21

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Periodic multiphase solutions of the Kadomsev–Petviashvili equation

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Abstract. N-phase solutions of the Kadomsev-Petviashvili (KP) equation, that are periodic in space variables x and y, were obtained and effectively investigated using the Schottky uniformisation, of which a short description is given. Many wave patterns are represented graphically as contour plots and as isometric projections for different parameter values of two-, three- and four-phase solutions of the KP equation.

1. Introduction

It is well known that the Kadomsev-Petviashvili (KP) equation

$$\tfrac{3}{4}u_{yy} = \frac{\partial}{\partial x}[u_t - \tfrac{1}{4}(6uu_x + u_{xxx})] \tag{1}$$

describes the evolution of long water waves of finite amplitude if they are weakly two-dimensional, as well as any weakly non-linear and weakly two-dimensional physical process [1]. The present paper is devoted to a new method for constructing its periodic solutions

$$u(x, y, t) = u(x+2\pi, y, t) = u(x, y+2\pi, t).$$

The main tool in solving the general periodic problem for the KP equation is the theta-function solution. These solutions were found by Krichever [2] using the method of finite-gap integration theory invented by Novikov, Dubrovin, Matveev, Its and others in the mid-1970s [3]. Despite the fact that the explicit theta-function formula is well known for these solutions, their investigation is a serious problem due to the complicated parametrisation. Several papers deal with the problem of the effective construction of the theta-function solutions, of which [4] suggests 'an algebra-geometric' effectivisation (by effectivisation we mean the concrete description of solutions) and [5] are devoted to 'a physical' effectivisation. We mention that Novikov's hypothesis which is the main idea in the algebra-geometric approach has lead to the solution [4, 6] of a classical problem of algebraic geometry-the Schottky problem. Concerning the effective description of the theta-function solution using the substitution technique, only two- and three-phase interactions can be investigated.

We should mention also that a portrait of the two-phase solution of the KdV equation appeared in the book by Mumford [7], where a detailed treatment of theta functions and their link with integrable equations is given.

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The present paper is based on the automorphic approach method suggested by one of the authors in [8, 9]. This method makes use of the Schottky uniformisation, as effectively applied to calculate the theta-function solutions of the KdV equation in [10].

Let us remark that a similar problem was investigated in the [11], where the theta-function substitution technique was used to construct two-phase solutions of the KP equation. Our automorphic approach leads to a general result; namely, to a natural description of an arbitrary number of interacting phases and to an effective determination of the periodic solutions.

2. Theta-function solutions

The theta-function solutions of the KP equation were obtained by Krichever.

Let Γ be the compact Riemann surface of genus N with the fixed point P_∞ and k be the local parameter near P_∞, i.e. $k \to 0$, $P \to P_\infty$, $P \in \Gamma$. $\mathrm{d}u_1, \ldots, \mathrm{d}u_N$ are the holomorphic differentials normalised by

$$\int_{a_m} \mathrm{d}u_n = 2\pi \mathrm{i} \delta_{mn} \qquad n, m = 1, \ldots, N$$

in the fixed canonical basis of cycles $a_1, b_1, \ldots, a_N, b_N$. $\Omega(P)$ is the Abelian integral of the second kind, normalised by the conditions $\int_{a_m} \mathrm{d}\Omega = 0$, having a single pole at P_∞. The formulae

$$\mathrm{d}u_n = f_n(k)\,\mathrm{d}k \qquad n = 1, \ldots, N \qquad \Omega = k^{-1} - ck + \mathrm{O}(k^2) \tag{2}$$

holding in a neighbourhood of P_∞ define the constant c and the vectors $U, V, W \in \mathbb{C}^N$ by means of

$$\begin{aligned} &U_n = f_n(0) \qquad V_n = \frac{\mathrm{d}}{\mathrm{d}k} f_n(k)\Big|_{k=0} \\ &W_n = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}k^2} f_n(k)\Big|_{k=0} \qquad n = 1, \ldots, N. \end{aligned} \tag{3}$$

The Riemann theta function

$$\theta(z \mid B) = \sum_{k \in \mathbb{Z}^N} \exp[\tfrac{1}{2}(Bk, k) + (z, k)] \tag{4}$$

is determined by the period matrix $B_{nm} = \int_{b_m} \mathrm{d}u_n$ of Γ.

Theorem 1 (see [2]). The theta-function (N-phase wave) solutions of equation (1) are given by the formula

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \theta(Ux + Vy + Wt + D \mid B) + 2c \tag{5}$$

where $D \in \mathbb{C}^N$ is an arbitrary vector (see also [4]).

For applications in physics only the real solutions are needed. As is well known, there are two different types of the KP equation which cannot be transformed each into another by a real change of variables. Equation (1) is called the KP2 equation. The complex transformation of (1)

$$x \to \mathrm{i}x \qquad y \to \mathrm{i}y \qquad t \to \mathrm{i}t$$

yields the KPI equation

$$\tfrac{3}{4}u_{yy}=\frac{\partial}{\partial x}[u_t-\tfrac{1}{4}(6uu_x-u_{xxx})].$$

Recently Dubrovin and Natanzon [12] and Krichever [13] have isolated all real non-singular theta-function solutions of both the KPI and KP2 equations. For the KP2 equation their result can be formulated in the following manner:

Theorem 2 (see [12]). For real and non-singular solutions (5) of the KP2 equation it is necessary and sufficient that the triple (Γ, P_∞, k) and the vector D satisfy the following two conditions.

(i) Γ is an M-curve, i.e. there is an antiholomorphic involution $\tau\colon \Gamma\to\Gamma$, $\tau^2=1$, with the maximal possible number of real ovals $\Gamma_0,\ldots,\Gamma_N$ (a real oval is the connected set of fixed points of τ). Besides this the conditions $\tau P_\infty=P_\infty$ and $\tau^*k=\bar{k}$ hold. Here P_∞ is chosen to be at Γ_0.

(ii) If the basis of cycles is chosen so that $\tau a_i=a_i$, $\tau b_i=-b_i$, then D is purely imaginary and, on the contrary, for $\tau a_i=-a_i$, $\tau b_i=b_i$ it is real, $D\in\mathbb{R}^N$.

