Discrete Differential Geometry

Integrable Structure

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Chapter 7

Discrete Complex Analysis. Linear Theory

7.1. Basic notions of discrete linear complex analysis

Many constructions in discrete complex analysis are parallel to discrete differential geometry in the space of real dimension 2.

Recall that a harmonic function $u : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}$ is characterized by the relation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A conjugate harmonic function $v : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}$ is defined by the Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\,, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\,.$$

Equivalently, $f = u + iv : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial y} = i\,\frac{\partial f}{\partial x}\,.$$

The real and the imaginary parts of a holomorphic function are harmonic, and any real-valued harmonic function can be considered as a real part of a holomorphic function.

A standard classical way to discretize these notions is the following. A function $u : \mathbb{Z}^2 \to \mathbb{R}$ is called *discrete harmonic* if it satisfies the *discrete Laplace equation*

$$(\Delta u)_{m,n} = u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0.$$

A natural domain of a conjugate discrete harmonic function $v : (\mathbb{Z}^2)^* \to \mathbb{R}$ is the *dual lattice*; see Figure 7.1. The defining *discrete Cauchy-Riemann*

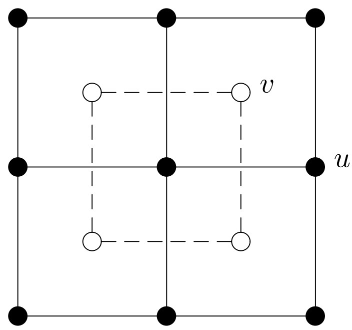


Figure 7.1. Regular square lattice and its dual.

equations read:

$$\begin{aligned} v_{m+1/2,n+1/2} - v_{m+1/2,n-1/2} &= u_{m+1,n} - u_{m,n}, \\ v_{m+1/2,n+1/2} - v_{m-1/2,n-1/2} &= -(u_{m,n+1} - u_{m,n}), \end{aligned}$$

with the natural indexing of the dual lattice; cf. Figure 7.2. The corre-

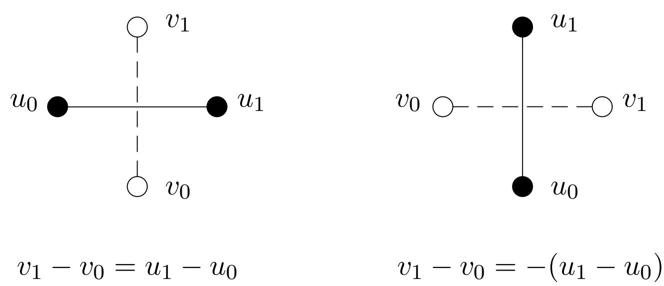


Figure 7.2. Discrete Cauchy-Riemann equations in terms of u, v.

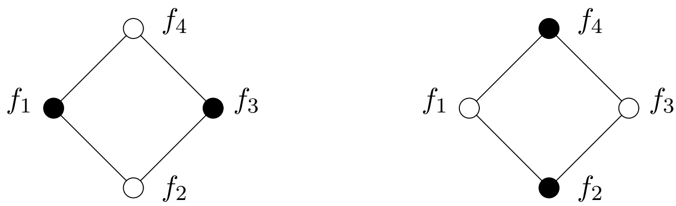
sponding *discrete holomorphic* function $f : \mathbb{Z}^2 \cup (\mathbb{Z}^2)^* \to \mathbb{C}$ is defined on the superposition of the original square lattice $\mathbb{Z}^2$ and the dual $(\mathbb{Z}^2)^*$, by the formula

$$f = \begin{cases} u, & \bullet, \\ iv, & \circ, \end{cases}$$

which comes to replace the smooth version $f = u + iv$. Remarkably, the discrete Cauchy-Riemann equation for f is one and the same for both pictures:

$$f_{m,n+1/2} - f_{m,n-1/2} = i(f_{m+1/2,n} - f_{m-1/2,n});$$

see Figure 7.3.



$$f_4 - f_2 = i(f_3 - f_1)$$

Figure 7.3. Discrete Cauchy-Riemann equations in terms of f.

This discretization of the Laplace and the Cauchy-Riemann equations apparently preserves the majority of important structural features. Its generalization for arbitrary graphs goes as follows.

Discrete harmonic functions can be defined for an arbitrary graph $\mathcal{G}$ with the set of vertices $V(\mathcal{G})$ and the set of edges $E(\mathcal{G})$.

Definition 7.1. (Discrete Laplacian and discrete harmonic functions) *For a given weight function $\nu : E(\mathcal{G}) \to \mathbb{R}_+$ on edges of $\mathcal{G}$, the discrete Laplacian is the operator acting on functions $f : V(\mathcal{G}) \to \mathbb{C}$ by*

$$(\Delta f)(x_0) = \sum_{x \sim x_0} \nu(x_0, x)(f(x) - f(x_0)), \tag{7.1}$$

where the summation is extended over the set of vertices x connected to x_0 by an edge. A function $f : V(\mathcal{G}) \to \mathbb{C}$ is called discrete harmonic (with respect to the weights ν) if $\Delta f = 0$.

The positivity of weights ν in this definition is important from the analytic point of view, since it guarantees, e.g., the maximum principle for the discrete Laplacian under suitable boundary conditions (so that discrete harmonic functions come as minimizers of a convex functional). However, from the pure algebraic point of view, one might consider at times also arbitrary real (or even complex) weights.

If $\mathcal{G}$ comes from a cellular decomposition of an oriented surface, let $\mathcal{G}^*$ be its dual graph, and let the quad-graph $\mathcal{D}$ be its double; see Section 6.4. Extend the weight function to the edges of $\mathcal{G}^*$ according to the rule

$$\nu(e^*) = 1/\nu(e). \tag{7.2}$$

Definition 7.2. (Discrete Cauchy-Riemann equations and discrete holomorphic functions) *A function $f : V(\mathcal{D}) \to \mathbb{C}$ is called discrete holomorphic (with respect to the weights ν) if for any positively oriented quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ (see Figure 7.4),*

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = i\nu(x_0, x_1) = -\frac{1}{i\nu(y_0, y_1)}. \tag{7.3}$$

These equations are called the discrete Cauchy-Riemann equations.

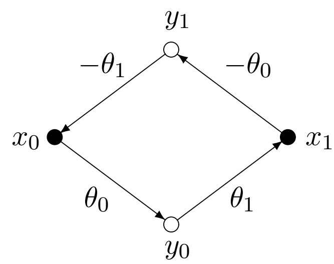


Figure 7.4. Positively oriented quadrilateral, with a labelling of directed edges.

The relation between discrete harmonic and discrete holomorphic functions is the same as in the smooth case. It is given by the following statement, which is a special case of Theorem 6.31.

Theorem 7.3. (Relation between discrete harmonic and discrete holomorphic functions)

a) *If a function $f : V(\mathcal{D}) \to \mathbb{C}$ is discrete holomorphic, then its restrictions to $V(\mathcal{G})$ and to $V(\mathcal{G}^*)$ are discrete harmonic.*

b) *Conversely, any discrete harmonic function $f : V(\mathcal{G}) \to \mathbb{C}$ admits a family of discrete holomorphic extensions to $V(\mathcal{D})$, differing by an additive constant on $V(\mathcal{G}^*)$. Such an extension is uniquely determined by a value at one arbitrary vertex $y \in V(\mathcal{G}^*)$.*

7.2. Moutard transformation for discrete Cauchy-Riemann equations

Observe that discrete Cauchy-Riemann equations (7.3) formally are not different from the Moutard equations (2.51) for T-nets. One only has to fix an orientation of all quadrilateral faces $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$. We assume that it is inherited from the orientation of the underlying surface.

One can now apply the Moutard transformation of Section 2.3.9 to discrete holomorphic functions. To this aim, one has to choose an orientation of *all* elementary quadrilaterals in Figure 7.5. This can be done, for example, as follows: for the quadrilaterals $(x_0^+, y_0^+, x_1^+, y_1^+) \in F(\mathcal{D}^+)$, choose an orientation to coincide with that of the corresponding $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$. For a "vertical" quadrilateral over an edge $(x, y) \in E(\mathcal{D})$, assume that $x \in V(\mathcal{G})$, $y \in V(\mathcal{G}^*)$, and choose the positive orientation corresponding to the cyclic order (x, y, y^+, x^+) of its vertices. Observe that under this convention, two opposite "vertical" quadrilaterals are always oriented differently.

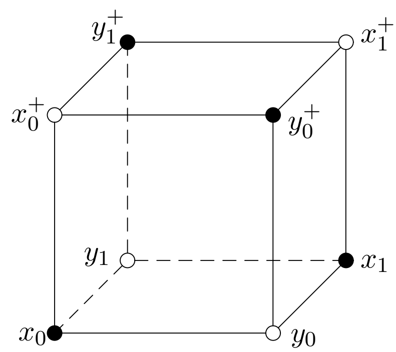


Figure 7.5. Elementary cube of **D**.

In the case of arbitrary quad-graphs, one has to generalize one more ingredient of the Moutard transformation, namely the data (MT_2^Δ).

Theorem 7.4. (Moutard transformation for discrete holomorphic functions) *On an arbitrary bipartite quad-graph $\mathcal{D}$, valid initial conditions for a Moutard transformation of the discrete Cauchy-Riemann equations consist of*

(MCR_1^Δ) *the value of f^+ at one point $x(0) \in V(\mathcal{D})$;*

(MCR_2^Δ) *the values of weights on "vertical" quadrilaterals (x, y, y^+, x^+) assigned to all edges (x, y) of a Cauchy path in $\mathcal{D}$.*

See Theorem 6.6 for necessary and sufficient conditions for a path to be a Cauchy path, i.e., to support initial data for a well-posed Cauchy problem. It is natural to assign the weights on the "vertical" quadrilaterals to the underlying edges of $\mathcal{D}$.

Weights ν on the faces of $\mathcal{D}$ together with the data (MCR_2^Δ) yield the transformed weights ν^+ on the faces of $\mathcal{D}^+$, as well as the weights over all edges of $E(\mathcal{D})$. This can be considered as a Moutard transformation for the *Cauchy-Riemann equations* on $\mathcal{D}$. Finding a *solution* $f : V(\mathcal{D}^+) \to \mathbb{C}$ of the transformed equations requires additionally the datum (MCR_1^Δ).

