REAL ALGEBROGEOMETRIC SOLUTIONS OF THE LANDAU–LIFSHITS EQUATION IN PRYM THETA FUNCTIONS

A. I. Bobenko

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INTRODUCTION

The goal of the present paper is the construction of finite-zone solutions of the Landau–Lifshits (L–L) equation

$$S_t = [S \times S_{xx}] + [S \times JS], \quad S_1^2 + S_2^2 + S_3^2 = 1, \qquad J = \mathrm{diag}\,(J_1, J_2, J_3), \quad J_1 \leqslant J_2 \leqslant J_3 \tag{0.1}$$

in the case of complete anisotropy (XYZ case), when all the quantities J_i are different.

Equation (0.1) is imbedded in the scheme of the method of the inverse scattering problem (MISP) and is a model equation with elliptic U–V pair (the spectral parameter varies on a torus), which was found independently by Sklyanin [1] and Borovik. They established that (0.1) is the compatibility condition for the system of linear equations

$$\frac{\partial \Psi}{\partial x} = U\Psi, \quad \frac{\partial \Psi}{\partial t} = V\Psi, \qquad U = -i \sum_{\alpha=1}^{3} S_\alpha w_\alpha(u)\, \sigma_\alpha, \tag{0.2}$$

$$V = 2iw_1(u)\, w_2(u)\, w_3(u) \sum_{\alpha=1}^{3} S_\alpha w_\alpha^{-1}(u)\, \sigma_\alpha - i \sum_{\alpha=1}^{3} [S \times S_x]_\alpha\, w_\alpha(u)\, \sigma_\alpha,$$

$$w_1(u) = \rho \frac{1}{\mathrm{sn}\,(u, k)}, \quad w_2(u) = \rho \frac{\mathrm{dn}\,(u, k)}{\mathrm{sn}\,(u, k)}, \quad w_3(u) = \rho \frac{\mathrm{cn}\,(u, k)}{\mathrm{sn}\,(u, k)}, \tag{0.3}$$

$$k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}, \quad \rho = \frac{1}{2}\sqrt{J_3 - J_1}, \quad w_\alpha^2 - w_\beta^2 = -\frac{1}{4}(J_\alpha - J_\beta), \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{0.4}$$

Here the spectral parameter u runs through a torus $\hat{T}$ with lattice 4K, 4iK' (4K, 4iK' are the periods of the Jacobi elliptic functions of modulus k), the formulas (0.3) are the explicit uniformizations of relations (0.4).

It is also shown in [1] that (0.1) in the "rapidly decreasing" case ($|S_3| \to 1$, $x \to \pm\infty$) describes a completely integrable Hamiltonian system, and an infinite series of commuting integrals of motion is calculated. However up until recently the difficulties in applying the MISP to (0.1) were not overcome. They were connected with the absence of an exact formulation of the matrix Riemann problem, corresponding to the elliptic bundle (0.2). The formulation of the Riemann problem needed was recently found by Mikhailov and it was investigated in [2, 3]. Based on this formulation, various procedures for "dressing" were proposed [4, 5] for (0.1).

In the case when two of the quantities J_i coincide, the elliptic pair (0.2) degenerates into a rational one, i.e., the functions $w_i(u)$ become rational functions of u, and u runs through a curve of genus zero. Finite-zoned solutions of such a degenerate case (XXZ) are constructed in [6].

In the present paper we construct finite-zoned solutions of the qualitatively more complex XYZ L–L equation, i.e., of the equation with nondegenerate elliptic U–V pair (0.2). The theory of finite-zoned integration of nonlinear equations, integrable by the MISP, is created in the papers of Novikov, Matveev, Its, Dubrovin, I. M. Krichever, and others, and is recounted in detail in the surveys [7-9].

We use an approach based on a synthesis of the scheme of Krichever, which is shown in finite-zoned integration, with general ideals of the method of the matrix Riemann problem

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which is the modern version of MISP [10]. This approach is based on a nonstandard version of the method of the matrix Riemann problem, extracted from the papers of Jimbo, Miwa, Ueno [12-13]. It was first demonstrated in Its [11] on the example of the nonlinear Schrödinger equation, and was later carried over to certain other equations, integrable by MISP [6]. Its advantage in comparison with Krichever's scheme is that it permits one to consider naturally in the axiomatics the Baker–Akhiezer function, the group of reductions of the corresponding U–V pair, and also is not based on deep algebrogeometric theorems (Riemann–Roch, problem of vanishing Jacobian). In the case considered this permits us not to investigate the problem of vanishing Jacobian on the Prymian.

Cherednik [14, 15] are devoted to the general theory of algebrogeometric solutions of equations integrable by the MISP with elliptic representation of curvature zero; however the explicit integration in theta-functions is not included in them. The important assertion that finite-zoned solutions of (0.1) can be expressed in terms of the Prym theta-functions was first made in [16]. Finally, exact formulas were recently constructed for solutions in terms of Riemann theta-functions of covering surfaces, and afterwards, applying the technique of reductions of theta-functions of symmetric surfaces [17, 18], expressions in terms of Prym theta-functions were found. However, these expressions differ by their extreme awkwardness and do not permit one to effectively extract real solutions.

In the present paper, thanks to a successful choice of basis of the homology group $H_1(\Gamma, Z)$, we succeed in constructing the Baker–Akhiezer function immediately in terms of the Prym theta-functions and Prym integrals. Conditions for the reality are effectively extracted by the technique of [20] for the compact formulas found for solutions of (0.1).

In the recent paper of Veselov [21], the integration of (0.1) for stationary solutions and simple waves is reduced, respectively, to the Neumann problem on motions of particles on the two-dimensional sphere and the classical integrable case of Clebsch of motion of a rigid body in an ideal fluid. In discussion of the results of [19] with S. P. Novikov, A. P. Veselov, and I. M. Krichever there arose an interesting question about the place of the solutions found in [21] among the whole set of finite-zoned solutions. The answer to this question is given in Sec. 5 of the present paper, where from the general finite-zoned formulas for solutions of (0.1) we find the known expressions for solutions of the Neumann and Clebsch problems.

The author expresses sincere thanks to A. R. Its for posing the problem and many helpful discussions, to V. B. Matveev for help in the study of the reductions of theta-functions, and also to S. P. Novikov, I. M. Krichever, A. P. Veselov, and R. F. Bikbaev for stimulating discussions.

1. Generalized Riemann Problem for the Landau–Lifshits Equation

Following the spirit of the papers [11-13] here we give the formulation of the Riemann problem found by Mikhailov for the L–L equation [2] in somewhat different form, which is most convenient, in our view, for finite-zoned integration.

Let $\Psi(u)$ be a matrix-valued (2×2) doubly periodic function ($4K$, $4iK'$ are the periods of the Jacobi elliptic functions of modulus k), which is meromorphic away from the points $\langle 0 \rangle$, and which has the following properties*:

$$\Psi(u) \underset{u\to 0}{=} (\Phi + \Phi_1 u + \ldots)\exp(-i\rho x\sigma_3 u^{-1} + 2\rho^2 it\sigma_3 u^{-2}), \tag{1.1}$$

is the asymptotic expansion of $\Psi(u)$ in a neighborhood of the point $u = 0$. One has the reductions

$$\sigma_1\Psi(u + 2iK')\, M_1(u) = \Psi(u), \tag{1.2}$$

$$\sigma_3\Psi(u + 2K)\, M_3(u) = \Psi(u), \tag{1.3}$$

where $M_1(u)$, $M_3(u)$ are matrices which are independent of x and t.