Figure 1 represents the four-phase periodic solution with the following values of the parameters:

$$B=-2\pi\begin{pmatrix}0.500 & 0.243 & -0.199 & -0.157\\ 0.243 & 0.800 & -0.127 & -0.102\\ -0.199 & -0.127 & 1.100 & 0.645\\ -0.157 & -0.102 & 0.645 & 1.500\end{pmatrix}$$

$$D=(0,0,0,0)\qquad c=-0.112$$

$$U=\mathrm{i}(-1.000,-1.000,2.000,2.000)$$

$$V=\mathrm{i}(1.000,2.000,1.000,2.000)$$

$$W=\mathrm{i}(-0.545,-2.939,-1.909,-7.388).$$

These values are obtained by the automorphic approach technique, which is described below.

Let us remark that N-phase wave solutions are important for handling the periodic problem.

Theorem 3 (see [13]). For any real non-singular periodic solution $u(x,y,t)$ of the KP2 equation there exists a sequence of N-phase wave solutions $u_N(x,y,t)$, $N=1,2,\ldots,$ uniformly converging to $u(x,y,t)$ with all derivatives for any x, y and for any interval $|t|<T_0$.

3. The Schottky uniformisation

Let $c_1, c_1',\ldots,c_N, c_N'$ be a set of $2N$ mutually disjoint Jordan curves on $\mathbb{C}$, which comprise the boundary of a $2N$-connected domain F (see figure 2). The linear transformation σ_n

$$\frac{\sigma_n z-B_n}{\sigma_n z-A_n}=\mu_n\frac{z-B_n}{z-A_n}\qquad |\mu_n|<1,\ n=1,\ldots,N$$

transforms the outside of a boundary curve c_n onto the inside of a boundary curve c'_n, $\sigma_n c_n = c'_n$. A_n and B_n are the fixed points of the loxodromic transformation σ_n.

The elements σ of the group PSL(2, $\mathbb{C}$) have the following representation:

$$\sigma z = \frac{\alpha z+\beta}{\gamma z+\delta} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{A-B}\begin{pmatrix} A\sqrt{\mu}-B/\sqrt{\mu} & -AB(\sqrt{\mu}-1/\sqrt{\mu}) \\ \sqrt{\mu}-1/\sqrt{\mu} & -B\sqrt{\mu}+A/\sqrt{\mu} \end{pmatrix}.$$

The centre of the isometric circle is given by

$$-\delta/\gamma = (B\sqrt{\mu}-A/\sqrt{\mu})(\sqrt{\mu}-1/\sqrt{\mu})^{-1}$$

and its radius equals $|\gamma|^{-1}$.

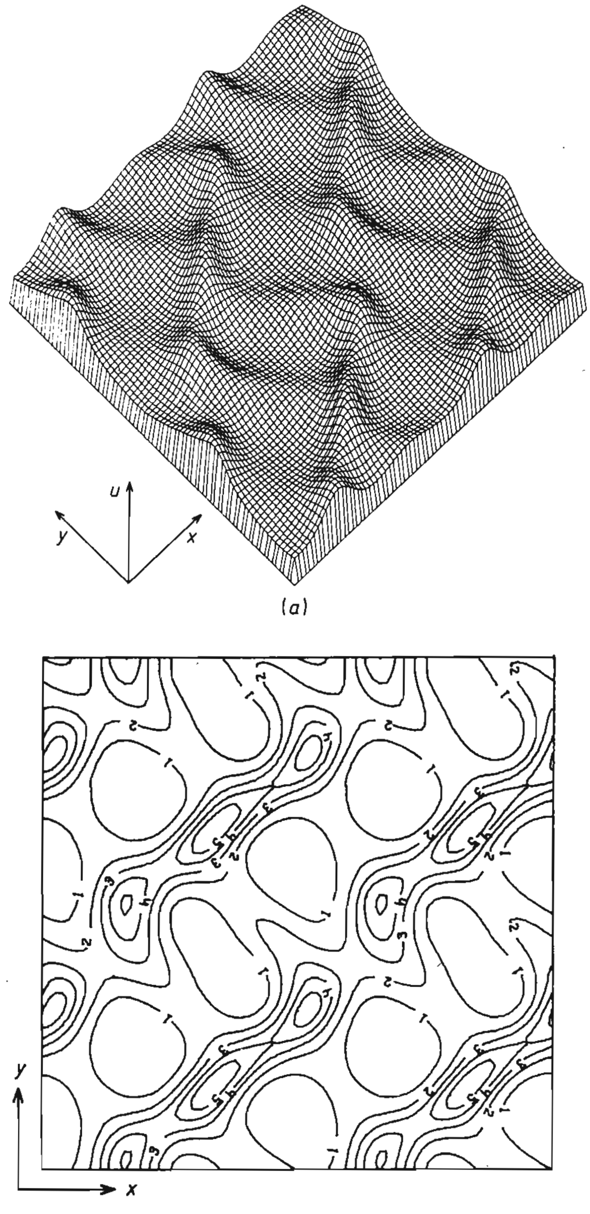


Figure 1. (*a*) The isometric projection of the surface $u(x, y, 0)$. This surface is represented as a set of lines which are produced by the intersection of the wave surface with planes being parallel to the x and y axes respectively. (*b*) The isolines $u(x, y, 0) = \text{constant}$, i.e. the contour plot of the same solution. The range of the variations of x and y equals two periods of the solution, i.e. $[0, 4\pi]$, $t = 0$.

