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REDUCTIONS OF RIEMANN THETA-FUNCTIONS OF GENUS g TO THETA-FUNCTIONS OF LOWER GENUS, AND SYMMETRIES OF ALGEBRAIC CURVES

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1. The reduction of general finite-zoned solutions of nonlinear equations of the Korteweg-de Vries (KdV) type to elliptic solutions has its roots in the last century, when three questions related to the theme of the present note were studied, largely as independent problems. (The theory of finite-zoned solutions and their history are described in the surveys [1]–[4].) These questions are: integration of Lamé equations and various generalizations by means of elliptic functions; reduction of abelian integrals of the first kind connected with an algebraic curve to elliptic integrals; and determination of conditions on a $g \times g$ matrix B giving the g-dimensional theta-function

$$\theta_g(\mathbf{x}|B) = \sum_{\mathbf{m}\in\mathbf{Z}^g} \exp\{\pi i(B\mathbf{m},\mathbf{m}) + 2\pi i(\mathbf{x},\mathbf{m})\}, \tag{1}$$

under which this theta-function reduces to θ-functions of lower dimension, in general without assuming that B is the matrix of b-periods of some Riemann surface. The most significant results on the first of these problems were obtained by Hermite and Picard; on the second, by Legendre, Jacobi, Hermite, Picard, Goursat, Königsberger, Boltz, Kovalevskaya, Poincaré and others; and the most interesting work on the third question was done by Appell [5]. Modern work on these topics, which has been carried out in the framework of the ideas and methods of finite-zone integration, is represented by the papers [6]–[9], [12] and [13].

In connection with this circle of questions, one of the authors (Matveev) has given convincing examples which show that one of the basic objects which account for reductions of Riemann theta-functions is the group G of birational automorphisms of an algebraic curve: the more "symmetrical" the curve, the smaller the number of parameters on which it depends (for fixed genus) and the greater the number of connections between entries in the matrix B leading to reductions. Here we shall discuss some reductions which are simple but important in the analysis of equations integrable by the inverse problem method; they are connected with hyperelliptic curves of small genus and the simplest genus 3 trigonal curves.

2. In all of our examples, the last column of B, which we shall denote by B_g, satisfies the condition $\hat{k}B_g = h$, where $\hat{k}$ is a rectangular $(g-1)\times g$ integer matrix, and h is an integer vector. Appell [5] proved that under this condition the g-dimensional theta-function (1) can be represented in the form

$$\theta_g(\mathbf{x}|B) = \sum_{\mathbf{t}\in\mathbf{Z}^g(n)} e^{\alpha}\theta_{g-1}(\mathbf{p}|A)\theta_1(\alpha_g|B_{gg}n_g^2). \tag{2}$$

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In (2), $\mathbf{Z}^g(n)$ denotes the set of vectors in $\mathbf{Z}^g$ whose components satisfy the inequalities $0 \leqslant t_\rho \leqslant n_\rho - 1$, $1 \leqslant \rho \leqslant g$. The number n_ρ is found from the relations

$$(3) \qquad n_j B_{jg} = q_j, \quad j = 1,2,\ldots,\nu; \quad n_{\nu+j} B_{\nu+j,g} = n_g B_{gg} + q_{\nu+j}, \\ j = 1,2,\ldots,g-\nu-1,$$

which hold (with integers g_j) by virtue of the condition $\hat{k}B_g = h$; here n_g is chosen to be the smallest possible value. θ_{g-1} and θ_1 denote theta-functions of the form (1) of dimension $g-1$ and 1, respectively, and the numbers α and α_ρ, the vector $\mathbf{p}$ and the matrix A are described by the formulas

$$\alpha = 2\pi i(\mathbf{t},\mathbf{x}) + \pi i(B\mathbf{t},\mathbf{t}),$$

$$\mathbf{p} = (\alpha_1, \alpha_2, \ldots, \alpha_\nu, \alpha_{\nu+1} - \alpha_g, \ldots, \alpha_{g-1} - \alpha_g),$$

$$\alpha_\rho = n_\rho x_\rho + \frac{1}{2}\frac{\partial(B\mathbf{t},\mathbf{t})}{\partial t_\rho}, \qquad \rho = 1,2,\ldots,g,$$

$$A_{ii} = n_i^2 B_{ii}, \quad i \leqslant \nu; \qquad A_{ii} = n_i^2 B_{ii} - n_g^2 B_{gg}, \quad i > \nu;$$

