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SCHOTTKY UNIFORMIZATION AND FINITE-GAP INTEGRATION

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1. The theory of finite-gap integration, which arose in the framework of the inverse problem method (IPM) in the mid-1970's in the works of Novikov, Dubrovin, Matveev, It·s, and others (see the survey [1]), allows the construction of multiphase solutions of nonlinear equations that are integrable by the IPM. The solutions are comparatively simply expressed using the Riemann theta-functions

$$\theta(z|B) = \sum_{m\in \mathbf{Z}^g} \exp\{\pi i\langle Bm, m\rangle + 2\pi i\langle z, m\rangle\}. \tag{1}$$

For example, the solution of the Kadomtsev-Petviashvili (KP) equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{1}{4}(6uu_x + u_{xxx})\right) \tag{2}$$

is given by the following expression [2] (see also [4]):

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta((Ux+Vy+Wt+D)/2\pi i|B) + 2c. \tag{3}$$

However, despite the simplicity of (3), extracting information about the solutions from it is complicated because U, V, W, and B are given implicitly. The actual parameter in (3) is a compact Riemann surface Γ of genus g and a point $P_0 \in \Gamma$. The constants U, V, W, and B are connected with each other; they are defined in the following way. Let $a_1, b_1, \ldots, a_g, b_g$ be a canonical basis of the cycles of Γ; $du_1, \ldots, du_g$ normalized holomorphic differentials; p a local parameter in a neighborhood of P_0, $p \to 0$ as $P \to P_0$; and $du_n(P) = f_n(p)\,dp$, $P \sim P_0$. Then

$$B_{nm} = \int_{b_m} du_n, \qquad \int_{a_m} du_n = \delta_{nm}, \tag{4}$$

$$U_n = 2\pi i f_n(p)|_{p=0}, \qquad V_n = 2\pi i \frac{d}{dp} f_n(p)\Big|_{p=0},$$

$$W_n = \pi i \frac{d^2}{dp^2} f_n(p)\Big|_{p=0}, \qquad \Omega(P) \to p^{-1} - cp + O(p^2) \quad \text{as } P \to P_0,$$

where $\Omega(P)$ is a normalized abelian integral of the second kind. If Γ is hyperelliptic and P_0 is a fixed point of the hyperelliptic involution, then $V = 0$ and (3) becomes the Matveev-It·s formula [3] for finite-gap solutions of the Korteweg-de Vries (KdV) equation $4u_t = 6uu_x + u_{xxx}$. In contrast to the KP equation, it is possible here to indicate explicitly a basis of cycles of Γ and expressions for du_n; however, in this case the analysis of a solution is very difficult.

Serious efforts have been made to obtain a more effective description of the finite-gap solutions. We have in mind the "algebro-geometric effectivization" of Dubrovin and Novikov [4] and the "physical effectivization" proposed in [8] and [9]. They are based on

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the substitution of expressions of type (3) in the equation; here the origin of the constants is "forgotten," and they are determined directly from the equation. Serious progress on this path has been made only for $g = 2$ (and for $g = 3$ for the KP equation [4]).

In this note we propose a universal (in relation to the value of the genus g) approach to this problem, based on Schottky's uniformization theory for Riemann surfaces. Here the constants in formulas of type (3) are expressed in terms of the uniformization parameters with the help of Poincaré series. In this way we succeed in effectively describing all the physically important real nonsingular solutions.

2. Let F be a $2g$-connected domain in $\overline{\mathbf{C}}$, bounded by $2g$ nonintersecting Jordan curves $C_1, C_1', \ldots, C_g, C_g'$. A marked Schottky group G is a free Kleinian group with a distinguished system of generators $\sigma_1, \ldots, \sigma_g$

$$\frac{\sigma_n z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n}, \qquad 0 < |\mu_n| < 1,$$