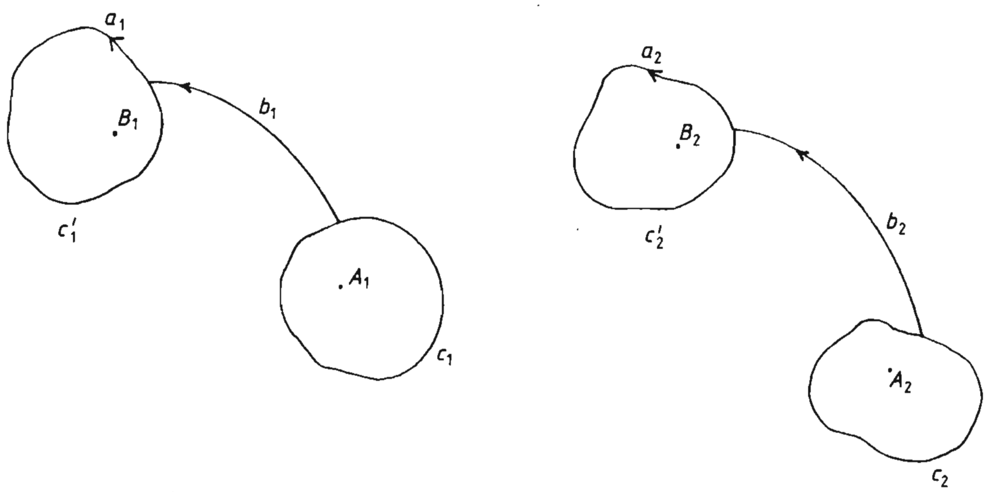


Figure 2. The fundamental domain *F*.

The collections of elements $\sigma_1, \ldots, \sigma_N$ generate a Schottky group G [14]. *F* is the fundamental domain of G. If all the boundary curves c_n, c_n' are circles then the Schottky group is called classical [15]. More generally, Schottky groups can be characterised as those finitely generated, discontinuous groups which are free and purely loxodromic [16]. It turns out to be equivalent to the previous one because any free system of generators of such a group gives rise to a fundamental domain *F* considered above [17]. Let $\Omega(\mathrm{G})$ be the set of discontinuity of G, then Ω/G is the compact Riemann surface of genus *N*.

According to the classical theorem [18] any compact Riemann surface Γ of genus *N* can be represented in this form. More precisely, let *N* homologically independent simple disjoint loops $v_1, \ldots, v_N$ be chosen on Γ. Γ, cut along these loops, is a plane region. It is mapped conformally to the fundamental domain *F* of the corresponding Schottky group G, v_n being mapped exactly at the curves c_n', c_n. If two loop systems $v_1, \ldots, v_N$ and $v_1', \ldots, v_N'$ generate the same subgroups in $H_1(\Gamma, \mathbb{Z})$ then they determine the same group *G* but with different choice of generators. A difference of the subgroups leads to a difference of the uniformising Schottky groups G and G$'$. Choose a canonical basis $H_1(\Gamma, \mathbb{Z})$ so that *a*-cycles coincide with the loops $v_n = a_n$. This canonical basis of cycles of Ω/G is presented in figure 2: a_n coincide with c_n' positively oriented, b_n runs on *F* between the points $z_n \in c_n$ and $\sigma_n z_n \in c_n'$, and *b*-cycles do not mutually intersect.

Denote by G_n the subgroup of G generated by σ_n. Cosets G/G_n and $\mathrm{G}_m\backslash\mathrm{G}/\mathrm{G}_n$ are sets of all elements $\sigma = \sigma_{i_1}^{j_1}, \ldots, \sigma_{i_k}^{j_k}$, $j_l \neq 0$ so that $i_k \neq n$ and for $\mathrm{G}_m\backslash\mathrm{G}/\mathrm{G}_n$ additionally $i_1 \neq m$. The following proposition is due to the classic papers [19, 20].

Proposition. If the series

$$\mathrm{d}u_n = \sum_{\sigma \in G/G_n} [(z - \sigma B_n)^{-1} - (z - \sigma A_n)^{-1}]\,\mathrm{d}z \tag{6}$$

are absolutely convergent then they define holomorphic differentials normalised in the basis shown in figure 2. The period matrix is given by

$$\begin{aligned} B_{nm} &= \sum_{\sigma \in G_m\backslash G/G_n} \ln\{B_m, A_m, \sigma B_n, \sigma A_n\} \qquad m \neq n \\ B_{nn} &= \ln \mu_n + \sum_{\sigma \in G_n\backslash G/G_n,\, \sigma \neq I} \ln\{B_n, A_n, \sigma B_n, \sigma A_n\} \end{aligned} \tag{7}$$

where $\{\ldots\}$ means the double-ratio

$$\{z_1, z_2, z_3, z_4\} = (z_1 - z_3)(z_2 - z_4)(z_1 - z_4)^{-1}(z_2 - z_3)^{-1}.$$

4. Convergence of Poincaré series

The series (6) are (-2)-dimensional Poincaré theta series (for a general theory of automorphic forms for Schottky groups see [21]). For a general Schottky group they can be absolutely divergent [22, 23]. However, if a Schottky group is classical and satisfies some restrictions then (-2)-dimensional theta series are convergent.

Assume that $2N-3$ circles $l_1, \ldots, l_{2N-3}$ can be fixed on the fundamental domain F so that the following conditions are satisfied.

(i) The circles $l_1, \ldots, l_{2N-3}, c_1, \ldots, c_N, c'_1, \ldots, c'_N$ are mutually disjoint.

(ii) The circles $l_1, \ldots, l_{2N-3}$ divide F into $2N-2$ regions $T_1, \ldots, T_{2N-2}$.

(iii) Each T_i has exactly three boundary circles (see figures 3 and 5).

Let us call these Schottky groups 'circle decomposable'. The Schottky condition (see [4, 24]) states that (-2)-dimensional Poincaré series corresponding to circle-decomposable Schottky groups are absolutely convergent.

In particular, each Schottky group having the invariant circle is always circle decomposable and the series are convergent [18, 19]. The convergence is also proved in the case when the circles $c_k, c'_k, k = 1, \ldots, N$, are small enough and far from each other (the corresponding estimates can be found in [19, 20]).

Let us pick out one of these regions T_i, $i = 1, \ldots, 2N-2$, (see figure 3). Consider any two circles of the boundary of T_i. Let R, r be their radii and e be the distance between their centres. So considering various pairs of circles we assign three numbers K_i^1, K_i^2, K_i^3 to each T_i:

$$K = (R^2 + r^2 - e^2)^2 (2Rr)^{-2} - 1.$$

Put $K = \min(K_1^1, K_1^2, \ldots, K_{2N-2}^2, K_{2N-2}^3)$. The proof [14, 24] of the Schottky convergence principle shows that the series converges better at larger k. The speed of convergence is characterised by the maximal K possible among various decompositions

$$K^* = \max_{l_1, l_2, \ldots, l_{2N-3}} K$$

and K^* depends on uniformisation itself only.

