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SOLUTIONS OF NONLINEAR EQUATIONS INTEGRABLE IN JACOBI THETA FUNCTIONS BY THE METHOD OF THE INVERSE PROBLEM, AND SYMMETRIES OF ALGEBRAIC CURVES

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ABSTRACT. A new approach is given for extracting from general formulas of finite-zone integration solutions of genus $g \geqslant 2$ expressible in terms of one-dimensional theta functions. As an application general formulas fo the type of the Lamb Ansatz for genus $g = 3$ are found for the sine-Gordon, nonlinear Schrödinger and Koretweg-de Vries equations, and the period matrices of some hyperelliptic curves are computed explicitly.

Bibliography: 35 titles.

§1. Introduction

In 1974 in the development of the method of the inverse problem for integrating nonlinear equations there arise a new direction—the theory of finite-zone solutions of equations of Korteweg-de Vries type. The main stages in the development of this direction, beginning with the pioneering work of S. P. Novikov [1], can be traced in the survey papers [2], [3], [5], [9] and [19].

The question of the superposition principle is one of the most interesting questions in the theory of finite-zone solutions and has so far not been completely investigated. The initial formulation of this question, due to Novikov [1], conjectures the possibility of representing any finite-zone solution of genus $g > 0$ by a suitable superposition of single-zone solutions (cnoidal waves). A satisfactory solution of the question of the superposition principle in this formulation has so far not been found.

In 1974 Dubrovin and Novikov [13] found a special two-zone solution $u(x,t)$ of the KdV equations with initial condition $u(x,0) = 6\wp(x)$,

$$u(x,t) = \sum_{i=1}^{3} 2\wp(x - x_i(t)),$$

in which the dependence of the functions x_i on t (which is rather involved) was found by direct substitution of an appropriate Ansatz into the KdV equation. In a certain sense this solution can be considered a "nonlinear superposition" of three cnoidal waves

$$2\wp(x - v_i t), \quad i = 1, 2, 3,$$

which are single-zone solutions of the KdV equation.

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Soon after the paper [1] was written A. R. It·s and one of the authors [4], [32] found an expression for finite-zone solutions of genus g of general position in terms of the g-dimensional Riemann theta function (see formula (7) below) of a hyperelliptic curve whose branch points are the edges of the zonal spectrum of the associated Schrödinger operator. Such solutions were then also found for the NS and sine-Gordon equations; the Toda lattice, and a number of others (in this regard see §2 and the survey papers cited above).

At present it has become clear that the construction of special classes of finite-zone solutions expressed in terms of elliptic functions is most conveniently carried out by means of an appropriate reduction of general formulas of the type (7).

Several methods of constructing such reductions have now been developed. The first of these was proposed by E. D. Belokolos and was developed by him and especially by V. Z. Ènol′skiĭ in a series of papers ([8], [10]–[12]). The authors' note [25] and the present paper are devoted to another method.

In [10] the idea was expressed of obtaining elliptic solutions from general finite-zone solutions by means of a reduction of Abelian integrals to elliptic integrals. The reduction of Abelian integrals to elliptic integrals was actively studied in the second half of the last century by Weierstrass, Jacobi, Hermite, and others. In [10]–[12] a corresponding procedure was effectively applied to extract the Lamb Ansatz and the simplest of its generalizations—special two-zone, periodic, elliptic solutions of the sine-Gordon equation—from a general formula for two-zone solutions of this equation. Belokolos and Ènol′skiĭ also indicated a general program for constructing such formulas in the theory of finite-zone solutions which is connected with the reduction of Abelian integrals. Ènol′skiĭ developed this program in application to the KdV equation with initial condition $6\mathcal{P}(x)$ and extracted from it still another representation of the Novikov-Dubrovin solution in terms of one-dimensional theta functions whose arguments depend linearly on x and t, and he also constructed new classes of solutions related to the Lamb Ansatz. Formulas which are interesting but not effective from the above viewpoint, describing the behavior of elliptic solutions of the KdV and Kadomtsev-Petviashvili equations in terms of the trajectories of the finite-dimensional dynamical systems of Moser-Calogero, were found in [7] and [16].

The program of [12] for extracting effective formulas in terms of one-dimensional theta functions from general solutions of higher genera is rather laborious and has so far not been developed for curves of genus $g > 2$. In connection with this the authors proposed an approach less general in principle but easily applicable to a number of situations pertaining to higher genera, which is based on the anaysis of algebraic curves with nontrivial groups G of birational automorphisms. In recent years such curves have been the subject of intense study (see [20] and [27]–[30]), containing the remarkable investigations of Klein, Hurwitz, Poincaré and a number of other mathematicians. In particular, a complete classification has now been obtained of curves with nontrivial automorphisms of genus $g \leqslant 3$, and there are also many interesting results for curves of higher genera (see [27] and the literature cited there). The basis of our approach is the study of a representation of the group G of a curve of genus g in a suitable basis of the homology group and a corresponding basis of the normalized Abelian differentials. This study makes it possible to reveal the symmetry of the matrix B and of the windings of the Jacobian corresponding to the appropriate nonlinear dynamics. Further, the required results are easily extracted from a theorem of Appell [14], a proof of which is presented below, on the reduction of a multidimensional theta function to one-dimensional theta functions. As applications we have considered only the simplest physically interesting examples pertaining to the NS, KdV, and sine-Gordon equations and containing, in particular, a generalization of the Lamb Ansatz to curves of genus 3.(1) Some results not contained in this paper and pertaining to curves of genus 5 and the simplest nonhyperelliptic curves can be found in the note [25], although applications to nonlinear equations were not considered there due to a lack of space. The recent two-part survey [34], [35] is also devoted to the application to nonlinear equations of the general theory of reduction of theta functions.

It is a pleasure for us to thank E. D. Belokolos and V. Z. Ènol'skiĭ for discussion of many of the questions connected with the theme of this paper.

§2. Finite-zone solutions of the nonlinear Schrödinger, Korteweg-de Vries, and sine-Gordon equations

In this section we recall the main concepts and objects of "finite-zone integration", a detailed description of which is contained in the surveys [2], [3], [9], and [19], and we present the final formulas for finite-zone solutions of the nonlinear Schrödinger equation (the case of repulsion)

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0, \tag{1}$$

the KdV equation

$$u_t = 6uu_x - u_{xxx} \tag{2}$$

and the sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u. \tag{3}$$

Let Γ be an arbitrary hyperelliptic surface of genus g given by the equation

$$\omega^2 = \prod_{i=1}^{2g+2} (z - z_i) = P_{2g+2}(z), \tag{4}$$

where the quantities z_i are pairwise distinct. On Γ it is always possible to choose (in a nonunique manner) a canonical basis of oriented cycles $a_1, \dots, a_g, b_1, \dots, b_g$ with matrix of intersection indices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. A basis of holomorphic differentials dU_j

$$dU_j = \sum_{k=1}^{g} c_{jk}\, dw_k, \quad dw_k = \frac{z^{k-1}\, dz}{\sqrt{P_{2g+2}(z)}}, \qquad k = 1, \dots, g,$$

on Γ is said to be normalized in the basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ if

$$\oint_{a_k} dU_j = \delta_{kj}.$$

The period matrix of the Riemann surface Γ (below we call it the *B-matrix*) is here defined by the formula

$$B_{kj} = \oint_{b_k} dU_j, \qquad j, k = 1, \dots, g.$$

It is known that this is a symmetric matrix with positive-definite imaginary part.