$\Psi(u)$ is holomorphic and invertible everywhere except for the points $a_1, \ldots, a_N$, in neighborhoods of which it can be represented in the form

$$\Psi(u) = \widehat{\Psi}(u)\,(u - a_j)^{T_j} C_j, \qquad j = 1, \ldots, N, \tag{1.4}$$

*We use the notation $\langle u \rangle = \{u,\ u + 2K,\ u + 2iK',\ u + 2K + 2iK'\}$.

where T_j are diagonal and C_j are invertible matrices, independent of x and t; T_j is independent of u; $\hat{\Psi}(u)$ is holomorphic and invertible in a neighborhood of a_j. We note that for T_j rational, and not integer-valued, the point a_j is a branch point, i.e., upon going around a small contour in the positive direction around a_j there occurs a right multiplication of the function $\Psi(u) \to \Psi(u)M_j$

$$M_j = C_j^{-1} \exp\{2\pi i T_j\} C_j. \tag{1.5}$$

In this case the Riemann problem should be understood as the problem for a branch. Later we shall consider it as a problem on the "upper" torus (cf. Sec. 4).

Further, suppose given on the torus $\hat{T}$ a system of contours $\mathscr{L}_i$ and conjugate matrices $G_i(u)$. The conjugation problem on $\mathscr{L}_i$ is

$$\Psi_-(u) = \Psi_+(u)\, G_i(u)\,|_{u \in \mathscr{L}_i}, \tag{1.6}$$

where Ψ_- and Ψ_+ are the boundary values of Ψ on the different shores of $\mathscr{L}$. The contour $\mathscr{L}_i$ and the matrices $G_i(u)$ should be chosen so that (1.6) is not contradicted by the reductions (1.2)-(1.3).

THEOREM 1. Let the function $\Psi(u)$ be constructed, satisfying (1.1)-(1.4) and (1.6). Then its logarithmic derivatives $\Psi_x\Psi^{-1}$ and $\Psi_t\Psi^{-1}$, up to summands proportional to the identity matrix, coincide with the U and V operators of the pair (0.2), and the vector $\mathbf{S} = (S_1, S_2, S_3)$, defined by

$$\sum_\alpha S_\alpha \sigma_\alpha = \Phi\sigma_3\Phi^{-1} \tag{1.7}$$

is a solution of the L–L equation (0.1). If in addition one has the reduction

$$\sigma_2\overline{\Psi(\bar{u})}\, M_R(u) = \Psi(u), \tag{1.8}$$

where $M_R(u)$ is a matrix which is independent of x and t, the vector S has real components.

Proof. We consider $\tilde{U} = \Psi_x\Psi^{-1}$, $\tilde{V} = \Psi_t\Psi^{-1}$. It follows from (1.4) that these functions do not have singularities at the points a_j and are single-valued on the torus $\hat{T}$ [cf. (1.5)], and also do not have jumps on $\mathscr{L}_i$ [cf. (1.6)]. Considering (1.2) and (1.3), and substituting the asymptotic expansion (1.1) into (0.2), we get (1.7).

It follows from (1.7) that

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad S_1 = \frac{CD - AB}{AD - BC}, \quad S_2 = -i\,\frac{CD + AB}{AD - BC}, \quad S_3 = \frac{AD + BC}{AD - BC}. \tag{1.9}$$

Definition 1.

$$\Lambda = \{a_j, T_j, C_j, j = 1, \ldots, N;\ \mathscr{L}_i, G_i(u), i = 1, \ldots, M\} \tag{1.10}$$

are called the *data of the generalized Riemann problem for the L–L equation.*

We note that Λ are not uniquely defined from the Ψ-function, for example, C_j can be multiplied on the left by a diagonal matrix (changing $\hat{\Psi}_j$).

2. The Surface Γ. Prym Differentials and Theta-Functions

In the case of rational U–V pairs (Korteweg–de Vries equation, nonlinear Schrödinger, sine-Gordon equations, XXZ–L–L equation, etc.), when the spectral parameter varies in the complex plane or another curve of genus 0 (cf. [6]) and the matrices T_j, corresponding to the branch points a_j, are half-integral, the Ψ-function (or Baker–Akhiezer function) is constructed on the hyperelliptic Riemann surface, which is a two-sheeted covering of the complex plane with branch points a_j. In the case considered of an elliptic U–V pair, the finite-zoned Ψ-function is defined on a branched covering of the torus.

The Riemann surface Γ is a two-sheeted covering of the torus T (its sides are equal to 2K, 4iK') which is "half" of a "big" torus $\hat{T}$, on which the Riemann problem is posed [thus in what follows we consider the reduction (1.3)]. To construct Γ we take two copies of the torus T and glue them along slits $[p_i, q_i]$, $[p_i + 2iK', q_i + 2iK']$, $i = 1,\ldots,g$. We get a surface of genus $\hat{g} = 2g + 1$ together with a canonical basis of cycles on it (the cycle b_0 consists of two components) pictured in Fig. 1. Here the part of the cycles lying on the "upper" torus (upper sheet) is drawn with a solid line, and the part lying on the "lower" sheet with a dashed one.

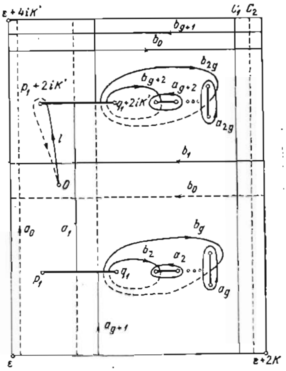

Fig. 1. Γ_1, a surface of type 1.

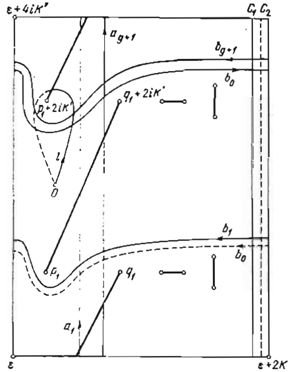

Fig. 2. Γ_2, a surface of type 2.

There exist two different coverings Γ_1 and Γ_2 of the torus T with the same branch points: Γ_1 is defined by

$$\omega_1^2 = R(u), \quad R(u) = B(u)\, B(u - 2iK'), \quad u \in T, \tag{2.1}$$

$$B(u) = \prod_{i=1}^{g} \left(\zeta(u - p_i) - \zeta(u - q_i) + 2\zeta\left(\frac{p_i - q_i}{2}\right)\right),$$

($\zeta(u)$ is the Weierstrass zeta-function, defined by the torus (cf. [22])), and Γ_2 by the same equation, where $p_1 \to p_1 + 2iK'$. In order to get Γ_2 from Γ_1, one should replace the slits $[p_1, q_1]$, $[p_1 + 2iK', q_1 + 2iK']$ by the slits $[p_1, q_1 + 2iK']$, $[p_1 + 2iK', q_1]$, i.e., for Γ_2 one slit intersects the real axis; $\omega_1(u)$ has poles at the branch points and zeros at the points $s_k = (p_k + q_k)/2$, $s_k + 2iK'$, $k = 1,\ldots,g$; $\omega_2(u)/\omega_1(u)$ is a nonsingle-valued function on T (it changes sign under the transformation $u \to u + 2K$): its zeros are at the points s_1, $s_1 + 2iK'$, and its poles are at $s_1 + iK'$, $s_1 + 3iK'$. In what follows, the notation Γ is used when we do not distinguish the coverings Γ_1 and Γ_2.