$A_{ij} = n_i n_j B_{ij}$, if i or $j \leqslant \nu$; $A_{ij} = n_i n_j B_{ij} - n_g^2 B_{gg}$, $i,j > \nu$.

3. In this section, using only considerations of symmetry, we show that the matrices of b-periods of certain families of hyperelliptic curves satisfy the Appell condition. Moreover, we shall choose these curves in such a way that Appell's theorem can be applied repeatedly, until the g-dimensional Riemann theta-function is split up into a finite sum of products of g one-dimensional theta-functions. As examples of this section we have, in particular, the determination of the physically interesting periodic and conditionally periodic solutions of the Todd chain, the nonlinear Shrödinger equation, the equation of an anisotropic XXZ Heisenberg ferromagnet, KdV equations, MKdV equations, and several other equations. We shall discuss those applications in another paper. Here we shall merely note the following important fact: the dependence on the space and time variables in the arguments of the corresponding one-dimensional theta-functions that arise when the finite-zoned solutions are split off, turns out to be linear.

a) We consider the curve $\omega^2 = P_3(z^2)$, where P_3 is a nonsingular polynomial of degree 3. The genus $g = 2$. We choose a canonical basis of cycles a_1, a_2, b_1, b_2 in such a way that they transform as follows under the automorphism $z \to -z$:

$$(4) \qquad a_1 = a_1', \quad a_2 = -a_1' - a_2', \quad b_1 = -b_2' + b_1', \quad b_2 = -b_2'.$$

Let the abelian integrals of the first kind $I_1(z)$ and $I_2(z)$ be normalized in the basis a_i, b_i, and let B be their matrix of b-periods. It is clear that the integrals $I_1'(z) = I_1(-z)$, and $I_2'(z) = I_2(-z)$ are normalized in the basis a_i', b_i' with the same matrix of b-periods. Using (4), we easily obtain the periods I_1' and I_2' in the original basis (their matrix of a-periods is not the identity matrix); finally, normalizing I_1' and I_2', we obtain a matrix of b-periods which must coincide with our original matrix B. This matrix equality is what leads to the necessary "specialness" condition on B:

$$(5) \qquad B = \begin{pmatrix} \beta & \alpha \\ \alpha & 2\alpha \end{pmatrix}.$$

a matrix which automatically falls within the applicability of Appell's theorem. By (2), the Riemann theta-function (1) can be represented as a sum of two terms, each of which is a product of one-dimensional θ-functions with periods 2α and $4\beta - 2\alpha$. This curve is not, of course, the only genus 2 curve with this structure for the matrix B. The decisive feature is the change of basis (4) of the cycles under the automorphism; this holds, for example, for a genus 2 curve having the automorphism $z \to sz^{-1}$. Such an example of a curve of the form $\omega^2 = P_5(z)$ was first considered by E. D. Belokolos and V. Z. Ènol'skiĭ [6]–[8] in connection with a generalization of the Lamb *ansatz* for the sine-Gordon equation. To show the agreement of their results with ours, it suffices to note that the matrices of b-periods

$$B_1 = \begin{pmatrix} \beta & \alpha \\ \alpha & 2\alpha \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \beta & \beta - \alpha \\ \beta - \alpha & \beta \end{pmatrix}, \tag{6}$$

are equivalent, i.e., they differ only by the choice of canonical basis for the curve. Thus, the case $B_{11} = B_{22}$ which was considered in [6]–[8] (see also [15], §5) in essence also falls within the applicability of Appell's theorem.