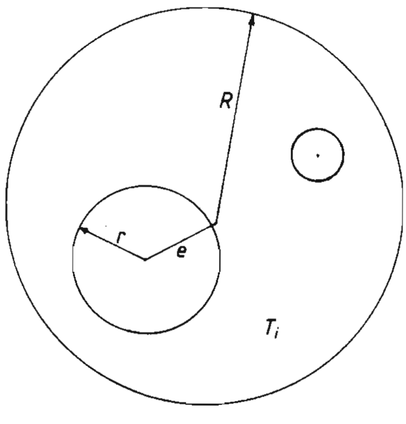


Figure 3. A region T_i having exactly three boundary circles.

5. Schottky uniformisation of the M-curves

In this section we show that any M-curve has the Schottky uniformisation Ω/G (i.e. one can choose the loops $v_1, \ldots, v_N$) so that:

(i) the series (7) determined by this uniformisation are absolutely convergent;

(ii) the set $S = \{A_1, B_1, \mu_1, \ldots, A_N, B_N, \mu_N\}$

of the uniformisation parameters can be explicitly described.

Let Γ be a M-curve, and $\Gamma_0, \ldots, \Gamma_N$ be its real ovals, which decompose Γ into two components Γ_+ and Γ_-. Each of these two components is a sphere with $N+1$ boundary curves. Consider the Fuchsian uniformisation H/G of the surface Γ_+, where H is the upper complex half-plane and G is the Fuchsian group of the second kind. The factor $\bar{H}/G$, with $\bar{H} = \{z \in \mathbb{C}, \operatorname{Im} z < 0\}$ is conformally equivalent to Γ_-. So the Schottky group G uniformising Γ is the Fuchsian group of the second kind. The loops v_n satisfy the condition $\tau v_n = -v_n$ (here the direction is changed). The circles c_n, c'_n, $n = 1, \ldots, N$ are orthogonal to the real axis. The complete description of the set S can be easily obtained as described by Natanzon [25]. For the generators shown on figure 4 it is [9]

$$B_N < B_{N-1} < \ldots < B_1 < A_1 < \ldots < A_N \qquad 0 < \sqrt{\mu_n} < 1;\ n = 1, \ldots, N$$
$$\{B_n, A_n, B_{n+1}, A_{n+1}\} > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}}\right)^2 \qquad n = 1, \ldots, N-1. \tag{8}$$

The (-2)-dimensional Poincaré theta series always converges for the Fuchsian groups of the second kind [18, 19].

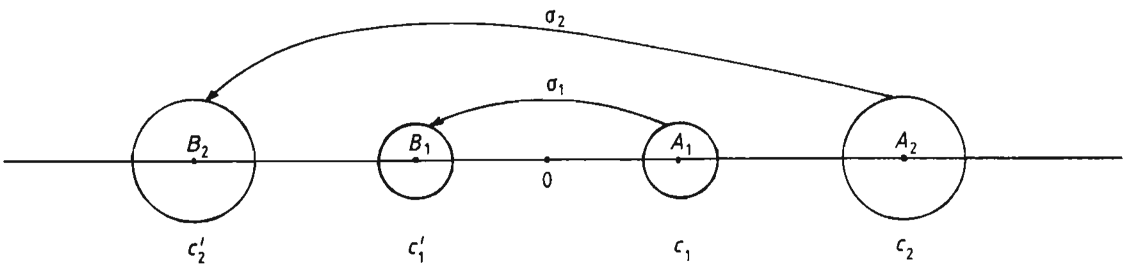


Figure 4. The fundamental domain F of UI.

To study small-amplitude waves of the KP2 equation it is more convenient to consider another Schottky uniformisation of the M-curve. Γ_+ can be always mapped to the upper half-plane with N discs removed whereby P_∞ is mapped to ∞ [18]. Then the group G is described as follows (see figure 5):

$$B_n = \bar{A}_n \qquad \operatorname{Im} A_n > 0 \qquad 0 < \sqrt{\mu_n} < 1;\ n = 1, \ldots, N. \tag{9}$$

In this case c_n and c'_n are the isometric circles of the transformations σ_n and σ_n^{-1}. c_n and c'_n are mutually complex conjugated. Since their centres and radii are known (see § 3) it is easy to write down the conditions for the circles to be disjoint:

$$\left|\frac{\bar{A}_n\sqrt{\mu_n} - A_n/\sqrt{\mu_n}}{\sqrt{\mu_n} - 1/\sqrt{\mu_n}} - \frac{\bar{A}_m\sqrt{\mu_m} - A_m/\sqrt{\mu_m}}{\sqrt{\mu_m} - 1/\sqrt{\mu_m}}\right| > -\frac{2 \operatorname{Im} A_n}{\sqrt{\mu_n} - 1/\sqrt{\mu_n}} - \frac{2 \operatorname{Im} A_m}{\sqrt{\mu_m} - 1/\sqrt{\mu_m}}. \tag{10}$$

These inequalities together with (9) determine S. However, for this uniformisation the convergency of the series (6) can be proved not for any point of S but for its subset of circle-decomposable groups. In particular, the series always converge when $N = 2$.

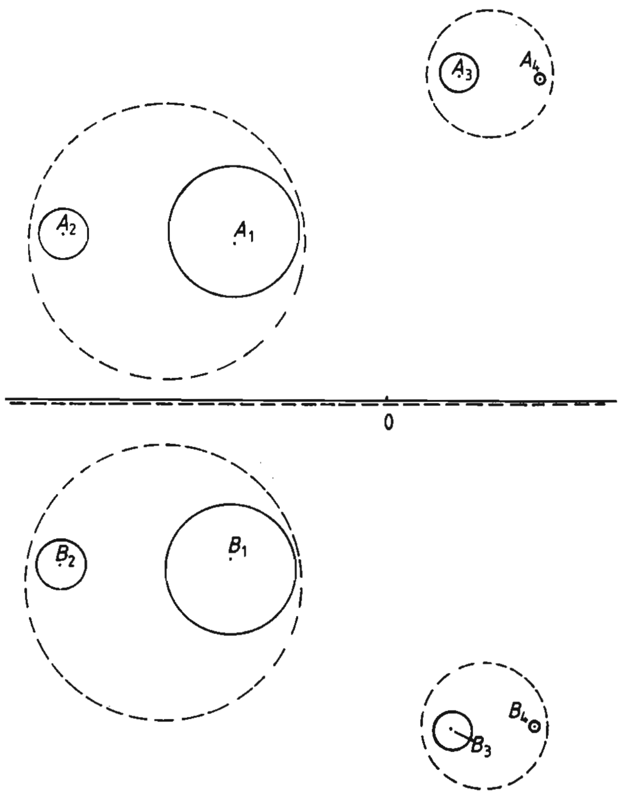


Figure 5. The fundamental domain F of UII. Broken lines show the circle decomposition. The corresponding Schottky group leads to the solution presented in figure 1.

