PERIODIC FINITE-ZONE SOLUTIONS OF THE SINE-GORDON EQUATION

A. I. Bobenko

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Presently, within the framework of the theory of finite-zone integration, one has constructed wide classes of solutions of nonlinear equations, integrable by the method of the inverse problem. However, practically, the question of the identification of physically interesting periodic solutions in the continual models (for g > 1) has not been investigated. The general finite-zone solutions of genus g of the sine-Gordon (SG) equation $u_{tt} - u_{xx} = \sin x$ are parametrized by $3g - 1$ independent parameters [1, 2]; the periodicity condition leads to additional $g - 1$ constraints. In the case g = 2, the four-parameter class of periodic solutions of the SG equation gives the known Lamb ansatz [3, 4, 8]. In the present paper, making use of reduction technique of the Riemann theta-function [5], we construct a six-parameter family of periodic solutions for g = 3.

We consider the hyperelliptic Riemann surface Γ_3: $\omega^2 = z\prod_{i=1}^{2g}(z-E_i)\ (g=3)$ with the indicated choice of cycle basis (see Fig. 1).

$\Omega_1(P)$, $\Omega_2(P)$, $P \in \Gamma$ are normalized Abelian integrals of the second kind with singularities at the points z = 0 and z = ∞;

$$\Omega_j(P) \to z^{1/2}/4 + \ldots,\quad z \to \infty,\quad \Omega_j(P) \to (-1)^j z^{-1/2}/4 + \ldots,\quad z \to 0,\ j = 1, 2;$$

V and **W** are the corresponding vectors of their b-periods. Assume that the cycle C, going around the cut [0, ∞] on Γ, is equal to $C = \sum_{i=1}^{g} m_i a_i$. Then

$$u(x, t) = \frac{2}{i}\ln\left\{\frac{\Theta((\mathbf{V}x + \mathbf{W}t)/2\pi + \zeta + \mathbf{M})}{\Theta((\mathbf{V}x + \mathbf{W}t)/2\pi + \zeta)}\right\}, \tag{1}$$

where $\mathbf{M} = {}^1/_2\,(m_1, \ldots, m_g)$, $\Theta(z)$ is the Riemann theta function, and $\zeta \in \mathbb{C}^g$.

Assume that the curve Γ_3 admits the automorphism λ: $(z, \omega) \to (z^{-1}, \omega)$, i.e., $E_{7-k} = E_k^{-1}$, k = 1, ..., 6. The following fact is known [6]: if Γ of genus $g = 2\hat{g} + n - 1$ has an involution λ with n pairs of fixed points, then the g-dimensional Riemann theta function of the curve Γ can be expressed by the $\hat{g}$-dimensional Riemann theta function of the curve Γ/λ and by the $(\hat{g} + n - 1)$-dimensional Prym theta function.

In the present case ($\Gamma = \Gamma_3$, $\hat{g} = 2$, n = 0) it is convenient to apply the technique of [5], based on Appel's theorem [7]. In the canonical basis of $H_1(\Gamma)$, indicated in Fig. 1, the

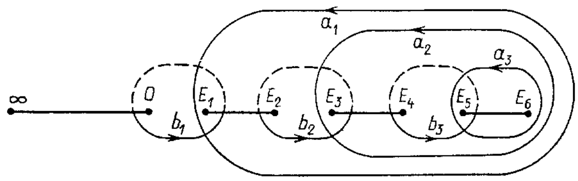

Fig. 1

V. A. Steklov Institute of Mathematics, Leningrad Branch, Academy of Sciences of the USSR.
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automorphism λ has the following representation:

$$\mathbf{b} = \Lambda\lambda\mathbf{b}, \ \mathbf{a} = (\Lambda^t)^{-1}\lambda\mathbf{a}, \ \Lambda = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

(the equality is in $H_1(\Gamma)$). Here $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{a} = (a_1, a_2, a_3)$ are the cycles of Fig. 1, $\lambda\,\mathbf{b}$, $\lambda\,\mathbf{a}$ are the cycles obtained from them under the action of the involution λ. It is more convenient to operate in an other canonical basis $\mathbf{a}'$, $\mathbf{b}'$: $\mathbf{b}' = \Phi\mathbf{b}$, $\mathbf{a}' = (\Phi^t)^{-1}\mathbf{a}$. Here $\mathbf{b}' = \Lambda'\lambda\mathbf{b}'$, $\mathbf{a}' = (\Lambda'^t)^{-1}\lambda\mathbf{a}'$, $\Lambda' = \Phi\Lambda\Phi^{-1}$;

$$\Phi = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Lambda' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Similarly to [5], we obtain a restriction on the matrix of the periods in the basis $\mathbf{a}'$, $\mathbf{b}'$, $B = \Lambda' B \Lambda'^t$, whence

$$B = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & \delta \\ 0 & \delta & 2\delta \end{pmatrix}. \tag{2}$$

We have $\Omega_1(P) = -\Omega_1(\lambda P)$, $\Omega_2(P) = \Omega_2(\lambda P)$ since for these normalized integrals of the second kind the singularities are identical. Consequently, also the vectors of their b-periods coincide:

$$\mathbf{V} = -\Lambda'\mathbf{V}, \ \mathbf{W} = \Lambda'\mathbf{W}, \ \mathbf{V} = (0, v, 2v), \quad \mathbf{W} = (w_1, w_2, 0). \tag{3}$$