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(1) Periodic finite-zone solutions of genus 3 of the sine-Gordon equation are singled out in [33] by means of the technique developed in this paper.

The most important object in the subsequent exposition—the Riemann theta function with characteristics $[\alpha, \beta]$ ($\alpha, \beta \in \mathbf{R}^g$)—is constructed on the basis of the B-matrix

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\mathbf{x}|B) = \sum_{\mathbf{m}\in\mathbf{Z}^g} \exp\{\pi i\langle B(\mathbf{m}+\alpha), \mathbf{m}+\alpha\rangle + 2\pi i\langle \mathbf{m}+\alpha, \mathbf{x}+\beta\rangle\}, \quad (5)$$

where $\mathbf{x} = (x_1, \dots, x_g) \in \mathbf{C}^g$, the summation goes over the integral g-dimensional lattice, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product. For brevity we write $\theta(\mathbf{x}|B) \equiv \theta[{}^0_0](\mathbf{x}|B)$.

On the surface Γ we define the normalized Abelian integrals of second kind $\Omega_1(P)$ and $\Omega_2(P)$, $P \in \Gamma$, by the following conditions:

a) The Ω_i have zero a-periods.

b) The Ω_i have singularities only at the points $\infty^{\pm}$ (these are two points on Γ corresponding to $z \to \infty$, $\omega = \pm\sqrt{P_{2g+2}(z)}$) and the asymptotics

$$\Omega_1 \to \pm(z - E/2 + O(z^{-1})), \qquad z \to \infty^{\pm},$$
$$\Omega_2 \to \mp(2z^2 + N/2 + O(z^{-1})), \qquad z \to \infty^{\pm}.$$

We denote the vectors of b-periods of the integrals Ω_1 and Ω_2 by $\mathbf{V} = (V_1, \dots, V_g)$ and $\mathbf{W} = (W_1, \dots, W_g)$ respectively:

$$V_i = \oint_{b_i} d\Omega_1, \qquad W_i = \oint_{b_i} d\Omega_2.$$

If now all branch points on Γ are real,(²) then the solution of the NS equation (1) is given by

$$\psi(x,t) = S\frac{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \mathbf{D} - \mathbf{C}|B)}{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \mathbf{D}|B)} e^{iEx+iNt}, \quad (6)$$

where $\mathbf{C} = (C_1, \dots, C_g)$, $C_i = 2\int_{\infty^-}^{\infty^+} dU_i$, $2\,\mathrm{Re}\,\mathbf{D} \in \mathbf{Z}^g$, S is a constant determined from the asymptotics of an Abelian integral of third kind [3], [26], and the quantities E and N were defined above.

Solutions of the KdV equation are constructed on the basis of the vectors of b-periods $\mathbf{V}$ and $\mathbf{W}$ of the Abelian integrals Ω_1 and Ω_2 of second kind having a single pole at a branch point according to the formula (see [4] and [32])

$$u(x,t) = -2(\partial^2/\partial x^2)\ln\theta(\mathbf{V}x + \mathbf{W}t + \mathbf{D}|B) + \text{const}. \quad (7)$$

A different situation corresponds to the case of the sine-Gordon equation (3). Usually finite-zone solutions of the sine-Gordon equation are constructed on the basis of a hyperelliptic Riemann surface with a single fixed cut from 0 to ∞ (see [6], [3] and [15]). Below we shall only consider Riemann surfaces with finite cuts, and we therefore here formulate a procedure conformally equivalent to that described in [3] and [6] for constructing solutions parametrized by surfaces of the form (4). In order to see the validity of this reformulation, it is necessary to consider a conformal mapping under which the points ∞ and 0 go over into the points z_1 and z_2 respectively (see below).

(²) In This paper we restrict ourselves to the case of hyperelliptic curves with real branch points (although this restriction is obviously not essential for all subsequent constructions), since only such curves correspond to real solutions of the KdV equation, and we wish to construct physically interesting solutions in terms of Jacobi theta functions simultaneously for all three equations (1)–(3).

We define normalized Abelian integrals of second kind Ω_1 and Ω_2 by the following conditions:

a) They have zero a-periods.

b) They have singularities only at the points z_1 and z_2 of the following form:([3])

$$\begin{aligned} &\Omega_j \to p(z) + \cdots, \quad z \to z_1, \\ &\Omega_1 \to \tfrac{1}{16}(-1)^j p^{-1}(z) + \cdots, \quad z \to z_2, \qquad j = 1,2, \\ &p(z) = \tfrac{1}{4}\sqrt{(z - z_2)/(z - z_1)}. \end{aligned} \tag{8}$$

As in the case of the NS and KdV equations, we denote their vectors of b-periods by $\mathbf{V}$ and $\mathbf{W}$ respectively. We choose a basis of cyles on Γ so that the cycle $\mathscr{C}$ going from z_1 to z_2 on the upper sheet and then from z_2 to z_1 on the lower sheet is homologous to a combination of a-cycles, and we denote the integral over this cycle by

$$2\Delta_j = \oint_{\mathscr{C}} dU_j = M_j, \quad \mathbf{M} \in \mathbf{Z}^g, \quad \mathscr{C} = \sum_{i=1}^{g} M_i a_i \tag{9}$$

(the last equality holds in the homology group $H_1(\Gamma, Z)$). The function

$$u(x,t) = \frac{2}{i} \ln \frac{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \zeta + \Delta)}{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \zeta)} \tag{10}$$

is then a solution of the sine-Gordon equation [3], [6], [15]. We note that the solution (10) can be written in terms of theta functions with characteristics

$$u(x,t) = \frac{2}{i} \ln \frac{\theta\begin{bmatrix} 0 \\ M/2 \end{bmatrix}(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \zeta)}{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \zeta)}.$$

§3. Period matrices of algebraic curves possessing a nontrivial group of automorphisms

Suppose on a curve Γ with a fixed canonical basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ there acts an automorphism τ which transforms the cycles as follows:

$$a_i = \sum_{k=1}^{g} Q_{ik} \tau a_k, \qquad b_i = \sum_{k=1}^{g} T_{ik} \tau b_k, \tag{11}$$

where T and Q are integral matrices whose inverses are also integral. The basis of cycles τa_i, τb_i, $i = 1, \ldots, g$, is, in turn, also canonical, and the connection between the bases a, b and τa, τb is therefore given by a symplectic matrix (see, for example, [9]); the symplectic condition in the present case reduces to the equality

$$Q = (T')^{-1}, \tag{12}$$

where T' denotes the transpose of the matrix T.