Note that the system of weights ν is highly redundant, due to (7.2). To fix the ideas in writing the equations, we stick to the weights assigned to the "black" diagonals of the quadrilateral faces of the complex $\mathbf{D}$. On the ground floor, these are the edges of the "black" graph $\mathcal{G}$; on the first floor, these are the edges of the "black" graph which is a copy of $\mathcal{G}^*$; and for the "vertical" faces, these are the edges (x, y^+), with $x \in V(\mathcal{G})$ and $y \in V(\mathcal{G}^*)$. Needless to say that the latter weights can be assigned to the quad-graph edges $(x, y) \in E(\mathcal{D})$. So, we write the discrete Cauchy-Riemann equations as follows:

$$f(y_1) - f(y_0) = i\nu(x_0, x_1)\big(f(x_1) - f(x_0)\big), \tag{7.4}$$

$$f(x_0^+) - f(x_1^+) = i\nu(y_0^+, y_1^+)\big(f(y_1^+) - f(y_0^+)\big), \tag{7.5}$$

$$f(x^+) - f(y) = i\nu(x, y^+)\big(f(y^+) - f(x)\big). \tag{7.6}$$

Denote, for the sake of brevity,

$$\nu = \nu(x_0, x_1), \quad \nu^+ = \nu(y_0^+, y_1^+), \quad \mu_{jk} = \nu(x_j, y_k^+).$$

Regarding the weights ν, μ_{00}, and μ_{01} as the input of the Moutard transformation on an elementary hexahedron of $\mathbf{D}$, its output consists of the weights ν^+, μ_{10}, and μ_{11}, given by (cf. (2.59))

$$\nu^+\nu = -\mu_{11}\mu_{00} = -\mu_{10}\mu_{01} = \frac{\nu\mu_{00}\mu_{01}}{\mu_{00} - \mu_{01} - \nu}. \tag{7.7}$$

This transformation is well defined for real weights ν, μ_{jk}, but it *does not* preserve, in general, positivity of the weights ν.

To give a different form of this transformation, observe that the relation $\mu_{11}\mu_{00} = \mu_{10}\mu_{01}$ for each elementary quadrilateral (x_0, y_0, x_1, y_1) of $\mathcal{D}$ yields the existence of the function $\theta : V(\mathcal{D}) \to \mathbb{C}$, defined up to a constant factor, such that $i\mu_{jk} = \theta(y_k)/\theta(x_j)$ (see Exercise 7.1). Moreover, choosing $\theta(x_0)$ real at some point $x_0 \in V(\mathcal{G})$, one sees that θ takes real values on $V(\mathcal{G})$ and imaginary values on $V(\mathcal{G}^*)$. An easy computation shows that the last equation in (7.7) is equivalent to

$$\theta(y_1) - \theta(y_0) = i\nu(x_0, x_1)\big(\theta(x_1) - \theta(x_0)\big),$$

so that the function θ is discrete holomorphic with respect to the weights ν. For the transformed weights ν^+ one finds:

$$\nu^+\nu = \frac{\theta(y_0)\theta(y_1)}{\theta(x_0)\theta(x_1)}. \tag{7.8}$$

Conversely, an arbitrary discrete holomorphic function $\theta : V(\mathcal{D}) \to \mathbb{C}$ defines, via (7.8), a Moutard transformation of the discrete Cauchy-Riemann equations. It should be mentioned that the data (MCR_2^Δ) can be reformulated in terms of the function θ:

(MCR_2^Δ) the values of θ at all vertices along a Cauchy path in $\mathcal{D}$.

Remark. A Moutard transformation for discrete Cauchy-Riemann equations yields, by restriction to the "black" graphs, a sort of Darboux transformation of arbitrary discrete Laplacians on $\mathcal{G}$ into discrete Laplacians on $\mathcal{G}^*$.

7.3. Integrable discrete Cauchy-Riemann equations

We now turn to a useful question of "stationary points" of the Moutard transformation discussed in the previous section. More precisely, this is the question about conditions on the weights $\nu : E(\mathcal{G}) \to \mathbb{R}_+$ such that there exists a Moutard transformation for which the opposite faces of any elementary hexahedron of $\mathbf{D}$ (see Figure 7.5) carry *identical* equations.

Theorem 7.5. (Integrability of discrete Cauchy-Riemann equations) *A system of discrete Cauchy-Riemann equations with the weight function $\nu : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{R}_+$ satisfying (7.2) admits a Moutard transformation into itself if and only if for all $x_0 \in V(\mathcal{G})$ and all $y_0 \in V(\mathcal{G}^*)$ the following conditions are fulfilled:*

$$\prod_{e \in \text{star}(x_0;\mathcal{G})} \frac{1+i\nu(e)}{1-i\nu(e)} = 1, \qquad \prod_{e^* \in \text{star}(y_0;\mathcal{G}^*)} \frac{1+i\nu(e^*)}{1-i\nu(e^*)} = 1. \tag{7.9}$$

Proof. Opposite faces of $\mathcal{D}$ and $\mathcal{D}^+$ carry identical equations if $\nu^+\nu = 1$ in (7.7). Clearly, this yields also $\mu_{11}\mu_{00} = \mu_{10}\mu_{01} = -1$, which means that the opposite "vertical" faces also support identical equations (recall that opposite "vertical" faces carry different orientations). Moreover, given $\nu = \nu(x_0, x_1)$ for an elementary quadrilateral (x_0, y_0, x_1, y_1) of $\mathcal{D}$, we find that the input data μ_{00}, μ_{01} of the Moutard transformation should be related as follows:

$$\frac{\nu\mu_{00}\mu_{01}}{\mu_{00}-\mu_{01}-\nu} = 1 \quad \Leftrightarrow \quad \mu_{01} = \frac{\mu_{00}-\nu}{\mu_{00}\nu+1} = \begin{pmatrix} 1 & -\nu \\ \nu & 1 \end{pmatrix}[\mu_{00}],$$

where the standard notation for the action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{C}$ by Möbius transformations is used. This means that all the weights on the vertical faces of a "stationary" Moutard transformation are completely defined by just one of them, so that such transformations form a one-parameter family. To derive a condition for ν for the existence of a "stationary" Moutard transformation, consider a flower of quadrilaterals (x_0, y_{k-1}, x_k, y_k) around $x_0 \in V(\mathcal{G})$ (see Figure 6.5). In the natural notation, we find:

$$\mu_{0,k} = \frac{\mu_{0,k-1}-\nu_k}{\mu_{0,k-1}\nu_k+1} = \begin{pmatrix} 1 & -\nu_k \\ \nu_k & 1 \end{pmatrix}[\mu_{0,k-1}].$$

Running around x_0 should for any μ_{00} return its value, which means that the matrix product

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \prod_k^{\curvearrowleft} \begin{pmatrix} 1 & -\nu_k \\ \nu_k & 1 \end{pmatrix}$$

should be proportional to the identity matrix. This matrix product is easily computed (see Exercise 7.2):

$$A = \frac{1}{2}\Big(\prod_k (1+i\nu_k) + \prod_k (1-i\nu_k)\Big), \quad B = \frac{1}{2i}\Big(\prod_k (1+i\nu_k) - \prod_k (1-i\nu_k)\Big),$$

and the condition $B = 0$ is equivalent to the first equality in (7.9). The second condition in (7.9) is proved similarly, by considering a flower of quadrilaterals around $y_0 \in V(\mathcal{G}^*)$. □

Thus, the existence of a "stationary" Moutard transformation singles out a special class of discrete Cauchy-Riemann equations, which have to be considered as 2D systems with the 3D consistency property; see Section 6.7. In other words, such Cauchy-Riemann equations should be termed *integrable*. The main difference as compared with the examples in Section 6.7 is that discrete Cauchy-Riemann equations naturally depend on the *orientation* of the elementary quadrilaterals, and that their parameters ν are apparently assigned not to the edges of the quad-graph, but rather to the diagonals of its faces.

The integrability condition (7.9) admits a nice geometric interpretation. It is convenient (especially for positive real-valued ν) to use the notation

$$\nu(e) = \tan\frac{\phi(e)}{2}, \quad \phi(e) \in (0,\pi). \tag{7.10}$$

The condition $\nu(e^*) = 1/\nu(e)$ is translated into

$$\phi(e^*) = \pi - \phi(e), \tag{7.11}$$

while the condition (7.9) says that for all $x_0 \in V(\mathcal{G})$ and all $y_0 \in V(\mathcal{G}^*)$,

$$\prod_{e\in \mathrm{star}(x_0;\mathcal{G})} \exp(i\phi(e)) = 1, \qquad \prod_{e^*\in \mathrm{star}(y_0;\mathcal{G}^*)} \exp(i\phi(e^*)) = 1. \tag{7.12}$$

These conditions should be compared with conditions characterizing the angles $\phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to (0,\pi)$ of a rhombic embedding of a quad-graph $\mathcal{D}$, which consist of (7.11) and

$$\sum_{e\in \mathrm{star}(x_0;\mathcal{G})} \phi(e) = 2\pi, \qquad \sum_{e^*\in \mathrm{star}(y_0;\mathcal{G}^*)} \phi(e^*) = 2\pi, \tag{7.13}$$

for all $x_0 \in V(\mathcal{G})$ and all $y_0 \in V(\mathcal{G}^*)$. Thus, the integrability condition (7.12) says that the system of angles $\phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to (0,\pi)$ comes from a realization of the quad-graph $\mathcal{D}$ as a *rhombic ramified embedding* in $\mathbb{C}$. Flowers of such an embedding can wind around its vertices more than once.

Another formulation of the integrability conditions is given in terms of the *edges* of the rhombic realizations.

Theorem 7.6. (Integrable Cauchy-Riemann equations in terms of rhombic edges) *Integrability condition* (7.9) *for the weight function* $\nu : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{R}_+$ *is equivalent to the following: there exists a labelling of directed edges of* $\mathcal{D}$, $\theta : \vec{E}(\mathcal{D}) \to \mathbb{S}^1$, *such that, in the notation of Figure* 7.4,

$$\nu(x_0, x_1) = \frac{1}{\nu(y_0, y_1)} = i\,\frac{\theta_0 - \theta_1}{\theta_0 + \theta_1}. \tag{7.14}$$

Under this condition, the 3D consistency of the discrete Cauchy-Riemann equations is assured by the following values of the weights ν *on the diagonals of the vertical faces of* $\mathbf{D}$*:*

$$\nu(x, y^+) = i\,\frac{\theta - \lambda}{\theta + \lambda}, \tag{7.15}$$

where $\theta = \theta(x, y)$, *and* $\lambda \in \mathbb{C}$ *is an arbitrary number which is interpreted as the label assigned to all vertical edges of* $\mathbf{D}$*:* $\lambda = \theta(x, x^+) = \theta(y, y^+)$.

So, integrable discrete Cauchy-Riemann equations can be written in a form with parameters assigned to *directed* edges of $\mathcal{D}$:

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = \frac{\theta_1 - \theta_0}{\theta_1 + \theta_0}, \tag{7.16}$$

where

$$\theta_0 = p(y_0) - p(x_0) = p(x_1) - p(y_1), \quad \theta_1 = p(y_1) - p(x_0) = p(x_1) - p(y_0),$$

and $p : V(\mathcal{G}) \to \mathbb{C}$ is a rhombic realization of the quad-graph $\mathcal{D}$. Since

$$\frac{\theta_1 - \theta_0}{\theta_1 + \theta_0} = \frac{p(y_1) - p(y_0)}{p(x_1) - p(x_0)},$$

we see that for a discrete holomorphic function $f : V(\mathcal{G}) \to \mathbb{C}$, the quotient of diagonals of the f-image of any quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ is equal to the quotient of diagonals of the corresponding rhombus.

