega}_\pm = 0\right\} \tag{69}$$

have the following properties:

(1) *they consist of two ovals* $S_\pm = S_\pm^1 \cup S_\pm^2$;
(2) *the* $S_\pm$ *are invariant with respect to the involution* $\check{t} \to -\check{t}$;
(3) *the projections* $\check{s}(S_\pm^1)$, $\check{s}(S_\pm^2)$ *to the* $\check{s}$*-plane are curves connecting* (e_1, ∞) *and* $(e_2, \bar{e}_2)$ *respectively;*
(4) *the* $\check{s}(S_\pm^1)$ *lie on the real axis, and* $\check{s}(S_\pm^2)$ *intersect the real axis at one point only.*

Let us choose $\check{X}_+$ and $\check{Y}_-$ such that (see Figure 5)

$$\int_\beta d\check{\omega}_+ = \int_\alpha d\check{\omega}_- = 4\pi i. \tag{70}$$

Then all periods of $d\check{\omega}_\pm$ are imaginary and proportional to $2\pi i$; i.e., the intrinsic periodicity conditions are satisfied. Let us take some "rational" point P^0 on $S_\pm^2$

$$\int_\infty^{P^0} d\check{\omega}_\pm = \pi i p, \qquad p = n/m \in \mathbf{Q},\ n, m \in \mathbf{Z}. \tag{71}$$

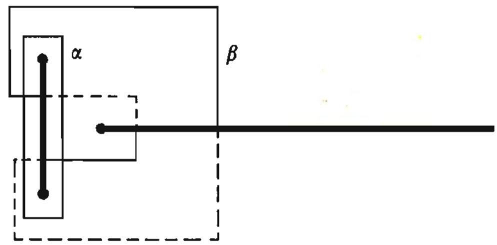

FIGURE 5

The transformation

$$\check{s} \to s = 2(\check{s} - P_R^0)/P_I^0, \qquad P^0 = P_R^0 + iP_I^0, \tag{72}$$

reduces the curve (68) to the form (63) with both intrinsic and extrinsic periodicity conditions satisfied by the differential $d\hat{\omega}_\pm = m\sqrt{P_I^0/2}\, d\check{\omega}_\pm$, $(X_+ = m\sqrt{P_I^0/2}\check{X}_+$, $Y_- = m\sqrt{P_I^0/2}\check{Y}_-)$.

The restriction (64) can be rewritten in terms of the modulus of the elliptic curve (68)

$$q_p(\delta) = \frac{\int_{l_\pm^1} \check{t}_2^{-1}\, d\check{s}}{\int_{l_\pm^2} \check{t}_2^{-1}\, d\check{s}} \in \overline{\mathbf{Q}}, \qquad \check{t}_2^2 = \check{s}(\check{s} - e^{i\delta})(\check{s} - e^{-i\delta})(\check{s} - P^0)(\check{s} - \bar{P}^0), \tag{73}$$

where $l_\pm^{1,2}$ are the same as in Figure 4. Elliptic curves (68) with rational $q_p(\delta)$ determine Willmore tori with umbilics. The condition (73) is nondegenerate since $q_p(\delta)$ is an analytic function different from a constant. The last fact for the plus family u_+ is proved in the appendix by consideration of the limit $\delta \to 0$.

Finally, the simplest Willmore tori with umbilic lines are described by the following theorem.

THEOREM 12. *Elliptic curve (68) with the conditions (70, 71, 73) satisfied generate a one-parameter family of Willmore tori with umbilic lines. As above, they are described by the formulas (12, 13, 36, 44) where the spectral curve (62) is of genus 3 and is related to the elliptic curve (68) by (63, 72). The mean curvature lines of these tori are closed.*

11. Branch point case. The formulas for the immersion obtained are not valid when the point

$$\lambda = e^{2i\phi} = E_1$$

is a branch point of the spectral curve. In this case columns in (33) coincide and $\det \Phi$ vanishes.

