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INTEGRABLE SURFACES

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Let $\Omega \subset R^2 = C$ be a domain and $F: \Omega \to R^n$ be a conformal parametrization of a surface in R^n: $\langle F_z, F_z\rangle = \langle F_{\bar z}, F_{\bar z}\rangle = 0$, $z = x + iy$, $(x, y) \in \Omega$; $4e^u dz d\bar z$ be the induced metric on Ω, $\langle F_z, F_{\bar z}\rangle = 2e^u$. We complete F_x, F_y to a basis in R^n by vectors $i = 1, \ldots, m$; $m = n - 2$, orthogonal to the surface:

$$\langle N_i, N_j\rangle = \delta_{ij},\ \langle N_i, F_x\rangle = \langle N_i, F_y\rangle = 0.$$

This basis satisfies the Gauss–Weingarten equations:

$$\sigma_z = U\sigma,\ \sigma_{\bar z} = V\sigma,\quad \sigma = \begin{pmatrix} F_z \\ F_{\bar z} \\ N \end{pmatrix},\ N = \begin{pmatrix} N_1 \\ \vdots \\ N_m \end{pmatrix},$$

$$U = \begin{pmatrix} u_z & 0 & A \\ 0 & 0 & B \\ -e^{-u}B^T/2 & -e^{-u}A^T/2 & C \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 & B \\ 0 & u_{\bar z} & \bar A \\ -e^{-u}\bar A^T/2 & -e^{-u}B^T/2 & \bar C \end{pmatrix},$$

where $A_i = \langle F_{zz}, N_i\rangle$, $B_i = \langle F_{z\bar z}, N_i\rangle$, $C_{ij} = \langle N_{iz}, N_j\rangle$. The compatibility conditions

$$U_{\bar z} - V_z + [U, V] = 0 \tag{1}$$

are the Gauss–Peterson–Codazzi equations of the surface.

We call a surface <u>integrable</u> if (1) is integrable in the usual sense of the theory of solitons, i.e., the equations can be represented as compatibility conditions (1) with $U(\lambda)$, $V(\lambda)$, depending on an additional (spectral) parameter λ. One should note that in this formulation the problem has already been studied in [1-3] where in particular a number of examples were constructed.

<u>THEOREM.</u> Surfaces with mean curvature vector $H = e^{-u}B/2$ such that $H_z + HC = 0$ are integrable. The transformation $A \to \lambda A$, $\bar A \to 1/\lambda\, \bar A$ turns (1) into the usual representation of zero curvature with spectral parameter.

<u>Example 1.</u> Surfaces of constant mean curvature (CMC) (H = const) in R^3. Here $C = 0$, the equations (1) have the form

$$u_{z\bar z} + 2H^2 e^u - |A|^2 e^{-u}/2 = 0,\quad A_{\bar z} = 0.$$

<u>Example 2.</u> Surfaces of CMC in S^3, $\langle F, F\rangle = 1$. One can choose $N_2 = F$, whence $C = 0$, $A_2 = \langle F_{zz}, F\rangle = 0$, $H = (h, -1)$ = const. The equations (1) have the form

$$u_{z\bar z} + 2e^u(1 + h^2) - |A_1|^2 e^{-u}/2 = 0,\ A_{1\bar z} = 0.$$

<u>Example 3.</u> Minimal surfaces in $S^{n-1} \subset R^n$, $\langle F, F\rangle = 1$. The equation of minimality of the surface in S^n has the following form: $F_{z\bar z} = -2e^u F$. Choosing $N_m = F$, we have $H = (0, \ldots, 0, -1)$, $A_m = \langle F_{zz}, F\rangle = 0$, $C_{mi} = \langle F_z, N_i\rangle = 0$. In the new notation $\alpha = (A_1, \ldots, A_{m-1})$, $Q_{pq} = C_{pq}$, $p, q = 1, \ldots, m-1$ we get

$$\begin{gathered} u_{z\bar z} + 2e^u - \alpha\bar\alpha^T e^{-u}/2 = 0, \\ \alpha_{\bar z} + \alpha\bar Q = 0,\quad \bar\alpha_z + \bar\alpha Q = 0, \\ Q_{\bar z} - \bar Q_z + (\bar\alpha^T\alpha - \alpha^T\bar\alpha)e^{-u}/2 + [Q, \bar Q] = 0. \end{gathered} \tag{2}$$

In the following two examples we consider surfaces in Lorentz space $R^{n,1}$.