We now suppose that the holomorphic differentials $dU_j(P)$, $j = 1, \ldots, g$, are normalized in the basis a, b, and their period matrix is equal to B. The differentials $\tau^* dU_j(P) = dU_j(\tau P)$ are then obviously normalized in the basis τa, τb with the same period matrix.

([3]) To a different choice of local parameters in neighborhoods of the points z_1 and z_2 there correpond solutions of the sine-Gordon equation differing only by a Lorentz transformation. With this special choice of the function $p(z)$ the interesting effect of separation of the dynamics in x and t is observed (see §6).

Using (11) and (12), it is easy to find the a- and b-periods of $\tau^*dU_j(P)$ in the original basis a, b:

$$(T')^{-1}, \qquad TB$$

(the first matrix is the matrix of a-periods of the differentials $\tau^*dU_j(P)$ whose rows are indexed by cycles and whose columns are indexed by differentials; the second matrix is the matrix of B-periods). The Abelian differentials

$$d\hat{U}_j = \sum_{k=1}^{g} T_{jk}\tau^*dU_k$$

are normalized in the basis a, b with period matrix equal to TBT', whence from the uniqueness of the choice of normalized holomorphic differentials it follows that $d\hat{U}_j$, $j = 1, \ldots, g$, and

$$B = TBT'. \tag{13}$$

Equality (13) gives conditions on the form of the B-matrix of a curve possessing a nontrivial automorphism. If on the curve there acts a group of automorphisms each of which possesses property (11) with generators $\tau_1, \ldots, \tau_N$ whose representation in the basis of the homology group is given by the generating matrices $T_1, \ldots, T_N$, then the B-matrix of such a Riemann surface must satisfy the system of equalities

$$B = T_j B T_j', \qquad j = 1, \ldots, N. \tag{14}$$

In this section we considered only the special case of automorphisms posessing property (11) from which there followed the linear relations (13) and (14) for the B-matrix. Examples of curves with automorphisms of a completely different type are presented in the Appendix. It turns out that for these curves, proceeding only from similar symmetry considerations, one can obtain the numerical value of the B-matrices. For some nonhyperellipc curves the numerical values of the B-matrices are given in [25].

§4. Appell's theorem

There are several methods of representing theta functions (5) with a special form of the B-matrix in terms of theta functions of lower dimension (see, for example, [14] and [17]). The method of proof of these decompositions is the same in all cases: using the special form of the B-matrix, it is possible to replace summation over the lattice $\mathbf{Z}^g$ in (5) by summation over a lattice of smaller dimension. In this section we present a theorem of Appell [14], which describes what is in our view one of the most convenient methods of reducing a g-dimensional theta function (a priori in no way associated with a Riemann surface but constructed on the basis of an arbitrary symmetric matrix with positive-definite imaginary part) to a $(g-1)$-dimensional theta function (5). We first present a theorem on the reduction of a theta function with zero characteristics.

Suppose the last column of the matrix B satisfies the relations

$$\begin{aligned} &n_j B_{jg} = q_j, \qquad j = 1, \ldots, \nu, \\ &n_j B_{jg} = n_g B_{gg} + q_j, \qquad j = \nu + 1, \ldots, g-1. \end{aligned} \tag{15}$$

Here the n_k and q_j are integers. It is always possible to choose $n_g > 0$ and $n_j > 0$, $j = 1, \dots, \nu$. Suppose now that $n_k < 0$ for $k > \nu$. We carry out the following transformations: we change the signs of x_k (the argument of the theta function $\theta(\mathbf{x}|B)$), m_k (the summation index in (16)), B_{kj} and B_{jk}, $j = 1, \dots, g$, $j \neq k$. Here the theta function

$$\theta(\mathbf{x}|B) = \sum_{\mathbf{n} \in \mathbf{Z}^g} \exp\{\pi i \langle B\mathbf{m}, \mathbf{m} \rangle + 2\pi i \langle \mathbf{x}, \mathbf{m} \rangle\} \tag{16}$$

does not change, while n_k in (15) changes sign and becomes positive. Thus, we may always assume below that in (15) the quantities n_k are already natural numbers, i.e., the change described above has already been carried out. Moreover, we choose $n_k \in \mathbf{N}$ to be the smallest possible for which the equalities (15) are satisfied; in particular, if $B_{kg} = 0$, then we assume that $n_k = 1$.

THEOREM 1 (APPELL).

$$\theta_g(\mathbf{x}|B) = \sum_{\mathbf{t} \in \mathbf{Z}^g(\hat{n})} e^{\alpha} \theta_{g-1}(\mathbf{p}|A) \theta_1\left(\alpha_g | B_{gg} n_g^2\right),$$

where summation over $\mathbf{t} \in \mathbf{Z}^g(\hat{n})$ *denotes the finite sum of* $\mathbf{t}$; $0 \leqslant t_\rho \leqslant n_\rho - 1$, $\rho = 1, \dots, g$; θ_g, θ_{g-1} *and* θ_1 *are, respectively, the g-dimensional,* $(g-1)$*-dimensional, and one-dimensional theta functions* (16), *and the parameters are given by the following expressions*:

$$\begin{gathered}
\alpha = 2\pi i \langle \mathbf{t}, \mathbf{x} \rangle + \pi i \langle B\mathbf{t}, \mathbf{t} \rangle, \\
\mathbf{p} = \left(\alpha_1, \dots, \alpha_\nu, \alpha_{\nu+1}, \dots, \alpha_{g-1} - \alpha_g\right), \\
\alpha_\rho = n_\rho x_\rho + \tfrac{1}{2} n_\rho \partial \langle B\mathbf{t}, \mathbf{t} \rangle / \partial t_\rho, \qquad \rho = 1, \dots, g, \\
A_{ii} = n_i^2 B_{ii}, \quad i \leqslant \nu; \qquad A_{ii} = n_i^2 B_{ii} - n_g^2 B_{gg}, \quad i > \nu, \\
A_{ij} = n_i n_j B_{ij} \quad \text{if } i \text{ or } j \leqslant \nu, \\
A_{ij} = n_i n_j b_{ij} - n_g^2 B_{gg}, \qquad i, j > \nu.
\end{gathered} \tag{17}$$

This theorem was proved by Appell [14]. Below we shall use the special case of this theorem in which $q_j = 0$, $j = 1, \dots, g$.

