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All constant mean curvature tori in R^3, S^3, H^3 in terms of theta-functions

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1 Introduction

Let us consider a small domain Π in the complex plane, $z\in\Pi$ and an immersion $F:\Pi\to R^3$ of Π into Euclidean space. It is well known that if F is a conformal parametrization of a surface with constant mean curvature (CMC) and Π is a neighborhood of a nonumbilic point then the metric $g=4e^u dz d\bar{z}$ induced on Π via F satisfies the elliptic sinh-Gordon equation

$$u_{z\bar{z}}+\sinh u=0. \tag{1.1}$$

Conversely, every solution of the Eq. (1.1) gives rise to CMC immersion.

It is very important that here the elliptic sinh-Gordon equation may be applied not only in differential geometry in the small but also in differential geometry in the large. In particular, special doubly periodic solutions of this equation describe all CMC tori.

Wente was the first [48] to show the existence of compact surfaces of genus one in R^3 with CMC, he constructed a countable number of isometrically distinct examples using a special solution of (1.1). These examples solved the long standing problem of Hopf [31]: Is a compact CMC surface in R^3 necessarily a round sphere?

Important previous results were due to Alexandrov [3] and Hopf [31]. Alexandrov showed that if a CMC surface is embedded then it must be a round sphere. Hopf showed that an immersion of S^2 into R^3 with CMC, must be a round sphere.

Wente's paper was followed by the series [1, 43, 45, 49] where Wente tori were investigated in detail and other examples of CMC tori were constructed. All these modern results were obtained with the help of analytical investigation of special solutions of (1.1). In particular, Abresch described [1] all CMC tori having one family of planar curvature lines in terms of elliptic integrals. Walter [45] obtained more explicit representation of these tori in terms of elliptic and theta-functions of Jacobi type.

At last, a general classification of all CMC tori in R^3 was given by Pinkall and Sterling [40]. This classification is based on a simple but very important theorem which is implicit in [40]: All nonsingular doubly periodic solutions of the Eq. (1.1) are of the finite-gap type.

The paper [40] was the starting point of the author's research in this field. Notice that (1.1) is one of the real versions of the sine-Gordon equation which is well known in the theory of integrable nonlinear equations – soliton theory (see, for example [21, 24]). This theory has revealed many interesting properties of the sine-Gordon as well as other integrable equations. In particular, the periodic problem for the sine-Gordon equation has been solved via the method of finite-gap integration (for the history of the problem and detailed references see [21, 22]). The main achievement of this theory is the construction of new explicit solutions that are called finite-gap solutions and are defined as the stationary solutions with respect to one of the "higher" flows of the integrable equation regarded as an infinite-dimensional Hamiltonian system. In the general case they are quasiperiodic and can be expressed in terms of Riemann theta-functions.

Due to the theorem of Pinkall and Sterling mentioned above the problem of construction of all CMC tori reduces to the following one: Find all finite-gap solutions of (1.1) that determine CMC tori and describe explicitly all these tori. As a matter of fact, to obtain a CMC torus we need a solution $u(z, \bar{z})$ of (1.1) having the property that not only $u(z, \bar{z})$ but also the immersion $F(z, \bar{z})$ are doubly periodic functions of $(x, y) \in R^2$, $z = x + iy$. To solve this problem two different techniques were suggested in [12, 40]. The paper [40] uses systems of ordinary differential equations (ODE) to study the finite-gap solutions of the elliptic sinh-Gordon equation and hence of the CMC tori. As compared with the ODE method an approach suggested by the author [12] and based on the Baker-Akhiezer (B-A) function technique has essential advantages. It enables one to perform explicitly the integration of the Gauss-Weingarten equations and in this way yields a theta-functional formula for the immersion $F(z, \bar{z})$ (see Sect. 6) induced by general finite-gap solution of (1.1). In general these immersions are quasiperiodic. The explicit formula for F gives rise to the very important periodicity conditions (and to a complete description of CMC tori) which could hardly have been obtained via the ODE technique.

In Sect. 7 we derive a formula for areas of CMC tori in R^3 as well as for areas of CMC tori in S^3 and H^3 (see below).

A conformal parametrization $F: \Pi \rightarrow S^3$ of a CMC surface in S^3 gives rise to a metric which also satisfies the Eq. (1.1). In Sects. 8 and 9 we construct all CMC tori in S^3 in quite the same way as CMC tori in R^3. In particular, the final theta-functional formula for the immersion F is derived in Sect. 9.

Of greatest interest is the special case of CMC surfaces in S^3 with the zero mean curvature. These surfaces are called minimal. Whereas the case $H = 0$ of minimal surfaces in R^3 induces the degeneration of the elliptic sinh-Gordon equation to the Liouville equation and there are no compact minimal surfaces in R^3, minimal surfaces in S^3 are described by the same Eq. (1.1). In Sect. 10, as a special case of CMC tori, we obtain all minimal tori in S^3 and also give an expression for their area which is a deep characteristic of special interest. Also, the well known connection between minimal surfaces in S^3 and Willmore surfaces in R^3 is discussed.

All compact CMC tori in H^3 are constructed in Sect. 11. By the maximum principle there are no compact CMC surfaces in H^3 with $|H| \leqq 1$ (or in R^3 with

$H=0$). For the CMC tori with $|H|>1$ the corresponding Gauss-Peterson-Codazzi equation is also the elliptic sinh-Gordon equation. So with minor modifications the B-A function approach gives rise to explicit formulae for immersions in H^3.

Note that CMC tori in S^3 and H^3 with spherical curvature lines similar to Wente tori in R^3 were recently constructed by Walter [46].

There are a lot of interesting conjectures on CMC tori in particular on minimal tori in S^3 (see Sect. 10). Although now explicit formulae are already known, the investigation of CMC tori remains a serious problem. As a matter of fact, the actual parameter in these formulae is a hyperelliptic Riemann surface C which makes a parametrization to be rather complicated. This problem is well known in the finite-gap integration theory as the effectivization problem [20]. A new approach to the effective construction of finite-gap solutions was suggested in [10]. It is based on the Schottky uniformization of the corresponding Riemann surface C and was used in the papers [14, 15] to draw the plots of the finite-gap solutions of the Korteweg-de Vries and Kadomsev-Petviashvili equations. An application of this approach to the elliptic sinh-Gordon equation is presented in Sect. 12. It indicates a direct way for investigation and drawing pictures of CMC tori.

CMC tori are singled out from general quasiperiodic immersions by the periodicity conditions, which are in fact the conditions on the corresponding hyperelliptic Riemann surface C of genus g. The existence of C satisfying the periodicity conditions must be proved. For $g=2$ it is done in Sect. 13. We hope that the arguments in Sects. 9 and 13 convince readers that for every g ($g \geqq 2$ in R^3 and H^3 cases) there are CMC tori corresponding to hyperelliptic Riemann surfaces of genus g. But, of course, without a rigorous mathematical proof this statement remains to be a conjecture.

One should note that the idea of the application of soliton equations to differential geometry is not new. It was already discussed in [6, 41, 44], where some examples of integrable Gauss-Peterson-Codazzi equations were constructed (in particular, for minimal surfaces in H^4 [6]). Recently, the integrability of nonlinear equations corresponding to minimal surfaces in S^n and H^n was proved [13], which allows one to apply the technique of the present paper to derive explicit formulae for all minimal tori in S^n, H^n.

After this paper was finished the author learned of two remarkable papers by Abresch [2] and Hitchin [30]. Abresch described [2] by the ODE method all CMC tori of $g=2$ in R^3. In this paper [30] Hitchin classified all harmonic immersions of torus in S^3. He showed also the finiteness of genus of the corresponding spectral curve C. The periodicity condition and the formula for areas of minimal tori in S^3 in slightly different terms compared with the present paper were also obtained and several examples of low genus g minimal tori in S^3 were constructed.

2 CMC tori in R^3. Analytical statement of the problem

Our aim is to construct all compact smooth (of the class C^3) CMC immersions of genus one in R^3.

Let T^2 be a compact smooth surface of genus one in R^3. A metric Ω on T^2 induced by an immersion $T^2 \to R^3$ generates a complex structure with respect to which the metric is diagonal $\Omega = 4e^u dz d\bar{z}$. On the other hand, an arbitrary Riemann surface of genus one is conformally equivalent to a factor $C/\tilde{\Lambda}$ over a lattice $\tilde{\Lambda}$

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generated by two vectors; one of them may set equal to unity and the second is called the module. It is more convenient for the present to leave the z-scale unfixed. So for every compact smooth torus T^2 in R^3 a conformal parametrization of T^2 by a doubly periodic function $F: C/\Lambda \to R^3$ always exists. Here Λ is a lattice generated by some vectors $Z_1 = X_1 + iY_1, Z_2 = X_2 + iY_2$. Ω is diagonal in z-variable that gives for the vector function $F(z, \bar{z})$

$$\langle F_z, F_{\bar{z}} \rangle = 2e^u, \quad \langle F_z, F_z \rangle = \langle F_{\bar{z}}, F_{\bar{z}} \rangle = 0. \tag{2.1}$$

Here we use the canonical scalar product

$$\langle a, b \rangle = \sum_{k=1}^{3} a_k b_k$$

and denote $z = x + iy$, $\bar{z} = x - iy$.

The vectors F_z and $F_{\bar{z}}$ are tangent to the surface. They may be supplemented with the normal N

$$\langle F_z, N \rangle = \langle F_{\bar{z}}, N \rangle = 0, \quad \langle N, N \rangle = 1. \tag{2.2}$$

Variation of this basis with respect to motion along the surface is described by the Gauss-Weingarten equations

$$\sigma_z = \mathcal{U}\sigma, \quad \sigma_{\bar{z}} = \mathcal{V}\sigma, \quad \sigma = \begin{pmatrix} F_z \\ F_{\bar{z}} \\ N \end{pmatrix}, \tag{2.3}$$

$$\mathcal{U} = \begin{pmatrix} u_z & 0 & A \\ 0 & 0 & B \\ -\frac{e^{-u}}{2}B & -\frac{e^{-u}}{2}A & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & B \\ 0 & u_{\bar{z}} & \bar{A} \\ -\frac{e^{-u}}{2}\bar{A} & -\frac{e^{-u}}{2}B & 0 \end{pmatrix}, \tag{2.4}$$

where $A = \langle F_{zz}, N \rangle$, $B = \langle F_{z\bar{z}}, N \rangle$.

The first and the second quadratic forms are as follows:

$$\langle dF, dF \rangle = 4e^u dz d\bar{z},$$

$$-\langle dF, dN \rangle = A dz dz + \bar{A} d\bar{z} d\bar{z} + 2B dz d\bar{z},$$

which imply that the mean and the Gauss curvatures (k_1, k_2 are the principal curvatures) are given by

$$H = (k_1 + k_2)/2 = Be^{-u}/2,$$

$$K = k_1 k_2 = e^{-2u}(B^2 - A\bar{A})/4.$$

The compatibility condition for (2.3), (2.4)

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z + [\mathcal{U}, \mathcal{V}] = 0$$

(the Gauss-Peterson-Codazzi equation) looks as follows:

$$\begin{aligned} u_{z\bar{z}} + e^{-u}(B^2 - A\bar{A})/2 &= 0, \\ A_{\bar{z}} + u_z B - B_z &= 0, \\ \bar{A}_z + u_{\bar{z}} B - B_{\bar{z}} &= 0. \end{aligned} \tag{2.5}$$

We study CMC tori. Condition $H=\text{const}$ yields $B=2He^u$ and Eqs. (2.5) shows that A is an analytic function of z, $A_{\bar z}=0$. So we see that A is a doubly periodic analytic function, i.e. an elliptic function. Remind that we study compact smooth tori, therefore, A is finite. The elliptic function without poles is a constant $\langle F_{zz}, N\rangle=\text{const}$.

Lemma 2.1. *Compact smooth CMC tori in R^3 have no umbilic points.*

Proof. An umbilic point $k_1=k_2$ is a point where $A(z)$ vanishes. $A(z)$ is a bounded elliptic function which implies that $A\equiv\text{const}$ identically. But the $A\equiv 0$ case is explicitly solved and corresponds to the standard immersion of sphere [31].

Finally, we have for CMC tori

$$A=\text{const}\neq 0\,.$$

Now let us fix scales in R^3 and C in such a way that $H=1/2$, $|\langle F_{zz}, N\rangle|=1$. So for coefficients A, B we have

$$B=e^u\,,\qquad A=e^{i\varphi}\,,\qquad \varphi\in R\,,\qquad \varphi=\text{const}\,. \tag{2.6}$$

Certainly a simple rotation of a basis in R^2 yields $A=1$. In this case parameter curves $x=\text{const}$, $y=\text{const}$ are curvature lines on the surfaces. But we shall use the normalization (2.6). Equation (2.5) becomes the elliptic sinh-Gordon equation

$$u_{z\bar z}+\sinh u=0 \tag{2.7}$$

and the basis σ satisfies (2.3) with

$$\mathscr{U}=\begin{pmatrix} u_z & 0 & e^{i\varphi} \\ 0 & 0 & e^u \\ -1/2 & -e^{-u+i\varphi}/2 & 0 \end{pmatrix},\quad \mathscr{V}=\begin{pmatrix} 0 & 0 & e^u \\ 0 & u_{\bar z} & e^{-i\varphi} \\ -e^{-u-i\varphi}/2 & -1/2 & 0 \end{pmatrix}. \tag{2.8}$$

Let us consider the matrix

$$\phi=\begin{pmatrix} F_{1z} & F_{2z} & F_{3z} \\ F_{1\bar z} & F_{2\bar z} & F_{3\bar z} \\ N_1 & N_2 & N_3 \end{pmatrix},$$

where F_i, N_i are the coordinates of the vectors F, N in some fixed basis of R^3. Rotation of this fixed basis leads to the transformation $\phi\to\phi\omega$, $\omega\in O(3)$. ϕ satisfies the following equations

$$\phi_z=\mathscr{U}\phi\,,\qquad \phi_{\bar z}=\mathscr{V}\phi\,, \tag{2.9}$$

$$\bar\phi=\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}\phi\,,\quad \phi\phi^T=\begin{pmatrix} 0 & 2e^u & \\ 2e^u & 0 & \\ & & 1 \end{pmatrix} \tag{2.10}$$

with $\mathscr{U}$, $\mathscr{V}$ of the form (2.8).