A standard construction of zero curvature representation for 3D consistent equations, given in Theorem 6.4, leads in the present case to the following result.

Theorem 7.7. (Zero curvature representation of discrete Cauchy-Riemann equations) *The discrete Cauchy-Riemann equations* (7.16) *admit a zero curvature representation with spectral parameter dependent* 2×2

matrices along $(x,y) \in \vec{E}(\mathcal{D})$ *given by*

$$L(y,x,\alpha;\lambda) = \begin{pmatrix} \lambda+\theta & -2\theta(f(x)+f(y)) \\ 0 & \lambda-\theta \end{pmatrix}, \tag{7.17}$$

where $\theta = p(y) - p(x)$.

Linearity of the discrete Cauchy-Riemann equations is reflected in the triangular structure of the transition matrices.

Also, all constructions of Section 6.8 can be applied to integrable discrete Cauchy-Riemann equations. In particular, for weights coming from a quasicrystallic rhombic embedding of the quad-graph $\mathcal{D}$, with labels $\Theta = \{\pm\theta_1, \ldots, \pm\theta_d\}$, discrete holomorphic functions can be extended from the corresponding surface $\Omega_{\mathcal{D}} \subset \mathbb{Z}^d$ to its hull, preserving discrete holomorphy. Here we have in mind the following natural definition:

Definition 7.8. (Discrete holomorphic functions on $\mathbb{Z}^d$) *A function* $f : \mathbb{Z}^d \to \mathbb{C}$ *is called discrete holomorphic if it satisfies, on each elementary square of* $\mathbb{Z}^d$, *the equation*

$$\frac{f(\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k) - f(\boldsymbol{n})}{f(\boldsymbol{n}+\boldsymbol{e}_j) - f(\boldsymbol{n}+\boldsymbol{e}_k)} = \frac{\theta_j+\theta_k}{\theta_j-\theta_k}. \tag{7.18}$$

For discrete holomorphic functions in $\mathbb{Z}^d$, the transition matrices along the edges $(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_k)$ of $\mathbb{Z}^d$ are given by

$$L_k(\boldsymbol{n};\lambda) = \begin{pmatrix} \lambda+\theta_k & -2\theta_k(f(\boldsymbol{n}+\boldsymbol{e}_k)+f(\boldsymbol{n})) \\ 0 & \lambda-\theta_k \end{pmatrix}. \tag{7.19}$$

All results of this section hold also in the case of generic complex weights ν, which leads to $\theta \in \mathbb{C}$ and to parallelogram realizations of $\mathcal{D}$.

7.4. Discrete exponential functions

An important class of discrete holomorphic functions is built by discrete exponential functions. We define them for an arbitrary rhombic embedding $p : V(\mathcal{D}) \to \mathbb{C}$. Fix a point $x_0 \in V(\mathcal{D})$. For any other point $x \in V(\mathcal{D})$, choose some path $\{\mathfrak{e}_j\}_{j=1}^n \subset \vec{E}(\mathcal{D})$ connecting x_0 to x, so that $\mathfrak{e}_j = (x_{j-1}, x_j)$ and $x_n = x$. Let the slope of the j-th edge be $\theta_j = p(x_j) - p(x_{j-1}) \in \mathbb{S}^1$. Then

$$e(x;z) = \prod_{j=1}^{n} \frac{z+\theta_j}{z-\theta_j}.$$

Clearly, this definition depends on the choice of the point $x_0 \in V(\mathcal{D})$, but not on the path connecting x_0 to x.

An extension of the discrete exponential function from $\Omega_{\mathcal{D}}$ to the whole of $\mathbb{Z}^d$ is given by the following simple formula:

$$e(\boldsymbol{n}; z) = \prod_{k=1}^{d} \Big(\frac{z+\theta_k}{z-\theta_k}\Big)^{n_k}. \tag{7.20}$$

The discrete Cauchy-Riemann equations for the discrete exponential function are easily checked: they are equivalent to a simple identity

$$\Big(\frac{z+\theta_j}{z-\theta_j}\cdot\frac{z+\theta_k}{z-\theta_k}-1\Big)\Big/\Big(\frac{z+\theta_j}{z-\theta_j}-\frac{z+\theta_k}{z-\theta_k}\Big)=\frac{\theta_j+\theta_k}{\theta_j-\theta_k}.$$

At a given $\boldsymbol{n} \in \mathbb{Z}^d$, the discrete exponential function is rational with respect to the parameter z, with poles at the points $\epsilon_1\theta_1, \dots, \epsilon_d\theta_d$, where $\epsilon_k = \operatorname{sign} n_k$.

Equivalently, one can identify the discrete exponential function by its initial values on the axes:

$$e(n\boldsymbol{e}_k; z) = \Big(\frac{z+\theta_k}{z-\theta_k}\Big)^{n}. \tag{7.21}$$

Another characterization says that $e(\cdot; z)$ is the Bäcklund transformation of the zero solution of discrete Cauchy-Riemann equations on $\mathbb{Z}^d$, with the "vertical" parameter z.

We now show that the discrete exponential functions form a basis in some natural class of functions (growing not faster than exponentially).

Theorem 7.9. (Discrete exponentials form a basis of discrete holomorphic functions) *Let f be a discrete holomorphic function on $V(\mathcal{D}) \sim V(\Omega_{\mathcal{D}})$, satisfying*

$$|f(\boldsymbol{n})| \le \exp(C(|n_1| + \cdots + |n_d|)), \qquad \forall \boldsymbol{n} \in V(\Omega_{\mathcal{D}}), \tag{7.22}$$

with some $C \in \mathbb{R}$. Extend it to a discrete holomorphic function on the hull $\mathcal{H}(V(\Omega_{\mathcal{D}}))$. There exists a function g defined on the disjoint union of small neighborhoods around the points $\pm\theta_k \in \mathbb{C}$ and holomorphic on each of these neighborhoods, such that

$$f(\boldsymbol{n}) - f(\boldsymbol{0}) = \frac{1}{2\pi i}\int_{\Gamma} g(\lambda)e(\boldsymbol{n};\lambda)d\lambda, \qquad \forall \boldsymbol{n} \in \mathcal{H}(V(\Omega_{\mathcal{D}})), \tag{7.23}$$

where Γ is a collection of $2d$ small loops, each running counterclockwise around one of the points $\pm\theta_k$.

Proof. The proof is constructive and consists of three steps.

(i) Extend f from $V(\Omega_{\mathcal{D}})$ to $\mathcal{H}(V(\Omega_{\mathcal{D}}))$; inequality (7.22) propagates in the extension process, if the constant C is chosen large enough.

(ii) Introduce the restrictions $f_n^{(k)}$ of $f : \mathcal{H}(V(\Omega_{\mathcal{D}})) \to \mathbb{C}$ to the coordinate axes:

$$f_n^{(k)} = f(n\boldsymbol{e}_k), \qquad a_k(\Omega_{\mathcal{D}}) \le n \le b_k(\Omega_{\mathcal{D}}).$$

(iii) Set $g(\lambda) = \sum_{k=1}^{d}(g_k(\lambda) + g_{-k}(\lambda))$, where the functions $g_{\pm k}(\lambda)$ vanish everywhere except in small neighborhoods of the points $\pm\theta_k$, respectively, and are given there by convergent series

$$g_k(\lambda) = \frac{1}{2\lambda}\left(f_1^{(k)} - f(\mathbf{0}) + \sum_{n=1}^{\infty}\Big(\frac{\lambda - \theta_k}{\lambda + \theta_k}\Big)^n \big(f_{n+1}^{(k)} - f_{n-1}^{(k)}\big)\right), \tag{7.24}$$

and a similar formula for $g_{-k}(\lambda)$. Formula (7.23) is then easily verified by computing the residues at $\lambda = \pm\theta_k$ (see Exercise 7.5).

□

It is important to observe that the data $f_n^{(k)}$, necessary for the construction of $g(\lambda)$, are *not among* the values of f on $V(\mathcal{D}) \sim V(\Omega_{\mathcal{D}})$ known initially, but are encoded in the extension process.

7.5. Discrete logarithmic function

We now define the discrete logarithmic function on a rhombic quad-graph $\mathcal{D}$. Fix some point $x_0 \in V(\mathcal{D})$, and set

$$\ell(x) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\log(\lambda)}{2\lambda}\, e(x;\lambda)d\lambda, \qquad \forall x \in V(\mathcal{D}). \tag{7.25}$$

Here the integration path Γ is the same as in Theorem 7.9, and fixing x_0 is necessary for the definition of the discrete exponential function on $\mathcal{D}$. To make (7.25) a valid definition, one must specify a branch of $\log(\lambda)$ in a neighborhood of each point $\pm\theta_k$. This choice depends on x, and is done as follows.

Assume, without loss of generality, that the circular order of the points $\pm\theta_k$ on the positively oriented unit circle $\mathbb{S}^1$ is the following: $\theta_1, \dots, \theta_d$, $-\theta_1, \dots, -\theta_d$. We set $\theta_{k+d} = -\theta_k$ for $k = 1, \dots, d$, and then define θ_r for all $r \in \mathbb{Z}$ by $2d$-periodicity. For each $r \in \mathbb{Z}$, assign to $\theta_r = \exp(i\gamma_r) \in \mathbb{S}^1$ a certain value of the argument $\gamma_r \in \mathbb{R}$: choose a value γ_1 of the argument of θ_1 arbitrarily, and then extend it according to the rule

$$\gamma_{r+1} - \gamma_r \in (0, \pi), \qquad \forall r \in \mathbb{Z}.$$

Clearly, $\gamma_{r+d} = \gamma_r + \pi$, and therefore also $\gamma_{r+2d} = \gamma_r + 2\pi$. It will be convenient to consider the points θ_r, supplied with the arguments γ_r, as belonging to the Riemann surface Λ of the logarithmic function (a branched covering of the complex λ-plane).

For each $m \in \mathbb{Z}$, define the "sector" U_m on the plane $\mathbb{C}$ carrying the quad-graph $\mathcal{D}$ as the set of all points of $V(\mathcal{D})$ which can be reached from x_0 along paths with all edges from $\{\theta_m, \ldots, \theta_{m+d-1}\}$. Two sectors U_{m_1} and U_{m_2} have a nonempty intersection if and only if $|m_1 - m_2| < d$. The union $U = \bigcup_{m=-\infty}^{\infty} U_m$ is a branched covering of the quad-graph $\mathcal{D}$, and it serves as the domain of the discrete logarithmic function.