Γ has an involution $\tau P = P + 2iK'$, which does not change the sheets, and also an involution $\pi P = P^*$ which commutes with it and does change the sheets (in what follows an asterisk will denote the point on the other sheet). The involution τ has no fixed points. We give some facts from the theory of surfaces having such an involution, following [17].

A basis $a_0, b_0, a_1, b_1, \ldots, a_{\hat{g}-1}, b_{\hat{g}-1}$ of the homology group $H_1(\Gamma, \mathbb{Z})$ can be chosen so that τ acts on it as follows (such a basis is indicated in Figs. 1 and 2):

$$\tau a_0 = a_0, \quad \tau b_0 = b_0, \quad \tau a_i = a_{i+g}, \tag{2.2}$$

$$\tau b_i = b_{i+g}, \quad i = 1, \ldots, g$$

[equality in $H_1(\Gamma, \mathbb{Z})$]. Here we denote by τa the cycle obtained from the cycle a by the action of τ. Consequently, for the corresponding normalized $\left(\int_{a_i} du_j = \delta_{ij}\right)$ holomorphic differentials one has

$$\tau^* du_0(P) = du_0(\tau P) = du_0(P), \tag{2.3}$$

$$\tau^* du_i = du_{i+g}, \quad i = 1, \ldots, g,$$

and the period matrix has the simple structure:

$$B = \begin{pmatrix} T_{00} & T_0 & T_0 \\ T_0^t & M & M' \\ T_0^t & M' & M \end{pmatrix}, \tag{2.4}$$

where $T_{00} \in \mathbb{C}$, T_0 is a g-dimensional row, and M and M' are matrices of size $g \times g$.

LEMMA 1. $\Pi = M - M'$ is a symmetric matrix with positive-definite imaginary part.

Definition 2. The holomorphic differentials dw_i, $i = 1, \ldots, g$, are called the Prym differentials of the curve Γ with respect to the involution τ, normalized in the basis (2.2), if

$$\tau^* dw_i = -dw_i, \int_{a_i} dw_j = \delta_{ij}, \qquad i, j = 1, \ldots, g. \tag{2.6}$$

*

LEMMA 2.

$$dw_i = du_i - du_{i+g}, \qquad i = 1, \ldots, g. \tag{2.7}$$

Here the matrix Π of (2.5) is equal to $\Pi_{ij} = \int_{b_i} dw_j, \int_{b_0} dw_j = 0$.

LEMMA 3.

$$\pi^* dw_i = -dw_i, \qquad i = 1, \ldots, g. \tag{2.8}$$

Proof. The involution $\pi P = P^*$ acts on the cycles of Figs. 1 and 2 as follows:

$$\pi a_1 = a_{g+1}, \quad \pi a_i = -a_i, \qquad i = 2, \ldots, g. \tag{2.9}$$

Using the definition of the Prym differentials (2.6), we get that $-dw_i$ are the normalized Prym differentials in the basis πa, πb. From this and from their uniqueness the assertion of the lemma follows.

Definition 3. The Abelian differential $d\omega_Q^{(n)}$ is called *a normalized Prym differential of the second kind, if*

$$\tau^* d\omega_Q^{(n)} = -d\omega_Q^{(n)}, \quad \int_{a_i} d\omega_Q^{(n)} = 0, \quad i = 1, \ldots, g, \tag{2.10}$$

and $d\omega_Q^{(n)}$ has a unique pole at the point Q of multiplicity $n + 1$ [and consequently, by (2.10), the same kind of pole at the point τQ] with principal part of the form

$$d\omega_Q^{(n)} = (z^{-n-1} + O(1))\, dz, \tag{2.11}$$

where z is a local variable in the neighborhood of Q ($z \to 0$, $P \to Q$).

The vector of b-periods of such a differential satisfies

$$\int_{b_i} d\omega = \int_{\tau b_i} \tau^* d\omega = -\int_{b_{i+g}} d\omega, \quad \int_{b_0} d\omega = 0. \tag{2.12}$$

Definition 4. The *Prym theta-function with characteristics* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{R}^g$ is the function $f(P)$, $P \in \Gamma$, equal to

$$f(P) = \theta[\boldsymbol{\alpha}, \boldsymbol{\beta}]\left(\int_{P_0}^{P} d\mathbf{w} + \mathbf{D} \mid \Pi\right), \tag{2.13}$$

where $P_0 \in \Gamma$ is a fixed point, $d\mathbf{w} = (dw_1, \ldots, dw_g)$, $\mathbf{D} \in \mathbf{C}^g$, Π is the matrix of (2.5),

$$\theta[\boldsymbol{\alpha}, \boldsymbol{\beta}](\mathbf{z} \mid \Pi) = \sum_{\mathbf{m} \in Z^g} \exp\{\pi i \langle \Pi(\mathbf{m} + \boldsymbol{\alpha}), \mathbf{m} + \boldsymbol{\alpha} \rangle + 2\pi i \langle \mathbf{z} + \boldsymbol{\beta}, \mathbf{m} + \boldsymbol{\alpha} \rangle\}, \tag{2.14}$$

$$\theta(\mathbf{z} \mid \Pi) \equiv \theta[0, 0](\mathbf{z} \mid \Pi), \quad \mathbf{z} \in \mathbf{C}^g,$$

the summation is over the whole g-dimensional integral lattice, $\langle\, ,\, \rangle$ denotes the ordinary scalar product.

We shall not cite here the usual properties of nonsingle-valuedness of the function $f(P)$, with the help of which it is easy to prove the following:

LEMMA 4. For a vector D in general position, the function (2.13) vanishes at precisely $2g = \hat{g} - 1$ points of Γ.

3. Construction of the Baker–Akhiezer Function. Solution of the Landau–Lifshits Equation

We consider the normalized Prym integrals of the second kind (the integrals of the Prym differentials of the second kind) $\Omega_1(P)$ and $\Omega_2(P)$

*Numbering as in Russian original — Publisher.

$$\tau^* d\Omega_i = -d\Omega_i, \quad \int_{a_j} d\Omega_i = 0; \quad i = 1, 2, \quad j = 1, \ldots, g, \tag{3.1}$$

with singularities at the points 0 and $0^* = \pi 0$ of the form

$$\begin{gathered} \Omega_1(P) \underset{P \to 0}{=} -\rho u^{-1} - a + O(u), \quad \Omega_1(P) \underset{P \to 0^*}{=} \rho u^{-1} + a + O(u), \\ \Omega_2(P) \underset{P \to 0}{=} 2\rho^2 u^{-2} + b + O(u), \quad \Omega_2(P) \underset{P \to 0^*}{=} -2\rho^2 u^{-2} - b + O(u). \end{gathered} \tag{3.2}$$