where σ_n maps the exterior of C_n into the interior of C_n', $\sigma_n C_n = -C_n'$. A_n lies inside C_n, and B_n lies inside C_n'. If C_n and C_n' are circles, then the Schottky group is called classical [7]. F is a fundamental domain of this group. The limit set of G is a Cantor set; the domain of discontinuity Ω is connected. The quotient space Ω/G is a compact Riemann surface of genus g. On it we choose a canonical basis of cycles in the following way: the cycle a_n coincides with the curve C_n', oriented in a positive direction, and b_n goes in F from the point $z_n \in C_n$ to the point $\sigma_n z_n \in C_n'$, and the b-cycles do not intersect each other. According to the classical theorem of cuts [5], in this form we can represent any marked compact Riemann surface Γ of genus g. Thus, to each marked Riemann surface Γ there is associated a point $(A_1, \ldots, A_g, B_1, \ldots, B_g, \mu_1, \ldots, \mu_g) \in \mathbf{C}^{3g}$. The conjugate Schottky groups in $\mathrm{PSL}(2, \mathbf{C})$ lead to conformally equivalent Riemann surfaces, so that one usually considers normalized Schottky groups. For a uniformization of Γ with a marked point $P_0 \in \Gamma$ it is convenient to choose the normalization $A_1 = 1$, $B_1 = -1$, $P_0 = \infty$. Then the points $(A_2, \ldots, A_g, B_2, \ldots, B_g, \mu_1, \ldots, \mu_g)$ form a subset $S \subset \mathbf{C}^{3g-2}$.

We denote by G_n the smallest subgroup G containing an element σ_n. Then G/G_n and $G_m \backslash G/G_n$ consist of the elements $\sigma = \sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}$, $j_l \neq 0$, where $i_k \neq n$, and for $G_m \backslash G/G_n$ in addition $i_1 \neq m$.

LEMMA. *If the Poincaré series of dimension* (-2)

$$du_n = \frac{1}{2\pi i} \sum_{\sigma \in G/G_n} \left(\frac{1}{z - \sigma B_n} - \frac{1}{z - \sigma A_n} \right) dz \tag{5}$$

are absolutely convergent, then they define holomorphic differentials of the surface Γ, *normalized in the basis of cycles indicated above. The period matrix is given by the expression*

$$\begin{aligned} B_{nm} &= \frac{1}{2\pi i} \sum_{\sigma \in G_m \backslash G/G_n} \ln\{B_m, A_m, \sigma B_n, \sigma A_n\}, \qquad m \neq n, \\ B_{nn} &= \frac{\ln \mu_n}{2\pi i} + \frac{1}{2\pi i} \sum_{\sigma \in G_n \backslash G/G_n,\ \sigma \neq I} \ln\{B_n, A_n, \sigma B_n, \sigma A_n\}, \end{aligned} \tag{6}$$

where the curly brackets indicate the cross-ratio

$$\{z_1, z_2, z_3, z_4\} = (z_1 - z_3)(z_2 - z_4)/(z_1 - z_4)(z_2 - z_3).$$

Actually the proof of this theorem is contained in the old papers [10]–[12].

Thus, for the purposes of finite-gap integration we need to solve two problems:

1) Describe the subset S explicitly.

2) Prove the absolute convergence of the series (5).

It is apparently impossible to give a general solution to these two problems in the general case. (In particular, it is known that not every Schottky group is classical [7], and that a Poincaré series of dimension (-2) may not converge absolutely [15].) However, in the case of real Riemann surfaces, which is most important for applications, the situation is more favorable.

A Riemann surface Γ is called an *M-curve* if there is an antiholomorphic involution $\tau: \Gamma \to \Gamma$, $\tau^2 = 1$, on it, having $g+1$ fixed ovals. The ovals divide an M-curve into two components Γ_+ and Γ_-, homeomorphic to a sphere with $g+1$ holes. An M-curve can be uniformized by a Fuchsian Schottky group. In fact, we construct a Fuchsian uniformization $\Gamma_+ = H/G$, where $H = \{z \in \mathbf{C}, \operatorname{Im} z > 0\}$ and G is a Fuchsian group of the second kind. Such groups have been thoroughly studied [14], [13]. One can choose a fundamental polygon F_+ with boundary $c'_g l'_{g-1} c'_{g-1} \cdots c'_1 l_0 c_1 l_1 c_2 \cdots c_g l_g$, where l_n and l'_n are segments of the real axis and c_n and c'_n are half-circles, $0 \in l_0$, and $\infty \in l_g$. The hyperbolic transformations σ_n that map c_n onto c'_n form a free (from the relations) system of generators of the group G. The fixed points of σ_n are ordered in the following way:

$$(7) \qquad -\infty < B_g < \cdots < B_1 < 0 < A_1 < \cdots < A_g < +\infty, \quad A_n, B_n \in \mathbf{R}.$$