THEOREM 1a. *Suppose the last column of the B-matrix satisfies* (15) *and* $q_j = 0$, $j = 1, \dots, g$. *Then*

$$\theta_g(\mathbf{x}|B) = \sum_{\mathbf{t} \in \mathbf{Z}^g(\hat{n})} \theta_{g-1}\begin{bmatrix} \boldsymbol{\gamma} \\ 0 \end{bmatrix}(\mathbf{f}|A) \theta_1 \begin{bmatrix} \delta \\ 0 \end{bmatrix}\left(n_g x_g | n_g^2 B_{gg}\right), \tag{18}$$

where $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{g-1})$, $\gamma_i = t_i / n_i$, $i = 1, \dots, g-1$, $\delta = \Sigma_{\nu+1}^g (t_i / n_i)$, *the matrix A is given, as before, by* (17), *and*

$$\begin{aligned}
f_j &= n_j x_j, & j &= 1, \dots, \nu, \\
f_j &= n_j x_j - n_g x_g, & j &= \nu + 1, \dots, g-1.
\end{aligned} \tag{19}$$

PROOF OF THEOREM 1a. We shall describe the change of the summation lattice in the series (16). We set $m_\rho = n_\rho \mu_\rho + t_\rho$, $\rho = 1, \dots, g$, $0 \leqslant t_\rho \leqslant n_\rho - 1$ (μ_ρ and t_ρ are the integers obtained on dividing m_ρ by n_ρ) or $\mathbf{m} = \hat{n}\boldsymbol{\mu} + \mathbf{t}$, where $\hat{n} = \operatorname{diag}(n_1, \dots, n_g)$. Here the vector $\mathbf{m}$ runs through the entire lattice $\mathbf{Z}^g$ if μ runs through $\mathbf{Z}^g$, while $\mathbf{t} \in \mathbf{Z}^g(\hat{n}) = \{\mathbf{t}: 0 \leqslant t_\rho \leqslant n_\rho - 1\}$. We have

$$\begin{aligned}\theta_g(\mathbf{x}|B) &= \sum_{\mathbf{m}\in\mathbf{Z}^g} \exp\left\{\pi i \sum_{i,j=1}^{g} B_{ij}m_im_j + 2\pi i \sum_{j=1}^{g} x_jm_j\right\} \\ &= \sum_{\mathbf{t}\in\mathbf{Z}^g(\hat{n})}\sum_{\mu\in\mathbf{Z}^g} \exp\left\{\pi i \sum_{i,j=1}^{g} B_{ij}n_in_j\left(\mu_i + \frac{t_i}{n_i}\right)\left(\mu_j + \frac{t_j}{n_j}\right)\right. \\ &\qquad\left. + 2\pi i \sum_{j=1}^{g} x_jn_j\left(\mu_j + \frac{t_j}{n_j}\right)\right\} \\ &= \sum_{\mathbf{t}}\sum_{\mu} \exp\left\{\pi i\left[B_{gg}n_g^2\left(\mu_g + \frac{t_g}{n_g}\right)^2 + 2\sum_{i=1}^{g-1} B_{ig}n_in_g\left(\mu_i + \frac{t_i}{n_i}\right)\left(\mu_g + \frac{t_g}{n_g}\right)\right.\right. \\ &\qquad\left. + \sum_{i,j=1}^{g-1} B_{ij}n_in_j\left(\mu_i + \frac{t_i}{n_i}\right)\left(\mu_j + \frac{t_j}{n_j}\right)\right] \\ &\qquad + 2\pi i\left[n_gx_g\sum_{j=\nu+1}^{g}\left(\mu_j + \frac{t_j}{n_j}\right) + \sum_{j=1}^{\nu} n_jx_j\left(\mu_j + \frac{t_j}{n_j}\right)\right. \\ &\qquad\left.\left. + \sum_{j=\nu+1}^{g-1}(n_jx_j - n_gx_g)\left(\mu_j + \frac{t_j}{n_j}\right)\right]\right\}.\end{aligned}$$

Using (15) for $q_j = 0$ and the definitions of the matrix A of (17) and the vector f of (19), we obtain

$$\begin{aligned}\theta_g(\mathbf{x}|B) = \sum_{\mathbf{t}}\sum_{\mu}\exp&\left\{\pi i\left[B_{gg}n_g^2\left(\sum_{j=\nu+1}^{g}\mu_j + \sum_{j=\nu+1}^{g}\left(\frac{t_j}{n_j}\right)\right)^2\right.\right. \\ &\left. + \sum_{i,j=1}^{g-1} A_{ij}\left(\mu_j + \frac{t_i}{n_i}\right)\left(\mu_j + \frac{t_j}{n_j}\right)\right] \\ &\left. + 2\pi i\left[n_gx_g\left(\sum_{j=\nu+1}^{g}\mu_j + \sum_{j=\nu+1}^{g}\left(\frac{t_j}{n_j}\right)\right) + \sum_{j=1}^{g-1} f_j\left(\mu_j + \frac{t_j}{n_j}\right)\right]\right\}.\end{aligned}$$

Passing now from summation on $\mu = (\mu_1, \ldots, \mu_g) \in \mathbf{Z}^g$ to summation on $\mu' = (\mu_1, \ldots, \mu_{g-1}, \sum_{\nu+1}^{g}\mu_j)$, we immediately obtain the assertion of Theorem 1a:

$$\begin{aligned}\theta_g(\mathbf{x}|B) &= \sum_{\mathbf{t}}\sum_{\mu'\in\mathbf{Z}^g}\exp\left\{\pi i\left[B_{gg}n_g^2(\mu'_g + \delta)^2 + \sum_{i,j=1}^{g-1} A_{ij}(\mu'_i + \gamma_i)(\mu'_j + \gamma_j)\right.\right. \\ &\qquad\left.\left. + 2\pi i\left[n_gx_g(\mu'_g + \delta) + \sum_{j=1}^{g-1} f_j(\mu_j + \gamma_j)\right]\right]\right\} \\ &= \sum_{\mathbf{t}} \theta_{g-1}\begin{bmatrix}\boldsymbol{\gamma}\\ 0\end{bmatrix}(\mathbf{f}|A)\,\theta_1\begin{bmatrix}\boldsymbol{\delta}\\ 0\end{bmatrix}(n_gx_g|n_g^2B_{gg}).\end{aligned}$$

COROLLARY. *Under the conditions of Theorem* 1a

$$\theta_g\begin{bmatrix}\alpha\\0\end{bmatrix}(\mathbf{x}|B) = \sum_{\mathbf{t}\in\mathbf{Z}^g(\hat{n})} \theta_{g-1}\begin{bmatrix}\varphi\\0\end{bmatrix}(\mathbf{f}|A)\theta_1\begin{bmatrix}\psi\\0\end{bmatrix}(n_g x_g|n_g^2 B_{gg})e^{-2\pi i(B\alpha,t)}, \tag{20}$$

where $\varphi_i = (t_i+\alpha_i)/n_i$, $\psi = \Sigma_{\nu+1}^g(t_j+\alpha_j)/n_j$, *and the quantities f and A are given by* (19) *and* (17).