The definition (7.25) of the latter should be read as follows: for $x \in U_m$, the poles of $e(x;\lambda)$ are exactly the points $\theta_m, \ldots, \theta_{m+d-1} \in \Lambda$. The integration path Γ consists of d small loops on Λ around these points, and $\arg(\lambda) = \Im \log(\lambda)$ takes values in a small open neighborhood (in $\mathbb{R}$) of the interval

$$[\gamma_m, \gamma_{m+d-1}] \tag{7.26}$$

of length less than π. If m increases by $2d$, the interval (7.26) is shifted by 2π. As a consequence, the function ℓ is discrete holomorphic, and its restriction to the set $V(\mathcal{G})$ of "black" points is discrete harmonic everywhere on U except at the point x_0:

$$\Delta \ell(x) = \delta_{x_0 x}. \tag{7.27}$$

Thus, the functions g_k in the integral representation (7.23) of an arbitrary discrete holomorphic function, defined originally in disjoint neighborhoods of the points α_r, in the case of the discrete logarithmic function are actually restrictions of a single analytic function $\log(\lambda)/(2\lambda)$ to these neighborhoods. This allows one to deform the integration path Γ into a connected contour lying on a single leaf of the Riemann surface of the logarithm, and then use standard methods of complex analysis to obtain asymptotic expressions for the discrete logarithmic function. In particular, one can show that at the "black" points of $V(\mathcal{G})$,

$$\ell(x) \sim \log|x - x_0|, \qquad x \to \infty. \tag{7.28}$$

Properties (7.27), (7.28) characterize the *discrete Green's function* on $\mathcal{G}$. Thus:

Theorem 7.10. (Discrete Green's function) *The discrete logarithmic function on $\mathcal{D}$, restricted to the set of vertices $V(\mathcal{G})$ of the "black" graph $\mathcal{G}$, coincides with discrete Green's function on $\mathcal{G}$.*

Now we extend the discrete logarithmic function to $\mathbb{Z}^d$, which will allow us to gain significant additional information about it. In addition to the unit vectors $\boldsymbol{e}_k \in \mathbb{Z}^d$ (corresponding to $\theta_k \in \mathbb{S}^1$), we introduce their opposites $\boldsymbol{e}_{k+d} = -\boldsymbol{e}_k$, $k \in [1, d]$ (corresponding to $\theta_{k+d} = -\theta_k$), and define $\boldsymbol{e}_r$ for all

$r \in \mathbb{Z}$ by $2d$-periodicity. Then

$$S_m = \bigoplus_{r=m}^{m+d-1} \mathbb{Z}\boldsymbol{e}_r \subset \mathbb{Z}^d \tag{7.29}$$

is a d-dimensional octant containing exactly the part of $\Omega_{\mathcal{D}}$ which is the P-image of the sector $U_m \subset \mathcal{D}$. Clearly, only $2d$ different octants appear among the S_m (out of 2^d possible d-dimensional octants). Define $\widetilde{S}_m$ as the octant S_m equipped with the interval (7.26) of values for $\Im \log(\theta_r)$. By definition, $\widetilde{S}_{m_1}$ and $\widetilde{S}_{m_2}$ intersect if the underlying octants S_{m_1} and S_{m_2} have a nonempty intersection spanned by the common coordinate semiaxes $\mathbb{Z}\boldsymbol{e}_r$, and the $\Im \log(\theta_r)$ for these common semiaxes match. It is easy to see that $\widetilde{S}_{m_1}$ and $\widetilde{S}_{m_2}$ intersect if and only if $|m_1 - m_2| < d$. The union $\widetilde{S} = \bigcup_{m=-\infty}^{\infty} \widetilde{S}_m$ is a branched covering of the set $\bigcup_{m=1}^{2d} S_m \subset \mathbb{Z}^d$.

Definition 7.11. (Discrete logarithmic function on $\mathbb{Z}^d$) *The discrete logarithmic function on $\widetilde{S}$ is given by the formula*

$$\ell(\boldsymbol{n}) = \frac{1}{2\pi i} \int_\Gamma \frac{\log(\lambda)}{2\lambda} e(\boldsymbol{n}; \lambda) d\lambda, \qquad \forall \boldsymbol{n} \in \widetilde{S}, \tag{7.30}$$

where for $\boldsymbol{n} \in \widetilde{S}_m$ the integration path Γ consists of d loops around the points $\theta_m, \ldots, \theta_{m+d-1}$ on Λ, and $\Im \log(\lambda)$ on Γ is chosen in a small open neighborhood of the interval (7.26)*.*

The discrete logarithmic function on $\mathcal{D}$ can be described as the restriction of the discrete logarithmic function on $\widetilde{S}$ to a branched covering of $\Omega_{\mathcal{D}} \sim \mathcal{D}$. This holds for an *arbitrary* quasicrystallic quad-graph $\mathcal{D}$ whose set of edge slopes coincides with $\Theta = \{\pm\theta_1, \ldots, \pm\theta_d\}$.

Now we are in a position to give an alternative definition of the discrete logarithmic function. Clearly, it is completely characterized by its values $\ell(n\boldsymbol{e}_r)$, $r \in [m, m+d-1]$, on the coordinate semiaxes of an arbitrary octant $\widetilde{S}_m$. Let us stress once more that the points $n\boldsymbol{e}_r$ do not lie, in general, on the original quad-surface $\Omega_{\mathcal{D}}$.

Theorem 7.12. (Values of discrete logarithmic function on coordinate axes) *The values $\ell_n^{(r)} = \ell(n\boldsymbol{e}_r)$, $r \in [m, m+d-1]$, of the discrete logarithmic function on $\widetilde{S}_m \subset \widetilde{S}$ are given by:*

$$\ell_n^{(r)} = \begin{cases} 2\Big(1 + \frac{1}{3} + \cdots + \frac{1}{n-1}\Big), & n \text{ even}, \\ \log(\theta_r) = i\gamma_r, & n \text{ odd}. \end{cases} \tag{7.31}$$

Here the values $\log(\theta_r) = i\gamma_r$ *are chosen in the interval* (7.26)*.*

Proof. Comparing formula (7.30) with (7.24), we see that the values $\ell_n^{(r)}$ can be obtained from the expansion of $\log(\lambda)$ in a neighborhood of $\lambda = \theta_r$

into the power series with respect to the powers of $(\lambda-\theta_r)/(\lambda+\theta_r)$. This expansion reads:

$$\log(\lambda)=\log(\theta_r)+\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\Big(\frac{\lambda-\theta_r}{\lambda+\theta_r}\Big)^n.$$

Thus, we come to a simple difference equation

$$n(\ell_{n+1}^{(r)}-\ell_{n-1}^{(r)})=1-(-1)^n, \tag{7.32}$$

with the initial conditions

$$\ell_0^{(r)}=\ell(\mathbf{0})=0,\qquad \ell_1^{(r)}=\ell(\boldsymbol{e}_r)=\log(\theta_r), \tag{7.33}$$

which yield (7.31). □

Observe that values (7.31) at even (resp. odd) points imitate the behavior of the real (resp. imaginary) part of the function $\log(\lambda)$ along the half-lines $\arg(\lambda)=\arg(\theta_r)$. This can be easily extended to the whole of $\widetilde{S}$. Restricted to the black points $\boldsymbol{n}\in\widetilde{S}$ (those with $n_1+\cdots+n_d$ even), the discrete logarithmic function models the real part of the logarithm. In particular, it is real-valued and does not branch: its values on $\widetilde{S}_m$ depend on m (mod $2d$) only. In other words, it is a well-defined function on S_m. On the contrary, the discrete logarithmic function restricted to the white points $\boldsymbol{n}\in\widetilde{S}$ (those with $n_1+\cdots+n_d$ odd) takes purely imaginary values, and increases by $2\pi i$ as m increases by $2d$. Hence, this restricted function models the imaginary part of the logarithm.

It turns out that recurrence relations (7.32) are characteristic for an important class of solutions of the discrete Cauchy-Riemann equations, namely for the *isomonodromic* solutions. In order to introduce this class, recall that discrete holomorphic functions in $\mathbb{Z}^d$ possess a zero curvature representation with transition matrices (7.19). The *moving frame* $\Psi(\cdot,\lambda):\mathbb{Z}^d\to \mathrm{GL}(2,\mathbb{C})[\lambda]$ is defined by prescribing some $\Psi(\mathbf{0};\lambda)$, and by extending it recurrently according to the formula

$$\Psi(\boldsymbol{n}+\boldsymbol{e}_k;\lambda)=L_k(\boldsymbol{n};\lambda)\Psi(\boldsymbol{n};\lambda). \tag{7.34}$$

Finally, define the matrices $A(\cdot;\lambda):\mathbb{Z}^d\to \mathrm{gl}(2,\mathbb{C})[\lambda]$ by

$$A(\boldsymbol{n};\lambda)=\frac{d\Psi(\boldsymbol{n};\lambda)}{d\lambda}\Psi^{-1}(\boldsymbol{n};\lambda). \tag{7.35}$$

These matrices satisfy a recurrence relation, which is obtained by differentiating (7.34),

$$A(\boldsymbol{n}+\boldsymbol{e}_k;\lambda)=\frac{dL_k(\boldsymbol{n};\lambda)}{d\lambda}L_k^{-1}(\boldsymbol{n};\lambda)+L_k(\boldsymbol{n};\lambda)A(\boldsymbol{n};\lambda)L_k^{-1}(\boldsymbol{n};\lambda), \tag{7.36}$$

and therefore they are determined uniquely upon fixing some $A(\mathbf{0};\lambda)$.

Definition 7.13. (Isomonodromy) *A discrete holomorphic function $f : \mathbb{Z}^d \to \mathbb{C}$ is called isomonodromic if, for some choice of $A(\mathbf{0};\lambda)$, the matrices $A(\boldsymbol{n};\lambda)$ are meromorphic in λ, with poles whose positions and orders do not depend on $\boldsymbol{n} \in \mathbb{Z}^d$.*

This term originates in the theory of integrable nonlinear differential equations, where it is used for solutions with a similar analytic characterization.

It is clear how to extend Definition 7.13 to functions on the covering $\widetilde{S}$. In the following statement, we restrict ourselves to the octant $S_1 = (\mathbb{Z}_+)^d$ for notational simplicity.

Theorem 7.14. (Discrete logarithmic function is isomonodromic) *For a proper choice of $A(\mathbf{0};\lambda)$, the matrices $A(\boldsymbol{n};\lambda)$ at any point $\boldsymbol{n} \in (\mathbb{Z}_+)^d$ have simple poles only:*

$$A(\boldsymbol{n};\lambda) = \frac{A^{(0)}(\boldsymbol{n})}{\lambda} + \sum_{l=1}^{d} \Big(\frac{B^{(l)}(\boldsymbol{n})}{\lambda+\theta_l} + \frac{C^{(l)}(\boldsymbol{n})}{\lambda-\theta_l} \Big), \tag{7.37}$$

with

$$A^{(0)}(\boldsymbol{n}) = \begin{pmatrix} 0 & (-1)^{n_1+\cdots+n_d} \\ 0 & 0 \end{pmatrix}, \tag{7.38}$$

$$B^{(l)}(\boldsymbol{n}) = n_l \begin{pmatrix} 1 & -(\ell(\boldsymbol{n}) + \ell(\boldsymbol{n}-\boldsymbol{e}_l)) \\ 0 & 0 \end{pmatrix}, \tag{7.39}$$

$$C^{(l)}(\boldsymbol{n}) = n_l \begin{pmatrix} 0 & \ell(\boldsymbol{n}+\boldsymbol{e}_l) + \ell(\boldsymbol{n}) \\ 0 & 1 \end{pmatrix}. \tag{7.40}$$

At any point $\boldsymbol{n} \in \widetilde{S}$, the following constraint holds:

$$\sum_{l=1}^{d} n_l \Big(\ell(\boldsymbol{n}+\boldsymbol{e}_l) - \ell(\boldsymbol{n}-\boldsymbol{e}_l) \Big) = 1 - (-1)^{n_1+\cdots+n_d}. \tag{7.41}$$

Proof. The proper choice of $A(\mathbf{0};\lambda)$ mentioned in the Theorem, can be read off formula (7.38):

$$A(\mathbf{0};\lambda) = \frac{1}{\lambda}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The proof consists of two parts.