We call the two Schottky uniformisations of the M-curves, as described above, to be the uniformisations I and II (UI and UII). The loops v_n of UII (figure 5) are chosen uniquely—they are the real ovals of τ without P_∞. In the UI case v_n can be chosen in many various ways. One of the possible choices leads to figure 4. The most natural is the choice v_n when the value K^* is maximal and the series (6) are the most rapidly convergent ones.

6. Formulae for solutions

Let us return to the KP2 equation. The local parameter in the neighbourhood of $P_\infty = \infty$ is equal to $k = z^{-1}$. Then we have from (3) and (6)

$$U_n = \sum_{\sigma \in G/G_n} (\sigma A_n - \sigma B_n) \qquad V_n = \sum_{\sigma \in G/G_n} [(\sigma A_n)^2 - (\sigma B_n)^2]$$
$$W_n = \sum_{\sigma \in G/G_n} [(\sigma A_n)^3 - (\sigma B_n)^3]. \tag{11}$$

The Abelian integral of the second kind (2) is expressed as follows:

$$\Omega(z) = \sum_{\sigma \in G} (\sigma_z - \alpha/\gamma).$$

Then, applying (2), we get

$$c = \sum_{\sigma \in G, \sigma \neq I} \gamma^{-2}. \tag{12}$$

The KP equation allows the following transformation. If $u(x, y, t)$ is a solution then

$$\tilde{u}(x, y, t) = u(x - \tfrac{3}{2}\alpha t, y, t) - \alpha \tag{13}$$

is also a solution of the KP equation.

For UII the vectors U, V, W, D are purely imaginary. The periodic condition leads to the restriction of the parameters

$$\mathrm{i}U_n, \mathrm{i}V_n \in \mathbb{Z} \qquad n = 1, \ldots, N. \tag{14}$$

Let us fix the solution in the class (13) normalised by the condition

$$\int_0^{2\pi} u(x, y, t)\, \mathrm{d}x = 0.$$

As a result we obtain

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \theta(Ux + Vy + \tilde{W}t + D|B) \qquad \tilde{W} = W - 3cU. \tag{15}$$

In UI case $D \in \mathbb{R}^N$ is a real vector and all series are convergent. So the following theorem is proved.

Theorem 4.

(UI). All real non-singular theta-function solutions of the KP2 equation are given by the formulae (7), (11), (12) and (15), where the parameters A_n, B_n, μ_n belong to the set (8) and $D \in \mathbb{R}^N$ is an arbitrary real vector.

(UII). For circle-decomposable groups (in particular any two-phase solution) real non-singular theta-function solutions of the KP2 equation can be described by the formulae (7), (11), (12) and (15), where the parameters A_n, μ_n belong to the set (9), (10) and D is an arbitrary purely imaginary vector.

When the generators are chosen as in figure 4 then all the U_n are arranged as follows:

$$0 < U_1 < U_2 < \ldots < U_N.$$

We remark also that the simple periodic condition (14) shows that the UII representation is more convenient to isolate the periodic solutions. In this case A_n, μ_n are natural convenient parameters of the solution, because for a given Γ UII is unique and we have a one-to-one correspondence between $A_n, \mu_n \in S$ and solutions of the KP2 equation.

The same results for the KP1 equation were obtained in [9].

7. Multi-soliton solutions and small-amplitude waves

From the general N-phase wave solution we can arrive by a limiting procedure to two kinds of simply described degenerate solutions, namely to the multisoliton solutions and to small-amplitude waves.

Carrying out the limit

$$\mu_n \to 0 \qquad n = 1, \ldots, N \tag{16}$$

the circles c_n and c'_n collapse to the points A_n and B_n respectively. The corresponding N-phase solution becomes the multisoliton solution. Let us describe this limiting process in detail.

In the degenerate case (16) for all non-identical mappings σ and arbitrary $a, b \in F$ the equality $\sigma a = \sigma b$ holds. Therefore in the series (7), (11), (12) members corresponding to $\sigma = 1$ are non-zero only, i.e.

$$\begin{aligned} &\operatorname{Re} B_{nn} \to -\infty \qquad B_{nm} \to \ln\{B_m, A_m, B_n, A_n\} \qquad n \neq m, \qquad c \to 0 \\ &U_n \to A_n - B_n \qquad V_n \to A_n^2 - B_n^2 \qquad W_n \to A_n^3 - B_n^3. \end{aligned} \tag{17}$$

The parameter D in solution (15) is arbitrary. Suppose D_n is equal to

$$D_n = -\tfrac{1}{2} B_{nn} + \eta_n + \mathrm{o}(1) \tag{18}$$

where η_n are finite constants. Then it can be seen that the argument of the exponential function in the series (4) is given by the formula

$$\tfrac{1}{2} \sum_n B_{nn} k_n (k_n - 1) + \sum_{n<m} B_{nm} k_n k_m + \sum_n k_n (U_n x + V_n y + \tilde{W}_n t + \eta_n + \mathrm{o}(1)).$$

Since all terms of the series with $k_n \neq 0, 1$ are identically zero, this series is finite. Let $\{0, 1\}^N$ be the set of all N-dimensional vectors with coordinates equal to 0 and 1, then the limit (17) leads to

$$\theta(Ux + Vy + \tilde{W}t + D \mid B) \to \theta(x, y, t)$$

where

$$\theta(x, y, t) = \sum_{k \in \{0,1\}^N} \prod_{n<m} \left(\frac{(B_m - B_n)(A_m - A_n)}{(B_m - A_n)(A_m - B_n)} \right)^{k_n k_m} \exp\left(\sum_n k_n [(A_n - B_n)x + (A_n^2 - B_n^2)y + (A_n^3 - B_n^3)t + \eta_n] \right).$$

Finally the solution of the KP equation is given by

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \ln \theta(x, y, t). \tag{19}$$

(see figure 6).