b) *The curve* $\omega^2 = P_2(z^3)$. The covering group is cyclic. The genus $g = 2$. We choose a canonical basis in such a way that the action of the automorphism $z \to ze^{2\pi i/3}$ on the basis cycles has the form

$$b_1 = b_2' - b_1', \quad b_2 = -b_2', \quad a_1 = a_2', \quad a_2 = -a_1' - a_2'.$$

Here the matrix B is of the form (5) with $\beta = 2\alpha$.

c) *The curve* $\omega^2 = zP_2(z^2)$. The genus $g = 2$. Under the automorphism $z \to -z$ the basis cycles are transformed according to the rule

$$a_1 = -b_2', \quad a_2 = -b_1' + b_2', \quad b_1 = a_2' + a_1', \quad b_2 = a_1'.$$

The matrix B is of the form (5) with $\beta = (\alpha^2 - 1)/2\alpha$. This example is a special case of the two-zone Lamé curve. The general case of a two-zone Lamé curve and the correspondence between this type of result and the formulas of Dubrovin and Novikov [12] for the dynamics of a two-zone Lamé potential have recently been worked out by V. Z. Ènol'skiĭ, who made essential use of a reduction of the corresponding abelian integrals to elliptic integrals by means of Hermite substitutions (personal communication at a conference on "Quantum solitons" in October 1982).

d) *The curve* $\omega^2 = (z^4 - (a^2 + s^2a^{-2})z^2 + s^2)(z^4 - (b^2 + s^2b^{-2})z^2 + s^2)$. The genus $g = 3$. The covering group is a dihedral automorphism group with generators $z \to -z$, $z \to sz^{-1}$. The canonical basis is chosen so that under the mapping $z \to -z$, $a_i \to a_i'$, $b_i \to b_i'$:

$$b_1 = b_2' - b_3', \qquad b_2 = b_1' - b_3', \qquad b_3 = -b_3',$$
$$a_1 = a_2', \qquad a_2 = a_1', \qquad a_3 = -a_1' - a_2' - a_3'.$$

and under the mapping $z \to sz^{-1}$, $a_i \to a_i''$, $b_i \to b_i''$:

$$b_1 = b_2'', \quad b_2 = b_1'', \quad b_3 = b_3'', \quad a_1 = a_2'', \quad a_2 = a_1'', \quad a_3 = a_3''.$$

Here the matrix B has the form

$$B = \begin{pmatrix} \alpha & \gamma & \beta \\ \gamma & \alpha & \beta \\ \beta & \beta & 2\beta \end{pmatrix}. \tag{7}$$

e) *The curve* $\omega^2 = P_2(z^4)$. The covering group is a cyclic group with generator $z \to iz$ of order 4. In a suitable basis the matrix B has the form (7) with $\gamma = \alpha - \beta$. In examples d) and e), if we apply Appell's theorem twice, we see that the original θ-function can be reduced to one-dimensional ones (here one must take into account the equivalence of B-matrices in a) when using Appell's theorem the second time).

f) *The curve*

$$\omega^2 = \left(z^4 - (a^2 + a^{-2})z^2 + 1\right)$$
$$\times\left(z^4 - 2\frac{a^4 - 6a^2 + 1}{(a^2+1)^2} + 1\right)\left(z^4 + 2\frac{a^4 + 6a^2 + 1}{(a^2-1)^2}z^2 + 1\right).$$

The genus $g = 5$. The covering group is nonabelian; namely, it is the tetrahedral group. It has generators $z \to T(z) = (i - z)/(i + z)$ and $z \to U(z) = i(z - 1)/(z + 1)$ or order 3. We note that $UT(z) = -z$ and $TU(z) = z^{-1}$. With a good choice of canonical basis, B has the form

$$B = \begin{pmatrix} \alpha & \beta & \beta & 2\beta & -4\beta \\ \beta & \alpha & \beta & -4\beta & 2\beta \\ \beta & \beta & \alpha & 2\beta & 2\beta \\ 2\beta & -4\beta & 2\beta & -8\beta & 4\beta \\ -4\beta & 2\beta & 2\beta & 4\beta & -8\beta \end{pmatrix}.$$

If we apply Appell's theorem four times, we can represent the original θ-function as a sum of 64 terms, each of which (up to an exponential factor) is a product of one-dimensional theta-functions with periods -8β, $4\alpha + 8\beta$, -24β, $16\alpha + 32\beta$, $16\alpha + 32\beta$.

