ON THE FINITE-GAP INTEGRATION OF THE LANDAU-LIFSHITZ EQUATION

A.I. BOBENKO
Leningrad Branch of the Steklov Mathematical Institute, Leningrad, USSR

Abstract By means of a new method, a synthesis of the ideas of the Krichever scheme and the Riemann matrix problem, general finite-gap solutions of the XXZ and XYZ Landau-Lifshitz equations are constructed. The final formulae for the finite-gap solutions in terms of the Riemann and Prym theta-functions are obtained. All real solutions of the XXZ case are isolated.

The Landau-Lifshitz (LFTL) equation,

$$S_t = SxS_{xx} + SxJS, |S| = 1, \ J = \text{diag}\ (J_1, J_2, J_3), \qquad (1)$$

can be investigated on the basis of the inverse scattering method (ISM). E.K. Sklyanin [7] and A.E. Borovik have independently found that (1) is a compatibility condition for theset of two linear equations

$$\Psi_x = U(u)\Psi, \Psi_t = V(u)\Psi \to U_t - V_x + |U+V| = 0, \qquad (2)$$

where the spectral parameter U is a point on an elliptic curve.

However, for a long time, explicit solutions of the equation had been obtained by direct methods, not by using the ISM algorithm or by methods that used this algorithm only partially (one can find the corresponding literature and physics in A.M. Kosevich's review [5]). Recently, A.V. Borisov and the author [4] independently suggested the "dressing-up" procedure using the Riemann problem formulation for (1) by A.V. Mikhailov. Here, on the basis of reference 3 and 6, we shall present the finite-gap solutions of the LL equation. For the corresponding literature, see the lists of references of those papers.

If two of the constants J_i coincide (XXZ case), than the elliptic pair (2) degenerates, becomping a rational pair, i.e. the parameter u is a point on a curve of genus zero. The finite-gap solutions of the XXZ LL equation were found by A.R. Its, R.F. Bikbaev, and the author. [3] To construct the real finite-gap solutions of the XXZ LL equation with the external field

$$S_t = SxS_{xx} + SxJS + SxH, \quad J = \mathrm{diag}(0,0,\varepsilon), H = (0,0,H(t)), \tag{3}$$

we have to consider the hyperelliptic surface Γ of genus g,

$$\omega^2 = (z^2 - a^2) \prod_{i=1}^{g} (z - c_i)(z - \bar{c}_i), \quad a = i\sqrt{\varepsilon}/2$$

Let $dU_i(p), p\varepsilon\Gamma$ be the basis of normalized holomorphic Abelian differentials. Then a_i, b_i, $i = 1, \dots g$ form the canonical homology basis of $H_1(\Gamma)$. Also let the cycle a_1 enclose the cut $|-a,a|$ and other a cycles enclose other cuts. $\theta(z)$ is a corresponding Riemannian theta-function. Ω_1 and Ω_2 are the b-period vectors of the normalized Abelian integrals of the second kind, $\Omega_1(p)$ and $\Omega_2(p)$, which are defined by their singularities at $\infty^\pm$ $(\infty^\pm : z\to\infty,\ \omega/z^{g+1}\to\pm 1)\Omega_1(p)\to\mp z+\ldots\Omega_2(p)\to\pm 2z^2+\ldots \to\infty\pm$.

Theorem. The real finite-gap solutions of the LL equation (3) ($\varepsilon<0$, easy plane and $\varepsilon>0$, easy axis anisotropy) are given by the formulae

$$S_1 = \frac{CD-AB}{AD-BC}, \quad S_2 = i\frac{CD+AB}{AD-BC}, \quad S_3 = \frac{AD+BC}{AD-BC},$$

$$A=\theta(\Omega+\zeta)\exp\left(i\int^t H(s)ds\right), B=\theta(\Omega+\zeta+r)\exp\left(i\int^t H(s)ds\right),$$

$$C = \theta(\Omega+\zeta+n),\ D = -\theta(\Omega+\zeta+n+r),$$

$$\Omega=(\Omega_1 x+\Omega_2 t)/2\pi,\ n=(1/2,0,\ldots,0),\zeta\in C^g,\mathrm{Jm}\,\zeta=-\mathrm{Jm}\,r/2,$$

$$r = \int_{\infty^-}^{\infty^+} dU,$$ where the path of integration crosses the cut $|-a,a|$.

By the technique used in references 1 and 2, it is easy to isolate some interesting classes of solutions expressed in terms of elliptic functions; for example, the periodic solutions ($g = 1,2,3$).

The precise formulae for the finite-gap solutions of the XYZ LL equation (all the constants J_i are different and the U-V pair is elliptic) were recently found by R.F. Bikbaev and the author.[6] The corresponding finite-gap Ψ-function is defined on the elliptic-hyperelliptic surface Γ, i.e., a two-sheeted ramified covering of some torus $T=\Gamma/\pi$

(the automorphism π transposes the sheets of Γ). In addition, Γ has the automorphism $\tau:\Gamma\to\Gamma$ without fixed points with the properties that $\tau^2 = I$ and $\tau\pi = \pi\tau$. The genus of Γ is equal to $g = 2n+1$, where n is a dimension of the Prym type, $P = \{A-\tau A, A\in J(\Gamma)\}$ ($J(\Gamma)$ is the Jacobi type of Γ and τA denotes the induced action τ on $J(\Gamma)$). The Ψ-function has the following structure:

$$\Psi(p) = \begin{pmatrix} \psi(p) & \psi(\pi p) \\ \Psi(\pi\tau p) & \Psi(\tau p) \end{pmatrix}.$$

One can find all the formulae in reference 6. We must note further only that simple arguments show that the dynamics in x,t and the higher flows are restricted from $J(\Gamma)$ to P, remaining linear on it. So the finite-gap solutions expressed in terms of n-dimensional Prym theta-functions are determined by $3n$ parameters.

REFERENCES

1. M.V. Babich, A.I. Bobenko and V.B. Matveev. Dokl. Akad. Nauk SSSR, 272, N 1, p. 13-17 (1983) (in Russian).
2. E.D. Belokolos and V.Z. Enol'skii. Teoret. i Mat. Fiz. 53, N 2, p. 271-282 (1982) (in Russian).
3. R.F. Bikbaev, A.I. Bobenko and A.R. Its. Dokl. Akad. Nauk SSSR, 272, p. 1314-1317 (1983) (in Russian).
4. A.I. Bobenko. Landau-Lifshitz equation. Zap. Nauch. Semin. LOMI, 123, p.58-66 (1983) (in Russian).
5. A.M. Kosevich. Fiz. Met. i Metalloved., 53, N3, p. 420-445 (1982) (in Russian).

6. R.F. Bikbaev and A.I. Bobenko. On the finite-gap integration of the Landau-Lifshitz equation. (Preprint LOMI, E-8-83, 1983) p. 27.
7. E.K. Sklyanin. On complete integrability of the Landau-Lifshitz equation. (Preprint LOMI, E-3-79, 1979) p. 33.