Simple regularization gives a correct answer. Let $p = \sqrt{\lambda - E_1}$ be a local parameter at $\lambda = e^{2i\phi}$. We take the function

$$\Psi_{bp} = \begin{pmatrix} \partial\psi_1/\partial p & \psi_1 \\ \partial\psi_2/\partial p & \psi_2 \end{pmatrix}(\lambda = E_1) = \frac{1}{2}\Psi\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix}\Bigg|_{p=0} \tag{74}$$

instead of (33). It satisfies the equations (30) and

$$\det \Psi_{bp} = \frac{\partial}{\partial p} \det \Psi, \tag{75}$$

where $\det \Psi$ is given by (35).

The same transformation (34) provides us with a regularized function Φ:

$$\Phi_{bp} = \frac{1}{\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm)} \times \begin{bmatrix} ve^{\pi ik/2}\dfrac{\partial}{\partial p}\{\theta(\Omega_\pm + u)e^{\omega_\pm} - \log\theta(D+u)\} & ve^{\pi ik/2}\theta(\Omega_\pm - u)e^{-\omega_\pm} \\ \dfrac{\partial}{\partial p}\left\{\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm + u)e^{\omega_\pm} - \log\theta(D+u)\right\} & -\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm - u)e^{-\omega_\pm} \end{bmatrix}, \tag{76}$$

$$\det \Phi_{bp} = -2ve^{\pi ik/2}c_{bp}\frac{\theta(\Omega_\pm)}{\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm)},$$

$$c_{bp} = \frac{\dfrac{\partial}{\partial p}\left\{\theta\begin{bmatrix} \Delta_1/2 \\ 0 \end{bmatrix}(u)\theta\begin{bmatrix} 0 \\ \Delta_2/2 \end{bmatrix}(u) - \log(\theta(D-u)\theta(D+u))\right\}}{\theta\begin{bmatrix} \Delta_1/2 \\ 0 \end{bmatrix}(0)\theta\begin{bmatrix} 0 \\ \Delta_2/2 \end{bmatrix}(0)}.$$

As before (44), we choose the analytic square root (13)

$$\Delta_{bp} = 2e^{\pi ik/2}|c_{bp}|\frac{\theta(\Omega_\pm)}{\theta\begin{bmatrix} \Delta_1/2 \\ \Delta_2/2 \end{bmatrix}(\Omega_\pm)}. \tag{77}$$

THEOREM 13. *If $\lambda = e^{2i\phi}$ is a branch point of the spectral curve, then the corresponding Willmore surface is described by the formulas (12, 13, 76, 77). This* *immersion is doubly periodic if and only if there exist on X two independent abelian differentials of the second kind* $d\hat{\omega}^1_{\pm}$ *and* $d\hat{\omega}^2_{\pm}$ *with the following properties:*

(i) *singularities of the form (48) (with* Z_1 *and* Z_2 *respectively),*
(ii) *all periods being integer multiples of* $2\pi i$,
(iii) *(extrinsic periodicity) vanishing at* $\lambda = e^{2i\phi}$

$$d\hat{\omega}^{1,2}_{\pm}(\lambda = e^{2i\phi}) = 0. \tag{78}$$

Proof. As above, (ii) is the intrinsic periodicity condition. It guarantees the double periodicity of the metric and of the theta functions and their derivatives in (12, 13, 76). Formulas (12, 13, 76) show that there is a linear term (in z and $\bar{z}$) in $G_1 + iG_2$ coming from the derivative $\partial\omega_{\pm}/\partial p$ in (76). Generally, the surface determined by (12, 13, 76) with intrinsic periodicity conditions satisfied possesses two translational periods parallel to the absolute. The condition of vanishing of these periods is (iii).