(i) First, one proves the claim for the points of the coordinate semi-axes. For any $r = 1, \ldots, d$, construct the matrices $A(n\boldsymbol{e}_r;\lambda)$ along the r-th coordinate semi-axis via formula (7.36) with transition matrices (7.19). This formula shows that the singularities of $A(n\boldsymbol{e}_r;\lambda)$ are poles at $\lambda = 0$ and at $\lambda = \pm\theta_r$, and that the pole $\lambda = 0$ remains simple for all $n > 0$. By a direct computation and induction, one shows that it is exactly the recurrence relation (7.32) for $f_n^{(r)} = f(n\boldsymbol{e}_r)$ which assures that the poles $\lambda = \pm\theta_r$ remain simple for all $n > 0$. Thus, (7.37) holds on the r-th coordinate semiaxis, with $B^{(l)}(n\boldsymbol{e}_r) = C^{(l)}(n\boldsymbol{e}_r) = 0$ for $l \neq r$.

(ii) The second part of the proof is conceptual, and is based upon the multidimensional consistency only. Proceed by induction, filling out the hull of the coordinate semiaxes: each new point is of the form $\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_k$, $j \neq k$, with three points $\boldsymbol{n}$, $\boldsymbol{n} + \boldsymbol{e}_j$, and $\boldsymbol{n} + \boldsymbol{e}_k$ known from the previous steps, where the statements of the proposition are assumed to hold. Suppose that (7.37) holds at $\boldsymbol{n} + \boldsymbol{e}_j$, $\boldsymbol{n} + \boldsymbol{e}_k$. The new matrix $A(\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_k;\lambda)$ is obtained by two alternative formulas,

$$A(\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_k;\lambda) = \frac{dL_k(\boldsymbol{n} + \boldsymbol{e}_j;\lambda)}{d\lambda}\, L_k^{-1}(\boldsymbol{n} + \boldsymbol{e}_j;\lambda) + L_k(\boldsymbol{n} + \boldsymbol{e}_j;\lambda)A(\boldsymbol{n} + \boldsymbol{e}_j;\lambda)L_k^{-1}(\boldsymbol{n} + \boldsymbol{e}_j;\lambda), \tag{7.42}$$

and the other with k and j interchanged. Equation (7.42) shows that all poles of $A(\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_k;\lambda)$ remain simple, with the possible exception of $\lambda = \pm\theta_k$, whose orders might increase by 1. The same statement holds with k replaced by j. Therefore, *all* poles remain simple, and (7.37) holds at $\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_k$. Formulas (7.38)–(7.40) and constraint (7.41) follow by direct computations based on (7.42). $\square$

7.6. Exercises

7.1. Let $\mathcal{D}$ be a bipartite quad-graph, with black vertices x_j and white vertices y_j. Let $\mu : E(\mathcal{D}) \to \mathbb{C}$ be a function such that, for any elementary quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$,

$$\mu(x_0, y_0)\mu(x_1, y_1) = \mu(x_0, y_1)\mu(x_1, y_0).$$

Show that there exists a function $\theta : V(\mathcal{D}) \to \mathbb{C}$ such that for every edge $(x, y) \in E(\mathcal{D})$ we have $i\mu(x, y) = \theta(y)/\theta(x)$. If μ is real-valued, then one can assume that θ takes real values at black points and imaginary values at white points.

7.2. Prove by induction that the entries of the matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \overset{\curvearrowleft}{\prod_k} \begin{pmatrix} 1 & -\nu_k \\ \nu_k & 1 \end{pmatrix}$$

are given by

$$A = \frac{1}{2}\Big(\prod_k (1+i\nu_k) + \prod_k (1-i\nu_k)\Big), \quad B = \frac{1}{2i}\Big(\prod_k (1+i\nu_k) - \prod_k (1-i\nu_k)\Big).$$

7.3. Check that the function $f : \mathbb{Z}^2 \to \mathbb{C}$ given by $f(m,n) = (m\theta_1 + n\theta_2)^2$ satisfies the discrete Cauchy-Riemann equation

$$\frac{f(m+1,n+1) - f(m,n)}{f(m+1,n) - f(m,n+1)} = \frac{\theta_1+\theta_2}{\theta_1-\theta_2}.$$

Generalize this function ("discrete z^2") for $\mathbb{Z}^d$ and for arbitrary quad-graphs $\mathcal{D}$.

7.4. Find the "discrete z^3", i.e., the function $f : \mathbb{Z}^2 \to \mathbb{C}$ which is polynomial in m, n of degree 3, with cubic terms $(m\theta_1+n\theta_2)^3$, and satisfying the discrete Cauchy-Riemann equations.

7.5. Prove that for the functions $g_k(\lambda)$ from (7.24),

$$\mathrm{Res}_{\lambda=\theta_k} \Big(\frac{\lambda+\theta_k}{\lambda-\theta_k}\Big)^n g_k(\lambda) = f_n^{(k)} - f(\mathbf{0}).$$

7.6. Estimate the difference $\ell_n^{(k)} - \log n$ for the values given in (7.31), for n even.

7.7. Bibliographical notes

Section 7.1: Basic notions of discrete linear complex analysis. The standard discretization of harmonic and holomorphic functions on the regular square grid goes back to Ferrand (1944) and Duffin (1956). This discretization of the Cauchy-Riemann equations apparently preserves the majority of important structural features. A pioneering step in the direction of further generalization of the notions of discrete harmonic and discrete holomorphic functions was undertaken by Duffin (1968), where the combinatorics of $\mathbb{Z}^2$ was given up in favor of arbitrary planar graphs with rhombic faces. A far reaching generalization of these ideas was given by Mercat (2001), who extended the theory to discrete Riemann surfaces.

Section 7.2: Moutard transformation for discrete Cauchy-Riemann equations. For general Moutard transformations see the bibliographical note to Section 2.3 and Exercise 2.27. A further discussion of the Darboux transformation for discrete Laplace operators induced by the Moutard transformation for discrete Cauchy-Riemann equations can be found in Doliwa-Grinevich-Nieszporski-Santini (2007).

Section 7.3: Integrable discrete Cauchy-Riemann equations. Condition (7.13) on the system of angles $\phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to (0,\pi)$ characterizing rhombic embedding was given in Kenyon-Schlenker (2004). Theorems

7.5, 7.6 characterizing 3D consistent (integrable) Cauchy-Riemann equations and their zero curvature representation from Theorem 7.7 are from Bobenko-Mercat-Suris (2005).

Section 7.4: Discrete exponential functions. A discrete exponential function on $\mathbb{Z}^2$ was defined and studied in Ferrand (1944) and Duffin (1956). It was generalized for quad-graphs $\mathcal{D}$ in Mercat (2001) and Kenyon (2002). The question whether discrete exponential functions form a basis in the space of discrete holomorphic functions on $\mathcal{D}$ (Theorem 7.9) was posed in Kenyon (2002) and answered in Bobenko-Mercat-Suris (2005).

Section 7.5: Discrete logarithmic function. The discrete logarithmic function on a rhombic quad-graph $\mathcal{D}$ was introduced in Kenyon (2002). Also the asymptotics (7.28) as well as Theorem 7.10 were proven in that paper. All other results in this section, starting with the extension of the discrete logarithmic function to $\mathbb{Z}^d$, are from Bobenko-Mercat-Suris (2005). For the theory of isomonodromic solutions of differential equations and its application to integrable systems see Fokas-Its-Kapaev-Novokshenov (2006). Isomonodromic constraint (7.41) was found in Nijhoff-Ramani-Grammaticos-Ohta (2001), with no relation to the discrete logarithmic function.

Chapter 8

Discrete Complex Analysis. Integrable Circle Patterns

8.1. Circle patterns

The idea that circle packings and, more generally, circle patterns serve as a discrete counterpart of analytic functions is by now well established. We give here a presentation of several results in this area, which treat the interrelations between circle patterns and integrable systems.

Definition 8.1. (Circle pattern) *Let $\mathcal{G}$ be an arbitrary cell decomposition of an open or closed disk in $\mathbb{C}$. A map $z : V(\mathcal{G}) \to \mathbb{C}$ defines a circle pattern with combinatorics of $\mathcal{G}$ if the following condition is satisfied. Let $y \in F(\mathcal{G}) \sim V(\mathcal{G}^*)$ be an arbitrary face of $\mathcal{G}$, and let $x_1, x_2, \ldots, x_n$ be its consecutive vertices. Then the points $z(x_1), z(x_2), \ldots, z(x_n) \in \mathbb{C}$ lie on a circle, and their circular order is just the listed one. We denote this circle by $C(y)$, thus putting it into a correspondence with the face y, or, equivalently, with the respective vertex of the dual cell decomposition $\mathcal{G}^*$.*

As a consequence of this condition, if two faces $y_0, y_1 \in F(\mathcal{G})$ have a common edge (x_0, x_1), then the circles $C(y_0)$ and $C(y_1)$ intersect in the points $z(x_1), z(x_2)$. In other words, the edges from $E(\mathcal{G})$ correspond to pairs of neighboring (intersecting) circles of the pattern. Similarly, if several faces $y_1, y_2, \ldots, y_m \in F(\mathcal{G})$ meet in one point $x_0 \in V(\mathcal{G})$, then the corresponding circles $C(y_1), C(y_2), \ldots, C(y_m)$ also have a common intersection point $z(x_0)$. A finite piece of a circle pattern is shown in Figure 8.1.

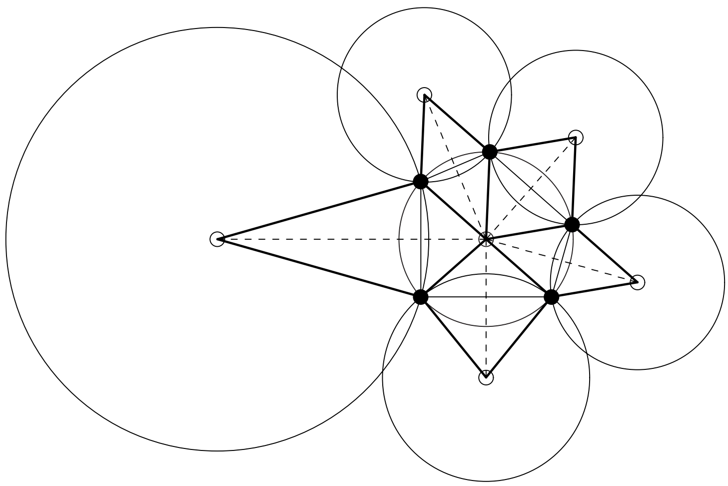


Figure 8.1. Circle pattern.