Extending the action of G to the lower half-plane $\bar{H}$, we observe that $\Gamma_- = \bar{H}/G$, and the fundamental polygon $F_- = \bar{F}_+$ is the reflection of F_+ relative to the real axis. $F = F_+ \cup F_-$, bounded by the circles $C_n = c_n \cup \overline{c_n}$ and $C'_n = c'_n \cup \overline{c'_n}$, is the fundamental domain of the Schottky group with generators $\sigma_n: C_n \to C'_n$, uniformizing the M-curve $\Gamma = \Gamma_+ \cup \Gamma_-$, and $l_0, l_1 \cup l'_1, \ldots, l_g$ are real ovals. The restrictions on the parameters μ_n

$$(8) \qquad \{B_{n+1}, A_{n+1}, B_n, A_n\} > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}}\right)^2, \qquad n = 1, \ldots, g-1,$$

are a consequence of the hyperbolicity of the elements $\sigma_{n+1}\sigma_n^{-1}$. From the analysis of the invariant lines of Fuchsian groups, following [6] and [13], one can show that relations (7) and (8) completely determine the set S. The problem of the convergence of the series (5) can also be solved successfully, since for a Fuchsian group of the second kind the Poincaré series of dimensions (-2) are absolutely convergent [5], [10].

If Γ is an M-curve, P_0 lies on a real oval, $\tau^* p = \bar{p}$, $\tau b_n = b_n$, $\tau a_n = -a_n$, and $D \in \mathbf{R}^g$, then (3) gives a real nonsingular solution of equation (2) [4]. Recently B. A. Dubrovin and S. M. Natanzon showed that the conditions are also necessary (private communication). Now setting $P_0 = \infty \in F$ and $p = z^{-1}$, we obtain from (4) and (6) that

$$(9) \qquad \begin{aligned} U_n &= \sum_{\sigma \in G/G_n} (\sigma A_n - \sigma B_n), \qquad V_n = \sum_{\sigma \in G/G_n} ((\sigma A_n)^2 - (\sigma B_n)^2), \\ W_n &= \sum_{\sigma \in G/G_n} ((\sigma A_n)^3 - (\sigma B_n)^3), \qquad c = \sum_{\sigma \neq I} \gamma^{-2}, \quad \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det \sigma = 1. \end{aligned}$$

Thus we have the following theorem.

THEOREM. *All the real nonsingular finite-gap solutions of equation* (2) *are given by* (3), (6), *and* (9), *in which the parameters* A_n, B_n, *and* μ_n *satisfy* (7) *and* (8).

Let $B_n = -A_n$; then one can choose F to be bounded by isometric circles of the transformations σ_n. F is symmetric relative to the involution $\pi z = -z$, $\pi F = F$. The group G is a subgroup of index 2 of the group with generators π and α_n satisfying $\alpha_n^2 = 1$ and $\sigma_n = \alpha_n \pi$, $n = 1, \ldots, g$. The corresponding surface $\Gamma = \Omega/G$ is hyperelliptic. The points of intersection of C_n with the real axis, and also 0 and ∞ (the fixed points of α_n

and π) are fixed points of the hyperelliptic involution. Thus, all the hyperelliptic M-curves are uniformized. Precisely they define the finite-gap potentials of the Schrödinger operator and the real nonsingular finite-gap solutions of the KdV equation [3]. In this case $V = 0$, and the period matrix

$$(10)\qquad \begin{aligned} B_{nm} &= \frac{1}{2\pi i} \sum_{\sigma \in G_m \backslash G/G_n} n \left(\frac{A_m - \sigma A_n}{A_m - \sigma(-A_n)} \right)^2, \\ B_{nn} &= \frac{\ln \mu_n}{2\pi i} + \frac{1}{2\pi i} \sum_{\sigma \in G_n \backslash G/G_n,\ \sigma \neq I} \ln \left(\frac{A_n - \sigma A_n}{A_n - \sigma(-A_n)} \right)^2. \end{aligned}$$