For solution (19) to be real and non-singular we should apply the limiting procedure described above to the UI type solution, i.e. put A_n, B_n real.

The limits (18) and (19) for the UII-type case lead to singular solutions. To obtain UII-type degenerate non-singular real solutions we need to choose a finite parameter D, so that the conditions of theorem 3 are satisfied. Then

$$\begin{aligned} &\theta(Ux + Vy + \tilde{W}t + D \mid B) \\ &\quad \to 1 + \sum_{n=1}^{N} \sqrt{\mu_n} \, [\exp(U_n x + V_n y + \tilde{W}_n t + D_n) + \exp(-U_n x - V_n y - \tilde{W}_n t - D_n)] \\ &B_{nn} \to \ln \mu_n \\ &u(x, y, t) \to -4 \sum_n \sqrt{\mu_n} \, u_n^2 \cos(u_n x + v_n y + w_n t + d_n) \\ &U_n \to \mathrm{i} u_n = A_n - \bar{A}_n \qquad V_n \to \mathrm{i} v_n = A_n^2 - \bar{A}_n^2 \qquad c \to 0 \\ &\tilde{W}_n \to \mathrm{i} w_n = A_n^3 - \bar{A}_n^3 \qquad D_n = \mathrm{i} d_n. \end{aligned} \tag{20}$$

Clearly (20) represents a linear superposition of N non-interacting Fourier modes of small amplitude (see figure 7).

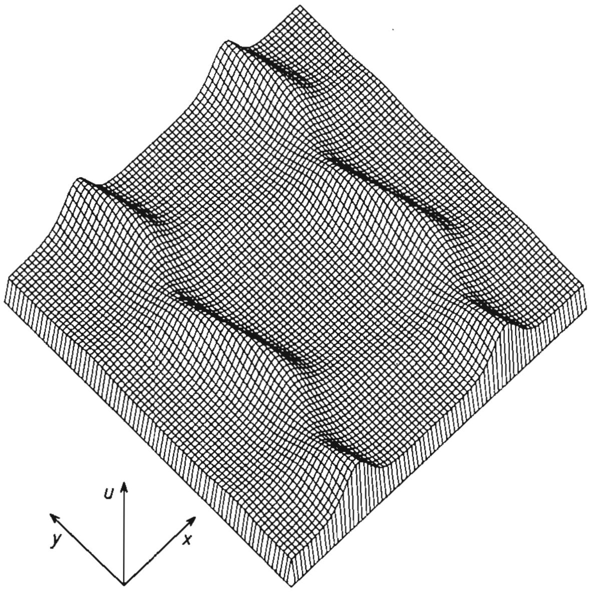


Figure 6. The isometric projection of the big-amplitude two-phase periodic solution in the soliton regime. Soliton structure is evident. It has been calculated by UII with the parameters: $U = \mathrm{i}(1.000, -1{,}000)$, $V = \mathrm{i}(0.000, -1.000)$, $c = -0.217$, $W = \mathrm{i}(-0.171, -0.823)$, $D = (0, 0)$, $B = -2\pi \begin{pmatrix} 0.400 & -0.226 \\ -0.226 & 0.600 \end{pmatrix}$, $t = 0$, $x, y \in [0, 4\pi]$.

So far as μ_n and c_n are small the solution is described by the linear limit (20). When phase amplitudes and, consequently, the dimension of the circles c_n, c'_n increase, the phases start to interact but the solution remains described by the general UII formula. At last, with further amplitude increase we reach the near-soliton regime and the UI description becomes more natural (see figure 8).

8. Method of calculation

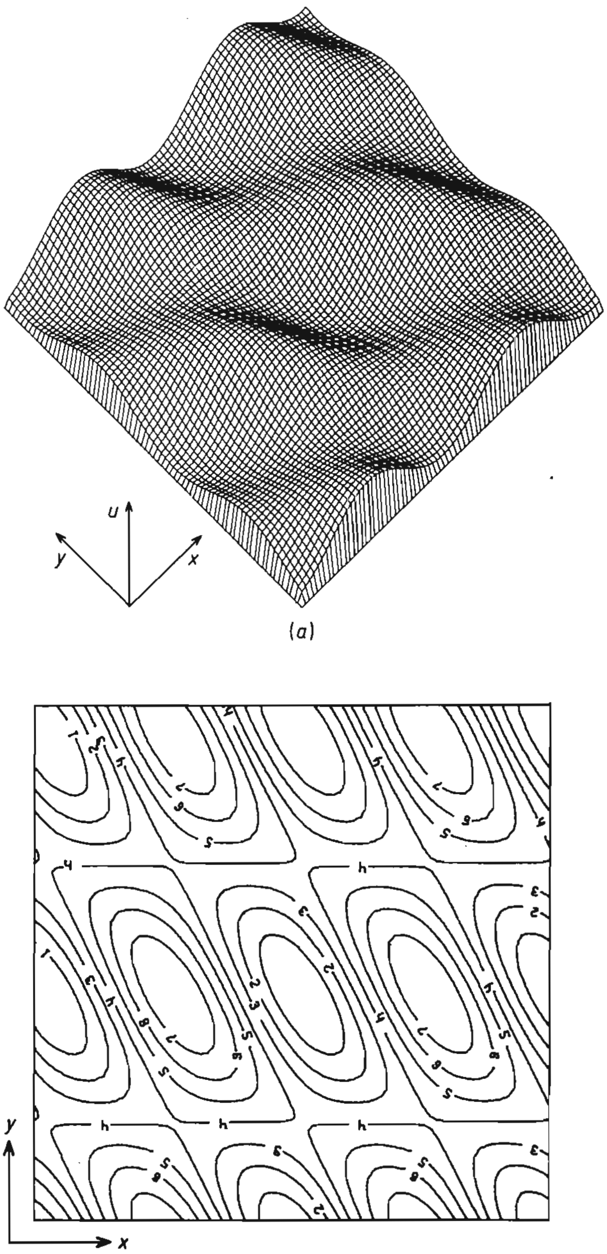
In this section we present the calculation of the parameters of the solution. The calculations are done using formula (15). For convenience approximately all quantities are given in the text of this paper with accuracy $\pm 10^{-3}$, however they were found with accuracy $\pm 10^{-5}$.