Our strategy will be as follows. First of all we give a simple proof of the Pinkall-Sterling theorem that all doubly periodic solutions of (2.7) are of the finite-gap type. Then for a general finite-gap solution of (2.7) with the help of the B-A functions we construct the matrix ϕ satisfying all the conditions (2.9), (2.10) and also the vector-function $F(z,\bar{z})$. After that we single out the case when $u(z,\bar{z})$, $\phi(z,\bar{z})$ and also $F(z,\bar{z})$ are doubly periodic functions of (x,y). As a result all compact smooth CMC tori in R^3 are constructed.

3 Doubly periodic solutions of the elliptic sinh-Gordon equation

Now we start investigation of nonsingular doubly periodic solutions of the Eq. (2.7). First of all note that the derivative in (2.7) is well defined since $u\in C^2(R^2)$ by virtue of the smoothness of torus $F\in C^3(R^2)$. The fact that Eq. (2.7) is elliptic in turn implies that u is real-analytic, $u(z,\bar{z})\in C^\infty(R^2)$, so the derivatives of u with respect to z and $\bar{z}$ are well defined for arbitrary order.

Equation (2.7) is the compatibility condition

$$U_{\bar{z}}-V_z+[U,V]=0$$

for the system of two linear differential equations [24]

$$\Psi_z=U\Psi,\qquad \Psi_{\bar{z}}=V\Psi, \tag{3.1}$$

$$U=\frac{1}{2}\left(\begin{array}{c|c}-u_z & -iv\\ \hline -iv & u_z\end{array}\right);\qquad V=\frac{1}{2iv}\begin{pmatrix}0 & e^{-u}\\ e^u & 0\end{pmatrix} \tag{3.2}$$

with an auxiliary parameter v.

The $U-V$ pair (3.1), (3.2) satisfies the following reductions

$$\begin{aligned}&\Psi(v)\to\sigma_3\Psi(-v),\\ &\Psi(v)\to R\overline{\Psi(\bar{v}^{-1})},\qquad R=\begin{pmatrix}0 & -e^{-u/2}\\ e^{u/2} & 0\end{pmatrix},\end{aligned} \tag{3.3}$$

where the arrow means that both the left and the right sides are solutions of the system (3.1), (3.2) with the same u. Here and below σ_α denote the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Equation (2.7) is completely integrable and possesses an infinite series of conservation laws which in turn determine commuting "higher" flows. "Higher" flows are introduced in a standard way [24]. Set $z=z_1$ and consider an infinite series of new variables $z_2,\ldots,z_n,\ldots$. The Ψ-function as a function of z_n satisfies an infinite series of differential equations

$$\Psi_{z_n}=U_n\Psi, \tag{3.4}$$

where U_n is a matrix polynomial of degree $2n-1$ with respect to v. Coefficients of U_n are certain polynomials in derivatives $u_z,u_{zz},\ldots,u_z^{(n)}$ of various orders. Formula (I.III.3.31) of the book [24] present the generating function of all U_n for the U-operator, which is equivalent to (3.2). The explicit form of U_n in (3.4) is a simple corollary of this formula. Since it is not essential for us we do not present it here.

In this way the solution of the elliptic sinh-Gordon equation becomes dependent on z_n variables. The second reduction (3.3) implies that

$$\Psi_{\bar{z}_n}=V_n\Psi,$$

where V_n is a polynomial of degree $2n-1$ with respect to v^{-1} and

$$U_n(v)=R_{z_n}R^{-1}+R\overline{V_n(\bar{v}^{-1})}R^{-1}. \tag{3.5}$$

We use the normalization $V_n(v=\infty)=0$. Putting $v=0$ in (3.5) we obtain the "higher" sinh-Gordon equations

$$u_{z_n}=P_n(u_z,\ldots,u_z^{(n)}),$$

where P_n are coefficients in $U_n: 2U_n(0)=-P_n\sigma_3$. As was mentioned above, P_n are certain polynomials of derivatives $u_z^{(m)}$. All P_n were calculated in [40], where a geometrical interpretation of "higher" flows as Jacobi fields was suggested.

Finally, in real variables x_n, y_n $(z_n=x_n+iy_n)$ we have a nonsingular solution

$$u(x_1,y_1,\ldots,x_n,y_n,\ldots) \tag{3.6}$$

of all "higher" sinh-Gordon equations. Moreover, the function (3.6) is real-analytic with respect to all variables. From now on we denote "higher" times $x_1,y_1,\ldots,x_n,y_n,\ldots$ by t_i, $i=1,2,\ldots$.

Since all "higher" flows commute, the set of solutions stationary with respect to some flow $u_{t_i}=0$ is invariant with respect to other flows including x, y flows. In the theory of solitons these solutions are called the finite-gap solutions (or algebro-geometric, multi-phase, theta-functional) [21, 22].

Now we are in a position to formulate the Pinkall-Sterling theorem (see [40]).

Theorem 3.1. *All nonsingular doubly-periodic solutions of* (2.7) *are stationary solutions of a higher elliptic sinh-Gordon flow.*

Proof. Concerning the partial derivatives $v_i=\frac{\partial}{\partial t_i}u(t_1,\ldots)$ we have

$$(\partial_z\partial_{\bar z}+\cosh u)v_i=0. \tag{3.7}$$

A spectrum of the operator (3.7) on the torus R^2/Λ is discrete, so v_i are linearly dependent. Finally, a "higher" time t exists with respect to which $u(t_1,\ldots,t_n,\ldots)$ is stationary $u_t=0$ and hence $u(z,\bar z)$ is of the finite-gap type.

4 Formulae for the finite-gap solutions and the Baker-Akhiezer function

The Eq. (2.7) is one of the real versions of the sine-Gordon equation, finite-gap solutions of which were found first by Kozel and Kotlyarov [32]. Later similar results were obtained by McKean [36]. Krichever's scheme was applied to this equation by Its [35] who suggested the B-A function for it (see also [18]). All real nonsingular finite-gap solutions of (2.7) were obtained in [12].

Let $u(z,\bar z)$ be the finite-gap solution of the equation (2.7) which is stationary with respect to the "higher" time t. The t-evolution of the Ψ-function is determined by the polynomial $W(v)$ of degree $2N-1$ in both v and $1/v$

$$\Psi_t=W\Psi.$$

The corresponding stationary equations are as follows

$$-W_z+[U,W]=0, \qquad -W_{\bar z}+[V,W]=0. \tag{4.1}$$

Let Ψ_0 be the function reducing W to the diagonal form $W=\Psi_0\hat{\mu}\Psi_0^{-1}$, where $\hat{\mu}$ is the diagonal matrix of eigenvalues. Due to (4.1) the eigenvalues $\hat{\mu}$ and hence the characteristic polynomial $\det(W(v)-\mu I)=0$ do not depend on $z, \bar{z}$. The matrix W always can be fixed traceless so we see that the hyperelliptic curve C

$$\mu^2=\det(W(v)) \tag{4.2}$$

is independent of $z, \bar{z}$. It is called the spectral curve. Reduction (3.3) shows that (4.2) reduces to the form $\mu^2=P(\lambda)$, $\lambda=v^2$.

The B-A function is the common solution of the equations

$$W\psi=\mu\psi\,, \qquad \psi_z=U\psi\,, \qquad \psi_{\bar{z}}=V\psi\,.$$

It is an analytic function on the Riemann surface (4.2). The values of ψ on first and second sheets of the covering $C\to v$ substituted in corresponding columns form the matrix Ψ satisfying (3.1) and

$$W\Psi=\Psi\hat{\mu}\,, \qquad \hat{\mu}=\mu\sigma_3\,.$$

Known analytical properties of ψ yield an explicit theta-functional formula for Ψ and u.

Our strategy will be as follows. We start with a formulation of well known results on theta-function integration of the sine-Gordon equation (see [18, 21, 32, 35, 36]). There we present complex valued finite-gap solutions. After that we specify parameters of solutions in order to make them real-valued solutions of the Eq. (2.7).

Consider the Riemann surface of the hyperelliptic curve C defined by the equation

$$\tilde{\mu}^2=\lambda\prod_{i=1}^{2g}(\lambda-\lambda_i)\,, \tag{4.3}$$

$C\to C/\pi$; $\pi:(\lambda,\mu)\to(\lambda,-\mu)$ is a double cover of λ-plane with $0, \infty, \lambda_i$ being the ramification points. Starting from here we suppose C to be nonsingular curve $\lambda_i\neq\lambda_j$. It appears that the case of singular spectral curves, considered in Appendix, does not result CMC tori.

Let $a_n, b_n, n=1, \ldots, g$, be the canonical basis of cycles and du_n be the normalized holomorphic abelian differentials on C $\int_{a_m} du_n=2\pi i\delta_{mn}$. B is the associated period matrix $B_{mn}=\int_{b_m} du_n$ defining the Riemann theta-function

$$\theta(z)=\sum_{m\in Z^g}\exp(\tfrac{1}{2}\langle Bm,m\rangle+\langle z,m\rangle)\,, \qquad z\in C^g$$

which is periodic with periods $2\pi i Z^g$

$$\theta(z+2\pi iN)=\theta(z)\,. \tag{4.4}$$

A contour $\mathscr{L}$ on C such that

$$\mathscr{L}=a_1+\ldots+a_g$$

fixes a branch of $v=\sqrt{\lambda}$ on $C\backslash\mathscr{L}$ (see Fig. 1). Cycle $\mathscr{L}$ defines the unramified covering $\hat{C}\to C$, where $\hat{C}$ may be thought of as the Riemann surface of the function v on C. v is a multivalued function on C, it equires a factor $(-1)^{\langle\gamma,\mathscr{L}\rangle}$ upon a circuit of γ, where $\langle\gamma,\mathscr{L}\rangle$ is the intersection number of γ and $\mathscr{L}$.

We need also the abelian integrals of the second kind Ω_1, Ω_2 normalized by the condition

$$\int_{a_n} d\Omega_i = 0, \quad i=1,2, \quad n=1,\dots,g \tag{4.5}$$

and the following asymptotic behaviour at the singularities

$$d\Omega_1 \to dv, \quad v \sim \infty; \quad d\Omega_2 \to -dv/v^2, \quad v \sim 0. \tag{4.6}$$

Remind that we have fixed certain branch of v on C. Periods of Ω_1 and Ω_2 upon the b-cycles we denote as

$$U_n = \int_{b_n} d\Omega_1, \quad V_n = \int_{b_n} d\Omega_2. \tag{4.7}$$

Now we consider z and $\bar{z}$ as independent complex variables.

Theorem 4.1. *The complex finite-gap solutions of the Eq. (2.7) are given by the formula*

$$u(z, \bar{z}) = 2\log \frac{\theta(\Omega)}{\theta(\Omega + \Delta)}. \tag{4.8}$$

The B-A function $\psi(z, \bar{z}, P)$ is as follows:

$$\psi_1 = \frac{\theta(u+\Omega)\theta(D)}{\theta(u+D)\theta(\Omega)} \exp\left(-\frac{i}{2}(z\Omega_1 + \bar{z}\Omega_2)\right),$$

$$\psi_2 = \frac{\theta(u+\Omega+\Delta)\theta(D)}{\theta(u+D)\theta(\Omega+\Delta)} \exp\left(-\frac{i}{2}(z\Omega_1 + \bar{z}\Omega_2)\right).$$

Here $\Omega = -\frac{i}{2}(Uz + V\bar{z}) + D$, u is an Abel's map $u = \int_\infty^P du$, $P = (\lambda, \mu) \in C$, $\Omega_i = \int_\infty^P d\Omega_i$. Vectors $U, V, \Delta \in C^g$ are as follows $U = (U_1, \dots, U_g)$, $V = (V_1, \dots, V_g)$, $\Delta = \pi i(1, \dots, 1)$. $D \in C^g$ is arbitrary and the paths of integration in u and Ω_i are identical.

Note that the sign of v in U and V is determined by the analytic continuation of the fixed branch along the same path as u. ψ_1 and $\psi_2 v$ are the single-valued functions on C. The divisor of poles $\mathscr{D}$ of ψ_1 as well as ψ_2 is of degree g and is determined by the theta-function in denominators of ψ. Both v and ψ are the single-valued functions on $\hat{C}$.