To the Riemann theta function, defined by the B-matrix (2), one can apply Appel's theorem [5, 7]:

$$\Theta(\mathbf{x} \mid B) = \theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{matrix} x_1 \\ 2x_2 - x_3 \end{matrix} \middle| A\right)\theta\,[0 \mid 0]\,(x_3 \mid 2\delta) + \theta\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}\left(\begin{matrix} x_1 \\ 2x_2 - x_3 \end{matrix} \middle| A\right)\theta\,[1/2 \mid 0]\,(x_3 \mid 2\delta), \tag{4}$$

where $\theta\,[\alpha \mid \beta]\,(\mathbf{x} \mid B)$ is the Riemann theta function with characteristics α, β; $\mathbf{x} = (x_1, x_2, x_3)$;

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & 4\gamma - 2\delta \end{pmatrix}. \tag{5}$$

The cycle C, going around the cut $[0, \infty]$, is equal to $C = a_2 = a_1' + a_2'$, therefore $M = {}^1/_2 \cdot (1, 1, 0)$. Inserting (2)-(5) into formula (1), we obtain the solution of the SG equation:

$$u(x, t) = \frac{2}{i}\ln\left\{\left(\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{matrix} z_1 + 1/2 \\ z_2 \end{matrix} \middle| A\right)\theta\,[0 \mid 0]\,(z_3 \mid 2\delta)\right.\right.$$
$$\left.\left. - \theta\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}\left(\begin{matrix} z_1 + 1/2 \\ z_2 \end{matrix} \middle| A\right)\theta\,[1/2 \mid 0]\,(z_3 \mid 2\delta)\right) \middle/ \left(\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| A\right)\theta\,[0 \mid 0]\,(z_3 \mid 2\delta) + \theta\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}\left(\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| A\right)\theta\,[1/2 \mid 0]\,(z_3 \mid 2\delta)\right)\right\}, \tag{6}$$

where $z_1 = w_1 t/2\pi + \zeta_1$, $z_2 = w_2 t/\pi + 2\zeta_2 - \zeta_1$, $z_3 = vx/\pi + \zeta_3$. In expression (6), only the one-dimensional theta function depends on x; consequently, solution (6) is periodic.

In the described situation, one can apply the reduction technique of Abelian integrals to elliptic ones, presented in [9]. In this case it turns out that one Abelian integral of genus one reduces to an elliptic one, and the other two to integrals of a curve of genus $g = 2$, the constants v, δ are elliptic, while w_1, w_2, α, β, γ are expressed in terms of a curve of genus $g = 2$.

Making use of the results of [8], from (6) one can easily separate the real periodic finite-zone solutions of genus $g = 3$.

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THREE PROBLEMS OF ARONSZAJN IN MEASURE THEORY

V. I. Bogachev

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In this paper we solve three problems posed in [1]. En route we prove an assertion (the corollary of Theorem 1) which may be regarded as the answer to a well-known question raised by Gel'fand [2]. The approach to these problems is based on the theory of differentiable measures developed in [3-6].

1. Notations and Terminology. Henceforth X and $\mathscr{B}(X)$ denote, respectively, a separable Banach space and the σ-algebra of Borel subsets of X. By a measure on X we mean a countably additive function (not necessarily nonnegative) on $\mathscr{B}(X)$ with values in R. The symbol $|\mu|$ denotes the total variation of the measure μ [7]. A measure μ on X is said to be continuous in the direction of the vector $h \in X$ if $\lim_{t\to 0} \mu(A + th) = \mu(A)$ for all $A \in \mathscr{B}(X)$ [8]. Measure μ is said to be differentiable in the direction of h (in the sense of Skorohod [5]) if for any continuous bounded function f: X → R the limit $\lim_{t\to 0}\frac{1}{t}\int(f(x+th)-f(x))\mu(dx)$ exists. In this case a measure $d_h\mu$ exists, called the derivative of μ in the direction of h, such that the indicated limit equals $\int f(x)d_h\mu(dx)$ [6]. The infinite differentiability in the direction of h is defined naturally. A measure will be said to be densely differentiable if it is differentiable in the direction of all vectors of some sequence with dense linear span. A measure is quasi-invariant if it is equivalent to its translates by the elements belonging to a dense linear subspace. For each sequence $\{a_n\} \subset X$ we denote by $\mathscr{U}\{a_n\}$ the collection of all sets $B \in \mathscr{B}(X)$ such that $B = \bigcup_n B_n$, where $B_n \in \mathscr{B}(X)$ and $\mathrm{mes}\,((B_n + x) \cap R^1 a_n) = 0$. $\forall n, \forall x$ (mes denotes the standard Lebesgue measure on the line $R^1 a_n$); in other words, every section of the set B_n by a line parallel with a_n has measure zero. Let $\mathscr{U} = \bigcap \mathscr{U}\{a_n\}$, where the intersection is taken over all sequences $\{a_n\}$ with dense linear span. The sets in collection $\mathscr{U}$ are referred to as exceptional. Measure μ is said to be absolutely continuous with respect to $\mathscr{U}\{a_n\}$ if $\mu(A) = 0$ $\forall A \in \mathscr{U}\{a_n\}$, and singular with respect to $\mathscr{U}\{a_n\}$ if there is an $A \in \mathscr{U}\{a_n\}$ such that $|\mu|(A) = |\mu|(X)$ [1]. A nonzero measure is said to be exceptional if it is singular with respect to all classes $\mathscr{U}\{a_n\}$, where $\{a_n\}$ has a dense linear span, whereas μ(A) = 0 for all $A \in \mathscr{U}$ [1]. Therefore, an exceptional measure is "concentrated" on each of the collections $\mathscr{U}\{a_n\}$, but vanishes on their intersections. The class $\mathscr{U}$ was introduced in [1], where it was shown that for X finite-dimensional, $\mathscr{U}$ coincides with the σ-algebra of Borel subsets with zero Lebesgue measure. In the general case $\mathscr{U}$ retains some features of this σ-algebra. For example, every Lipschitz function f: X → R is differentiable everywhere except for the points of an exceptional set [1].

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