Given a circle pattern with combinatorics of $\mathcal{G}$, we can extend the function z to the vertices of the dual graph, setting

$$z(y) = \text{ center of the circle } C(y), \quad y \in F(\mathcal{G}) \simeq V(\mathcal{G}^*).$$

After this extension, the map z is defined on all of $V(\mathcal{D}) = V(\mathcal{G}) \sqcup V(\mathcal{G}^*)$, where $\mathcal{D}$ is the double of $\mathcal{G}$. Consider a face of the double. Its z-image is a quadrilateral of the *kite* form, whose vertices correspond to the intersection points and the centers of two neighboring circles C_0, C_1 of the pattern. Denote the radii of C_0, C_1 by r_0, r_1, respectively. Let x_0, x_1 correspond to the intersection points, and let y_0, y_1 correspond to the centers of the circles. Give the circles C_0, C_1 a positive orientation (induced by the orientation of the underlying $\mathbb{C}$), and let $\phi \in (0, \pi)$ stand for the intersection angle of these oriented circles. This angle ϕ is equal to the kite angles at the "black" vertices $z(x_0), z(x_1)$; see Figure 8.2, where the complementary angle $\phi^* = \pi - \phi$ is also shown. It will be convenient to assign the intersection angle $\phi = \phi(e)$ to the "black" edge $e = (x_0, x_1) \in E(\mathcal{G})$, and to assign the complementary angle $\phi^* = \phi(e^*)$ to the dual "white" edge $e^* = (y_0, y_1) \in E(\mathcal{G}^*)$. Thus, the function $\phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to (0, \pi)$ satisfies (7.11).

The geometry of Figure 8.2 yields following relations. First of all, the cross-ratio of the four points corresponding to the vertices of a quadrilateral face of $\mathcal{D}$ is expressed through the intersection angle of the circles C_0, C_1:

$$q(z(x_0), z(y_0), z(x_1), z(y_1)) = \exp(2i\phi^*). \tag{8.1}$$

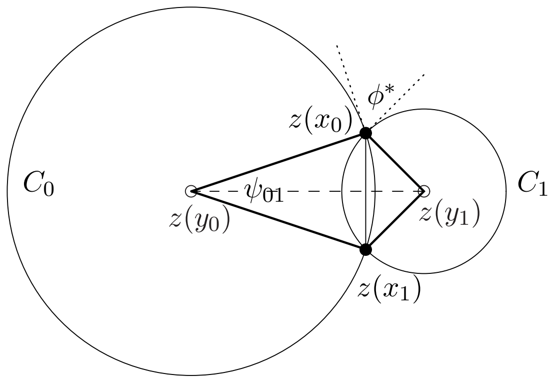


Figure 8.2. Two intersecting circles.

Furthermore, running around a "black" vertex of $\mathcal{D}$ (a common intersection point of several circles of the pattern), we see that the sum of the consecutive kite angles vanishes $\pmod{2\pi}$, hence:

$$\prod_{e \in \mathrm{star}(x_0;\mathcal{G})} \exp(i\phi(e)) = 1, \quad \forall x_0 \in V(\mathcal{G}). \tag{8.2}$$

Finally, let ψ_{01} be the angle of the kite $(z(x_0), z(y_0), z(x_1), z(y_1))$ at the "white" vertex $z(y_0)$, i.e., the angle between the half-lines from the center $z(y_0)$ of the circle C_0 to the intersection points $z(x_0), z(x_1)$ with its circle C_1. It is not difficult to calculate this angle:

$$\exp(i\psi_{01}) = \frac{r_0 + r_1 \exp(i\phi^*)}{r_0 + r_1 \exp(-i\phi^*)}. \tag{8.3}$$

Running around the "white" vertex of $\mathcal{D}$, we come to the relation

$$\prod_{j=1}^{m} \frac{r_0 + r_j \exp(i\phi_j^*)}{r_0 + r_j \exp(-i\phi_j^*)} = 1, \quad \forall y_0 \in V(\mathcal{G}^*), \tag{8.4}$$

where the product is extended over all edges $e_j^* = (y_0, y_j) \in \mathrm{star}(y_0; \mathcal{G}^*)$, and $\phi_j^* = \phi(e_j^*)$, while r_j are the radii of the circles $C_j = C(y_j)$.

8.2. Integrable cross-ratio and Hirota systems

Our main interest is in the circle patterns with prescribed combinatorics and with prescribed intersection angles for all pairs of neighboring angles. According to formula (8.1), prescribing all intersection angles amounts to prescribing cross-ratios for all quadrilateral faces of the quad-graph $\mathcal{D}$. Thus, we come to the study of cross-ratio equations on arbitrary quad-graphs.

Let there be given a function $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ satisfying the condition

$$Q(e^*) = 1/Q(e), \qquad \forall e \in E(\mathcal{G}). \tag{8.5}$$

Definition 8.2. (Cross-ratio system) *The cross-ratio system on $\mathcal{D}$ corresponding to the function Q consists of the following equations for a function $z : V(\mathcal{D}) \to \mathbb{C}$, one for any quadrilateral face (x_0, y_0, x_1, y_1) of $\mathcal{D}$:*

$$q(z(x_0), z(y_0), z(x_1), z(y_1)) = Q(x_0, x_1) = 1/Q(y_0, y_1). \tag{8.6}$$

An important distinction from the discrete Cauchy-Riemann equations is that the cross-ratio equations actually do not depend on the orientation of quadrilaterals.

We have already encountered 3D consistent cross-ratio systems on $\mathbb{Z}^d$ in Section 6.7 (see equation (6.33)), in the version with labelled edges. A natural generalization to the case of arbitrary quad-graphs is this:

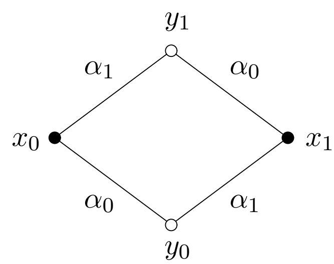


Figure 8.3. Quadrilateral, with a labelling of undirected edges.

Definition 8.3. (Integrable cross-ratio system) *A cross-ratio system is called integrable if there exists a labelling $\alpha : E(\mathcal{D}) \to \mathbb{C}$ of* undirected *edges of $\mathcal{D}$ such that the function Q admits the following factorization (in the notation of Figure* 8.3*):*

$$Q(x_0, x_1) = \frac{1}{Q(y_0, y_1)} = \frac{\alpha_0}{\alpha_1}. \tag{8.7}$$

Clearly, integrable cross-ratio systems are 3D consistent (see Theorem 4.26), admit *Bäcklund transformations*, and possess *zero curvature representation* with the transition matrices (6.47). It is not difficult to give an equivalent reformulation of the integrability condition (8.7).

Theorem 8.4. (Integrability condition of a cross-ratio system) *A cross-ratio system with the function $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ is integrable if* *and only if for all $x_0 \in V(\mathcal{G})$ and for all $y_0 \in V(\mathcal{G}^*)$ the following conditions are fulfilled:*

$$\prod_{e \in \mathrm{star}(x_0;\mathcal{G})} Q(e) = 1, \qquad \prod_{e^* \in \mathrm{star}(y_0;\mathcal{G}^*)} Q(e^*) = 1. \tag{8.8}$$

For a labelling of undirected edges $\alpha : E(\mathcal{D}) \to \mathbb{C}$, we can find a labelling $\theta : \vec{E}(\mathcal{D}) \to \mathbb{C}$ of directed edges such that $\alpha = \theta^2$. The function $p : V(\mathcal{D}) \to \mathbb{C}$ defined by $p(y) - p(x) = \theta(x, y)$ gives, according to (8.8), a parallelogram realization (ramified embedding) of the quad-graph $\mathcal{D}$. The cross-ratio equations are written as

$$q(z(x_0), z(y_0), z(x_1), z(y_1)) = \frac{\theta_0^2}{\theta_1^2} = q(p(x_0), p(y_0), p(x_1), p(y_1)); \tag{8.9}$$

in other words, for any quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$, the cross-ratio of the vertices of its image under the map z is equal to the cross-ratio of the vertices of the corresponding parallelogram. In particular, one always has the *trivial* solution $z(x) \equiv p(x)$ for all $x \in V(\mathcal{D})$.

A very useful transformation of the cross-ratio system is given by the following construction.

Definition 8.5. (Hirota system) *For a given labelling of directed edges $\theta : \vec{E}(\mathcal{D}) \to \mathbb{C}$, the Hirota system consists of the following equations for the function $w : V(\mathcal{D}) \to \mathbb{C}$, one for every quadrilateral face $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$:*

$$\theta_0 w(x_0)w(y_0) + \theta_1 w(y_0)w(x_1) - \theta_0 w(x_1)w(y_1) - \theta_1 w(y_1)w(x_0) = 0. \tag{8.10}$$

Note that the Hirota equation coincides with equation (6.30) of Section 6.7 (by the way, this shows that also in that previous version it was natural to assign parameters to *directed* edges). In terms of the parallelogram realization $p : V(\mathcal{D}) \to \mathbb{C}$ of the quad-graph $\mathcal{D}$ corresponding to the labelling θ, equation (8.10) reads:

$$\begin{aligned} w(x_0)w(y_0)\big(p(y_0) - p(x_0)\big) + w(y_0)w(x_1)\big(p(x_1) - p(y_0)\big) \\ + w(x_1)w(y_1)\big(p(y_1) - p(x_1)\big) + w(y_1)w(x_0)\big(p(x_0) - p(y_1)\big) = 0. \end{aligned} \tag{8.11}$$

Obviously, a transformation $w \mapsto cw$ on $V(\mathcal{G})$ and $w \mapsto c^{-1}w$ on $V(\mathcal{G}^*)$ with a constant $c \in \mathbb{C}$, hereafter called a black-white scaling, maps solutions of the Hirota system into solutions. A relation between the cross-ratio and the Hirota system is based on the following observation:

Theorem 8.6. (Relation between cross-ratio and Hirota systems) *Let $w : V(\mathcal{D}) \to \mathbb{C}$ be a solution of the Hirota system. Then the relation*

$$z(y) - z(x) = \theta(x, y)w(x)w(y) = w(x)w(y)\big(p(y) - p(x)\big) \tag{8.12}$$

for all directed edges $(x, y) \in \vec{E}(\mathcal{D})$ *defines a unique (up to an additive constant) function* $z : V(\mathcal{D}) \to \mathbb{C}$ *which is a solution of the cross-ratio system* (8.9). *Conversely, for any solution* z *of the cross-ratio system* (8.9), *relation* (8.12) *defines a unique (up to a black-white scaling) function* $w : V(\mathcal{D}) \to \mathbb{C}$*; this function* w *solves the Hirota system* (8.10).