For the calculation the method of successive approximations is used. The computing program consists of two parts. In the first part we find the approximated parameters A_n, μ_n and in the second calculate all parameters B_{nm}, U_n, V_n and W_n. We propose a solution of the KP equation with parameters in the neighbourhood of given B^0_{nn}, V^0_n, V^0_n. Let us show in which manner the parameters A_n, μ_n emerge from B^0_{nn}, U^0_n, V^0_n.

Substituting the given values B^0_{nn}, U^0_n, V^0_n into the formulae

$$B_{nn} = \ln \mu_n \qquad U_n = A_n - \bar{A}_n \qquad V_n = A_n^2 - \bar{A}_n^2 \tag{21}$$

yields the first approximation to the values for A_n, μ_n. Suppose A_n, μ_n to be fixed; then we calculate the values of $\tilde{B}_{nn}$, $\tilde{U}_n$, $\tilde{V}_n$ using formulae (7) and (11). From the entire series in (7) and (11) terms with $L \leqslant 6$ are used only (if σ is an element of the group G generated by σ_{n_k}, $\sigma = \sigma^{j_1}_{n_1}, \ldots, \sigma^{j_k}_{n_k}$, then L is equal $L = |j_1| + \ldots + |j_k|$). The sum rule is: if the sum of all terms with $L = \tilde{L}$ is greater than 10^{-6} then the calculation



(a)

(b)

Figure 7. (*a*) The isometric projection and (*b*) the contour plot of the small-amplitude periodic solution representing two non-interacting phases. It has been calculated by UII with parameters: $U=\mathrm{i}(1.000,\ 1.000)$, $V=\mathrm{i}(0.000,\ 1.000)$, $c=-0.00001$, $W=\mathrm{i}(-0.250,\ 0.500)$, $D=(0, 0)$, $B=-2\pi\begin{pmatrix}2.000 & 0.256\\ 0.256 & 2.000\end{pmatrix}$, $t=0$, $x, y\in[0, 4\pi]$.

is continued over all terms with $L=\tilde{L}+1$ and so on, otherwise the sum procedure is terminated. These values are then compared with the initial data B^0_{nm}, U^0_n, V^0_n. If the errors

$$\Delta B_{nn}=|\tilde{B}_{nn}-B^0_{nn}| \qquad \Delta U_n=|\tilde{U}_n-U_n| \qquad \Delta V_n=|\tilde{V}_n-V^0_n| \tag{22}$$

do not exceed 10^{-5} then the iteration process is finished and the appropriate parameters A_n, μ_n are found. If not, then we repeat this procedure with corrected values of A_n, μ_n. These corrections are evident from (21) and (22). At the end of this process we

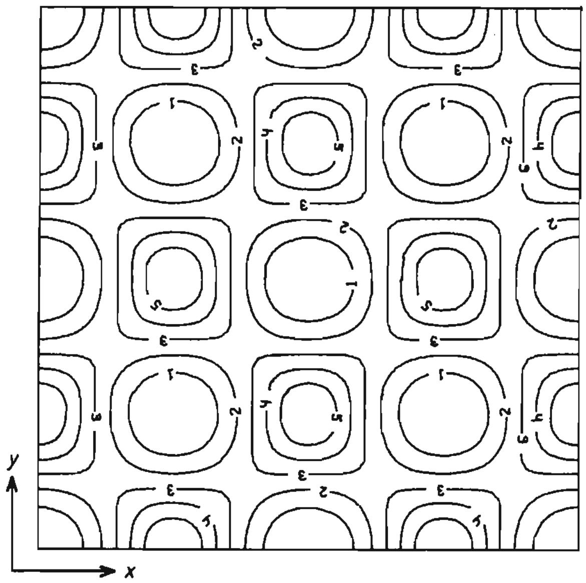


Figure 8. Intermediate case. The phases slightly interact. The parameters are equal: $U = \mathrm{i}(1.000,\ 1.000)$, $V = \mathrm{i}(1.000,\ -1.000)$, $c = -0.008$, $W = \mathrm{i}(0.500,\ 0.500)$, $D = (0, 0)$, $B = -2\pi\begin{pmatrix} 1.000 & -0.111 \\ -0.111 & 1.000 \end{pmatrix}$, $t = 0$, $x, y \in [0, 4\pi]$.

have A_n, μ_n with an error that never exceeds 10^{-5}. For given parameters A_n, μ_n, the parameters B_{nm}, U_n, V_n, W_n can be determined from (14). At this step the infinite series are replaced by finite sums over all terms with $L \leqslant 6$. In all investigated cases the differences between B^0_{nn}, U^0_n, V^0_n and final B_{nn}, U_n, V_n do not exceed 10^{-5}.

The computer plots of some KP2 solutions and their parameters are shown in figures 1, 6–8. We now focus our interest on some important cases. At first we show how the wave pattern changes on transition from a two-phase solution to a three-phase solution. The two-phase solution has parameters:

$$U = \mathrm{i}(1.000,\ 1.000) \qquad V = \mathrm{i}(1.000,\ 2.000)$$

$$W = \mathrm{i}(0.523, 2.912) \qquad D = (0, 0) \qquad c = -0.105$$

$$B = -2\pi\begin{pmatrix} 0.500 & 0.242 \\ 0.242 & 0.800 \end{pmatrix}$$

i.e. this solution is well approximated by a periodic soliton interaction (see figure 9).