ψ_1 and ψ_2 may be combined into the matrix-valued function of v. We have fixed a branch of v on C so that there is 1:1 correspondence between points of $C \backslash \mathscr{L}$ and values of v. Let us consider two paths: ℓ from ∞ to $P = (v, \mu)$ such that $\langle \mathscr{L}, \ell \rangle$ is even and ℓ^* from ∞ to πP such that $\langle \mathscr{L}, \ell^* \rangle$ is odd. Note that the values of the function v determined by analytic continuation along these paths coincide $v_\ell = v_{\ell^*}$ (we denote f_ℓ the analytic continuation of the function f along the path ℓ).

Let us denote $\psi_i = \psi_{i\ell}$, $\psi_i^* = \psi_{i\ell^*}$. It is evident that the function

$$\Psi(v) = \begin{pmatrix} \psi_1 & \psi_1^* \\ \psi_2 & \psi_2^* \end{pmatrix}$$

satisfies the reduction

$$\Psi(-v) = \sigma_3 \Psi(v) \sigma_1 \tag{4.9}$$

[compare with (3.3)]. $\ell^* - \pi\ell$ is a cycle and $\langle \mathscr{L}, \ell^* - \pi\ell \rangle$ is odd due to invariance of $\mathscr{L} = \pi\mathscr{L}$. Hence we have $\psi_{i\ell^*} = (-1)^{i+1}\psi_{i\pi\ell}$. If we put u be an Abel's map of the point v.

$$u = \int_\ell du = \int_\infty^v du\,,$$

then $\int_{\pi\ell} du = -u$. Denote also $\Omega_i = \int_\ell d\Omega_i$.

We can summarize the above arguments in the following way. The function

$$\Psi(v) = \begin{pmatrix} \dfrac{\theta(u+\Omega)}{\theta(\Omega)} & \dfrac{\theta(-u+\Omega)}{\theta(\Omega)} \\ \dfrac{\theta(u+\Omega+\Delta)}{\theta(\Omega+\Delta)} & -\dfrac{\theta(-u+\Omega+\Delta)}{\theta(\Omega+\Delta)} \end{pmatrix} \begin{pmatrix} \dfrac{\theta(D)}{\theta(D+u)} e^{\omega} & 0 \\ 0 & \dfrac{\theta(D)}{\theta(D-u)} e^{-\omega} \end{pmatrix}, \tag{4.10}$$

$$\omega = -\frac{i}{2}(\Omega_1 z + \Omega_2 \bar{z})$$

is a solution of the Eqs. (3.1), (3.2) with $u(z, \bar{z})$ given by (4.8). Here $\langle \ell, \mathscr{L} \rangle$ is even.

Asymptotic behaviour of $\Psi(v)$ at the essential singularity points are as follows:

$$\begin{aligned} \Psi &= \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + B/v + \dots \right) \exp\left(-\frac{i}{2} v z \sigma_3 \right), \\ \Psi &= (Q + \dots) \exp\left(-\frac{i}{2v} \bar{z} \sigma_3 \right) \end{aligned} \tag{4.11}$$

with $Q = \sigma_3 Q \sigma_1$, $B = -\sigma_3 B \sigma_1$ due to (4.9).

From now on we put z and $\bar{z}$ be mutually conjugated. For the described complex finite-gap solutions to be real-valued solutions of the Eq. (2.7) C necessarily has an antiholomorphic involution. Indeed, the second reduction (3.3) yields $\det W(v) = \overline{\det W(\bar{v}^{-1})}$. Hence the spectral curve C (4.2) has an involution

$$\tau : \lambda \to \frac{1}{\bar{\lambda}}. \tag{4.12}$$

More exactly, let us consider the Riemann surface C with all branch points λ_i divided into pairs $(|\lambda_i| \neq 1)$

$$\bar{\lambda}_{2n+1} = \lambda_{2n}^{-1}, \quad n = 1, \dots, g. \tag{4.13}$$

Canonical basis of cycles can be choosen in such a way that τ acts on it as follows (see Fig. 1):

$$\tau a_n = -a_n, \quad \tau b_n = b_n - a_n + \sum_{i=1}^{g} a_i.$$

Then the period matrix of C is equal to

$$B = B_R + \pi i(I - \mathbb{1}),$$

where B_R is a real matrix and $\mathbb{1}_{mn} = 1$, $I_{mn} = \delta_{mn}$. Respectively, theta-function satisfies the following conjugation condition

$$\overline{\theta(z)} = \theta(\bar{z}).$$

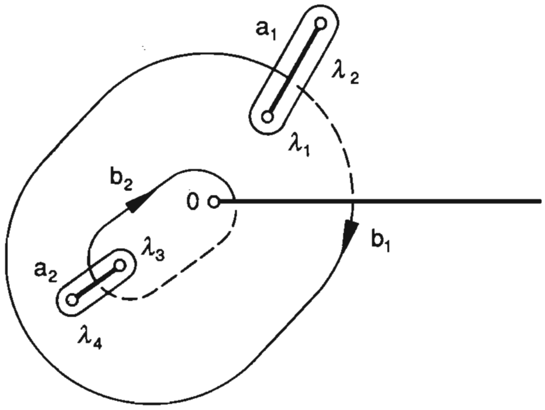

Fig. 1. The contour $\mathscr{L}$ consists of g components each being an oval lying over the cut $[\lambda_{2i-1}, \lambda_{2i}]$. $\mathscr{L}$ is invariant with respect to both π and τ

We fix $\mathscr{L}$ so that $\tau\mathscr{L}=\mathscr{L}$ then for the fixed branch of v we have

$$\overline{\tau^* v} = v^{-1},$$

hence

$$\tau^* d\Omega_1 = \overline{d\Omega_2}. \tag{4.14}$$

Periods (4.6) of $d\Omega_i$ are mutually conjugated

$$\bar{U} = V,$$

therefore, the vector Ω in (4.8), (4.10) is imaginary.

A standard technique [19] yields the following.

Theorem 4.2 [12]. *All real finite-gap solutions of the Eq.* (2.7) *are given by the formula* (4.8), *where C is a real curve with branch points satisfying the condition* (4.13) *and D is an arbitrary imaginary vector.*

Remark. A simple analysis of possible singularities of $u(z,\bar{z})$ shows that all real finite-gap solutions given by the formula (4.8) are nonsingular.

Remark. It is evident that the hyperelliptic curve $\mu^2 = P(\lambda)$ (4.2) with all different branch points is always of odd genus. Just this case was considered in [40]. At the same time, we see that the finite-gap solutions are defined by the curves (4.3) of arbitrary genera. In this way CMC tori determined by C of arbitrary genera are obtained (see below). The evenness of genus is irrelevant. For coinciding branch points the geometric genus of the spectral curve drops down and becomes even.

We can obtain also a useful formula for

$$d = \psi_1 \psi_2^* - \psi_1^* \psi_2, \tag{4.15}$$

analyzing its analytical properties. $d(P)$ is a meromorphic function on C with divisor of poles $\mathscr{D} + \pi\mathscr{D}$ and divisor of zerous $\lambda_1 + \ldots + \lambda_{2g}$. Here $\mathscr{D}$ is the divisor of poles of $\psi(P)$ on C coinciding with g zerous of $\theta(u+D)$. In addition $vd(P)$ is a single-valued function on C and asymptotics (4.11) yields $d(\infty) = -2$. Hence $d(P)$ is a rational function of v, it can be expressed in terms of theta-functions as follows:

$$d = -2 \frac{\theta^2(D)}{\theta(0)\theta(\Delta)} \cdot \frac{\theta(u)\theta(u+\Delta)}{\theta(u-D)\theta(u+D)}. \tag{4.16}$$

At last, consider an action of the real reduction on Ψ. A set of fixed points of τ, $|\lambda|=1$ consists of one (if g is even) or two (if g is odd) real ovals. Let ℓ' be a path connecting the points ∞ and v which does not intersect $\mathscr{L}$ and ℓ'' be a similar path from ∞ to $\bar{v}^{-1}$. A conjugation law for holomorphic differentials $\tau^* du=\overline{du}$ shows that

$$\overline{\int_\infty^v du}=\int_\infty^{\bar{v}^{-1}} du-\Delta \quad (\operatorname{mod} 2\pi i Z^g).$$

So we see that the second reduction (3.3) is valid. More exactly

$$\Psi(v)=\begin{pmatrix} 0 & -e^{-u/2} \\ e^{u/2} & 0 \end{pmatrix} \overline{\Psi(\bar{v}^{-1})} \begin{pmatrix} 0 & -\dfrac{1}{\bar{\alpha}'} \\ \bar{\alpha}'' & 0 \end{pmatrix}, \tag{4.17}$$

$$\alpha'=\frac{\theta(u'-D)}{\theta(u'-D+\Delta)}, \quad \alpha''=\frac{\theta(u''-D)}{\theta(u''-D+\Delta)}, \quad u'=\int_{\ell'} du, \quad u''=\int_{\ell''} du. \tag{4.18}$$

5 CMC tori in R^3. Formulae for moving frame

Let $u(z,\bar{z})$ be the real finite-gap solution (4.8) of the Eq. (2.7) and ψ be the corresponding B-A function. Comparing differential equations (2.9) for ϕ with those for the B-A function (3.1), (3.2) we obtain the following

Lemma 5.1. *The function*

$$\tilde{\varphi}(v)=\begin{pmatrix} iv\psi_2^2 \\ ie^u\psi_1^2/v \\ \psi_1\psi_2 \end{pmatrix}, \qquad v^2=e^{i\varphi}$$

is a solution of the differential equations (2.9).

As a corollary ϕ can be presented as follows

$$\phi=\begin{pmatrix} iv & & \\ & ie^u/v & \\ & & 1 \end{pmatrix}\phi_0 M, \quad \phi_0=\begin{pmatrix} \psi_2^2 & \psi_2\psi_2^* & \psi_2^{*2} \\ \psi_1^2 & \psi_1\psi_1^* & \psi_1^{*2} \\ \psi_1\psi_2 & (\psi_1\psi_2^*+\psi_1^*\psi_2)/2 & \psi_1^*\psi_2^* \end{pmatrix}, \tag{5.1}$$

where matrix M is independent of $z,\bar{z}$.

M is determined by the two conditions (2.10). We see that $|v|=1$. Under this condition the complex conjugate B-A function is given by the simple rule (4.17). Taking it into account we can formulate conditions (2.10) in terms of M

$$MM^T=\frac{2}{d^2}\begin{pmatrix} & & -1 \\ & 2 & \\ -1 & & \end{pmatrix}, \quad \bar{M}=\begin{pmatrix} & & -\alpha^{-2} \\ & \bar{\alpha}\alpha^{-1} & \\ -\bar{\alpha}^2 & & \end{pmatrix} M,$$

where d and $\alpha=\alpha(v)$ are given by the expressions (4.15), (4.18). Due to the same Eqs. (4.17) for determinant (4.15) we have

$$\alpha d=\bar{\alpha}\bar{d}.$$

So the conditions for $N = Md$ are as follows:

$$NN^T = \begin{pmatrix} & & -2 \\ & 4 & \\ -2 & & \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} & & -(\alpha\bar{\alpha})^{-1} \\ & 1 & \\ -\alpha\bar{\alpha} & & \end{pmatrix} N.$$

These equations can be easily solved and finally, we obtain the following expression for M

$$M = \frac{1}{d} \begin{pmatrix} |\alpha|^{-1} & & \\ & 1 & \\ & & |\alpha| \end{pmatrix} \begin{pmatrix} 1 & & i \\ & 2 & \\ -1 & & i \end{pmatrix} \omega, \tag{5.2}$$

where $\omega \in O(3)$ is an arbitrary rotation. The freedom in (5.2) due to ω corresponds to the so far unspecified basis in R^3.

Theorem 5.2. *A moving frame $F_z, F_{\bar{z}}, N$ corresponding to the finite-gap solution* (4.8) *of the elliptic sinh-Gordon equation is equal to*

$$\Phi = \frac{\theta(0)\theta(\Delta)}{2\theta(u)\theta(u+\Delta)} \mathscr{F} \begin{pmatrix} -1 & & i \\ & 2 & \\ 1 & & i \end{pmatrix} \omega, \tag{5.3}$$

$$\mathscr{F} = \operatorname{diag}(ie^{i\varphi/2}, ie^{-i\varphi/2}, 1)\tilde{\mathscr{F}} \operatorname{diag}(e^{is}, 1, e^{-is}),$$

$$\tilde{\mathscr{F}} = \begin{pmatrix} A_+^2/A^2 & -A_+A_-/A^2 & A_-^2/A^2 \\ B_+^2/A^2 & B_+B_-/A^2 & B_-^2/A^2 \\ A_+B_+/(AB) & (A_+B_- - A_-B_+)/(2AB) & -A_-B_-/(AB) \end{pmatrix}, \tag{5.4}$$

$$\begin{aligned} &A = \theta(\Omega+\Delta), \quad A_+ = \theta(\Omega+\Delta+u), \quad A_- = \theta(\Omega+\Delta-u), \\ &B = \theta(\Omega), \quad B_+ = \theta(\Omega+u), \quad B_- = \theta(\Omega-u), \end{aligned} \tag{5.5}$$

$$s = 2(c_2 y - c_1 x), \quad \int_\ell d\Omega_1 = c_1 + ic_2, \quad u = \int_\ell du, \tag{5.6}$$

where ℓ is a path on C from ∞ to $P = (\nu, \mu)$, $\nu = e^{i\varphi/2}$, which does not intersect $\mathscr{L}$.