In particular, the trivial solution $z(x) = p(x)$ of the cross-ratio system corresponds to the trivial solution of the Hirota system, $w(x) \equiv 1$ for all $x \in V(\mathcal{D})$. By a direct computation one can establish the following fundamental property.

Theorem 8.7. (Integrability of Hirota system) *The Hirota system* (8.10) *is 3D consistent.*

As a usual consequence, the Hirota system admits Bäcklund transformations and possesses zero curvature representation with transition matrices along the edge $(x, y) \in \vec{E}(\mathcal{D})$ given by

$$L(y, x, \theta; \lambda) = \begin{pmatrix} 1 & -\theta w(y) \\ -\lambda\theta/w(x) & w(y)/w(x) \end{pmatrix}, \tag{8.13}$$

where $\theta = p(y) - p(x)$.

8.3. Integrable circle patterns

Returning to circle patterns, let $\{z(x) : x \in V(\mathcal{G})\}$ be the intersection points of the circles of a pattern, and let $\{z(y) : y \in V(\mathcal{G}^*)\}$ be their centers. Due to (8.1), the function $z : V(\mathcal{D}) \to \mathbb{C}$ satisfies a cross-ratio system with $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{S}^1$ defined as $Q(e) = \exp(2i\phi(e))$. Because of (8.2), the first of the integrability conditions (8.8) is fulfilled for an *arbitrary* circle pattern. Therefore, integrability of the cross-ratio system for circle patterns with prescribed intersection angles $\phi : E(\mathcal{G}^*) \to (0, \pi)$ is equivalent to

$$\prod_{e^* \in \mathrm{star}(y_0; \mathcal{G}^*)} \exp(2i\phi(e^*)) = 1, \quad \forall y_0 \in V(\mathcal{G}^*). \tag{8.14}$$

This is equivalent to the existence of the edge labelling $\alpha : E(\mathcal{D}) \to \mathbb{C}$ such that, in the notation of Figure 8.2,

$$\exp(2i\phi^*) = \frac{\alpha_0}{\alpha_1}. \tag{8.15}$$

Moreover, one can assume that the labelling α takes values in $\mathbb{S}^1$.

Our definition of integrable circle patterns will require somewhat more than integrability of the corresponding cross-ratio system.

Definition 8.8. (Integrable circle pattern) *A circle pattern with prescribed intersection angles* $\phi : E(\mathcal{G}^*) \to (0, \pi)$ *is called integrable if*

$$\prod_{e^* \in \text{star}(y_0; \mathcal{G}^*)} \exp(i\phi(e^*)) = 1, \quad \forall y_0 \in V(\mathcal{G}^*), \tag{8.16}$$

i.e., if for any circle of the pattern the sum of its intersection angles with all neighboring circles vanishes $\pmod{2\pi}$.

This requirement is equivalent to a somewhat sharper factorization than (8.15), namely, to the existence of a labelling of the directed edges $\theta : \vec{E}(\mathcal{D}) \to \mathbb{S}^1$ such that, in the notation of Figure 8.2,

$$\exp(i\phi) = \frac{\theta_1}{\theta_0} \quad \Leftrightarrow \quad \exp(i\phi^*) = -\frac{\theta_0}{\theta_1}. \tag{8.17}$$

(Of course, the last condition yields (8.15) with $\alpha = \theta^2$.) The parallelogram realization $p : V(\mathcal{D}) \to \mathbb{C}$ corresponding to the labelling $\theta \in \mathbb{S}^1$ is actually a *rhombic* one.

Theorem 8.9. (Isoradial integrability criterion) *Combinatorial data* $\mathcal{G}$ *and intersection angles* $\phi : E(\mathcal{G}) \to (0, \pi)$ *belong to an integrable circle pattern if and only if they admit an isoradial realization. In this case, the dual combinatorial data* $\mathcal{G}^*$ *and intersection angles* $\phi : E(\mathcal{G}^*) \to (0, \pi)$ *admit a realization as an isoradial circle pattern, as well.*

Proof. The rhombic realization $p : V(\mathcal{D}) \to \mathbb{C}$ of the quad-graph $\mathcal{D}$ corresponds to a circle pattern with the same combinatorics and the same intersection angles as the original one and with all radii equal to 1, and, simultaneously, to an analogous dual circle pattern. □

Consider a rhombic realization $p : V(\mathcal{D}) \to \mathbb{C}$ of $\mathcal{D}$. Solutions $z : V(\mathcal{D}) \to \mathbb{C}$ of the corresponding integrable cross-ratio system which come from integrable circle patterns are characterized by the property that the z-image of any quadrilateral (x_0, y_0, x_1, y_1) from $F(\mathcal{D})$ is a *kite* with the prescribed angle ϕ at the black vertices $z(x_0)$, $z(x_1)$ (cf. Figure 8.2). It turns out that the description of this class of kite solutions admits a more convenient analytic characterization in terms of the corresponding solutions $w : V(\mathcal{D}) \to \mathbb{C}$ of the Hirota system defined by (8.12).

Theorem 8.10. (Circle pattern solutions of Hirota system) *The solution* z *of the cross-ratio system corresponds to a circle pattern if and only if the solution* w *of the Hirota system, corresponding to* z *via* (8.12), *satisfies the condition*

$$w(x) \in \mathbb{S}^1, \quad w(y) \in \mathbb{R}_+, \quad \forall x \in V(\mathcal{G}),\ y \in V(\mathcal{G}^*). \tag{8.18}$$

The values $w(y) \in \mathbb{R}_+$ *have then the interpretation of the radii of the circles* $C(y)$, *while the (arguments of the) values* $w(x) \in \mathbb{S}^1$ *measure the rotation of the tangents to the circles intersecting at* $z(x)$ *with respect to the isoradial realization of the pattern.*

Proof. As is easily seen, the kite conditions are equivalent to

$$\frac{|w(x_0)|}{|w(x_1)|} = 1 \quad \text{and} \quad \frac{w(y_0)}{w(y_1)} \in \mathbb{R}_+.$$

This yields (8.18), possibly upon a black-white scaling. □

The conditions (8.18) form an *admissible reduction* of the Hirota system with $\theta \in \mathbb{S}^1$, in the following sense: if any three of the four points $w(x_0)$, $w(y_0)$, $w(x_1)$, $w(y_1)$ satisfy the condition (8.18), then so does the fourth one. This is immediately seen, if one rewrites the Hirota equation (8.10) in one of the two equivalent forms:

$$\frac{w(x_1)}{w(x_0)} = \frac{\theta_1 w(y_1) - \theta_0 w(y_0)}{\theta_1 w(y_0) - \theta_0 w(y_1)} \quad \Leftrightarrow \quad \frac{w(y_1)}{w(y_0)} = \frac{\theta_0 w(x_0) + \theta_1 w(x_1)}{\theta_0 w(x_1) + \theta_1 w(x_0)}. \tag{8.19}$$

As a consequence of this remark, we obtain Bäcklund transformations for integrable circle patterns.

Theorem 8.11. (Bäcklund transformations of integrable circle patterns) *Let all* $\theta \in \mathbb{S}^1$, *and let* $p : V(\mathcal{D}) \to \mathbb{C}$ *be the corresponding rhombic realization of* $\mathcal{D}$. *Let the solution* $w : V(\mathcal{D}) \to \mathbb{C}$ *of the Hirota system correspond to a circle pattern with combinatorics of* $\mathcal{G}$, *i.e., satisfy* (8.18). *Consider its Bäcklund transformation* $w^+ : V(\mathcal{D}) \to \mathbb{C}$ *with an arbitrary parameter* $\lambda \in \mathbb{S}^1$ *and with an arbitrary initial value* $w^+(x_0) \in \mathbb{R}_+$ *or* $w^+(y_0) \in \mathbb{S}^1$. *Then*

$$w^+(x) \in \mathbb{R}_+, \quad w^+(y) \in \mathbb{S}^1, \quad \forall x \in V(\mathcal{G}),\ y \in V(\mathcal{G}^*), \tag{8.20}$$

so that w^+ *corresponds to a circle pattern with combinatorics of* $\mathcal{G}^*$, *which we call a Bäcklund transform of the original circle pattern.*

We close this section by mentioning several Laplace type equations which can be used to describe integrable circle patterns. First of all, the restriction of the function z to $V(\mathcal{G})$ (i.e., to the intersection points of the circles) satisfies the equations

$$\sum_{k=1}^{n} \frac{\alpha_k - \alpha_{k+1}}{z(x_k) - z(x_0)} = 0.$$

Here $z(x_0)$ is any intersection point where n circles $C(y_1), \dots, C(y_n)$ meet, $z(x_k)$ is the second intersection point of $C(y_k)$ with $C(y_{k+1})$ for each k, and the α_k are the labels on the edges $(x_0, y_k) \in E(\mathcal{D})$. Analogously, the restriction of the function z to $V(\mathcal{G}^*)$ (i.e., to the centers of the circles) satisfies the equation

$$\sum_{j=1}^{m} \frac{\alpha_{j-1} - \alpha_j}{z(y_j) - z(y_0)} = 0.$$

Here $z(y_0)$ is the center of any circle $C(y_0)$ that intersects the m circles $C(y_1), \ldots, C(y_m)$ with centers at the points $z(y_j)$; the intersection of $C(y_0)$ with $C(y_j)$ consists of two points $z(x_{j-1})$, $z(x_j)$, and α_j are the labels on the edges $(y_0, x_j) \in E(\mathcal{D})$. These two Laplace type equations follow from the first claim of Theorem 6.31 applied to the cross-ratio system, which is nothing but the case $(\mathrm{Q1})_{\delta=0}$ of Theorem 6.32.

A similar construction can be applied to the Hirota system, written in the three-leg form (8.19). Again, it yields two multiplicative Laplace type equations — on $\mathcal{G}$ and on $\mathcal{G}^*$. It is instructive to look at the equation on $\mathcal{G}^*$ (for the radii $r_j = w(y_j)$ of the circles):

$$\prod_{j=1}^{m} \frac{\theta_j r_j - \theta_{j-1} r_0}{\theta_j r_0 - \theta_{j-1} r_j} = 1.$$

Due to (8.17), this equation can be written in terms of the intersection angles ϕ_j of $C(y_0)$ with $C(y_j)$, and it takes the form of (8.4). Interestingly, the latter equation holds for *any* circle pattern and is not specific for integrable ones (as opposed to the similar Laplace type equation on $\mathcal{G}$).

8.4. z^a and $\log z$ circle patterns

Due to the 3D consistency of the cross-ratio and the Hirota systems, we can follow the procedure of Section 6.8 and extend solutions of these systems from a quasicrystallic quad-graph $\mathcal{D}$, realized as a quad-surface $\Omega_{\mathcal{D}} \subset \mathbb{Z}^d$, to the whole of $\mathbb{Z}^d$ (more precisely, to the hull of $\Omega_{\mathcal{D}}$). Then, one can ask about isomonodromic solutions. This leads to discrete analogs of the power function. Naturally, these discrete power functions are defined on the same branched covering $\widetilde{S}$ of the set $\bigcup_{m=1}^{2d} S_m \subset \mathbb{Z}^d$ as the discrete logarithmic function of Section 7.5.