Let us investigate the solution with one more phase, i.e. the genus-3 solution (figure 10) with parameters:

$$U = \mathrm{i}(1.000, 1.000, 2.000) \qquad V = \mathrm{i}(1.000, 2.000, 1.000)$$

$$W = \mathrm{i}(0.514, 2.926, -2.104) \qquad D = (0, 0, 0) \qquad c = -0.111$$

$$B = -2\pi\begin{pmatrix} 0.500 & 0.242 & 0.289 \\ 0.242 & 0.800 & 0.191 \\ 0.289 & 0.191 & 1.100 \end{pmatrix}.$$

In figure 10 the disturbance due to third wave is evident, though it has a small amplitude.

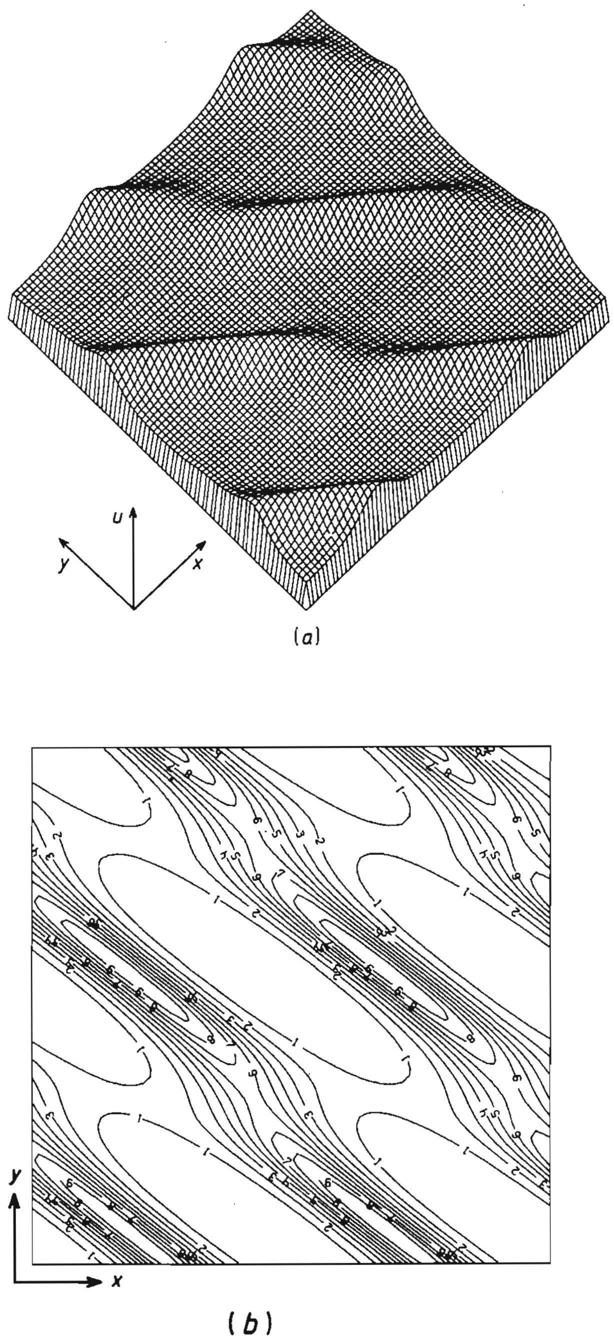


Figure 9. (*a*) The isometric projection and (*b*) contour plot of the two-phase solution. The range of the variations of x and y equals $[0, 4\pi]$, i.e. two periods of the solution, and $t = 0$.

Uniformisation parameters for all cases described in this paper are given in the appendix.

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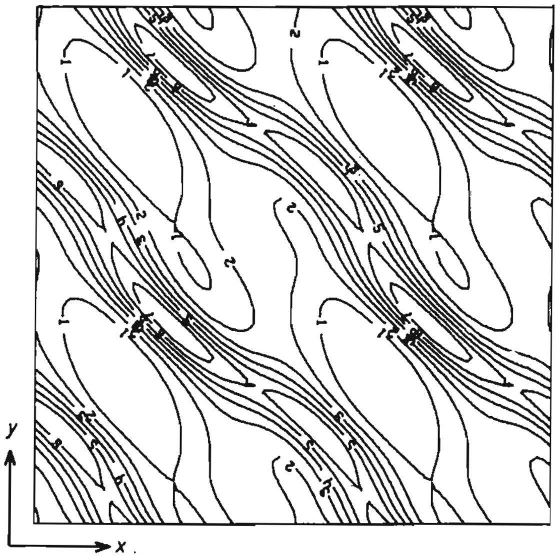

Figure 10. The contour plot of the three-phase solution. The range of the variations of x and y are equals $[0, 4\pi]$, i.e. two periods of the solution, and $t = 0$.

Appendix

The uniformisation parameters for the solutions in figures 1 and 5–10 are as follows.

Figures 1 and 5.

$$A_1 = -0.499 + i0.502 \qquad A_2 = -1.045 + i0.533$$
$$A_3 = 0.266 + i1.059 \qquad A_4 = 0.486 + i1.045$$
$$\mu_1 = 0.042 \qquad \mu_2 = 0.005 \qquad \mu_3 = 0.001 \qquad \mu_4 = 0.000\,07.$$

Figure 6.

$$A_1 = -0.023 + i0.507 \qquad A_2 = 0.588 + i0.556$$
$$\mu_1 = 0.755 \qquad \mu_2 = 0.014.$$

Figure 7.

$$A_1 = 0.000 + i0.500 \qquad A_2 = 0.500 + i0.500$$
$$\mu_1 = 0.000\,003 \qquad \mu_2 = 0.000\,003.$$

Figure 8.

$$A_1 = 0.499 + i0.501 \qquad A_2 = -0.499 + i0.501$$
$$\mu_1 = 0.002 \qquad \mu_2 = 0.002.$$

Figure 9.

$$A_1 = 0.493 + i0.503 \qquad A_2 = 1.042 + i0.533$$
$$\mu_1 = 0.042 \qquad \mu_2 = 0.005.$$

Figure 10.

$$A_1 = 0.498 + i0.500 \qquad A_2 = 1.045 + i0.532$$

$$A_3 = 2.580 + i1.104$$

$$\mu_1 = 0.418 \qquad \mu_2 = 0.005 \qquad \mu_3 = 0.0006.$$

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