§5. Solution of the NS equation in Jacobi theta functions. Genus $g = 2, 3$

We shall treat in detail the solution of the NS equation corresponding to the curve

$$\omega^2 = (z^2 - v^2)(z^2-u^2)(z^2-s^2) \tag{21}$$

with real u, v and s. This curve, which possesses the automorphism $\tau(z,\omega) = (-z,\omega)$, together with a canonical basis of cycles, is shown in Figure 1 (the solid straight lines indicate the cuts; in the picture of the cycles the solid line corresponds to a path on the upper sheet, while the dashed line corresponds to a path on the lower sheet). It will be convenient for us to use the basis of cycles shown in Figure 1.

The automorphism τ acts on these cycles in precisely the manner considered in §3, i.e., the cycles τa are expressed only in terms of the a-cycles shown in Figure 1, while the cycles τb are expressed only in terms of b-cycles. This imposes the condition (13) on the form of the B-matrix. In the present case

$$B = TBT^t, \qquad T = \begin{pmatrix}-1 & 1\\0 & 1\end{pmatrix} \Rightarrow B = \begin{pmatrix}\beta & \alpha\\ \alpha & 2\alpha\end{pmatrix}. \tag{22}$$

A B-matrix of this structure is covered by Theorem 1a (here $\nu = 0$, $n_1 = 2$ and $n_2 = 1$); applying it, we verify the validity of the decomposition

$$\theta(\mathbf{x}|B) = \theta(2x_1 - x_2|4\beta - 2\alpha)\theta(x_2|2\alpha) + \theta\begin{bmatrix}1/2\\0\end{bmatrix}(2x_1 - x_2|4\beta-2\alpha)\theta\begin{bmatrix}1/2\\0\end{bmatrix}(x_2|2\alpha). \tag{23}$$

The expression (6) for a solution of the NS equation contains the b-periods of the Abelian integrals of second kind Ω_1 and Ω_2; the symmetry of the curve also imposes conditions on them, since the points of singularities of these integrals are fixed under the automorphism τ.

We consider the Abelian integral $\Omega_1(z,\omega)$; it has zero a-periods, the vector of b-periods $\mathbf{V}$, and the singularities $\pm z$ at the points $\infty^\pm$. The integral $-\tau^*\Omega_1(z,\omega) = -\Omega_1(\tau(z,\omega)) = -\Omega_1(-z,\omega)$ also has zero a-periods and singularities $\pm z$ at the points $\infty^\pm$, while its b-periods in the basis τb are equal to $-\mathbf{V}$; in the original basis they are therefore equal to $-T\mathbf{V}$; hence, by the uniqueness of prescribing an Abelian integral of second kind with zero a-periods, on the basis of the form of the singularity, $\Omega_1(z,\omega) = -\Omega_1(-z,\omega)$, and we have

$$\mathbf{V} = -T\mathbf{V} \Rightarrow \mathbf{V} = \begin{pmatrix}v\\0\end{pmatrix}. \tag{24}$$

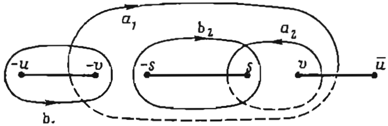

FIGURE 1

It is established in a similar way that

$$\mathbf{W} = T\mathbf{W} \Rightarrow \mathbf{W} = \begin{pmatrix} w \\ 2w \end{pmatrix}, \tag{25}$$

since the asymptotics of the integral Ω_2 under the automorphism τ remains unchanged. For the vector $\mathbf{C}$ (see (6)) we have

$$\mathbf{C} = 2\int_{\infty^-}^{\infty^+} d\mathbf{U} = 2\int_{\infty^-}^{\infty^+} \tau^* d\mathbf{U} = 2T^{-1}\int_{\infty^-}^{\infty^+} d\mathbf{U} = T^{-1}\mathbf{C} \Rightarrow \mathbf{C} = \begin{pmatrix} c \\ 2c \end{pmatrix}, \tag{26}$$

since the path from ∞^- to ∞^+ passing through the point 0 is invariant under the action of τ.

We now consider the antiholomorphic involution $\pi(z, \omega) = (\bar{z}, \bar{\omega})$. It is not hard to see that $\pi a_i = a_i$ and $\pi b_i = -b_i$, $i = 1, 2$. The Abelian integrals $U_j(z, \omega)$ are normalized in the basis a, b; therefore, $\overline{\pi^* U_j(z, \omega)} = \overline{U_j(\bar{z}, \bar{\omega})}$ are normalized in the basis πa, πb, and their period matrix in this basis is equal to $\bar{B}$; going back to a and b, we see that $B = -\bar{B}$, i.e., the B-matrix is pure imaginary. If we also use the fact that $\overline{\pi^*\Omega_k(z, \omega)} = \omega_k(z, \omega)$, then this implies that $\mathbf{V} = -\bar{\mathbf{V}}$ and $\mathbf{W} = -\bar{\mathbf{W}}$; it is established in a similar way that $\mathbf{C}$ is real.

Thus, from (16) and (23)–(26) we obtain *a solution of the NS equation in Jacobi theta functions*:

$$\begin{aligned}
\psi(x, t) = Se^{iEx+iNt}\{&\theta_3(z_1|\tau_1)\theta_3(z_2 - c|\tau_2) + \theta_2(z_1|\tau_1)\theta_2(z_2 - c|\tau_2)\}: \\
&\{\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2) + \theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)\}, \\
&z_1 = vx/\pi + d_1,\ z_2 = wt/\pi + d_2,\ \tau_1 = 4\beta - 2\alpha,\ \tau_2 = 2\alpha, \\
&S,\ N,\ E \in \mathbf{R};\ iv,\ iw,\ c \in \mathbf{R};\ 2\operatorname{Re} d_1, 2\operatorname{Re} d_2 \in \mathbf{Z}.
\end{aligned} \tag{27}$$

Here $\theta_3(z) = \theta[{}^0_0](z)$ and $\theta_2(z) = \theta[{}^{1/2}_0](z)$ are Jacobi theta functions.