Proof. We put $\nu = e^{i\varphi/2}$. Formulae (5.3), (5.4) are corollaries of (4.10), (4.16), (4.18), (5.1), (5.2). As compared with (5.4) $\mathscr{F}$ has to contain an additional right factor

$$\begin{pmatrix} e^{-i\beta} & & \\ & 1 & \\ & & e^{i\beta} \end{pmatrix}, \quad e^{i\beta} = \frac{\theta(u+D)}{\theta(u-D)} |\alpha|.$$

It can be considered as a constant rotation of a basis in R^3 and so can be included into $\omega \in O(3)$ which is arbitrary.

The normalization (4.5) implies $\int_\infty^0 d\Omega_i = 0$. Together with the equality (4.14) this yields

$$\int_\ell d\Omega_1 = \int_{\tau\ell} \overline{d\Omega_2} = \int_\ell \overline{d\Omega_2}$$

and as a final result we obtain the formula (5.6).

6 CMC tori in R^3. Formulae for immersion

Fortunately, it proved to be possible to make an explicit integration of expressions (5.4) for F_z, which leads to a formula for F. To perform this integration we use certain theta identities, which are special case of Fay's trisecant formula [25, 52]. Nevertheless, we prefer not to describe this specialization here but to derive the identities we need with the help of differential equations for the B-A function. It shows that the "integrability" of F_z is not accidental and makes the paper more self-contained.

The differential equations (3.1), (3.2) imply that

$$(\psi_1(v)\psi_2(v'))_z = -\frac{i}{2}(v'\psi_1(v)\psi_1(v') + v\psi_2(v)\psi_2(v')),$$

$$(\psi_1(v')\psi_2(v))_z = -\frac{i}{2}(v\psi_1(v)\psi_1(v') + v'\psi_2(v)\psi_2(v')),$$

where v and v' are two arbitrary values of the spectral parameter. Denote $\psi_i = \psi_i(v)$, then the limit $v' \to v$ yields

$$\psi_1^2 + \psi_2^2 = \frac{2i}{v}(\psi_1\psi_2)_z,$$

$$\psi_1^2 - \psi_2^2 = \frac{i}{v}(\psi_1\psi_2)_z + i\left(\psi_2\frac{\partial}{\partial v}\psi_1 - \psi_1\frac{\partial}{\partial v}\psi_2\right)_z.$$

Hence ψ_2^2 can be presented in form of a z-derivative

$$\psi_2^2 = \frac{i}{v}(\psi_1\psi_2)_z + i\left(\psi_2\frac{\partial}{\partial v}\psi_1 - \psi_1\frac{\partial}{\partial v}\psi_2\right)_z. \tag{6.1}$$

In terms of the theta-functions it looks as follows:

$$\frac{\theta^2(\Omega+\Delta+u)}{\theta^2(\Omega+\Delta)}e^{is} = \frac{i}{v}\left[\frac{\theta(\Omega+\Delta+u)\theta(\Omega+u)}{\theta(\Omega+\Delta)\theta(\Omega)}e^{is}\left(1+v\frac{\partial}{\partial v}\log\frac{\theta(\Omega+u)}{\theta(\Omega+\Delta+u)}\right)\right]_z.$$

Here u, Ω, s are as in Sect. 5.

In quite a similar way we have

$$\psi_2^{*2} = \frac{i}{v}(\psi_1^*\psi_2^*)_z + i\left(\psi_2^*\frac{\partial}{\partial v}\psi_1^* - \psi_1^*\frac{\partial}{\partial v}\psi_2^*\right)_z,$$

$$\frac{\theta^2(\Omega+\Delta-\mathrm{u})}{\theta^2(\Omega+\Delta)}e^{-is} = -\frac{i}{v}\left[\frac{\theta(\Omega+\Delta-u)\theta(\Omega-u)}{\theta(\Omega+\Delta)\theta(\Omega)}e^{-is}\left(1+v\frac{\partial}{\partial v}\log\frac{\theta(\Omega-u)}{\theta(\Omega+\Delta-u)}\right)\right]_z.$$

For the matrix elements $\mathscr{F}_{11}$ and $\mathscr{F}_{13}$ in (5.4) we have

$$\begin{aligned}\mathscr{F}_{11}&=-\left(\frac{A_+B_+}{AB}e^{is}\left(1+v\frac{\partial}{\partial v}\log\frac{B_+}{A_+}\right)\right)_z,\\ \mathscr{F}_{13}&=\left(\frac{A_-B_-}{AB}e^{-is}\left(1+v\frac{\partial}{\partial v}\log\frac{B_-}{A_-}\right)\right)_z.\end{aligned}\tag{6.2}$$

We put $\omega=I$ in (5.3). Then the formulae (5.3), (6.2) combined with the reality condition gives the final formulae for F_1, F_3 [see (6.11) below].

An expression for F_2 can be derived in quite a similar way by considering differential equations for products $\psi_i\psi_j^*$. But a more elegant formula is obtained using another trick based on consideration of a certain "higher" flow.

With respect to this "higher" time t a modified Ψ-function satisfies the equation

$$\Psi_t=W_0\Psi\tag{6.3}$$

with a specific W_0. Let v_0 denote a fixed value of v. The modified Ψ-function has quite the same analytical properties as those described in Sect. 4 and also two additional essential singularity points v_0 and $-v_0$. The asymptotics at v_0 is as follows

$$\Psi=R\exp\left(\frac{2v_0}{v-v_0}t\sigma_3\right),\quad R=\begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix}\tag{6.4}$$

and the asymptotics at $-v_0$ is determined by (6.4) and the reduction (4.9). These analytical properties lead to the following specialization of (6.3)

$$W_0=\frac{1}{v-v_0}w-\frac{1}{v+v_0}\sigma_3 w\sigma_3,\qquad w=2v_0R\sigma_3R^{-1}.$$

A solution Ψ of (6.3) is obtained by slightly altering the Ψ-function constructed in Sect. 4. The local parameters near the points v_0 and $-v_0$ of C are given by

$$k_{v_0}=\frac{v-v_0}{v+v_0},\qquad k_{-v_0}=\pi^*k_{v_0}=\frac{v+v_0}{v-v_0}.$$

Let us introduce a normalized (4.5) abelian differential of the second kind $d\Omega_3$ with the singularities

$$d\Omega_3=d\left(\frac{1}{k_{v_0}}\right),\quad v\sim v_0,\qquad d\Omega_3=-d\left(\frac{1}{k_{-v_0}}\right),\quad v\sim -v_0,$$

and b-periods

$$W=(W_1,\ldots,W_g),\qquad W_n=\int_{b_n}d\Omega_3.$$

Replacing the exponents in (4.10) by

$$\exp\left\{\left(-\frac{i}{2}(z\Omega_1+\bar{z}\Omega_2)+t\Omega_3\right)\sigma_3\right\},\qquad \Omega_3=\int_\infty^v d\Omega_3,$$

and replacing Ω in arguments of theta-functions by

$$\Omega\to-\frac{i}{2}(Uz+V\bar{z})+tW+D,$$

we obtain the Ψ-function we need. Note that an application of the anti-holomorphis involution τ to the local parameters gives

$$\tau^* k_{\nu_0} = -\bar{k}_{\nu_0}, \qquad \tau^* k_{-\nu_0} = -\bar{k}_{-\nu_0},$$

therefore, $W \in iR^g$ is imaginary.

Let us fix also a local parameter near $\nu = \infty$, $k_\infty = \nu^{-1}$. A trivial calculation gives for the t-derivative of the coefficient of the asymptotics (4.11)

$$B_{21t} = \frac{\partial}{\partial t}\frac{\partial}{\partial k_\infty} \log\theta(u + \Omega + \Delta)_{|u=0} + \gamma, \tag{6.5}$$

where γ is defined by the condition that $d\Omega_3 = \gamma dk_\infty$ at $\nu = \infty$. A reciprocity law for abelian differentials [20, 25] shows that Abel's map near $\nu = \infty$ is equal to

$$u = -Uk_\infty. \tag{6.6}$$

Hence the derivative $\partial/\partial k_\infty$ in (6.5) may be replaced by

$$\partial/\partial k_\infty \to -2i\partial/\partial z$$

and finally, we have

$$B_{21t} = -2i\frac{\partial}{\partial t}\frac{\partial}{\partial z}\log\theta(\Omega + \Delta) + \gamma. \tag{6.7}$$

On the other hand, substitution of the asymptotics (4.11) into the Eq. (6.3) yields

$$B_t = \frac{8\nu_0}{d(\nu_0)}\begin{pmatrix} -R_{11}R_{12} & R_{11}R_{12} \\ R_{21}R_{22} & R_{21}R_{22} \end{pmatrix},$$

where $d(\nu_0)$ defined by (4.15) already has been calculated (4.16). Identifying these two expressions for B_{21t} we obtain the following useful theta-functional equality

$$\begin{aligned} &\frac{\theta(\Omega + \Delta + u_0)\theta(\Omega + \Delta - u_0)}{\theta^2(\Omega + \Delta)} \\ &= \frac{1}{4\nu_0}\frac{\theta(u_0)\theta(u_0 + \Delta)}{\theta(0)\theta(\Delta)}\left(-2i\frac{\partial}{\partial z}\frac{\partial}{\partial t}\log\theta(\Omega + \Delta) + \gamma\right), \end{aligned} \tag{6.8}$$

where $u_0 = \int_\infty^{\nu_0} du$. Combined with (5.3), (5.4) it enables to integrate the formulae for F_{2z}

$$F_{2z} = -\frac{1}{2}\frac{\partial}{\partial z}\frac{\partial}{\partial t}\log\theta(\Omega + \Delta)_{|t=0} - \frac{\gamma i}{4}.$$

Let us fix some new notations. Denote the values of the differentials du_n, $d\Omega_1$ and the integral Ω_1 at the point $\nu = e^{i\varphi/2}$ by

$$du_n = \omega_n d\lambda, \qquad d\Omega_1 = \imath d\lambda, \qquad \int_{\ell} d\Omega_1 = c_1 + ic_2. \tag{6.9}$$

Denote also real and imaginary parts of the vector (4.7) by

$$U_n = \alpha_n + i\beta_n. \tag{6.10}$$

Theorem 6.1. *The immersion corresponding to the finite-gap solution* (4.8) *is determined by the following expression*

$$F_1 \pm iF_3 = \frac{\theta(0)\theta(\Delta)}{\theta(u)\theta(u+\Delta)} \frac{A_\pm B_\pm}{AB}\left(1 \mp \frac{1}{4}\frac{\partial}{\partial t}\log\frac{B_\pm}{A_\pm}\right)e^{\pm is},$$
$$F_2 = -\frac{1}{2}\frac{\partial}{\partial t}\log A - 2ize^{i\varphi}\imath + 2i\bar{z}e^{-i\varphi}\bar{\imath}, \tag{6.11}$$

where A, B, s *are defined in* (5.5) *and*

$$\frac{\partial}{\partial t}\theta(z) = -8e^{i\varphi}\sum_{n=1}^{g}\omega_n\frac{\partial}{\partial z_n}\theta(z), \qquad z=(z_1, \ldots, z_g). \tag{6.12}$$

All CMC *tori in* R^3 *are described by the doubly periodic immersions* (6.11). F *defined by* (6.11) *is doubly periodic granted a lattice* Λ *with the basical vectors* $(X_1, Y_1), (X_2, Y_2)$ *exists such that the matrix*

$$\frac{1}{2\pi}\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\begin{pmatrix} 2c_1 & \alpha_1 \ldots \alpha_g \\ -2c_2 & -\beta_1 \ldots -\beta_g \end{pmatrix} \tag{6.13}$$

is integer and

$$\imath = 0. \tag{6.14}$$

Proof. A reciprocity law for abelian differentials $d\Omega_1$ and $d\Omega_3$

$$\sum_C \operatorname{res}\Omega_3 d\Omega_1 = 0$$

shows that $d\Omega_1 = \frac{1}{2}\gamma dk_{v_0}$ at $v \sim v_0$. We also have $dk_{v_0} = d\lambda/(4e^{i\varphi})$. Combined with a reality condition it gives the formula for F_2 as a final result. A recipricity law for $d\Omega_3$ and du_n shows that an Abel's map near $v = v_0$ is equal to

$$du_n = -\frac{W_n}{2}dk_{v_0} = -\frac{W_n}{4v_0}dv.$$

Taking into account (6.9) we have (6.11), (6.12). The periodicity conditions (6.13), (6.14) are the evident corollary of (4.4).

Remark. A trivial solution $u=0$ is also doubly periodic and finite-gap. This case can be easily solved and yields a cylinder.

Remark. We see that there is no restriction on the parameter $D \in iR^g$ of the finite-gap solution. So we have commuting area-preserving (see below Sect. 7) flows acting on the sets of CMC tori.

As to the parameters $\lambda_{2n}, n=1, \ldots, g$ ($2g$ real parameters), X_1, X_2, Y_1, Y_2 we have $2g+4$ real conditions (6.13), (6.14). (The parameter φ is irrelevant. It can be fixed to be zero by a transformation of the curve C $\lambda_i \to e^{-i\varphi}\lambda_i$.) The condition (6.14) is the most essential among all periodicity conditions. Whereas (6.13) can be satisfied by a small variation of parameters, it is evident that the condition (6.14) cannot. For example, (6.14) proves to be unrealizable when $g=1$ and so there is no CMC torus determined by some one-gap solution. The simplest possible case is $g=2$. For any $g \geqq 2$ a countable number of Riemann surfaces C exists defining distinct CMC tori (see Sect. 13, where also the original Wente tori are singled out).