Here u is the natural projection of the point $P \in \Gamma$ to the torus T. $\mathbf{U} \in \mathbf{C}^g$ and $\mathbf{V} \in \mathbf{C}^g$ are the vectors of b-periods.

$$U_i = \int_{b_i} d\Omega_1, \quad V_i = \int_{b_i} d\Omega_2, \quad \mathbf{U} = (U_1, \ldots, U_g), \quad \mathbf{V} = (V_1, \ldots, V_g). \tag{3.3}$$

The functions $\psi_1(P)$, $\psi_2(P)$ are determined by the formulas

$$\begin{gathered} \psi_1(P) = \frac{\theta(\mathbf{w}(P) + \mathbf{\Omega} + \mathbf{A} \mid \Pi)}{\theta(\mathbf{w}(P) + \mathbf{A} \mid \Pi)} \exp\{i\Omega_1(P)x + i\Omega_2(P)t\}, \\ \psi_2(P) = \frac{\theta(\mathbf{w}(P) + \mathbf{\Omega} + \mathbf{A} + \mathbf{n} \mid \Pi)}{\theta(\mathbf{w}(P) + \mathbf{A} \mid \Pi)} \exp\{i\Omega_1(P)x + i\Omega_2(P)t\}, \quad A \in \mathbf{C}^g, \\ \mathbf{w}(P) = \int_{P_0}^{P} d\mathbf{w}, \quad \mathbf{\Omega} = (\mathbf{U}x + \mathbf{V}t)/2\pi, \quad \mathbf{n} = (1/2, 0, \ldots, 0). \end{gathered} \tag{3.4}$$

<u>LEMMA 5.</u> $\psi_1(P)$ is a single-valued function on Γ. $\psi_2(P)$ changes sign if the point P makes a circuit about the cycles b_1 and b_{g+1} [cf. the basis of $H_1(\Gamma, Z)$ of Figs. 1 and 2]; upon a circuit about the other cycles $\psi_2(P)$ does not change, i.e., $\psi_2(P)$ becomes single-valued on Γ with slits along the cycles $\mathscr{C}_1$ and $\mathscr{C}_2$, while $\psi_2^+(P) = -\psi_2^-(P)$, $P \in \mathscr{C}_i$; ψ_2^+ and ψ_2^- are the values of the function ψ_2 on different shores of the slits $\mathscr{C}_i$.

The *proof* goes in the standard way (cf. [9]), using the known nonsingle-valuedness of $\Omega_i(P)$ and the Prym theta-functions (2.13).

It follows from Lemma 4 that the functions $\psi_1(P)$ and $\psi_2(P)$ have identical divisors of poles on Γ:

$$D = \sum_{k=1}^{2g} \mu_k. \tag{3.5}$$

We consider the Riemann surface $\hat{\Gamma}$, which is a two-sheeted covering of the torus $\hat{T}$ with branch points $\langle p_i \rangle$, $\langle q_i \rangle$, $i = 1, \ldots, g$. The involution $\lambda P = P + 2K$ does not change the sheets of $\hat{\Gamma}$. It is obvious that $\hat{\Gamma}$ is an unbranched two-sheeted covering of the surface $\Gamma = \hat{\Gamma}/\lambda$. It is convenient to represent $\hat{\Gamma}$ as two copies of Γ, glued to one another along the slits $\mathscr{C}_i$ (the "upper" sheet with the "upper"). The functions ψ_1 and ψ_2 are naturally defined on $\hat{\Gamma}$ and are single-valued on it, where

$$\psi_1(\lambda P) = \psi_1(P), \quad \psi_2(\lambda P) = -\psi_2(P). \tag{3.6}$$

<u>LEMMA 6.</u>

$$\begin{gathered} \psi_1(\pi\tau P) = \psi_2(P)\, m(P), \quad \psi_2(\pi\tau P) = \psi_1(P)\, m(P), \\ m(P) = \theta(\mathbf{w}(P) + \mathbf{A})/\theta(\mathbf{w}(P) + \mathbf{A} + \mathbf{n}). \end{gathered} \tag{3.7}$$

<u>Proof.</u> We choose as initial point of integration a branch point (cf. Fig. 3):

$$\begin{gathered} \mathbf{w}(\pi\tau P) = \int_{P_0}^{\pi\tau P} d\mathbf{w} = \int_{\pi\tau P_0}^{P} \pi^*\tau^* d\mathbf{w} = \int_{\pi\tau P_0}^{P} d\mathbf{w} = \mathbf{w}(P) + \int_{\pi\tau P_0}^{P_0} d\mathbf{w} = \mathbf{w}(P) + \mathbf{n}, \\ \Omega_i(\pi\tau P) = \Omega_i(P) + \int_{\pi\tau P_0}^{P_0} d\Omega_i = \Omega_i(P), \end{gathered} \tag{3.8}$$

since from Lemma 3 and the analogous equation $\pi^* d\Omega_i = -d\Omega_i$ it follows (cf. Fig. 3) that $\int_{\pi\tau P_0}^{P_0} d\mathbf{w} = \frac{1}{2}, \int_{-a_0+a_1} d\mathbf{w} = \mathbf{n}, \int_{\pi\tau P_0}^{P_0} d\Omega_i = 0$. Substituting (3.8) into (3.4), we see that (3.7) is valid.

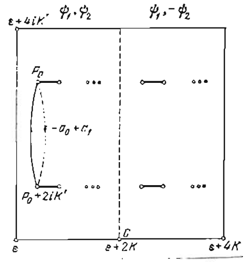

Fig. 3. The surface $\hat{\Gamma}$.

THEOREM 2. The function

$$\Psi(u)=\begin{pmatrix}\psi_1(P) & \psi_1(P^*)\\ \psi_2(P) & \psi_2(P^*)\end{pmatrix} \tag{3.9}$$

is a solution of the generalized Riemann problem for the L–L equation with data $\Lambda = \{\mu_j,\ \mu_j + 2K,\ T_j=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix},\ C_j=I$, if $\mu_j\in T_+$; $C_j=\sigma_1$, if $\mu_j\in T_-$, $j=1,\dots,2g$; $\langle p_j\rangle, \langle q_j\rangle,\ T_j=\begin{pmatrix}0 & 0\\ 0 & 1/2\end{pmatrix},\ C_j=\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix},\ j=1,\dots,g;\ \mathcal{L}=\bigcup_{j=1}^{g}\langle[p_j,q_j]\rangle,\ G=\sigma_1\}$. T_+, T_- are the upper and lower sheets of Γ, $\langle[p_j, q_j]\rangle = \bigcup_{\delta_1,\delta_2}[p_j+2K\delta_1+2iK'\delta_2,\ q_j+2K\delta_1+2iK'\delta_2]$, $u\in\hat{T}$ is the projection of the point $P\in\hat{\Gamma}$, $\delta_i=\{0,\ 1\}$, $i = 1,\ 2$.