We now consider the curve Γ_3 of genus 3

$$\omega^2 = \left(z^4 - (u^2 + u^{-2})z^2 + 1\right)\left(z^4 - (v^2 + v^{-2})z^2 + 1\right), \tag{28}$$

shown in Figure 2. It has the dihedral group of automorphisms with generators $\lambda(z, \omega) = (-z, \omega)$ and $\mu(z, \omega) = (z^{-1}, z^{-4}\omega)$. Under these automorphisms the canonical basis of cycles a'_i, b'_i, $i = 1, 2, 3$, transforms as follows:

$$b' = \Lambda\lambda b',\quad a' = (\Lambda'')^{-1}\lambda a',\quad b' = \mathrm{M}'\mu b',\quad a' = (\mathrm{M}'')^{-1}\mu a',$$

$$\Lambda' = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix},\qquad \mathrm{M}' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}. \tag{29}$$

Here $b' = (b'_1, b'_2, b'_3)$ denotes the vector of b-cycles shown in Figure 2, and $\mu b' = (\mu b'_1, \mu b'_2, \mu b'_3)$ denotes the vector of b-cyles obtained from b' under the action of the automorphism μ. Using the technique of §3, from this it is already possible to obtain conditions on the B-matrix of the curve Γ_3, but for subsequent application of Appell's theorem it will be convenient for us to go over to another canonical basis: $b = \Phi b'$,

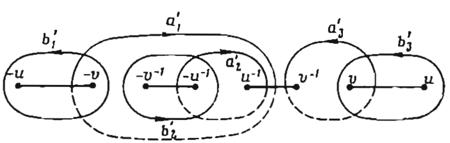

FIGURE 2

$a = (\Phi')^{-1}a'$, where

$$\Phi = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{30}$$

The picture of this basis looks more complex, but the B-matrix in it has an especially simple form.

This new basis a, b under the automorphisms λ and μ is transformed by matrices Λ and M which are obviously connected with Λ' and M' by the similarity transformation

$$\Lambda = \Phi\Lambda'\Phi^{-1}, \qquad M = \Phi M'\Phi^{-1};$$

$$b = \Lambda\lambda b, \quad b = M\mu b, \quad a = (\Lambda')^{-1}\lambda a, \quad a = (M')^{-1}\mu a,$$

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The conditions on the B-matrix computed in the basis a, b of (14) give it the following form:

$$B = \begin{pmatrix} 2\alpha & \alpha & 0 \\ \alpha & \gamma & \beta \\ 0 & \beta & 2\beta \end{pmatrix}. \tag{31}$$

We shall demonstrate the application of Theorem 1a to the theta function defined by the matrix (31). The last row satisfies the conditions of Theorem 1a ($\nu = 1$, $n_1 = 1$, $n_2 = 2$, $n_3 = 1$); therefore,

$$\theta(\mathbf{x}|B) = \theta(\mathbf{y}|A)\theta(x_3|2\beta) + \theta\begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix}(\mathbf{y}|A)\theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(x_3|2\beta), \tag{32}$$

where the first theta function in the product is two-dimensional, $\mathbf{y} = (y_1, y_2)$, $y_1 = x_1$, $y_2 = 2x_2 - x_3$, and the matrix A is given by

$$A = \begin{pmatrix} 2\alpha & 2\alpha \\ 2\alpha & 4\gamma - 2\beta \end{pmatrix}.$$

Formula (32) is not the final formula, since the two-dimensional theta function can be reduced to one-dimensional theta functions due to the fact that the first column of the matrix A satisfies Appell's theorem ($\nu = 0$, $n_1 = 1$, $n_2 = 1$) (in applying it, it is only necessary in the condition of the theorem to replace the last column by the first). Thus,

$$\begin{aligned} \theta(\mathbf{y}|A) &= \theta(y_1|2\alpha)\theta(y_2 - y_1|4\gamma - 2\beta - 2\alpha) \\ &= \theta(x_1|2\alpha)\theta(2x_2 - x_3 - x_1|4\gamma - 2\beta - 2\alpha). \end{aligned}$$

Applying Corollary 2, we obtain

$$\theta\begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix}(\mathbf{y}|A) = \theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(x_1|2\alpha)\theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(2x_2 - x_3 - x_1|4\gamma - 2\beta - 2\alpha),$$

since in (20) in this case the sum consists of one term corresponding to $t = (0, 0)$. We have finally

$$\begin{aligned} \theta(\mathbf{x}|B) = {} & \theta(x_1|2\alpha)\theta(x_3|2\beta)\theta(2x_2 - x_3 - x_1|4\gamma - 2\beta - 2\alpha) \\ & + \theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(x_1|2\alpha)\theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(x_3|2\beta) \\ & \times \theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(2x_2 - x_3 - x_1|4\gamma - 2\beta - 2\alpha). \end{aligned} \tag{33}$$

The points $\infty^{\pm}$—the points of the singularities of the Abelian integrals of second kind—are fixed under the automorphism λ; therefore, repeating word-for-word the arguments of the preceding section, we obtain the conditons on the vectors of b-periods $\mathbf{V} = -\Lambda\mathbf{V}$ and $\mathbf{W} = \Lambda\mathbf{W}$, whence

$$\mathbf{V} = \begin{pmatrix} v_1 \\ (v_1 + v_2)/2 \\ v_2 \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} 0 \\ w \\ 0 \end{pmatrix}. \tag{34}$$

It is similarly possible to find that $\mathbf{C} = (0, c, 0)$ up to the lattice of integers, which are inconsequential. Considering the anti-involution $\pi(z, \omega) = (\bar{z}, \bar{\omega})$, it is not hard to see that $\pi a_i = a_i$ and $\pi b_i = -b_i$, $i = 1, 2, 3$, the B-matrix is pure imaginary, and $\mathbf{V}$ and $\mathbf{W}$ are also imaginary.

The final expression for the solution of the NS equation obtained in Jacobi theta functions has the following form:

$$\begin{aligned}\psi(x, t) = Se^{iEx+iNt}\{ &\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2)\theta_3(z_3 - c|\tau_3) \\ &+\theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)\theta_2(z_3 - c|\tau_3)\} : \\ &\{\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2)\theta_3(z_3|\tau_3) + \theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)\theta_2(z_3|\tau_3)\},\end{aligned}$$

$$\begin{aligned}&z_1 = (v_1/2\pi)x + d_1,\ z_2 = (v_2/2\pi)x + d_2,\ z_3 = (w/\pi)t + d_3, \\ &\tau_1 = 2\alpha,\ \tau_2 = 2\beta,\ \tau_3 = 4\gamma - 2\beta - 2\alpha; \\ &S, E, N, iv_1, iv_2, iw, i\alpha, i\beta, i\gamma \in \mathbf{R}, \\ &2\operatorname{Re} d_1, 2\operatorname{Re} d_2, 2\operatorname{Re} d_3 \in \mathbf{Z}.\end{aligned} \tag{35}$$

§6. Elliptic solutions of the KdV and sine-Gordon equations

The proposed method of reducing general finite-zone solutions (corresponding to symmetric curves) to solutions in Jacobi theta functions was illustrated in §5 with the example of the NS equation. It obviously carries over to other nonlinear equations integrable by the method of the inverse problem, since finite-zone solutions of all these equations can be expressed in terms of Riemann theta functions. In this section we present the final expressions for solutions of the KdV and sine-Gordon equations corresponding to the curves Γ_2 and Γ_3 considered in §5.