7 Area of the CMC torus

Let Π be a fundamental region of the lattice Λ. It is a parallelogram with the sides determined by vectors with the coordinates $(X_1, Y_1), (X_2, Y_2)$. The immersion F is conformal (2.1) and therefore, the area of a constructed CMC torus T^2 is equal to

$$S = \int_\Pi |F_x|\,|F_y| dx dy = 4 \int_\Pi e^u dx dy, \tag{7.1}$$

where a domain of integration is the fundamental region on the z-plane.

Fortunately, it is possible to perform integration in (7.1) explicitly and hence to obtain a more effective formula for S. The substitution of the asymptotics (4.11) in the equation $\Psi_{\bar z} = V\Psi$ gives

$$B_{21\bar z} = e^u/2i.$$

On the other hand, a direct calculation analogous to (6.7) leads to the following formula for $B_{21\bar z}$

$$B_{21\bar z} = -2i \frac{\partial}{\partial z} \frac{\partial}{\partial \bar z} \log\theta(\Omega + \Delta) - \frac{i}{2} k,$$

where k is defined by the condition that $d\Omega_2 = k d(v^{-1})$ at $v = \infty$. A reciprocity law for $d\Omega_1, d\Omega_2$ allows us to define k in a more convenient way

$$d\Omega_1 = k dv \quad \text{at} \quad v = 0. \tag{7.2}$$

As a result the following representation for e^u is valid

$$e^u = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar z} \log\theta(\Omega + \Delta) + k. \tag{7.3}$$

This theta-functional equality is a special case of (6.8) when $v_0 \to 0$.

Then applying the Stokes formula to (7.1), (7.3) we obtain the following

Theorem 7.1. *The area of* CMC *torus is equal to*

$$S = 4kS(\Pi), \tag{7.4}$$

where $S(\Pi)$ is the area of fundamental parallelogram Π.

8 CMC tori in S^3. Analytical statement of the problem

Arguments of Sect. 2 show that there is a conformal parametrization $F: C/\Lambda \to S^3 \subset R^4$ for every smooth compact surface of genus one T^2 in a unit sphere. So for the vector function $F(z, \bar z) = (F_1, F_2, F_3, F_4)$ we have

$$\langle F, F\rangle = 1, \quad \langle F_z, F_z\rangle = \langle F_{\bar z}, F_{\bar z}\rangle = 0, \quad \langle F_z, F_{\bar z}\rangle = 2e^u,$$

where as before $\langle\,,\,\rangle$ means the canonical scalar product

$$\langle a, b\rangle = \sum_{k=1}^{4} a_k b_k.$$

Vectors $F, F_z, F_{\bar z}$ may be supplemented with a normal in such a way that

$$\langle F_z, N\rangle = \langle F_{\bar z}, N\rangle = \langle F, N\rangle = 0, \quad \langle N, N\rangle = 1.$$

Let

$$A=\langle F_{zz},N\rangle, \qquad B=\langle F_{z\bar{z}},N\rangle=2\tilde{H}e^{u}.$$

Then a variation of the basis $F, F_z, F_{\bar{z}}, N$ with respect to motion along the surface is described by the equations

$$\sigma_z=\mathcal{U}\sigma, \qquad \sigma_{\bar{z}}=\mathcal{V}\sigma, \qquad \sigma=(F,F_z,F_{\bar{z}},N)^T,$$

$$\mathcal{U}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & u_z & 0 & A \\ -2e^u & 0 & 0 & 2\tilde{H}e^u \\ 0 & -\tilde{H} & -Ae^{-u}/2 & 0 \end{pmatrix},$$

$$\mathcal{V}=\begin{pmatrix} 0 & 0 & 1 & 0 \\ -2e^u & 0 & 0 & 2\tilde{H}e^u \\ 0 & 0 & u_{\bar{z}} & \bar{A} \\ 0 & -\bar{A}e^{-u}/2 & -\tilde{H} & 0 \end{pmatrix}.$$

The CMC condition $\tilde{H}=\text{const}$ leads to the Gauss-Peterson-Codazzi equations

$$u_{z\bar{z}}+2(1+\tilde{H}^2)e^u-A\bar{A}e^{-u}/2=0, \qquad A_{\bar{z}}=0.$$

So $A(z)$ is an elliptic function. Finiteness of $A(z)$ implies $A(z)=\text{const}$. The case $A_1=0$ can be explicitely solved and corresponds to an immersion of the sphere. Finally, we have $A(z)=\text{const}\neq 0$.

Let us fix the conformal coordinate z so that $A=2\sqrt{1+\tilde{H}^2}\,e^{i\varphi}$. In terms of a new variable

$$w=\delta z, \qquad \delta=2\sqrt{1+\tilde{H}^2} \tag{8.1}$$

we have

$$u_{w\bar{w}}+\sinh u=0, \tag{8.2}$$

$$\sigma_w=\mathcal{U}\sigma, \qquad \sigma_{\bar{w}}=\mathcal{V}\sigma, \tag{8.3}$$

$$\mathcal{U}=\begin{pmatrix} 0 & \delta^{-1} & 0 & 0 \\ 0 & u_w & 0 & e^{i\varphi} \\ -2e^u/\delta & 0 & 0 & 2\tilde{H}e^u/\delta \\ 0 & -\tilde{H}/\delta & -e^{i\varphi-u}/2 & 0 \end{pmatrix},$$

$$\mathcal{V}=\begin{pmatrix} 0 & 0 & \delta^{-1} & 0 \\ -2e^u/\delta & 0 & 0 & 2\tilde{H}e^u/\delta \\ 0 & 0 & u_{\bar{w}} & e^{-i\varphi} \\ 0 & -e^{-i\varphi-u}/2 & -\tilde{H}/\delta & 0 \end{pmatrix}. \tag{8.4}$$

Let

$$\phi=\begin{pmatrix} F_1 & F_2 & F_3 & F_4 \\ F_{1z} & F_{2z} & F_{3z} & F_{4z} \\ F_{1\bar{z}} & F_{2\bar{z}} & F_{3\bar{z}} & F_{4\bar{z}} \\ N_1 & N_2 & N_3 & N_4 \end{pmatrix} \tag{8.5}$$

be a matrix with the coordinates of the vectors F, N in some fixed basis of R^4. Rotation of this fixed basis leads to the transformation $\phi \to \phi\omega$, $\omega \in O(4)$. ϕ satisfies the following equations

$$\phi_w = \mathcal{U}\phi, \qquad \phi_{\bar{w}} = \mathcal{V}\phi, \tag{8.6}$$

$$\bar{\phi} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \phi, \quad \phi\phi^T = \begin{pmatrix} 1 & & & \\ & 0 & 2e^u & \\ & 2e^u & 0 & \\ & & & 1 \end{pmatrix} \tag{8.7}$$

with $\mathcal{U}, \mathcal{V}$ of the form (8.4).

9 CMC tori in S^3. Formulae for immersion

Theorem 3.1 shows that it is sufficient to solve the problem (8.6), (8.7) for the finite-gap solutions only. Let $u(w, \bar{w})$ be a real finite-gap solution of the Eq. (8.2) and $\psi(v)$ be the respecting B-A function.

Let

$$v = \exp\left(i\frac{\pi}{4} - i\frac{h}{2} + i\frac{\varphi}{2}\right), \qquad v' = \exp\left(i\frac{3\pi}{4} + i\frac{h}{2} + i\frac{\varphi}{2}\right), \tag{9.1}$$

where h is defined by

$$2(1 + i\tilde{H}) = \delta e^{ih}. \tag{9.2}$$

A calculation proves that the function

$$\tilde{\phi} = g\phi_0, \qquad \phi_0 = \Psi(v) \otimes \Psi(v'),$$

$$g = \begin{pmatrix} 0 & ie^{-ih/2} & e^{ih/2} & 0 \\ 0 & 0 & 0 & 2e^{i\pi/4 + i\varphi/2} \\ -2e^{i\pi/4 - i\varphi/2 + u} & 0 & 0 & 0 \\ 0 & e^{-ih/2} & ie^{ih/2} & 0 \end{pmatrix} \tag{9.3}$$

satisfies the Eqs. (8.6). As a corollary ϕ can be presented in the form

$$\phi = \tilde{\phi} M,$$

where M is a matrix independent of $w, \bar{w}$.

With the help of the real reduction (4.17) (note that $|v| = |v'| = 1$) we can rewrite conditions (8.7) in terms of M

$$MM^T = \frac{i}{2dd'} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},$$

$$\bar{M} = i \begin{pmatrix} & & & -(\alpha\alpha')^{-1} \\ & & \bar{\alpha}'\alpha^{-1} & \\ & \bar{\alpha}(\alpha')^{-1} & & \\ -\bar{\alpha}\bar{\alpha}' & & & \end{pmatrix} M,$$

where $\alpha=\alpha(v)$, $\alpha=\alpha(v')$ are given by (4.18) and $d=d(v)$, $d'=d(v')$ are given by (4.15), (4.16). These equations can be easily solved and finally, we obtain the following expression for M

$$M=\frac{1}{2n}\begin{pmatrix}1&&&\\&\alpha'&&\\&&\alpha&\\&&&\alpha\alpha'\end{pmatrix}\begin{pmatrix}&&1&i\\1&-i&&\\-i&1&&\\&&i&1\end{pmatrix}\omega, \tag{9.4}$$

where

$$n^2=\alpha\alpha' dd' \tag{9.5}$$

and $\omega\in O(4)$ is arbitrary.

Formula (4.17) shows that

$$\alpha d=-(e^{u/2}|\psi_1|^2+e^{-u/2}|\psi_2|^2)$$

for $|v|=1$, hence n in (9.4) is real-valued.

Now we are in a position to formulate a desired

Theorem 9.1. *The immersion corresponding to the finite-gap solution is determined by the following expression*

$$\phi=mG\mathscr{A}\Theta\mathscr{B}T\omega, \tag{9.6}$$

$$\Theta=\begin{pmatrix}\theta(\Omega+u)&\theta(\Omega-u)\\ \theta(\Omega+\Delta+u)&-\theta(\Omega+\Delta-u)\end{pmatrix}\otimes\begin{pmatrix}\theta(\Omega+u')&\theta(\Omega-u')\\ \theta(\Omega+\Delta+u')&-\theta(\Omega+\Delta-u')\end{pmatrix},$$

$$G=\begin{pmatrix}0&i&1&0\\0&0&0&2e^{i\pi/4}\\-2e^{i\pi/4}&0&0&0\\0&1&i&0\end{pmatrix},\quad T=\begin{pmatrix}0&0&1&i\\1&-i&0&0\\-i&1&0&0\\0&0&i&1\end{pmatrix}.$$

$$\mathscr{A}=\operatorname{diag}\left(\frac{e^{-i\varphi/2}}{\theta(\Omega+\Delta)},\frac{e^{-ih/2}}{\theta(\Omega)},\frac{e^{ih/2}}{\theta(\Omega)},\frac{e^{i\varphi/2}}{\theta(\Omega+\Delta)}\right),$$

$$\mathscr{B}=\operatorname{diag}(e^{i(s+s')},e^{i(s-s')},e^{-i(s-s')},e^{-i(s+s')}),$$

$$m=\frac{\theta(0)\theta(\Delta)}{4\theta(\Omega+\Delta)(\theta(u)\theta(u')\theta(u+\Delta)\theta(u'+\Delta))^{1/2}},$$

$$s=\delta(c_2y-c_1x),\qquad s'=\delta(c_2'y-c_1'x),$$

$$\int_{\ell}d\Omega_1=c_1+ic_2,\qquad \int_{\ell'}d\Omega_1=c_1'+ic_2',$$

$$\Omega=-\frac{i\delta}{2}(Uz+\bar{U}\bar{z})+D,\qquad u=\int_{\ell}du,\qquad u'=\int_{\ell'}du,$$

where $\omega\in O(4)$ is arbitrary, ℓ and ℓ' are the paths on C from ∞ to v and v' (9.1), respectively, which do not intersect $\mathscr{L}$. $\mathscr{A}$ and $\mathscr{B}$ are diagonal matrices.

All CMC tori in S^3 are described by the doubly periodic immersions (9.6) *and also by the immersion in terms of elementary functions* ($u=0$)

$$\phi=\frac{1}{4}G\mathscr{A}_0\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\mathscr{B}T\omega\,,$$

$$\mathscr{A}_0=\operatorname{diag}(e^{-i\varphi/2},e^{-ih/2},e^{ih/2},e^{i\varphi/2})\,, \tag{9.7}$$

$$v=c_1+ic_2\,,\qquad v'=c_1'+ic_2'\,.$$

The immersion (9.6) *is doubly periodic with a lattice of periods Λ if and only if the matrix*

$$\frac{\delta}{2\pi}\begin{pmatrix}X_1 & Y_1\\ X_2 & Y_2\end{pmatrix}\begin{pmatrix}c_1+c_1' & c_1-c_1' & \alpha_1\ \dots\ \alpha_g\\ -c_2-c_2' & -c_2+c_2' & -\beta_1\ \dots\ -\beta_g\end{pmatrix} \tag{9.8}$$

is integer. The area of CMC torus is given by the formula (7.4).