Appendix

Let us consider the curve X_1 (63) with modulus

$$k = \sin(\tfrac{1}{2}\arg(e_1 - \bar{e}_2)) \in (0, 1)$$

and introduce the notations

$$p = k^{3/2}|e_2 - e_1|, e = e_1 - |e_2 - e_1|.$$

We consider the case

$$k \to 0, e \to 0, \qquad p \in \mathcal{D}_p = (1/2, 3/2). \tag{79}$$

The known analytic properties of the normalized ($\int_\alpha = 0$; see Figure 5) elliptic integral

$$\mathcal{F} = \frac{X}{2}\int^s \frac{s - s_0}{t_1}\, ds$$

with period $4\pi i = \int_\beta d\mathcal{F}$ allow us to represent it as

$$\mathcal{F} = 2K(Z(u, k) - ik\,\mathrm{cn}(u, k)) - \pi i, v = 2u + iK',$$

$$\mathrm{sn}(v, k) = ih/\sqrt{1 - 2kh}, \mathrm{cn}(v, k) = \sqrt{1 - 2kh + h^2}/\sqrt{1 - 2kh},$$

$$h = \sqrt{k}(s - e)/(2p).$$

Here we use the standard notations for the elliptic functions and integrals. ($Z(u, k)$

is the Jacobi zeta function.) For finite s

$$|s - e| < const,$$

where the constant is independent of k. This implies $h \to 0$ and the asymptotics ([15], page 906) for $(\mathscr{F}/\pi i)^2$ hold:

$$(\mathscr{F}/\pi i)^2 = k^2(1 - kh + h^2)(1 + O(k^{5/2})). \tag{80}$$

Requiring Re $\mathscr{F}$ $(s = 2i) = 0$ gives

$$e = -\sqrt{k}p(1 + O(k^2)), \tag{81}$$

which agrees with (79). Substituting (81) into (80) yields the following lemma.

LEMMA 3. *For $k \to 0$ it is always possible to choose the branch points ($e \approx -\sqrt{k}p$) such that $\mathscr{F}(s = 2i)$ is imaginary. In this case*

$$q_1 = \left(\frac{\mathscr{F}(s = 2i)}{\pi i}\right)^2 = k^2(1 + kO(1)), \tag{82}$$

where $O(1)$ is uniformly bounded for $p \in \mathscr{D}_p$.

Let us calculate now the asymptotics of the ratio (64) for $k \to 0$. We set

$$q_2 = \frac{N}{M} = \frac{\displaystyle\int_{2i}^{e_2} t_2^{-1}\, ds + \int_{2i}^{\bar{e}_2} t_2^{-1}\, ds}{\displaystyle\int_{e_1}^{e_2} t_2^{-1}\, ds + \int_{e_1}^{\bar{e}_2} t_2^{-1}\, ds}. \tag{83}$$

It is evident that rationality of q_2 is equivalent to rationality of (64). To calculate the numerator of (83), we make a fractional linear transformation $s \to s_1$, mapping the points $\pm 2i, e_2, \bar{e}_2$ to the points ± 2, $\pm p_1$ ($p_1 \in \mathbf{R}$) respectively. This transformation implies

$$\begin{gathered}
\frac{ds}{t_2} = c_1 \frac{ds_1}{\sqrt{(s_1^2 - 4)(s_1^2 - p_1^2)}} \sqrt{\frac{s_1 + ig_1}{s_1 + if_1}}, \\
c_1 = -(g_1^2 + 4)((g_1^2 + 4)((g_1 - a)^2 + b^2)(g_1 - e_1))^{-1/2}, \\
p_1 = \frac{b(g_1^2 + 4)}{(a - g_1)^2 + b^2}, f_1 = \frac{g_1^2 + 4}{e_1 - g_1} + g_1, \\
g_1 = r_1\left(1 + \sqrt{1 + \frac{4}{r_1^2}}\right), r_1 = \frac{b^2 + a^2 - 4}{2a},
\end{gathered} \tag{84}$$

where we use the notation

$$e_2 = a + ib.$$

Similarly, to calculate the denominator of (83) we make a fractional linear transformation $s \to s_2$, mapping the points $a \pm ib$, e_1, ∞ to the points $\pm ib$, $\pm f_2$ ($f_2 \in \mathbf{R}$) respectively. This transformation implies