The corresponding solutions of the KdV equation have the form

$$\begin{aligned}u(x, t) &= -2(\partial^2/\partial x^2)\ln\{\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2) + \theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)\} + \text{const}, \\ z_1 &= u_1 x + v_1 t + d_1,\ z_2 = u_2 x + v_2 t + d_2,\ \tau_1 = 4\beta - 2\alpha,\ \tau_2 = 2\alpha,\end{aligned} \tag{36}$$

for the curve Γ_2 of (21) of genus 2 and the form

$$\begin{aligned}u(x, t) = -2(\partial^2/\partial x^2)\ln\{&\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2)\theta_3(z_3|\tau_3) \\ &+\theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)\theta_2(z_3|\tau_3)\} + \text{const},\end{aligned}$$

$$\begin{aligned}&z_1 = u_1 x + v_1 t + d_1,\ z_2 = u_2 x + v_2 t + d_2,\ z_3 = u_3 x + v_3 t + d_3, \\ &\tau_1 = 2\alpha,\ \tau_2 = 2\beta,\ \tau_3 = 4\gamma - 2\beta - 2\alpha,\end{aligned} \tag{37}$$

for the curve Γ_3 of (28) of genus 3.

We shall consider the sine-Gordon equation in more detail. We take We take $z_1 = -v$ and $z_2 = v$ (see (9)); then $p(z) = \frac{1}{4}\sqrt{(z - v)/(z + v)}$. The cycle $\mathscr{C}$ joining the points z_1 and z_2 coincides with the cycle a_1 (see Figure 1). As a result we obtain the following solution of the sine-Gordon equation:

$$u(x,t)=\frac{2}{i}\ln\left\{\frac{\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2)-\theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)}{\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2)+\theta_2(z_1,\tau_1)\theta_2(z_2|\tau_2)}\right\},$$

$$z_1=vx/\pi+d_1,\ z_2=wt/\pi+d_2,\ \tau_1=4\beta-2\alpha,\ \tau_2=2\alpha,\ i\alpha,\ i\beta,\ iv,\ iw\in\mathbf{R}. \tag{38}$$

This solution follows from the Lamb substitution; it was studied by means of reduction of hyperelliptic integrals to elliptic integrals in [10] and [11].

In the case of the curve Γ_3 of (28) the situation is analogous. We take $z_1=-u^{-1}$ and $z_2=u^{-1}$; then $p(z)=\frac{1}{4}\sqrt{(z-u^{-1})/(z+u^{-1})}$. It is evident from Figure 2 that the cycle $\mathscr{C}$ coincides with the cycle $a_2'=a_1+a_2+a_3$ (the last equality follows from $a'=\Phi' a$, where Φ is given by (30)). A careful analysis of the automorphism $\lambda(z,\omega)=(-z,\omega)$, which does not change the sheets, shows that $p(-z)=p(\lambda z)=-p^{-1}(z)$. In analogy to (34), from this we obtain the following structure of the vectors $\mathbf{V}$ and $\mathbf{W}$:

$$\mathbf{V}=(0,v,0),\qquad \mathbf{W}=(w_1,(w_1+w_2)/2,w_2).$$

We use formula (33) describing the reduction of the corresponding theta function. The quantities x_i for the theta function in the numerator of (10) are

$$x_1=(w_1/2\pi)t+\tfrac{1}{2}+d_1,$$
$$x_2=((w_1+w_2)/4\pi)t+(v/2\pi)x+\tfrac{1}{2}+d_2,$$
$$x_3=(w_2/2\pi)t+\tfrac{1}{2}+d_3,$$

while for the theta function in the denominator

$$x_1=\frac{w_1}{2\pi}t+d_1,\quad x_2=\left(\frac{w_1+w_2}{4\pi}\right)t+\frac{v}{2\pi}x+d_2,\quad x_3=\frac{w_2}{2\pi}t+d_3.$$

Substituting these expressions into (33) and (10), we obtain a solution of the sine-Gordon equation in Jacobi theta functions:

$$\begin{aligned}u(x,t)=\frac{2}{i}\ln\Bigg\{&\left(\theta\begin{bmatrix}0\\1/2\end{bmatrix}(z_1|\tau_1)\theta\begin{bmatrix}0\\1/2\end{bmatrix}(z_2|\tau_2)\theta\begin{bmatrix}0\\0\end{bmatrix}(z_3|\tau_3)\right.\\&\left.+\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_1|\tau_1)\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_2|\tau_2)\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z_3|\tau_3)\right):\\&\times\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(z_1|\tau_1)\theta\begin{bmatrix}0\\0\end{bmatrix}(z_2|\tau_2)\theta\begin{bmatrix}0\\0\end{bmatrix}(z_3|\tau_3)\right.\\&\left.+\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z_1|\tau_1)\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z_2|\tau_2)\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z_3|\tau_3)\right)\Bigg\}\\=\frac{2}{i}\ln&\frac{\theta_4(z_1|\tau_1)\theta_4(z_2|\tau_2)\theta_3(z_3|\tau_3)-\theta_1(z_1|\tau_1)\theta_1(z_2|\tau_2)\theta_2(z_3|\tau_3)}{\theta_3(z_1|\tau_1)\theta_3(z_2|\tau_2)\theta_3(z_3|\tau_3)+\theta_2(z_1|\tau_1)\theta_2(z_2|\tau_2)\theta_2(z_3|\tau_3)},\end{aligned}$$

$$\begin{gathered}z_1=(w_1/2\pi)t+d_1,\ z_2=(w_2/2\pi)t+d_2,\\ z_3=(v/\pi)x+2d_2-d_1-d_3,\\ \tau_1=2\alpha,\ \tau_2=2\beta,\ \tau_3=4\gamma-2\beta-2\alpha;\\ iw_1,\ iw_2,\ iv,\ i\alpha,\ i\beta,\ i\gamma\in\mathbf{R},\end{gathered}\tag{39}$$

where α, β and γ in (38) and (39) are, as before, the elements of the B-matrices of (22) and (31). Formula (39) for the d satisfying the conditions $d_i + \bar{d}_i \equiv \frac{1}{2}$ (equality modulo the period lattice) describes real solutions of the sine-Gordon equation which are periodic in x with period $X = \pi\tau_3/v$. Such solutions are especially important for describing the Josephson effect.

In conclusion we note that although we have considered only curves with real branch points, using the results of [20] and [27], in a completely analogous way it is possible to obtain a complete classification of curves of genus $g = 3$ with the dihedral group of automorphisms which describe solutions of NS equations with attraction and the real solution of the sine-Gordon equation according to formulas (35) and (39).

§7. Concluding remarks

1. The technique proposed here for separating out from general finite-zone solutions the solutions expressible in terms of one-dimensional theta functions is of direct interest for applications in concrete physical problems. In particular, the real, periodic, smooth solutions of genus 3 of the sine-Gordon equation (38), (39) constructed above can be used in the theory of the Josephson effect (see [11]) and in the description of baroclinic wave packets in problems of atmospheric physics [18].