Proof. As compared with (9.6) ϕ_0 has an additional right factor

$$\operatorname{diag}\left(\frac{1}{\theta(D+u)\theta(D+u')},\frac{\alpha'}{\theta(D+u)\theta(D-u')},\frac{\alpha}{\theta(D-u)\theta(D+u')},\frac{\alpha\alpha'}{\theta(D-u)\theta(D-u')}\right).$$

The equality (4.17) with $|v|=|v'|=1$ shows that it is equal to

$$q^{-2}=\theta(D+u)\theta(D+u')\theta(D+\Delta-u)\theta(D+\Delta-u')\,. \tag{9.9}$$

$$q\operatorname{diag}(e^{i\gamma_1},e^{i\gamma_2},e^{-i\gamma_2},e^{-i\gamma_1})\,,$$

q is included in m and the diagonal matrix in (9.9) can be considered as a constant rotation of a basis in R^4. Therefore, it is natural to include it into ω. The factor δ is due to (8.1). A direct calculation proves (9.7).

It is evident that one can transform a quasiperiodic immersion into purely periodic one by a small variation of a correspondent curve C.

As previously in the R^3 case we have $2g+4$ conditions (9.8) on parameters λ_i ($2g$ real parameters) and X_1, X_2, Y_1, Y_2. So for any g a countable number of the curves C exists defining distinct CMC tori in S^3.

10 Minimal tori in S^3 and Willmore tori

Now we consider the case of minimal surfaces $\tilde{H}=0$. Lawson proved [33] that there exist closed orientable embedded minimal surfaces of arbitrary genus in S^3.

The simplest minimal torus in S^3 is the famous Clifford torus T^2_{Cl}

$$F_1^2+F_2^2=F_3^2+F_4^2=\tfrac{1}{2}\,.$$

In our normalization the Clifford embedding $F(z,\bar{z})$ is as follows

$$F_1+iF_2=\frac{1}{\sqrt{2}}e^{i2\sqrt{2}x}\,,\qquad F_3+iF_4=\frac{1}{\sqrt{2}}e^{i2\sqrt{2}y}$$

and the induced metric is trivial $u=0$. The area of the Clifford torus is equal to

$$S(T^2_{\mathrm{Cl}})=2\pi^2\,.$$

Minimal surfaces in S^3 are closely connected with so called Willmore surfaces in R^3 that are "as smooth as possible" compact surfaces of prescribed topological type [39]. Let $F: M^2 \to R^3$ be an immersion of an abstract compact surface M^2 into R^3. Consider the functional

$$W(F) = \int_{M^2} H^2 dS', \tag{10.1}$$

where H is the mean curvature and dS' is an ordinary 2-dimensional area measure. Because of the Gauss-Bonnet theorem this functional is equivalent to

$$\int_{M^2} (k_1^2 + k_2^2) dS'.$$

A surface in R^3 is called a Willmore surface if it is an extremal surface for the variational functional (10.1).

There is a well known connection between minimal surfaces in S^3 and Willmore surfaces [47].

Theorem 10.1. *Let X be compact minimal surface in S^3 and $\sigma: S^3 \setminus \text{point} \to R^3$ be stereographic projection. Then $\sigma(X)$ is a Willmore surface and*

$$W(\sigma(X)) = S(X),$$

where S is an area induced by embedding $S^3 \subset R^4$.

Many of the known examples of embedded Willmore surfaces come from compact minimal surfaces in S^3 in this way although there are Willmore tori which are not stereographic projections of minimal tori in S^3 [26, 27, 38]. Mention that Willmore tori in R^3 are in one to one correspondence with minimal tori in the Lorentzian 4-sphere [9, 23].

There are a lot of interesting results concerning Willmore surfaces. First of all for any closed orientable surface $W \geqq 4\pi$ [50] and the equality holds if and only if the surface is the standard (round) sphere. If F is an immersion of a compact surface into R^3 and there is a point $p \in R^3$ such that $F^{-1}(p)$ consists on n distinct points, then [34] $W(F) \geqq 4\pi n$. As corollaries we have that if $W < 8\pi$ then F is an embedding and that the similar results are valid also for areas of minimal surfaces in S^3.

For tori some additional more detailed results are known. It is proved that for any torus of revolution as well as for any tube $W \geqq 2\pi^2$ [50]. It is also known (see [50]) that for any knotted torus of bridge number n $W > 8\pi n$. If a torus T^2 is conformally equivalent to a flat torus with a lattice generated by $\{(1,0),(x,y)\}$, where $-1/2 \leqq x \leqq 1/2$, $\sqrt{1-x^2} \leqq y \leqq 1$ then $W(T^2) \geqq 2\pi^2$ [34]. Equality implies that T^2 is conformally equivalent to the square torus $x=0$, $y=1$.

There are interesting open problems concerning minimal and CMC tori in S^3.

Conjecture 1. *The area of any minimal torus T^2 in S^3*

$$S(T^2) \geqq 2\pi^2.$$

This problem is a special case of the famous Willmore conjecture that the absolute minimum of W over all tori is equal to $W(\sigma(T^2_{\mathrm{Cl}})) = 2\pi^2$. It is not equivalent to it since although the existence of a real-analytic immersion, minimizing W is proved [42], it is unknown if this immersion comes from a minimal torus in S^3. Due to the results of Li and Yau mentioned above [34] the Conjecture 1 is a corollary of the following one.

The Hsiang-Lawson conjecture. *The Clifford torus is the only embedded minimal torus in* S^3.

The Pinkall-Sterling conjecture [40]. *Every embedded* CMC *torus in* S^3 *is a torus of revolution.*

Returning to our formulae we should note that the minimal tori in S^3 are special case of CMC tori in S^3 with $\tilde{H}=0$.

Theorem 10.3. *All minimal tori in* S^3 *are the Clifford torus and tori described by the following formulae:*

$$\begin{aligned}
F_1+iF_2&=\varepsilon e^{i(s-s')}[\theta(\Omega+\Delta+u)\theta(\Omega-u')-i\theta(\Omega+u)\theta(\Omega+\Delta-u')],\\
F_1-iF_2&=\varepsilon e^{-i(s-s')}[\theta(\Omega-u)\theta(\Omega+\Delta+u')+i\theta(\Omega+\Delta-u)\theta(\Omega+u')],\\
F_3+iF_4&=\varepsilon e^{-i(s+s')}[\theta(\Omega-u)\theta(\Omega+\Delta-u')-i\theta(\Omega+\Delta-u)\theta(\Omega-u')],\\
F_3-iF_4&=\varepsilon e^{i(s+s')}[\theta(\Omega+\Delta+u)\theta(\Omega+u')+i\theta(\Omega+u)\theta(\Omega+\Delta+u')],
\end{aligned}\tag{10.2}$$

$$\varepsilon=\frac{\theta(0)\theta(\Delta)}{2\theta(\Omega)\theta(\Omega+\Delta)(\theta(u)\theta(u')\theta(u+\Delta)\theta(u'+\Delta))^{1/2}},$$

$$v=\exp(i\pi/4+i\varphi/2),\qquad v'=\exp(i3\pi/4+i\varphi/2),$$

granted the periodicity conditions (9.7) *with* $\delta=2$ *are valid (for all notations see Theorem 9.1). The area of the minimal torus* (10.2) *is equal to (if* $Q\neq 0$*)*

$$S=kN\frac{2\pi^2}{Q},\qquad Q=\begin{vmatrix} c_1 & c_1' \\ c_2 & c_2' \end{vmatrix}.\tag{10.3}$$

Here $N\in Z$ *and* k *is the coefficient* (7.2).

Proof. The area of a fundamental parallelogram up to a sign can be expressed as follows

$$S(\Pi)=\begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}.$$

Combined with (7.4) it gives (10.3). The formulae (10.2) are a particular case of (9.6).

Remark. We conjecture that Q never vanishes and (10.3) is universal. Applying the Stokes formula to (7.3) and to $\sinh u$ (8.2) we have

$$\int_\Pi e^u dxdy=kS(\Pi),\qquad \int_\Pi \sinh u\,dxdy=0.$$

Hence

$$\int_\Pi \cosh u\,dxdy/S(\Pi)=k\geq 1$$

and to verify Conjecture 1 it is necessary to estimate Q. At present this problem remains unsolved.

11 CMC tori in H^3

In this section we deal with all compact smooth CMC tori in H^3. All arguments are quite analogous to those of Sects. 8 and 9. We use the same notations unless explicitly stated otherwise.

A hyperbolic space H^3

$$\{F,F\}=-1\,, \qquad \{a,b\}=a_1b_1+a_2b_2+a_3b_3-a_4b_4 \tag{11.1}$$

is embedded into the Lorentz space $R^{3,1}$. Metric of $R^{3,1}$ induces positively definite metric $\{\,,\}$ on H^3.

Let $F(z,\bar{z}):C/\Lambda\to H^3\subset R^{3,1}$ be a conformal parametrization of a smooth compact torus in H^3. As in Sect. 8 it is possible to fix a frame $F, F_z, F_{\bar{z}}, N$ so that

$$\begin{gathered}\{N,F\}=\{N,F_z\}=\{N,F_{\bar{z}}\}=\{F_z,F_z\}=\{F_{\bar{z}},F_{\bar{z}}\}=0\,,\\ \{N,N\}=1\,, \qquad \{F_z,F_{\bar{z}}\}=2e^u\,.\end{gathered} \tag{11.1}$$

The equations for a moving frame $\sigma=(F,F_z,F_{\bar{z}},N)^T$ are analogous to those of Sect. 8 but the matrices $\mathscr{U}$ and $\mathscr{V}$ are slightly different

$$\mathscr{U}=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & u_z & 0 & A\\ 2e^u & 0 & 0 & 2\tilde{H}e^u\\ 0 & -\tilde{H} & -Ae^{-u}/2 & 0\end{pmatrix},$$

$$\mathscr{V}=\begin{pmatrix} 0 & 0 & 1 & 0\\ 2e^u & 0 & 0 & 2\tilde{H}e^u\\ 0 & 0 & u_{\bar{z}} & \bar{A}\\ 0 & -\bar{A}e^{-u}/2 & -\tilde{H} & 0\end{pmatrix},$$

$$A=\{F_{zz},N\}\,, \qquad 2\tilde{H}e^u=\{F_{z\bar{z}},N\}\,.$$

The CMC condition $\tilde{H}=\text{const}$ leads to the Gauss-Peterson-Codazzi equation

$$u_{z\bar{z}}+2(\tilde{H}^2-1)e^u-A\bar{A}e^{-u}/2=0\,, \qquad A_{\bar{z}}=0\,.$$

In the cases $|\tilde{H}|>1$, $|\tilde{H}|<1$, $|\tilde{H}|=1$ we have the elliptic sinh-Gordon, the elliptic cosh-Gordon and the Liouville equations, respectively. By the maximum principle there are no compact CMC surfaces in H^3 with $|\tilde{H}|\leqq 1$. So we restrict ourselves to the case $|\tilde{H}|>1$, although the finite-gap solutions of the elliptic cosh-Gordon equation are already known [4].

$A(z)$ is an elliptic function, that gives $A=\text{const}\neq 0$. Fixing the conformal coordinate z by the condition $A=2\sqrt{\tilde{H}^2-1}\,e^{i\varphi}$, in a new variable

$$w=\delta z\,, \qquad \delta=2\sqrt{\tilde{H}^2-1} \tag{11.3}$$

we have (8.2), (8.3) with $\mathscr{U}$, $\mathscr{V}$ being equal to

$$\begin{gathered}\mathscr{U}=\begin{pmatrix} 0 & \delta^{-1} & 0 & 0\\ 0 & u_w & 0 & e^{i\varphi}\\ 2e^u/\delta & 0 & 0 & 2\tilde{H}e^u/\delta\\ 0 & -\tilde{H}/\delta & -e^{i\varphi-u}/2 & 0\end{pmatrix},\\ \mathscr{V}=\begin{pmatrix} 0 & 0 & \delta^{-1} & 0\\ 2e^u/\delta & 0 & 0 & 2\tilde{H}e^u/\delta\\ 0 & 0 & u_w & e^{-i\varphi}\\ 0 & -e^{-i\varphi-u}/2 & -\tilde{H}/\delta & 0\end{pmatrix}.\end{gathered} \tag{11.4}$$

The problem is to find the matrix ϕ (8.5) which satisfies the following equations

$$\phi_w = \mathscr{U}\phi\,, \qquad \phi_{\bar w} = \mathscr{V}\phi\,, \tag{11.5}$$

$$\bar\phi = \begin{pmatrix}1&&&\\&0&1&\\&1&0&\\&&&1\end{pmatrix}\phi\,, \quad \phi\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}\phi^T = \begin{pmatrix}-1&&&\\&0&2e^u&\\&2e^u&0&\\&&&1\end{pmatrix} \tag{11.6}$$

with $\mathscr{U}, \mathscr{V}$ of the form (11.4).