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Example 4. Surfaces of CMC in H^3. The metric of $R^{3,1}$ induces a metric $\{,\}$ on H^3: $\{F, F\} = F_1^2 + F_2^2 + F_3^2 - F_4^2 = -1$. We fix $N_1 = N$, $N_2 = F$ just as for S^3, $\{N, N\} = 1$, $\{F_z, F_{\bar{z}}\} = 2e^u$. Then (1) is true but with the substitution $\langle,\rangle \to \{,\}$ for A, B, and C. For surfaces of CMC in H^3 we have $C = 0$,

$$A_2 = 0,\quad H = (h, 1) = \text{const},\quad u_{z\bar{z}} + 2(h^2 - 1)e^u - |A_1|^2 e^{-u}/2 = 0,\quad A_{1\bar{z}} = 0.$$

Example 5. Minimal surfaces in $H^{n-1} \subset R^{n-1,1}$, $\{F, F\} = F_1^2 + \ldots + F_{n-1}^2 - F_n^2 = -1$. The choice of $N_m = F$ and the equation of a minimal surface $F_{z\bar{z}} = 2e^u F$ give nonlinear equations almost the same as in the case of S^{n-1}, but the first equation of (2) looks somewhat different:

$$u_{z\bar{z}} - 2e^u - \alpha\bar{\alpha}^T e^{-u}/2 = 0.$$

For H^4 integrability is proved in [2].

One can simplify (1) by choosing the coordinate z and the N-basis in a special way. In Examples 1, 2, and 4, if Ω is a neighborhood of a nonumbilical point ($A \neq 0$) then thanks to the analyticity of the function A one can assure, by a suitable analytic substitution $z \to w(z)$ that $A = \langle F_{w,w}, N\rangle = 1$. The situation is also analogous for minimal surfaces in S^n, H^n. The equations (2) show that $\alpha\alpha^T$ is an analytic function. Hence a suitable analytic substitution $z \to w(z)$ carries the condition $\langle F_{zz}, F_{zz}\rangle \neq 0$ (or $\{F_{zz}, F_{zz}\} \neq 0$) into the condition $\alpha\alpha^T = 1$.

We consider the cases S^4 and S^5 in more detail. We fix an N-basis so that α_1, $i\alpha_2 \in R$ ($\alpha_3 = 0$ for S^5). Combined with $\alpha\alpha^T = 1$ this gives $\alpha_1 = \mathrm{ch}(v/2)$, $\alpha_2 = -i\,\mathrm{sh}(v/2)$, where $v(z, \bar{z})$ is a real-valued function. Thus, minimal surfaces in S^4 can be described by the equation*

$$u_{z\bar{z}} + 2e^u - \mathrm{ch}\, v \cdot e^{-u}/2 = 0,\quad v_{z\bar{z}} + \mathrm{sh}\, v \cdot e^{-u}/2 = 0.$$

This is a special reduction of a two-dimensional Toda chain [4]. For S^5 we have $Q_{12} = -iv_z/2$, $Q_{13} = w\,\mathrm{sh}(v/2)$, $Q_{23} = iw\,\mathrm{ch}(v/2)$, where $w(z, \bar{z})$ is a complex-valued function, and the equations (2) look as follows:

$$u_{z\bar{z}} + 2e^u - \mathrm{ch}\, v \cdot e^{-u}/2 = 0,\quad v_{z\bar{z}} + \mathrm{sh}\, v\,(e^{-u}/2 - w\bar{w}) = 0,$$
$$w_{\bar{z}} + (\bar{w}\,\mathrm{ch}\, v)_z = 0,\quad \bar{w}_z + (w\,\mathrm{ch}\, v)_{\bar{z}} = 0.$$

It is particularly important that the integrable differential equations given above are not only applicable to differential geometry in the small, but also to differential geometry in the large. In particular, to an immersed torus corresponds a doubly-periodic function $F(z, \bar{z})$. In addition the functions A, A_1, $\alpha\alpha^T$ in the examples are elliptic functions and the condition that they are not identically zero and are bounded leads to the same integrable equations. For example, tori of CMC in R^3 and S^3 can be described with the help of doubly-periodic solutions of the elliptic sh-Gordon equation so that the corresponding immersion function $F(z, \bar{z})$ is also double-periodic. With the help of such a solution a torus of CMC was first constructed in [5]. All such tori are classified in [6]. Finally, in [7], using finite-zone integration all tori of CMC in R^3 and S^3 are explicitly described in terms of theta-functions. The immersion function itself can be expressed here in terms of the Baker–Akhiezer function which made it possible to get formulas for $F(z, \bar{z})$.

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LITERATURE CITED

1. M. V. Savel'ev, Teor. Mat. Fiz., 60, No. 1, 9-23 (1984).
2. B. M. Barbashov and V. V. Nesterenko, Fiz. Elem. Chastits At. Yadra, 15, No. 5, 1032-1072 (1984).
3. A. Sym, Preprint IFT-3-85 (1985).
4. A. P. Fordy and J. Gibbons, Commun. Math. Phys., 77, No. 1, 21-31 (1980).
5. H. C. Wente, Pacific J. Math., 121, No. 1, 193-244 (1986).
6. U. Pinkall and I. Sterling, Preprint Techn. Univ. Berlin, No. 199 (1988).
7. A. I. Bobenko, Preprint LOMI, E-3-89 (1989).