2. The analysis which we have demonstrated here with the simplest examples of curves with nontrivial symmetry can be carried over easily to the majority of curves considered in [20], [27] and [28]. However, it becomes inconvenient to trace the mapping of cycles under the automorphisms by using a many-sheeted realization of the Riemann surface. In complex situations it may possibly turn out to be simpler to use the automorphic version of the theory of finite-zone integration proposed by one of the authors in [21] and to proceed as did Poincaré in [22], which contains an analysis of the Klein curve (this analysis is somewhat technically incomplete from the point of view of our requirements).

3. We note that, in all the examples we have considered, the Abelian integrals admit reductions to elliptic integrals which will be described in a separate paper. Here all the constants contained in (27), (33)–(35), (38) and (39) can be expressed in terms of elliptic integrals.

4. The study of the Hamiltonian aspect of the reductions considered here is of considerable interest; indeed, the fact that the Jacobian of a curve decomposes in the direct product of tori of smaller dimension on each of which the windings giving the dynamics remain rectilinear seems extremely interesting. Moreover, a nontrivial effect is observed in the formulas presented above for solutions of the NS and sine-Gordon equations: the variables x and t are separated, i.e., the solution can be expressed in terms of a finite sum of products of one-dimensional theta functions whose arguments depend only on x or only on t. This means that the dynamics in x is linearized not only on the Jacobian $J(\Gamma)$ of the curve Γ but in the situation considered also on the Prym variety of the curve Γ relative to the automorphism τ, while the dynamics in t is linearized on the quotient $J(\Gamma)/\tau$. It is possible that a thorough analysis of this phenomenon will make it possible to effectively compute the Moser-Calogero trajectories with elliptic pair interaction.

Appendix

Here we consider the curve R_2 given by the formula $\omega^2 = z^6 - 1$ of genus 2, shown together with a selected canonical basis of cycles in Figure 3, which possesses the automorphism $\tau(z, \omega) = (\varepsilon z, \omega)$, $\varepsilon = \exp(\pi i/3)$.

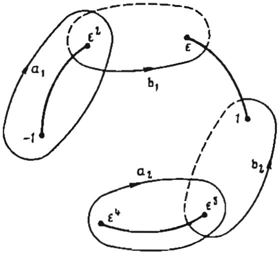

FIGURE 3

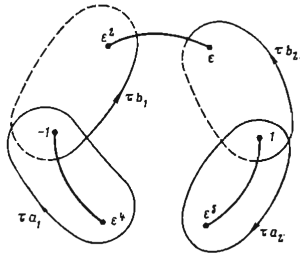

FIGURE 4

As before, we denote the cycles obtained from the original cycles under the action of τ by τa_i and τb_i. Under the mapping τ branch points go over into branch points, but cuts do not go over into cuts (Figure 4). Nevertheless, the curve thus obtained is equivalent to the original curve, and we can write out the relations we need which relate the "old" and "new" cycles. We shall indicate the simplest method of obtaining these relations. We make the following transformation of the curve shown in Figure 4: we "interchange" the upper and lower sheets of our hyperelliptic surface outside the unit circle. The cuts will then be located on the resulting curve in the same way as in Figure 3, and relations between the bases will then be especially graphic:

$$a_1 = -\tau b_1, \quad a_2 = \tau b_1 - \tau b_2, \quad b_1 = \tau a_1 + \tau a_2, \quad b_2 = \tau a_2.$$

Repeating verbatim the arguments of §4, we obtain the restriction on the B-matrix

$$B\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

from which, using the symmetry and the positive definiteness of the imaginary part, we obtain the numerical value of the B-matrix:

$$B = \frac{i}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The hyperelliptic curve B_3 of genus 3 (Figure 5) $\omega^2 = (z^8 - 1)$ provides still another example in which symmetry considerations make it possible to find numerical values of the B-matrix. This curve has the cyclic group of automorphisms with generator τ: $z \to \exp(\pi i/4)z$, $\omega \to \omega$, and the canonical basis shown in Figure 5.

The relations between the "old" and "new" cycles can be written in complete analogy to the preceding case, i.e., we make a rotation through an angle of $\pi/4$ and then "interchange" sheets outside the unit circle. We obtain the relations

$$b_1 = \tau a_1 + \tau a_2 + \tau a_3, \qquad a_1 = -\tau b_1 + \tau b_2 - \tau b_3,$$
$$b_2 = \tau a_2 + \tau a_1, \qquad a_2 = \tau b_1 - 2\tau b_2 + \tau b_3,$$
$$b_3 = -\tau a_1 + \tau a_3, \qquad a_3 = \tau b_2 - \tau b_3,$$

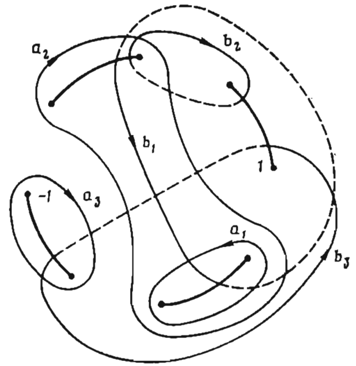

FIGURE 5

which lead to a nonlinear condition on the B-matrix computed in this basis:

$$B\begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

In obtaining a numerical expression for the B-matrix it is additionally necessary to use the conditions of symmetry and positive definiteness of the imaginary part. As a result we obtain the final value of the B-matrix:

$$B = \begin{pmatrix} 2\alpha & \alpha & 0 \\ \alpha & \gamma & \alpha \\ 0 & \alpha & 2\alpha \end{pmatrix}, \qquad \alpha = \frac{i}{\sqrt{2}}, \gamma = \frac{i}{\sqrt{2}} + \frac{i}{2}. \tag{40}$$

The B-matrices of the curves R_2 and R_3 considered in this appendix have the same structure as the B-matrices of the curves Γ_2 and Γ_3 of (22) and (31) studied in the main text; therefore the reductions of the Riemann theta functions are described by the same formulas.

We note finally that on the basis of the curves R_2 and R_3 formulas (27) and (35) define solutions of the NS equation with attraction in Jacobi theta functions and also real periodic solutions of the sine-Gordon equation in terms of Jacobi theta functions (38), (39). The latter possibility is especially interesting, since here, using the results of [15],([4]) it is possible to express all the undetermined constants in (39) in terms of elliptic theta constants and thus obtain formulas suitable for study on a computer.

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([4]) In applying the program of effectivization of Novikov and Dubrovin [9], [15] (see also [31]) to curves of genus $g > 2$ there arises a difficulty of principle in distinguishing B-matrices of hyperelliptic curves among all B-matrices. Our arguments make it possible to resolve this question in the special case of the curve R_3, since we have actually shown that the B-matrix (40) corresponds to a hyperelliptic curve, and hence a real periodic solution of the sine-Gordon equation of the form (39) is determined on the basis of it by the formulas of [15].

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