Proof. We recall that the function $\Psi(u)$ can be considered as a function on the upper sheet, so in (3.9), $P\in T_+$. From the definition of the integrals $\Omega_i(P)$ of (3.2) there follows the asymptotic behavior (1.1) of the Ψ-function. The reductions (1.2), (1.3) follow from (3.6), (3.7), while $M_3(u) = I, M_1(u)=\begin{pmatrix}0 & m^{-1}(u^*)\\ m^{-1}(u) & 0\end{pmatrix}$. The points μ_j and $\mu_j + 2K$ are simple poles [cf. (3.5)] of the functions ψ_1, ψ_2 [the representation (1.4) in them is obvious]. The points $\langle p_j\rangle$, $\langle q_j\rangle$ are branch points. The function

$$\hat{\Psi}(u) \underset{u\sim p_j}{=} \Psi(u)C_j^{-1}(u-p_j)^{-T_j} = -\frac{1}{2}\begin{pmatrix}\psi_1+\psi_1^*, & (\psi_1-\psi_1^*)/\sqrt{u-p_j}\\ \psi_2+\psi_2^*, & (\psi_2-\psi_2^*)/\sqrt{u-p_j}\end{pmatrix}$$

is holomorphic and invertible in a neighborhood of p_j. The corresponding monodromy matrix (1.5) is equal to $M_j = \sigma_1$, which is obvious from the form of the Ψ-function (3.9) itself also. On the slits $\mathcal{L}$ the function $\Psi(u)$, as a function on the upper sheet T_+, changes columns, so (1.6) is valid with $G = \sigma_1$. And finally, we note that at the remaining points of T_+ the function (3.9) has no singularities, since its components have no additional poles, and $\det\Psi(u)$ — zeros [all 8g poles μ_j, $\mu_j + 2K$, μ_j^*, $\mu_j^* + 2K$ on $\hat{\Gamma}$ are known as well as 8g (hence all!) zeros $\langle p_i\rangle$, $\langle q_i\rangle$ of the function $\det\Psi(u)$].

Calculating the first term of the asymptotic expansion (1.1) and cancelling A, B, C, D [cf. (1.9)] on common factors not appearing in the final answer for S (1.9), we get

THEOREM 3. The quantities

$$A=\theta(\Omega+\mathbf{D}\mid\Pi),\quad B=\theta(\Omega+\mathbf{D}+\mathbf{r}\mid\Pi)$$
$$C=\theta(\Omega+\mathbf{D}+\mathbf{n}\mid\Pi),\quad D=\theta(\Omega+\mathbf{D}+\mathbf{n}+\mathbf{r}\mid\Pi), \tag{3.10}$$
$$\Omega=(\mathbf{U}x+\mathbf{V}t)/2\pi,\quad \mathbf{D}\in\mathbf{C}^g,\quad \mathbf{r}=\int_0^{0^*}d\mathbf{w}\in\mathbf{C}^g,\quad \mathbf{n}=(1/2,0,\dots,0)$$

define by (1.9) a solution of the L–L equation. Here the path of integration in the integral defining $\mathbf{r}$ should not intersect the cycles $\mathscr{C}_1$ and $\mathscr{C}_2$ (cf. Figs. 1 and 2).

Remark. By analogous formulas one can give solutions of another familiar equation with elliptic U–V pair, the equations of an asymmetric chiral $O(3)$-field [26]. In this case $\Omega_2(P)$ is a Prym integral with a simple pole at some point $\eta \neq 0$.

4. Extraction of Real Solutions

In order to extract from all the solutions of the L–L equation found in Theorem 3.2, the real ones, according to Theorem 1 one should analyze the validity of the reduction (1.8) for the Ψ-function (3.9). However it is more convenient to apply the technique of [20], in which the real solutions of the sine-Gordon equation are extracted with the help of direct analysis of the formulas in the theta-functions for the solution of the nonlinear equation itself.

As a preliminary we formulate a simple assertion which can be verified by direct calculation.

LEMMA 7. In order that the solution $S(x, t)$, defined by (1.9), be real, it is necessary and sufficient that one have

$$-1 = (D\bar{C})/(\bar{A}B). \tag{4.1}$$

We consider a surface Γ (2.1), having the conjugation involution $\varphi(u, \omega) = (\bar{u}, \bar{\omega})$ which does not change the sheets of Γ. This means that the branch points p_k, q_k satisfy one of the three conditions:

1. $$\operatorname{Im} p_k = \operatorname{Im} q_k = \pm K'; \tag{4.2}$$
2. $$\bar{p}_k \equiv q_k + 2iK'; \tag{4.3}$$
3. $$p_k, q_k \in \mathbf{R}. \tag{4.4}$$

Successively we consider the two possible cases*:

1) Surface of type Γ_1 (i.e., there are no slits intersecting the real axis) and there exists a slit $[p_1, q_1]$ of type (4.2) (Fig. 1).

2) Surface of type Γ_2 and there exists a slit of type (4.2) (Fig. 2).

We give the calculation in detail for a surface of type 1 (Fig. 1), and for the rest we only formulate the final answer.

Let the surface Γ_1 (cf. Fig. 1) have ν slits $[p_1, q_1], \ldots, [p_\nu, q_\nu]$ of type (4.2) and $g - \nu$ slits $[p_{\nu+1}, q_{\nu+1}], \ldots, [p_g, q_g]$ of type (4.3). Under the action of the antiinvolution φ the cycles a_i, b_i go into the cycles φa_i, φb_i, connected with them by the relations

$$a_1 = -\varphi a_1, \quad a_i = -\varphi a_{g+i}, \qquad i = 2, \ldots, g, \tag{4.5}$$

$b_1 = \varphi b_1$, $b_i = \varphi b_{i+g}$, $b_j = \varphi b_{j+g} + \varphi a_{j+g}$, $i = 2, \ldots, \nu$; $j = \nu + 1, \ldots, g$. From this there follows the following law of conjugation for the corresponding normalized Prym differentials:

$$\overline{\varphi^* dw_1(P)} = \overline{dw_1(\bar{P})} = -dw_1(P), \quad \overline{\varphi^* dw_i(P)} = \overline{dw_i(\bar{P})} = dw_i(P). \tag{4.6}$$

For the matrix Π of (2.5), (2.7) here we get

$$\begin{aligned} &\bar{\Pi}_{11} = \overline{\int_{b_1} dw_1} = -\int_{\varphi b_1} dw_1 = -\Pi_{11}, \quad \bar{\Pi}_{1k} = \overline{\int_{b_1} dw_k} = \int_{\varphi b_1} dw_k = \Pi_{1k}, \quad k > 1, \\ &\bar{\Pi}_{ik} = \overline{\int_{b_i} dw_k} = \int_{\varphi b_i} dw_k = -\Pi_{ik}, \quad i, k > 1, i \leqslant \nu, \\ &\bar{\Pi}_{ik} = -\Pi_{ik} - \delta_{ik}, \quad i, k > \nu. \end{aligned} \tag{4.7}$$

The Prym differentials of the second kind [cf. (3.1)-(3.2)] $\overline{\varphi^* d\Omega_i}$ and $d\Omega_i$ coincide, since their singularities coincide, and the automorphisms φ and π commute

$$\overline{d\Omega_i(\bar{P})} = d\Omega_i(P), \quad i = 1, 2. \tag{4.8}$$

*Repeating the calculations of this section, one can show that the surfaces Γ with branch points of type (4.4) do not give real solutions of the L–L equation. In the case of discrete spectrum [1, 5], the points of the spectrum of type (4.4) are missing, and points of type (4.2) and (4.3) are classified, respectively, as solitons and breezers, so the absence of zones of type (4.4) is completely natural in the finite-zoned case.