$$\frac{ds}{t_2} = ic_2 \frac{ds_2}{\sqrt{(s_2^2 + b^2)(s_2^2 - f_2^2)}} \frac{s_2 - f_2}{\sqrt{(s_2 - a_2)^2 + b_2^2}}, \tag{85}$$

$$ic_2 = -i((a - e_1 + r_2)^2 + b^2)((f_2^2 + 4)((f_2 - a)^2 + b^2)(f_2 - e_1))^{-1/2},$$

$$r_2^2 = (e_1 - a)^2 + b^2, f_2 = -e_1 + r_2 + a,$$

$$a_2 = f_2 \frac{r_2^2 - e_1^2 - 4}{(r_2 - e_1)^2 + 4}, b_2 = \frac{4f_2 r_2}{(r_2 - e_1)^2 + 4}.$$

If (k, p) is such that $\operatorname{Re} \mathscr{F}(s = 2i) = 0$ and the conditions (79) hold, we have

$$c_1 = -k^{3/4}(3p)^{-1/2}(1 + 2kp^{-2}/3 + O(k^2)),$$

$$c_2 = p^{1/2}k^{1/4}(1 - kp^2/8 + O(k^2)),$$

$$g_1 = 4pk^{-3/2}(1 + kp^{-2} + O(k^2)),$$

$$f_1 = -\tfrac{4}{3}pk^{-3/2}(1 - kp^{-2} + O(k^2)),$$

$$f_2 = -2pk^{1/2}(1 + O(k^2)),$$

$$a_2 = p^3k^{-1/2}(1 - k(p^2/4 + 2p^{-2}) + O(k^2)),$$

$$b_2 = 2p^2k^{-1}(1 - kp^2/4 + O(k^2)).$$

Substituting the Taylor series for

$$\sqrt{\frac{s_1 + ig_1}{s_1 + if_1}}, \frac{s_2 - f_2}{\sqrt{(s_2 - a_2)^2 + b_2^2}}$$

in (84, 85) we represent the nominator and denominator of (83) in the form of series in k. The coefficients in these series are expressed in terms of complete elliptic integrals of the curves

$$w_1^2 = (s_1^2 - 4)(s_1^2 - p_1^2), \qquad w_2^2 = (s_2^2 + b^2)(s_2^2 - f_2^2).$$

Finally, we get

$$N = -4i\frac{k^{3/4}}{\sqrt{3p}}\left(1+\frac{k}{6p^2}+O(k^2)\right)\left(\frac{\sqrt{k}}{2p}K(k_1)-\frac{17}{32}k^{5/2}E(k_1)\right)$$

$$= -\frac{2}{\sqrt{3}}ik^{5/4}p^{-3/2}\log\frac{4p}{\sqrt{k}}\left(1+\frac{k}{12p}\left(5-\frac{3}{\log(4p/\sqrt{k})}\right)(1+O(k^2)\right),$$

$$M = -ip^{-3/2}k^{7/4}\left(1+\frac{kp^2}{2}\right)\left(K'(k)-\frac{\sqrt{k}}{2}(1-kp^2)E'(k)\right)(1+O(k^2))$$

$$= -ip^{-3/2}k^{9/4}\log\frac{4}{k}\left(1+\frac{\sqrt{k}}{2\log(4/k)}+\frac{kp^2}{2}\right)(1+O(k^{2-\varepsilon})),$$

where

$$k_1 = \sqrt{1-k_1'^2}, \qquad k_1' = 2/p_1 \to 0.$$

LEMMA 4. *In the conditions of the previous lemma the asymptotics for q_2 are valid:*

$$q_2 = \frac{4}{\sqrt{3}}k\left(1-\frac{\log(4p^2)-\sqrt{k}/2}{\log(4/k)}\right)(1+O(k)). \tag{86}$$

Formulas (82, 86) show that the map $(k, p) \to (q_1, q_2)$ is nondegenerate and there exist (k, p) generating rational pairs q_1, q_2. This completes the proof of the existence of the tori.

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