4. Nonhyperelliptic curves. For the trigonal genus 3 curves $x^4 + y^4 = 1$ and $y^3 + xy + x^3y = 0$ (the Klein curve), the period matrices of the abelian integrals of the first kind are given in [10], and in Poincaré's paper [11] for the Klein curve. Using results of Baker ([10], p. 260), one easily shows that, in his canonical basis for the curve $x^4 + y^4 = 1$, the matrix B has the form

$$B = \frac{1}{2 - 3i}\begin{pmatrix} 2 + 2i & -2 & 2i - 1 \\ -2 & 2 + 2i & 1 - i \\ 2i - 1 & 1 - i & 2 - i \end{pmatrix}.$$

One cannot immediately apply Appell's theorem to this matrix; however, if one takes another basis, obtained from the first one by the substitution $a_1' = a_3$, $a_2' = a_2 - a_1$, $a_3' = a_1$, $b_1' = b_3$, $b_2' = b_2$, $b_3' = b_1 + b_2$, then the matrix B now has the necessary structure:

$$B = \frac{1}{2 - 3i}\begin{pmatrix} 2 - i & 1 - i & i \\ 1 - i & 2 + 2i & 2i \\ i & 2i & 4i \end{pmatrix},$$

and from this it is clear that here as well our original θ-function can be reduced to one-dimensional ones.

The subsititution $x = -s$, $y = s^2(1 - t)^{-1}$ transforms the Klein curve to the form $s^7 = t(1 - t)^2$, which is a seven-fold covering of the plane with branch points 0, 1, ∞ and cuts along the intervals $[0, 1]$ and $[1, \infty]$. We choose a basis of cycles Ω_h, $h = 1, \dots, 6$, such that the cycle Ω_h goes around the point zero h times and the point one $3h$ times in the positive direction. The period matrix in this basis is given in [10]. A canonical basis can be obtained using the formulas

$$a_1 = \Omega_5 - \Omega_6, \quad a_2 = \Omega_6, \quad a_3 = \Omega_3 - \Omega_1 - \Omega_5 + \Omega_6,$$
$$b_1 = \Omega_6 - \Omega_1 - \Omega_5, \quad b_2 = \Omega_6 - \Omega_4, \quad b_3 = \Omega_4 - \Omega_2 + \Omega_3 - \Omega_1 - \Omega_5.$$

In this basis we have

$$B = \frac{1}{\tau + 1}\begin{pmatrix} \tau - 2 & -1 & \tau \\ -1 & 2\tau - 1 & -\tau \\ \tau & -\tau & 3\tau \end{pmatrix}, \qquad \tau = \frac{1}{4}(1 + i\sqrt{7}).$$

Obviously, the original θ-function again reduces to one-dimensional theta-functions.

5. For lack of space we are unable to include drawings of the corresponding Riemann surfaces with the cuts and canonical bases indicated. These will be given in a more detailed publication. Our results have a direct application to the program proposed by Novikov and Dubrovin [3] for making effective the formulas for finite-zone integration (in this connection, see also [8], [9] and [14]), and to the problem of distinguishing the matrices of b-periods of algebraic curves in the set of all Riemann matrices. When we formulated Appell's theorem, we ignored the possibility that the numbers n_ρ in the hypothesis of the theorem may, in general, turn out to be negative. The appropriate refinements of the theorem, along with various generalizations, can be found in [5].

In conclusion, we have the pleasure of thanking E. D. Belokolos and V. Z. Ènol'skiĭ. Our discussions with them on the circle of questions touched upon above, and our study of their articles [6] and [8], provided the essential stimulus for writing this paper.

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