Consequently, the b-periods of the differentials (4.8) coincide, and one has the following equation for the components of the vector $\boldsymbol{\Omega}$ [cf. (3.4)]: $\boldsymbol{\Omega} = (\Omega', \Omega'')$, $\Omega' \in \mathbf{C}$, $\Omega'' \in \mathbf{C}^{g-1}$, $x, t \in \mathbf{R}$,

$$\overline{\Omega'} = \overline{\int_{b_1} d\Omega} = \int_{\sigma b_1} d\Omega = \Omega', \quad \overline{\Omega''} = -\Omega''. \tag{4.9}$$

Analogously, for the vector $\mathbf{r} = (r', r'')$ we get (cf. Fig. 1)

$$\bar{r}' = \overline{\int_0^{0^*} dw_1} = \int_l^{\overline{\ }} dw_1 = -\int_{\varphi l} dw_1 = -r' + 1, \quad r' = r_0' + {}^1/_2, \quad \mathrm{Re}\, r_0' = 0,$$
$$\bar{r}'' = r'', \quad \varphi l = l - a_1 + a_0. \tag{4.10}$$

Finally, the law of conjugation for the Prym theta-functions defined by the matrix (4.7) looks like this:

$$\overline{\theta(z \mid \Pi)} = \overline{\theta((z', z'') \mid \Pi)} = \theta((\bar{z}', -\bar{z}'' + \lambda) \mid \Pi), \tag{4.11}$$

where $z = (z', z'')$, $z' \in \mathbf{C}$, $z'' \in \mathbf{C}^{g-1}$, $\lambda = {}^1/_2(0, \ldots, 0, 1, \ldots, 1)$ (zeros in the first $\nu - 1$ places).

Substituting (4.9)-(4.11) into condition (4.1),

$$-1 = \frac{D\bar{C}}{\bar{A}B} = \frac{\theta(\boldsymbol{\Omega} + \mathbf{D} + \mathbf{n} + \mathbf{r})\,\overline{\theta(\boldsymbol{\Omega} + \mathbf{D} + \mathbf{n})}}{\overline{\theta(\boldsymbol{\Omega} + \mathbf{D})}\,\theta(\boldsymbol{\Omega} + \mathbf{D} + \mathbf{r})} = \frac{\theta\left(\Omega' + D' + \frac{1}{2} + r_0' + \frac{1}{2},\ \Omega'' + D'' + r''\right)\theta\left(\Omega' + \bar{D}' + \frac{1}{2},\ \Omega'' - \bar{D}'' + \lambda\right)}{\theta(\Omega' + \bar{D}',\ \Omega'' - \bar{D}'' + \lambda)\,\theta\left(\Omega' + D' + r_0' + \frac{1}{2},\ \Omega'' + D'' + r''\right)},$$

we get that the vector $\mathbf{D} = (D', D'')$, defining the real solutions, should satisfy the equations

$$\mathrm{Im}\, D' = {}^1/_2\, \mathrm{Im}\,(\Pi_1' - r'), \quad \mathrm{Re}\, D'' = {}^1/_2\,(\Pi_1'' - r'' + \lambda + \delta), \tag{4.12}$$

where $\Pi'_1 = \Pi_{11}$, Π_1'' is the vector with components $\Pi_1'' = (\Pi_{21}, \Pi_{31}, \ldots, \Pi_{g1}) \in \mathbf{R}^{g-1}$, $\delta \in \mathbf{Z}^{g-1}/2\mathbf{Z}^{g-1}$ is an arbitrary $(g - 1)$-dimensional vector with coordinates 0 and 1.

In the case of a surface of type 2 (Fig. 2) we get analogously the same restriction (4.12) on D.

THEOREM 4. A surface Γ with branch points of type (4.2), (4.3), and $\mathbf{D} \in \mathbf{C}^g$, satisfying conditions (4.12), defines a real (and hence smooth!) solution of the L–L equation. For a fixed surface Γ of genus $2g + 1$ one constructs 2^{g-1} topologically distinct real components of solutions (by means of different choice of δ), which do not go into one another under evolution of the dynamical variable and not connected by trivial gauge transformations.

5. Solutions of Genus g = 1 and g = 2. Classical Integrable Neumann and Clebsch Problems

In the case of surfaces Γ of small genus one can carry out a more detailed analysis of solutions of the L–L equation. For this one uses the familiar formula for composition of theta functions (cf., e.g., [9])

$$\theta(z_1 \mid \Pi)\,\theta(z_2 \mid \Pi) = \sum \theta[\delta, 0](z_1 + z_2 \mid 2\Pi)\,\theta[\delta, 0](z_1 - z_2 \mid 2\Pi), \quad \delta \in \frac{1}{2}\mathbf{Z}^g/2\mathbf{Z}^g \tag{5.1}$$

where the summation is over all g-dimensional vectors δ, with coordinates 0, 1/2. Applying (5.1), we get the following expression for a solution of genus $g = 1$:

$$S_1 = -\frac{\theta[{}^1/_2, 0](\mathbf{z})\,\theta[{}^1/_2, 0](\mathbf{r})}{\theta[{}^1/_2, {}^1/_2](\mathbf{z})\,\theta[{}^1/_2, {}^1/_2](\mathbf{r})}, \quad S_2 = -i\,\frac{\theta[0, 0](\mathbf{z})\,\theta[0, 0](\mathbf{r})}{\theta[{}^1/_2, {}^1/_2](\mathbf{z})\,\theta[{}^1/_2, {}^1/_2](\mathbf{r})},$$
$$S_3 = \frac{\theta[0, {}^1/_2](\mathbf{z})\,\theta[0, {}^1/_2](\mathbf{r})}{\theta[{}^1/_2, {}^1/_2](\mathbf{z})\,\theta[{}^1/_2, {}^1/_2](\mathbf{r})}, \quad \mathbf{z} = 2\boldsymbol{\Omega} + 2\mathbf{D} + \mathbf{r}, \quad \mathbf{n} = {}^1/_2. \tag{5.2}$$

Here and later we use the notation $\theta[\alpha, \beta](\mathbf{x}) = \theta[\alpha, \beta](\mathbf{x} \mid 2\Pi)$. If $g = 2$ the formulas for solutions are more complicated:

$$S_1 = -\frac{\theta[({}^1/_2, 0), (0, 0)](\mathbf{z})\,\theta[({}^1/_2, 0), (0, 0)](\mathbf{r}) + \theta[({}^1/_2, {}^1/_2), (0, 0)](\mathbf{z})\,\theta[({}^1/_2, {}^1/_2), (0, 0)](\mathbf{r})}{\theta[({}^1/_2, 0), ({}^1/_2, 0)](\mathbf{z})\,\theta[({}^1/_2, 0), ({}^1/_2, 0)](\mathbf{r}) + \theta[({}^1/_2, {}^1/_2), ({}^1/_2, 0)](\mathbf{z})\,\theta[({}^1/_2, {}^1/_2), ({}^1/_2, 0)](\mathbf{r})},$$