3. CONCLUDING REMARKS. 1) The convergence of the series (5) has been proved not only for Fuchsian Schottky groups, but also in the case when the circles C_n are small and located far from each other [10]–[12]. The smaller C_n is, the more rapidly the series converge. The limit $\mu_n \to 0$ ($C_n \to A_n$) is particularly suitable for study. In this limit in the sums (6) and (9) there remain only the terms corresponding to $\sigma = I$, and the solution degenerates into a multisoliton solution. The KdV equation is invariant under the change of variables $x \to ix$, $t \to -it$, $u \to -u$. If we carry out the same change of variables and select $iD \in \mathbf{R}^g$, then we obtain a somewhat different formula for the same nonsingular real finite-gap solutions. With the aid of this formula, it is convenient to study the small amplitude limit.

2) Our expressions allow us to determine the physical characteristics of a multiphase solution such as the amplitudes, wave numbers and phase velocities of harmonics, which in the approach based on direct substitutions caused certain difficulties connected with the fact that the theta-function (1) admits a modular transformation which varies these characteristics [9]. In the small amplitude and large amplitude limits it is easy to see that the characteristics of the solution obtained by our approach are indeed physical.

3) Other real Riemann surfaces than M-curves occur in finite-gap integration. For example, the solutions of the sine-Gordon equation are parametrized by hyperelliptic real Riemann surfaces of nonseparating type. In this case the Schottky group G, as for the KdV equation, is given by the generators $\sigma_n = \alpha_n \pi$, $\alpha_n^2 = 1$; the difference is that the fixed points $\alpha_1, \dots, \alpha_k$, $k \leq g$, do not lie on the real axis, but are conjugate relative to it. $\sigma_1, \dots, \sigma_k$ map H into $\bar{H}$, and G is not a Fuchsian group. However, in this case too it is possible to prove the convergence of the series (5).

4) D. A. Kubenskiĭ and the author have carried out experimental computer computations of the parameters of finite-gap solutions of the KdV equation from (9) and (10). It turns out that the calculations can be carried out on practically the whole set S for the values $g = 4$ and even larger. The results will be given in a separate paper.

5) Our results permit us to make effective the classification [2] of commuting differential operators of relatively prime order with smooth real coefficients.

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BIBLIOGRAPHY

1. B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, Uspekhi Mat. Nauk **31** (1976), no. 1(187), 55–136; English transl. in Uspekhi Mat. Nauk **31** (1976).

2. I. M. Krichever, Uspekhi Mat. Nauk **32** (1977), no. 6(198), 183–208; English transl. in Russian Math. Surveys **32** (1977).

3. A. R. It·s and V. B. Matveev, Teoret. i Mat. Fiz. **23** (1975), 51–68; English transl. in Theoret. Math. Phys. **23** (1975).

4. B. A. Dubrovin, Uspekhi Mat. Nauk **36** (1981), no. 2(218), 11–80; English transl. in Russian Math. Surveys **36** (1981).

5. Lester R. Ford, *Automorphic functions*, McGraw-Hill, 1929.

6. S. M. Natanzon, Uspekhi Mat. Nauk **27** (1972), no. 4(166), 145–160; English transl. in Russian Math. Surveys **27** (1972).

7. Albert Marden, Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers), Academic Press, 1974, pp. 273–278.

8. Akira Nakamura, J. Phys. Soc. Japan **47** (1979), 1701–1705; **48** (1980), 1365–1370.

9. John P. Boyd, J. Mathematical Phys. **25** (1984), 3390–3401, 3402–3414, 3415–3423.

10. W. Burnside, Proc. London Math. Soc. **23** (1891/92), 49–88.

11. M. F. Baker, *Abel's theorem and the allied theory including the theory of the theta functions*, Cambridge Univ. Press, 1897.

12. F. Schottky, J. Reine Angew. Math. **101** (1887), 227–272.

13. Linda Keen, Ann. of Math. (2) **84** (1966), 404–420.

14. Robert Fricke and Felix Klein, *Vorlesungen über die Theorie der automorphen Funktionen.* Vols. I, II, Teubner, Leipzig, 1897, 1912; reprints, 1926, 1965.

15. P. J. Myrberg, Ann. Acad. Sci. Fenn. (A) **9** (1916/17), no. 4.

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