Let

$$v = \exp\left(-\frac{h}{2} + i\frac{\varphi}{2}\right), \qquad v' = \exp\left(\frac{h}{2} + i\frac{\varphi}{2}\right), \tag{11.7}$$

where h is defined as follows:

$$e^h = ((\tilde H + 1)/(\tilde H - 1))^{1/2}\,. \tag{11.8}$$

A calculation proves that ϕ can be presented in the form

$$\phi = g\Psi(v)\otimes\Psi(v')M\,, \tag{11.9}$$

$$g = \begin{pmatrix} 0 & e^{-h/2} & -e^{h/2} & 0 \\ 0 & 0 & 0 & 2ie^{i\frac{\varphi}{2}} \\ 2ie^{-i\varphi/2+u} & 0 & 0 & 0 \\ 0 & e^{-h/2} & e^{h/2} & 0 \end{pmatrix}, \tag{11.10}$$

where M is a matrix independent of $w, \bar w$.

Note that in the H^3 case we have $v' = v^{-1}$ (11.7). Combined with (4.17) this allows us to reformulate the conditions (11.6) in terms of M

$$M\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}M^T = \frac{1}{2dd'}\begin{pmatrix}&&&-1\\&&1&\\&1&&\\-1&&&\end{pmatrix},$$
$$\bar M = \begin{pmatrix}&&&-(\alpha\alpha')^{-1}\\&\bar\alpha'/\alpha'&0&\\&0&\bar\alpha/\alpha&\\-\bar\alpha\bar\alpha'&&&\end{pmatrix}M\,. \tag{11.11}$$

For all notations see Sect. 8. The solution of (11.11) is given by

$$M = \frac{1}{2n}\begin{pmatrix}1&&&\\&\alpha'&&\\&&\alpha&\\&&&\alpha\alpha'\end{pmatrix}\begin{pmatrix}1&i&&\\&&1&1\\&&1&-1\\-1&i&&\end{pmatrix}\omega\,,$$

where $\omega \in O(3,1)$ is arbitrary. n in this formula is defined by (9.5) and is real-valued since $\overline{\alpha d} = \alpha' d'$.

Theorem 11.1. *The immersion corresponding to the finite-gap solution is determined by the formula* (9.6), *where matrices G, T, ω and $\mathscr{A}$ are defined in a different way*

$$G = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2i \\ 2i & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & i & 0 & 0 \end{pmatrix}, \tag{11.12}$$

$$\mathscr{A} = \mathrm{diag}\left(\frac{e^{-i\varphi/2}}{\theta(\Omega+\Delta)}, \frac{e^{-h/2}}{\theta(\Omega)}, \frac{e^{h/2}}{\theta(\Omega)}, \frac{e^{i\varphi/2}}{\theta(\Omega+\Delta)} \right), \quad \omega \in O(3,1).$$

All CMC *tori in H^3 are described by the doubly periodic immersions* (9.6), (11.12). *These immersions are doubly periodic with a lattice of periods Λ if and only if the matrix*

$$\frac{\delta}{2\pi} \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix} \begin{pmatrix} 2c_1 & \alpha_1 \ldots \alpha_g \\ -2c_2 & -\beta_1 \ldots -\beta_g \end{pmatrix} \tag{11.13}$$

is integer and

$$\int_v^{v'} d\Omega_1 = 0, \tag{11.14}$$

where v and v' are given by (11.7).

Proof. Comparing with the S^3 case we have a different real condition for v, v': $\bar{v} = (v')^{-1}$. It leads to a different conjugation law

$$\int_{\ell} d\Omega_1 = \overline{\int_{\ell'} d\Omega_2}$$

and hence to the special periodicity condition (11.14). A direct calculation proves that there are no CMC tori corresponding to the trivial solution $u=0$.

As above in the R^3 case (11.14) proves to be unrealizable when $g=1$. For $g=2$ a countable number of C exists defining CMC tori (see Sect. 13).

12 Schottky uniformization and calculations

Now we have reached the end of our analytic calculation. To solve effectively the periodicity conditions we need the help of a computer.

A direct way to draw pictures of CMC tori with the help of the formulae obtained is to calculate the corresponding abelian integrals. However, this way is rather inconvenient because it is a problem to estimate values of abelian integrals and so the solution of the periodicity conditions is rather difficult. In addition, this method is very sensitive to the value of g. A change of g makes calculations start from the very beginning. Here we present a different approach [10], based on the Schottky uniformization of the corresponding Riemann surface.

Let $\mathcal{O}$ denote the unit circle $|z|=1$ in the z-plane (in this section we use the notation z for uniformizing variable). Let $C_1, \ldots, C_g$ be disjoint circles orthogonal

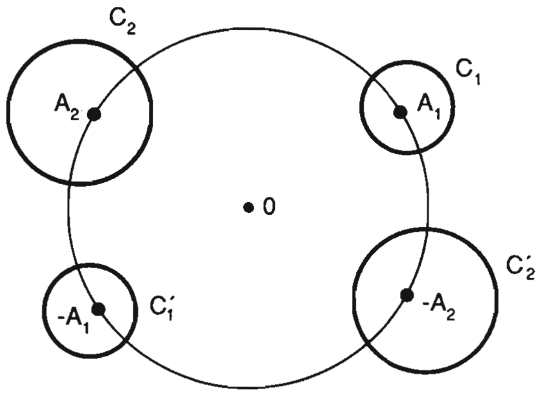

Fig. 2

to $\mathcal{O}$. The symmetry

$$\pi z = -z \tag{12.1}$$

transforms C_n onto a circle $C'_n = \pi C_n$. $C_1, C'_1, \ldots, C_g, C'_g$ comprise the boundary of a $2g$-connected domain F (see Fig. 2). The linear transformation σ_n

$$\frac{\sigma_n z + A_n}{\sigma_n z - A_n} = \mu_n \frac{z + A_n}{z - A_n}, \quad 0 < \mu_n < 1, \quad |A_n| = 1, \quad n = 1, \ldots, g, \tag{12.2}$$

maps the outside of the boundary circle C_n onto inside of the boundary circle C'_n, $\sigma_n C_n = C'_n$. A_n and $-A_n$ are the fixed points of the hyperbolic transformation σ_n and also A_n is inside of C_n.

The elements $\sigma_1, \ldots, \sigma_g$ generate a classical Schottky group [28], which in this case is also a Fuchsian group of the second kind. F is the fundamental domain of G. Let $\Omega(G)$ be the set of discontinuity of G, then Ω/G is a real hyperelliptic Riemann surface of genus g. As a matter of fact, (12.1) is a hyperelliptic involution with $2g+2$ fixed points: $0, \infty$ and the pairs of points given by $\sigma_n z = \pi z$,

$$(z - A_n)^2 = \mu_n (z + A_n)^2 .$$

Also we see that

$$\tau z = \bar{z}^{-1} \tag{12.3}$$

is an antiholomorphic involution of Ω/G. Fixed points of τ comprise a real oval (g is even) or two ovals (g is odd).

Every real hyperelliptic curve (4.3), (4.14) can be uniformized in the way presented above. It is proved by the usual technique of uniformization theory.

Choose a canonical basis of cycles of Ω/G such that the a_n's coincide with the C'_n's positively oriented, the b_n's join the points $z_n \in C_n$ and $\sigma_n z_n \in C'_n$, and b-cycles do not intersect each other.

Abelian differentials on Ω/G are well determined by the (-2) dimensional Poincaré theta series. For the Fuchsian group of the second kind these series converge [17, 28]. Denote by G_n the subgroup of G generated by σ_n. Cosets G/G_n and $G_m \backslash G/G_n$ are sets of all elements $\sigma = \sigma_{i_1}^{j_1} \ldots \sigma_{i_k}^{j_k}$ such that $i_k \neq n$ and for $G_m \backslash G/G_n$ additionally $i_1 \neq m$. The following theorem is due to the old papers [5, 16]:

Theorem 12.1. *The series*

$$du_n = \sum_{\sigma \in G/G_n} [(z-\sigma(-A_n))^{-1} - (z-\sigma A_n)^{-1}]dz$$

define holomorphic differentials normalized in the basis fixed above. The period matrix is given by

$$B_{nm} = \sum_{\sigma \in G_m \backslash G/G_n} \log\{-A_m, A_m, \sigma(-A_n), \sigma A_n\},$$

$$B_{nn} = \log \mu_n + \sum_{\sigma \in G_n \backslash G/G_n,\, \sigma \neq I} \log\{-A_n, A_n, \sigma(-A_n), \sigma A_n\},$$

where $\{\ \}$ *means the double-ratio*

$$\{z_1, z_2, z_3, z_4\} = (z_1 - z_3)(z_2 - z_4)(z_1 - z_4)^{-1}(z_2 - z_3)^{-1}.$$

Hence an Abel's map u in previous sections is equal to

$$u_n(z) = \int_\infty^z du_n = \int_0^z du_n + \pi i,$$

$$u_n(z) = \log \prod_{\sigma \in G/G_n} \{z, 0, \sigma(-A_n), \sigma A_n\} + \pi i.$$

G is a discrete subgroup of $\mathrm{PSL}(2, C)$. Denote the coefficients of $\sigma \in G$ as follows:

$$\sigma z = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}(2, C). \tag{12.4}$$

Since $|z| = 1$ is invariant we have $\gamma = \bar{\beta}$, $\delta = \bar{\alpha}$. The hyperelliptic involution (12.1) implies a group involution

$$\sigma \to \sigma^* = \pi \sigma \pi, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}. \tag{12.5}$$

The function $\lambda(z)$ from Sect. 4 has double zero at $z = 0$ and double pole at $z = \infty$. Hence it is given by

$$\lambda(z) = \frac{1}{q} \sum_{\sigma \in G} [(\sigma z)^2 - (\sigma 0)^2] \tag{12.6}$$

with a certain constant q. Due to (12.5) the following asymptotics hold

$$\lambda \to \frac{z^2}{q}, \quad z \sim \infty, \quad \lambda \to \frac{Q}{q} z^2, \quad z \sim 0, \quad Q = \sum_{\sigma \in G} (\delta^{-4}(1 - 2\beta\gamma)). \tag{12.7}$$

To define q let us identify the real reductions (4.12) and (12.3)

$$\overline{\lambda(z^{-1})} = \lambda^{-1}(\bar{z}).$$

Finally, we have

$$q = \sqrt{Q}. \tag{12.8}$$

The asymptotics (4.5), (12.7), (12.8) show that the abelian integral Ω_1 is equal to

$$\Omega_1(z) = \frac{1}{\sqrt{q}} \sum_{\sigma \in G} [\sigma z - \sigma 0]. \tag{12.9}$$

Hence the coefficient k in (7.2) is as follows:

$$k=\frac{1}{q}\sum_{\sigma\in G}\delta^{-2}.$$

At last, the reciprocity law (6.6) gives for the b-periods of Ω_1

$$U_n=\frac{1}{\sqrt{q}}\sum_{\sigma\in G/G_n}[\sigma(-A_n)-\sigma A_n]. \tag{12.10}$$

Remark. The coefficient q above is not essential. It may be put to be equal to unity $q=1$ by a simple scale transformation of the fundamental parallelogram Π (z-variable of previous sections).

Remark. Concerning the conjectures of Sect. 10, the case of CMC tori in S^3 is the most promising for calculation. The suggested method suits well the S^3 case whereas the periodicity conditions (6.14), (11.14) for R^3, H^3 are more complicated to take into account. The functions (12.6), (12.9), (12.10) are approximated by their terms with $\sigma=I$

$$\lambda\approx z^2, \qquad \Omega_1(z)\approx z, \qquad U_n\approx -2A_n.$$

This approximation allows effective resolution of the periodicity condition (9.8) with the help of the method of calculation suggested in [15].

Remark. The parameters $|A_n|=1$, $0<\mu_n<1$ in the formulae above are not arbitrary. The restriction is that they are induced by the corresponding mutually disjoint circles C_n, C_n'. Nevertheless, a domain of A, μ-variation $S=\{A_1,\mu_1,\ldots,A_g,\mu_g\}\subset(S^1\times[0,1])^g$ can be described explicitly via the usual technique [8, 37, 51].

13 Periodicity conditions for R^3, H^3 and Wente's tori

A direct calculation proves that for $g=1$ the conditions (6.14), (11.14) on the abelian differential $d\Omega_1$ are incompatible with the normalization (4.5). So there are no $g=1$ CMC tori in R^3, H^3.

The simplest possible case is $g=2$. Now we indicate the location of the original Wente's tori described explicitly by elliptic functions [1, 41] among all CMC tori. Let us consider the curve C of genus 2

$$\mu^2=\lambda(\lambda-\varrho e^{ik})(\lambda-\varrho e^{-ik})(\lambda-\varrho^{-1}e^{ik})(\lambda-\varrho^{-1}e^{-ik}) \tag{13.1}$$

having along with the involutions π and τ (see Sect. 4) one more involution

$$i:(\lambda,\mu)\to(\lambda^{-1},\mu\lambda^{-3}).$$

C covers two elliptic curves:

$$C'=C/i\pi, \qquad t'^2=(s-2)(s-E)(s-\bar{E}),$$

$$C''=C/i, \qquad t''^2=(s+2)(s-E)(s-\bar{E}),$$

$$s=\lambda+\lambda^{-1}, \qquad E=\varrho^{-1}e^{ik}+\varrho e^{-ik}, \qquad t'=(\lambda-1)\mu\lambda^{-2}, \qquad t''=(\lambda+1)\mu\lambda^{-2}.$$

The differentials (4.5), (4.6)

$$d\Omega_1=(\lambda^2+a\lambda+b)d\lambda/2\mu, \qquad d\Omega_2=-(1+a\lambda+b\lambda^2)d\lambda/2\lambda\mu$$

are the combinations

$$d\Omega_1=\tfrac{1}{2}(d\omega'+d\omega''), \qquad d\Omega_2=\tfrac{1}{2}(d\omega''-d\omega')$$

of the elliptic (in s, t variables) differentials

$$d\omega'=(\lambda+1)(\lambda+\lambda^{-1}+a+b-1)d\lambda/2\mu,$$
$$d\omega''=(\lambda-1)(\lambda+\lambda^{-1}+a-b+1)d\lambda/2\mu.$$

Put $\varphi=0$ in (6.11). The R^3 periodicity condition (6.14) becomes

$$d\Omega_1/d\lambda|_{\lambda=1}=0, \tag{13.2}$$

which yields $a+b=-1$. Finally, we see that (13.2) can be reformulated as the normalization condition (4.5)

$$\int_{\bar{E}}^{E} d\omega'=0 \tag{13.3}$$

for the elliptic differential

$$d\omega'=(s-2)ds/2t'$$

($d\omega''$ is normalized by a suitable value of $a-b$).