$$S_2 = -i\frac{\theta[(0,0),(0,0)](\mathbf{z})\,\theta[(0,0),(0,0)](\mathbf{r}) + \theta[(0,{}^1/_2),(0,0)](\mathbf{z})\,\theta[(0,{}^1/_2),(0,0)](\mathbf{r})}{\theta[({}^1/_2,0),({}^1/_2,0)](\mathbf{z})\,\theta[({}^1/_2,0),({}^1/_2,0)](\mathbf{r}) + \theta[({}^1/_2,{}^1/_2),({}^1/_2,0)](\mathbf{z})\,\theta[({}^1/_2,{}^1/_2),({}^1/_2,0)](\mathbf{r})},$$

$$S_3 = \frac{\theta[(0,0),({}^1/_2,0)](\mathbf{z})\,\theta[(0,0),({}^1/_2,0)](\mathbf{r}) + \theta[(0,{}^1/_2),({}^1/_2,0)](\mathbf{z})\,\theta[(0,{}^1/_2),({}^1/_2,0)](\mathbf{r})}{\theta[({}^1/_2,0),({}^1/_2,0)](\mathbf{z})\,\theta[({}^1/_2,0),({}^1/_2,0)](\mathbf{r}) + \theta[({}^1/_2,{}^1/_2),({}^1/_2,0)](\mathbf{z})\,\theta[({}^1/_2,{}^1/_2),({}^1/_2,0)](\mathbf{r})}, \quad (5.3)$$

where $n = (1/2, 0)$, i.e., one assumes a choice of canonical basis of cycles, analogous to that indicated in Figs. 1 and 2.

As shown in [21], solutions of the L–L equation which are independent of t are also solutions of the Neumann problem on motion of a particle on the two-dimensional sphere† [9], and those depending only on the combination x + vt (simple waves) to the classical integrable case of Clebsch of motion of a rigid body in a fluid [9, 21, 23]. All solutions of the L–L equation of genus g = 1 (5.2) are simple waves and consequently periodic solutions of the Clebsch case. It turns out that some of the solutions (5.3) are also simple waves. A necessary and sufficient condition for a solution (5.3) to be a simple wave is the proportionality of the vectors of b-periods of the Prym integrals $\Omega_1(P)$ and $\Omega_2(P)$. This, in its own right, is equivalent with the existence of the function

$$f(P) = v\,\Omega_1(P) - \Omega_2(P) \quad (5.4)$$

for some number $v \in \mathbb{C}$ [from (5.4) it follows that $vU - V = 0$, i.e., the solution (5.3) depends on x + vt].

We consider the Riemann surface Γ with branch points p_1, q_1, p_2, q_2, $p_1 + 2iK'$, $q_1 + 2iK'$, $p_2 + 2iK'$, $q_2 + 2iK'$ and the function $\omega(u) = \sqrt{R(u)}$ on it. This function is determined by the expression (2.1) and differs by sign on the upper and lower sheets of Γ; $\omega(u)$ has poles at all branch points and has simple zeros at the points $s_1 = (p_1 + q_1)/2$, $s_2 = (p_2 + q_2)/2$, $s_1 + 2iK'$, $s_2 + 2iK'$. Here $R(u) = R(u + 2iK')$. We construct the function f(P) (5.4) from the singularities of the integrals $\Omega_1(P)$, $\Omega_2(P)$ (3.2):

$$f(P) = \frac{N}{\omega(u)}\,g(u),$$
$$g(u) = c(\zeta(u) - \zeta(u - 2iK') - \zeta(2iK')) + 2(\wp(u) - \wp(u - 2iK')), \quad (5.5)$$

where $\zeta(u)$, $\wp(u)$ are the Weierstrass functions [22], defined by the lattice T, N is an inessential normalizing factor. Thus, f(P) has the singularities prescribed by (5.4) at the points 0, 0*, 2iK', (2iK')*. The condition of the absence of additional poles of f(P) distinguishes the surfaces Γ corresponding to simple waves. For this the zeros of the denominator of (5.5), $\omega(u)$, should be cancelled by zeros of the numerator, i.e.,

$$g(s_1) = 0, \quad g(s_2) = 0, \quad s_1 = (p_1 + q_1)/2, \; s_2 = (p_2 + q_2)/2. \quad (5.6)$$

The system (5.6) is solvable if and only if the centers of the zones are connected by the relation

$$s_2 = K - s_1 \quad (5.7)$$

[then the second equation of (5.6) is a consequence of the first]; here the rate of the simple wave is given by the following expression:

$$v(s_1) = 2\rho\left(\frac{\wp(s_1 - 2iK') - \wp(s_1)}{\zeta(s_1) - \zeta(s_1 - 2iK') - \zeta(2iK')} - \frac{\omega'(0)}{\omega(0)}\right). \quad (5.8)$$

Thus, we have proved

THEOREM 5. The solutions of the L–L equation of genus g = 1 are simple waves. If g = 2, simple waves are determined by surfaces Γ, the centers of whose zones s_1 and s_2 (5.6) satisfy (5.7). The solution is independent of t, if $(p + q)/2 = -iK'$ (g = 1) and $s_1 = -iK'$ (g = 2), and is independent of x, if $(p + q)/2 = K$ (q = 1).

From (5.3) it is easy to get Kötter's formulas for the Clebsch case [21, 23, 24]:

$$S_k = e_k\frac{\theta[i_k](z_0 + U\tau)\,\theta[j_k](w_0) - \theta[i_k](w_0)\,\theta[j_k](z_0 + U\tau)}{\theta[m](z_0 + U\tau)\,\theta[n](w_0) - \theta[m](w_0)\,\theta[n](z_0 + U\tau)},$$

†The fact that the stationary L–L equation is equivalent to the equation of motion of a point on the unit sphere in a force field with quadratic potential was first noted by V. M. Eleonskii etc. [Zh. Eksp. Teor. Fiz., 84, No. 2, 616-627 (1983); there are also references to earlier papers there].

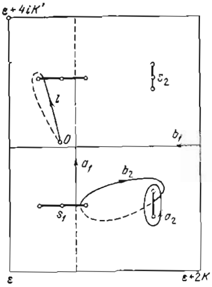

Fig. 4

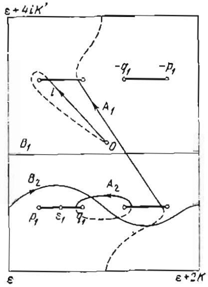

Fig. 5

$$\begin{gathered}\varepsilon_1=\varepsilon_3=1,\quad \varepsilon_2=-i,\quad \mathbf{z}_0+\mathbf{U}\tau=2\Omega+2\mathbf{D}+\mathbf{r},\quad \mathbf{w}_0=\mathbf{r}-\Pi\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix},\\ \tau=2(x+vt),\ m=[(0,0),(\tfrac12,0)],\ n=[(0,\tfrac12),(\tfrac12,0)],\\ i_1=[(\tfrac12,0),(0,0)],\ i_2=[(0,0),(0,0)],\quad i_3=[(\tfrac12,0),(\tfrac12,0)],\\ j_1=[(\tfrac12,\tfrac12),(0,0)],\ j_2=[(0,\tfrac12,(0,0)],\quad j_3=[(\tfrac12,\tfrac12),(\tfrac12,0)].\end{gathered} \tag{5.9}$$

For solutions independent of t, as is easy to see, $s_1=-iK'$, $s_2=K+iK'$ and Γ has the involution $\lambda u=-u$. Under the action of this involution the Prym differentials normalized in the basis of Fig. 4 change sign $\lambda^* dw=-dw$, so

$$\mathbf{r}=\int_l d\mathbf{w}=-\int_{\lambda l} d\mathbf{w}=-\mathbf{r}+\int_{a_1-a_2} d\mathbf{w}+\int_{b_1-b_3} d\mathbf{w}=-\mathbf{r}+\begin{pmatrix}1\\-1\end{pmatrix}+\Pi\begin{pmatrix}2\\0\end{pmatrix}, \tag{5.10}$$

since $\lambda l = l - a_1 + a_2 - b_1 + b_3 + a_0$ (cf. Fig. 4), whence

$$\mathbf{w}_0=\begin{pmatrix}1/2\\1/2\end{pmatrix}+2\Pi\begin{pmatrix}1/2\\1/2\end{pmatrix}. \tag{5.11}$$

This quantity is the half-period of the theta-function $\theta(x|2\Pi)$, so in (5.9) the half theta-constant $\theta[\alpha,\beta](\mathbf{w}_0)$ vanishes, and thus we get formulas for the solution of the Neumann problem analogous to those found by Veselov (cf. [9]):

$$\begin{aligned} S_1&=i\theta[(\tfrac12,0),(\tfrac12,0)](\mathbf{z}_0+2\mathbf{U}x)\theta[(0,0),(\tfrac12,\tfrac12)]/\theta(\mathbf{z}_0+2\mathbf{U}x)\theta[(\tfrac12,0),(0,\tfrac12)],\\ S_2&=\theta[(0,\tfrac12),(\tfrac12,0)](\mathbf{z}_0+2\mathbf{U}x)\theta[(\tfrac12,\tfrac12),(\tfrac12,\tfrac12)]/\theta(\mathbf{z}_0+2\mathbf{U}x)\theta[(\tfrac12,0),(0,\tfrac12)],\\ S_3&=-\theta[(\tfrac12,0),(0,0)](\mathbf{z}_0+2\mathbf{U}x)\theta[(0,0),(0,\tfrac12)]/\theta(\mathbf{z}_0+2\mathbf{U}x)\theta[(\tfrac12,0),(0,\tfrac12)].\end{aligned} \tag{5.12}$$

Here $\theta[\alpha,\beta]=\theta[\alpha,\beta](0|2\Pi)$.

And finally, we give a standing wave solution describing the process of interaction of two identical conoidal waves of genus $g=1$ (5.2), moving towards one another with different rates. Such a solution corresponds to a surface Γ with zones $[p,q]$, $[p+2iK', q+2iK']$, $[-p,-q]$, $[-p+2iK', -q+2iK']$ having an involution $\lambda u=-u$. We calculate the Prym matrix Π of this surface. In the basis of cycles A_i, B_i indicated in Fig. 5, the involution acts according to the rule

$$A_1=-\lambda A_1,\quad A_2=\lambda A_2,\ B_1=-\lambda B_1,\quad B_2=\lambda B_2. \tag{5.13}$$

In deriving (5.13), we formally took into consideration the equations $B_i=-B_{g+i}$, $A_i=-A_{g+i}$, which hold for the periods of all Prym differentials. From (5.13) it follows that the matrix†

$$\Pi=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \tag{5.14}$$

†For comparison we note that by the Wirtinger–Martens theorem the period matrix of an arbitrary Riemann surface Γ cannot be a block matrix.

is diagonal and the special structure of the vectors **U**, **V**, **r**

$$\mathbf{U}=\begin{pmatrix}u\\0\end{pmatrix},\quad \mathbf{V}=\begin{pmatrix}0\\v\end{pmatrix},\quad \mathbf{r}=\begin{pmatrix}\alpha+1/2\\s\end{pmatrix}, \tag{5.15}$$

since $\lambda^*d\Omega_1=-d\Omega_1$, $\lambda^*d\Omega_2=d\Omega_2$, $\lambda^*dw_1=-dw_1$, $\lambda^*dw_2=dw_2$]

$$\mathbf{r}=\int_l d\mathbf{w}=\int_{\lambda l}\lambda^*d\mathbf{w}=\int_{l-A_1-2B_1}d\mathbf{w}.$$

For diagonal Π the two-dimensional theta-function splits into a simple product of one-dimensional theta-functions. Substituting (5.14)-(5.15) into (3.10), we get the following solution of the L–L equation [cf. (1.9)]:

$$\begin{aligned} A&=\theta(ux/2\pi+D_1\mid\alpha)\,\theta(vt/2\pi+D_2\mid\beta),\\ B&=\theta(ux/2\pi+D_1+1/2\mid\alpha)\,\theta(vt/2\pi+D_2+s\mid\beta),\\ C&=\theta(ux/2\pi+D_1+1/2\mid\alpha)\,\theta(vt/2\pi+D_2+1/2\mid\beta),\\ D&=-\theta(ux/2\pi+D_1\mid\alpha)\,\theta(vt/2\pi+D_2+s+1/2\mid\beta). \end{aligned} \tag{5.16}$$

Such solutions were first considered by Bogdan [25].

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HIGHER-DIMENSIONAL ANALOGS OF THE THEOREMS OF NEWTON AND IVORY

A. D. Vainshtein and B. Z. Shapiro

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Well-known theorems of Newton and Ivory [1, 2] assert that the potential of a charged metallic ellipsoid equals a constant in the interior of the ellipsoid and is constant on the confocal ambient ellipsoids. In this paper we prove the analogs of these theorems for hyperboloids of arbitrary signature in Euclidean space of arbitrary dimension.

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1. Three-Dimensional Case. First, we state the result for the case of a one-sheeted hyperboloid in three-dimensional space. To this end, we include the given hyperboloid in a family of confocal surfaces. This family traces a net of orthogonal lines on the surface of the hyperboloid. The closed (open) lines of this net will be referred to as the parallels (respectively, meridians) of the ellipsoid. The family of meridians extends to a fibering of the simply connected domain bounded by the hyperboloid into open curves; these will be referred to as the meridians of the inner domain. Similarly, the family of parallels extends to a fibering of the outer, nonsimply connected domain bounded by the hyperboloid on closed curves, termed the parallels of the outer domain.

THEOREM 1. There exists a unique (modulo a constant factor) surface current, flowing along the meridians of the hyperboloid, which produces a magnetic field that vanishes in the inner domain and is directed along parallels in the outer domain of the hyperboloid. Similarly, there exists a unique (modulo a constant factor) surface current, flowing along the parallels of the hyperboloid, which produces a magnetic field that vanishes in the outer domain and is directed along meridians in the inner domain of the hyperboloid.

The magnetic field in the inner domain, outside of a charged conducting ellipsoid confocal to the given hyperboloid, coincides, up to its sign, with the electric field of the ellipsoid. Also, the magnetic field in the region of the outer domain between the sheets of a confocal two-sheeted hyperboloid coincides, up to its sign, to the electric field produced by two charges equal in magnitude, distributed on the sheets of the conducting two-sheeted hyperboloid.

The fields constructed in Theorem 1 also yield exact solutions of the problems of potential flow of an incompressible fluid through the inner domain of a triaxial hyperboloid and of the vortex-free flow around this hyperboloid.

Theorem 1 was communicated to the authors by Arnold (see [3, 4]), who asked whether this theorem may be generalized to higher dimensions and signatures. The theorems of Newton and

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