The condition (13.3) is equivalent to

$$\int_{e^{-i\theta}}^{e^{i\theta}} \frac{xdx}{\sqrt{x(x-e^{i\theta})(x-e^{-i\theta})}}=0, \tag{13.4}$$

where

$$\theta=\arg(E-2). \tag{13.5}$$

The integration contour is shown of Fig. 3. The substitution $x=e^{i\gamma}$ and the symmetry $\gamma\to 2\pi-\gamma$ transform Eq. (13.4) into the following one:

$$\int_{\pi}^{\theta} \frac{\cos\gamma d\gamma}{\sqrt{\cos\theta-\cos\gamma}}=0. \tag{13.6}$$

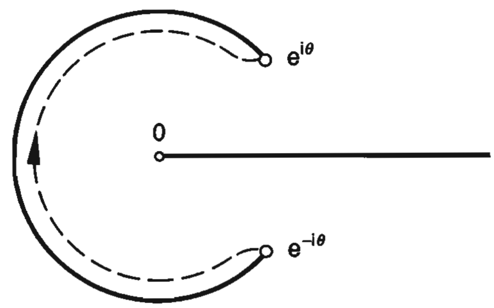

Fig. 3

It is evident that this equation has a unique solution θ. Combined with (13.5) it determines the function $\varrho(k)$

$$\operatorname{tg}\theta=\frac{(\varrho^{-1}-\varrho)\sin k}{(\varrho^{-1}+\varrho)\cos k-2}, \qquad 0<k<\theta. \tag{13.7}$$

Now the condition (13.2) is satisfied. As to other periodicity conditions (6.13) we use the evident symmetry relation

$$(\tau i)^* d\Omega_1=\overline{d\Omega_1}.$$

Hence, for the basis of Fig. 1 we have

$$\int_{b_1} d\Omega_1=-\overline{\int_{b_2} d\Omega_1}=\alpha+i\beta.$$

Since the value $\int_0^1 d\Omega_1=c_1\in R$ is real the fundamental parallelogram Π is a rectangle and the conditions (6.13) are equivalent to rationality of the point

$$c_1/\alpha\in Q. \tag{13.8}$$

An infinite number of points (13.8) on the curve (13.7) give the CMC tori in R^3 described in [1, 45].

The two-dimensional theta-function of the curve C reduces to the one-dimensional theta-functions of the curves C' and C''. In our approach the final formula in terms of the theta-functions of Jacobi type for the immersion are obtained by the technique of reduction of theta-functions suggested in [7]. In addition, the x and y variables are separated. They are arguments of theta-functions of C'' and C', respectively.

Let us make the similar calculations for the curve (13.1) in H^3 case. The corresponding tori in terms of elliptic integrals were constructed by Walter [46]. First of all, consider the specific H^3 periodicity condition (11.14). Put v (11.7) to be real ($\varphi=0$) and denote by $\tilde{L}$ the contour from $\lambda_0=v^2$ to $\bar{\lambda}_0^{-1}$. The involution i inverts $\tilde{L}$: $\tilde{L}=-i\tilde{L}$, therefore, (11.14) reduces to $\int_{\tilde{L}} d\omega'=0$ and finally, to

$$\int_L d\omega'=0, \tag{13.9}$$

where L is a contour from $\lambda=\lambda_0$ to $\lambda=1$. As in the R^3 case the additional condition on C is (13.3). The conditions (13.3), (13.9) are equivalent to

$$\begin{gathered}\int_{e^{-i\theta}}^{e^{i\theta}} \frac{(x-x_0)dx}{\sqrt{x(x-e^{i\theta})(x-e^{-i\theta})}}=\int_0^q \frac{(x-x_0)dx}{\sqrt{x(x-e^{i\theta})(x-e^{-i\theta})}}=0,\\ q=\frac{\lambda_0+\lambda_0^{-1}-2}{|E-2|}=\frac{2(\cosh h-1)}{\varrho+\varrho^{-1}-2\cos\theta}.\end{gathered} \tag{13.10}$$

The first Eq. (13.10) can be written in the following form

$$\int_\pi^\theta \frac{(\cos\gamma-x_0)d\gamma}{\sqrt{\cos\theta-\cos\gamma}}=0. \tag{13.11}$$

The investigation of Eqs. (13.10) and (13.11) in the limit $\varrho\to 0$ shows that there are CMC tori in H^3 with an arbitrary mean curvature $\tilde{H}$.

Remark. Immersions of rectangles Π are obtained in a similar way. As a matter of fact, if C is a curve (4.3) of an arbitrary genus with the involution $i:\lambda\to\lambda^{-1}$ then we have the coverings $C\to C'=C/i\pi$, $C\to C''=C/i$ and the corresponding reduction of abelian integrals and theta-functions [7]. In this way all solutions of the Dirichlet and Neumann zero boundary problems on rectangular for the elliptic sinh-Gordon equation were constructed in [11].

In the limit $|\lambda_n|\sim 1, n=1,\ldots,2g$ all zeros of $d\Omega_1$ are situated out the circle $|\lambda|\leqq 1$. At the same time, the examples of the present section show that for every $g\geqq 2$ there are Riemann surfaces such that $d\Omega_1$ have zeros lying inside the circle $|\lambda|\leqq 1$ (these surfaces are the small variations of considered surfaces of genus 2 with additional small cuts $[\lambda_{2i-1},\bar{\lambda}_{2i-1}^{-1}]$, $i=3,\ldots,g$ for $g>2$). So for every $g\geqq 2$ there is a $2g-2$ parametric family of C satisfying (6.14). A countable number of C determining CMC tori is singled out from this family by the condition (6.13).

The H^3 case is more difficult for analysis but the examples considered show that apparently there ase no objections to the condition (11.14) for $g\geqq 2$.

Appendix. Singular curves and dressing up procedure

Till now we restricted ourselves to the case of nonsingular spectral curves C, i.e. $\lambda_i \neq \lambda_j$ in (4.3). In this appendix we show that singular curves do not result in CMC tori. We do not construct here in full extent immersions corresponding to singular spectral curves. Our goal is to show that the periodicity conditions for singular curves have no solutions.

To describe all solutions $u(z,\bar{z})$ and also the corresponding Ψ-functions determined by the singular C's it is convenient to use a dressing up procedure (see, for example, [53]). If a certain solution of the sinh-Gordon equation and a corresponding solution Ψ_0 of the system (3.1) are known this procedure allows to construct new $u(z,\bar{z})$ and Ψ solving the same equations.

We describe the dressing up procedure in general complex case. Let $u_0(z,\bar{z})$ and $\Psi_0(z,\bar{z},v)$ be certain solutions of Eqs. (2.7), (3.1), and (3.2). Let us denote by

$$Q(v)=Q_N v^N+Q_{N-1}v^{N-1}+\ldots+Q_0$$

a polynomial of degree N with fixed leading coefficient ($Q_{\text{even}}=I$ and $Q_{\text{odd}}=\sigma_1$). This polynomial satisfies the following reduction

$$Q(-v)=\sigma_3 Q(v)\sigma_3\,, \tag{A.1}$$

i.e. Q_{even} are diagonal and Q_{odd} are anti-diagonal. We suppose also that all zeroes of $\det Q$ (we denote them by $v_1,\ldots,v_J$) do not depend on z and $\bar{z}$ and also a function

$$\Psi=Q\Psi_0$$

has regular singularities

$$\Psi=\hat{\Psi}(v-v_j)^{T_j}R_j\,,\qquad v\sim v_j \tag{A.2}$$

at all these points. Here $\hat{\Psi}$ and $\hat{\Psi}^{-1}$ are holomorphic functions in the neighborhood of v_j, T_j is a diagonal integer positive matrix, R_j is a constant invertible matrix and in addition

$$\sum_{j=1}^{J}\operatorname{tr}T_j=N\,.$$

Conditions (A.2) completely determine the coefficients of Q. Due to the reductions (A.1), (3.3) singularities of the same form (A.2) are also at the points $-v_j$, therefore, in (A.2) we set $v_i \neq -v_j$.

Theorem (Dressing up procedure). *The function $\Psi(z, \bar{z}, v)$ determined in this way is a solution of the system* (3.1), (3.2) *with some new $u(z, \bar{z})$, which is a solution of the sinh-Gordon equation.*

Proof. Let us consider

$$U = \Psi_z \Psi^{-1} = Q_z Q^{-1} + Q \Psi_{0z} \Psi_0^{-1} Q^{-1}.$$

This is a matrix of the same form as $U_0 = \Psi_{0z} \Psi_0^{-1}$. As a matter of fact U has a pole at $v = \infty$ of the same form as U_0. In addition it is a rational function of v without [due to the representation (A.2)] singularities at zeroes of $\det Q$. Condition (A.1) guarantees the validity of the reduction $U(-v) = \sigma_3 U(v) \sigma_3$. Arguments concerning the matrix $V = \Psi_{\bar{z}} \Psi^{-1}$ are exactly the same. As a corollary, the coefficients of matrices U and V determine a new solution of the sinh-Gordon equation.

Condition (A.2) is in fact a linear non-homogeneous system of equations determining the coefficients of matrix Q. Let us rewrite this system more explicitly in the case of simple zeroes of $\det Q$. In this case

$$(v - v_j)^{T_j} = \begin{pmatrix} v - v_j & 0 \\ 0 & 1 \end{pmatrix},$$

and the property of $\hat{\Psi}$ of being holomorphic gives

$$Q(v_j) \Psi_0(v_j) X_j = 0, \tag{A.3}$$

where X_j is a first column of matrix R_j^{-1}. We see that the matrix elements Q_n are rational functions of the matrix elements of $\Psi_0(v_1), \ldots, \Psi_0(v_J)$. This fact is also valid in case of multiple zeroes of $\det Q$.

All solutions of the sinh-Gordon equation stationary with respect to some higher flow are described in Sect. 4 (the spectral curve is nonsingular) or are obtained from solutions of Sect. 4 with the help of the dressing up procedure (the spectral curve is singular). In the last case the B-A function of nonsingular compactification C_0 of the spectral curve is taken as Ψ_0. This fact is common for all integrable equations and is well known in the theory of solitons.

The spectral curve of the elliptic sinh-Gordon equation possess the anti-holomorphic involution $v \to 1/\bar{v}$. It means that the regular singularities of Ψ-function (A.2) either lie on the unit circle or are arranged in pairs conjugated with respect to this circle. It can be shown that condition (A.3) is incompatible with the real reduction (4.17) in the first case, so only the second possibility is realized.

Finally, let $v_1, 1/\bar{v}_1, \ldots, v_m, 1/\bar{v}_m$ be the singularities of the spectral curve C. Denote its nonsingular compactification by C_0 and the corresponding B-A function by Ψ_0. The Ψ-function, corresponding to C is constructed by the dressing up procedure as a solution of the Eqs. (A.3). It is evident that Q depends rationally on values of the theta-functions and the exponents in expression (4.10) for Ψ_0 at the points $v_1, \ldots, 1/\bar{v}_m$. In particular, Q depends on all exponents

$$\exp\left\{\frac{i}{2}\left(z \int_{\infty}^{v_k} d\Omega_1 + \bar{z} \int_{\infty}^{v_k} d\Omega_2\right)\right\}. \tag{A.4}$$

We do not present here the formulae for immersions $F(z,\bar{z})$ in the singular case, which can be obtained exactly in the same way as above. The only important fact for us is that $F(z,\bar{z})$ is a quadratic function of Ψ. So the periodicity conditions for immersion determined by the curve C include the periodicity conditions for its nonsingular compactification C_0 discussed above in the main part of the paper. Beside this exponents (A.4) at all singularities v_k are required to be doubly-periodic with the same period lattice. For the nonsingular case the periodicity conditions of Sects. 6, 9, and 11 single out discrete set of spectral curves and fix the period lattice in 2-plane for every curve. Hence the condition for the exponents (A.4) to be doubly-periodic with the period lattice fixed by C_0 represent $2m$ complex equations on m complex parameters $v_1, \ldots, v_m$. With this count of equations and free parameters we finish our proof of non-existence of CMC tori corresponding to singular spectral curves.

The arguments presented above do not work for H^3 and the spectral curve with the only singularities at the points (11.7). An individual consideration gives the additional periodicity condition $d\Omega_1(v) = d\Omega_1(v') = 0$ in this case and so proves the non-existence.

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