The discrete cross-ratio system on $\mathbb{Z}^d$ reads:

$$q\big(z(\boldsymbol{n}), z(\boldsymbol{n}+\boldsymbol{e}_j), z(\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k), z(\boldsymbol{n}+\boldsymbol{e}_k)\big) = \theta_j^2/\theta_k^2, \tag{8.21}$$

and possesses the discrete zero curvature representation with transition matrices along the edges $(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_k)$ of $\mathbb{Z}^d$ given by

$$L_k(\boldsymbol{n}; \lambda) = \begin{pmatrix} 1 & z(\boldsymbol{n}) - z(\boldsymbol{n}+\boldsymbol{e}_k) \\ \lambda\theta_k^2/(z(\boldsymbol{n}) - z(\boldsymbol{n}+\boldsymbol{e}_k)) & 1 \end{pmatrix}. \tag{8.22}$$

Through the transformation

$$z(\boldsymbol{n}+\boldsymbol{e}_k)-z(\boldsymbol{n})=\theta_k w(\boldsymbol{n})w(\boldsymbol{n}+\boldsymbol{e}_k), \tag{8.23}$$

the solutions of the cross-ratio system are related to the solution of the Hirota system in $\mathbb{Z}^d$,

$$\begin{aligned}\theta_j w(\boldsymbol{n})w(\boldsymbol{n}+\boldsymbol{e}_j)+\theta_k w(\boldsymbol{n}+\boldsymbol{e}_j)w(\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k)&\\ -\theta_j w(\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k)w(\boldsymbol{n}+\boldsymbol{e}_k)-\theta_k w(\boldsymbol{n}+\boldsymbol{e}_k)w(\boldsymbol{n})&=0.\end{aligned} \tag{8.24}$$

The latter system possesses a discrete zero curvature representation with transition matrices along the edges $(\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_k)$ of $\mathbb{Z}^d$ given by

$$L_k(\boldsymbol{n};\lambda)=\begin{pmatrix} 1 & -\theta_k w(\boldsymbol{n}+\boldsymbol{e}_k)\\ \lambda\theta_k/w(\boldsymbol{n}) & w(\boldsymbol{n}+\boldsymbol{e}_k)/w(\boldsymbol{n})\end{pmatrix}. \tag{8.25}$$

Special solutions of these two systems on $\widetilde{S}$ are defined by the following choice of initial data.

Definition 8.12. (Discrete z^{2a}) *For $a\in(0,1)$, the discrete z^{2a} is the solution of the cross-ratio system on $\widetilde{S}$ defined by the values on the coordinate semiaxes $z_n^{(r)}=z(n\boldsymbol{e}_r)$, $r\in[m,m+d-1]$, which solve the recurrence relation*

$$n\,\frac{(z_{n+1}-z_n)(z_n-z_{n-1})}{z_{n+1}-z_{n-1}}=az_n \tag{8.26}$$

with the initial conditions

$$z_0^{(r)}=z(\boldsymbol{0})=0,\qquad z_1^{(r)}=z(\boldsymbol{e}_r)=\theta_r^{2a}=\exp(2a\log\theta_r), \tag{8.27}$$

where $\log\theta_r$ is chosen in the interval (7.26).

Definition 8.13. (Discrete w^{2a-1}) *For $a\in(0,1)$, the discrete w^{2a-1} is the solution of the Hirota system on $\widetilde{S}$ defined by the values on the coordinate semiaxes $w_n^{(r)}=w(n\boldsymbol{e}_r)$, $r\in[m,m+d-1]$, which solve the recurrence relation*

$$n\,\frac{w_{n+1}-w_{n-1}}{w_{n+1}+w_{n-1}}=\big(a-\tfrac12\big)\big(1-(-1)^n\big) \tag{8.28}$$

with the initial conditions

$$w_0^{(r)}=w(\boldsymbol{0})=0,\qquad w_1^{(r)}=w(\boldsymbol{e}_r)=\theta_r^{2a-1}=\exp((2a-1)\log\theta_r), \tag{8.29}$$

where $\log\theta_r$ is chosen in the interval (7.26).

By induction, one can derive the following explicit expressions for the solutions $z_n^{(r)}$:

$$z_{2n}^{(r)}=\prod_{k=1}^{n-1}\frac{k+a}{k-a}\cdot\frac{n}{n-a}\cdot\theta_r^{2a},\qquad z_{2n+1}^{(r)}=\prod_{k=1}^{n}\frac{k+a}{k-a}\cdot\theta_r^{2a}, \tag{8.30}$$

and for $w_n^{(r)}$:

$$w_{2n}^{(r)} = \prod_{k=1}^{n} \frac{k-1+a}{k-a}, \qquad w_{2n+1}^{(r)} = \theta_r^{2a-1}. \tag{8.31}$$

Observe the asymptotic relations for $n \to \infty$:

$$z_n^{(r)} = c(a)(n\theta_r)^{2a}\big(1 + O(n^{-1})\big), \tag{8.32}$$

$$w_{2n}^{(r)} = c(a) n^{2a-1}\big(1 + O(n^{-1})\big). \tag{8.33}$$

The main technical advantage of the w variables is seen from the following observation.

Theorem 8.14. (Discrete z^{2a} defines a circle pattern) *The function w^{2a-1} takes values in $\mathbb{R}_+$ at the white points and values in $\mathbb{S}^1$ at the black points. Therefore, the function z^{2a} defines a circle pattern.*

Proof. The claim for w^{2a-1} on the coordinate axes is obvious from the explicit formulas (8.31), and can be extended to the whole of $\widetilde{S}$ according to the remark after Theorem 8.10. The statement for z^{2a} is now a consequence of Theorem 8.10. $\square$

The restriction of z^{2a} to various quad-surfaces $\Omega_{\mathcal{D}}$ give the discrete analogs of the power function on the corresponding quasicrystallic quad-graphs $\mathcal{D}$ with the set $\Theta = \{\pm\theta_1, \dots, \pm\theta_d\}$ of edge slopes; see Figure 8.4. These pictures lead to the conjecture that the circle patterns z^{2a} are embedded. One possible approach to the analytic study of these patterns could be based on applying the well-developed techniques of the theory of isomonodromic solutions. For either of the systems one can introduce the moving frame as in (7.34):

$$\Psi(\boldsymbol{n} + \boldsymbol{e}_k; \lambda) = L_k(\boldsymbol{n}; \lambda)\Psi(\boldsymbol{n}; \lambda),$$

and define its logarithmic derivatives as in (7.35):

$$A(\boldsymbol{n}; \lambda) = \frac{d\Psi(\boldsymbol{n}; \lambda)}{d\lambda}\Psi^{-1}(\boldsymbol{n}; \lambda).$$

Theorem 8.15. (Discrete z^{2a} is isomonodromic) *Consider the solution of the cross-ratio system in $(\mathbb{Z}_+)^d$ with the initial data (8.30). For a proper choice of $A(\mathbf{0}; \lambda)$, the matrices $A(\boldsymbol{n}; \lambda)$ at any point $\boldsymbol{n} \in (\mathbb{Z}_+)^d$ have simple poles only:*

$$A(\boldsymbol{n}; \lambda) = \frac{A^{(0)}(\boldsymbol{n})}{\lambda} + \sum_{l=1}^{d} \frac{B^{(l)}(\boldsymbol{n})}{\lambda - \theta_l^{-2}}, \tag{8.34}$$

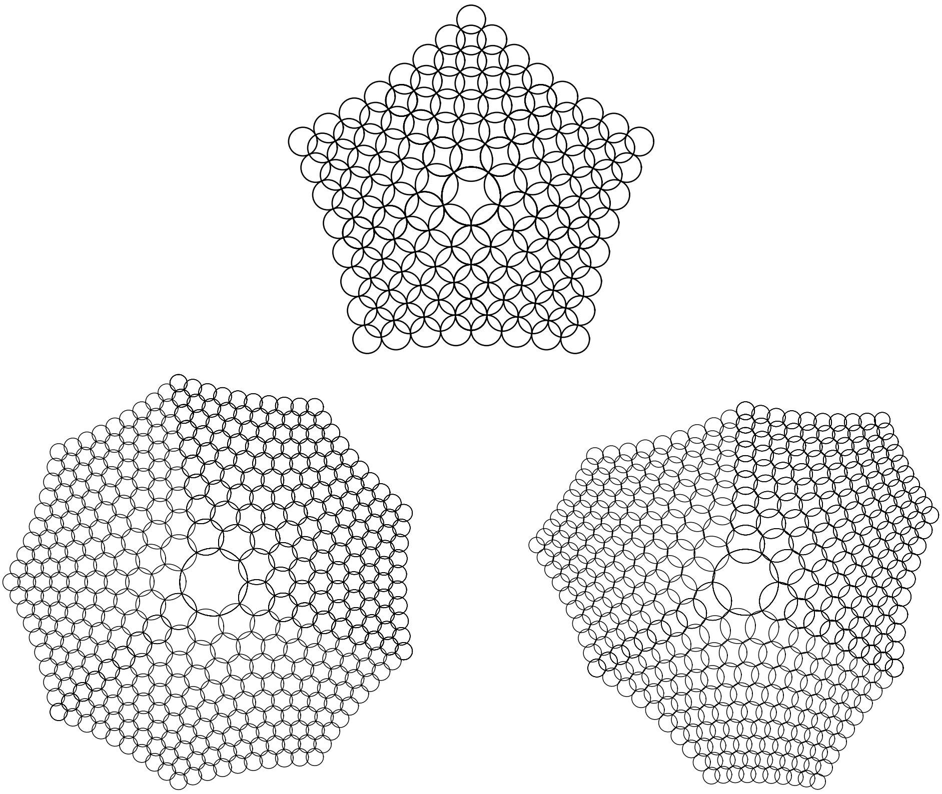


Figure 8.4. Circle patterns $z^{4/5}$ with combinatorics of the square grid, and $z^{2/3}$ with combinatorics of the regular hexagonal lattice (isotropic and nonisotropic).

with

$$A^{(0)}(\boldsymbol{n}) = \begin{pmatrix} -a/2 & -az(\boldsymbol{n}) \\ 0 & a/2 \end{pmatrix}, \tag{8.35}$$

$$\begin{aligned} B^{(l)}(\boldsymbol{n}) &= \frac{n_l}{z(\boldsymbol{n}+\boldsymbol{e}_l) - z(\boldsymbol{n}-\boldsymbol{e}_l)} \\ &\times \begin{pmatrix} z(\boldsymbol{n}+\boldsymbol{e}_l) - z(\boldsymbol{n}) & (z(\boldsymbol{n}+\boldsymbol{e}_l) - z(\boldsymbol{n}))(z(\boldsymbol{n}) - z(\boldsymbol{n}-\boldsymbol{e}_l)) \\ 1 & z(\boldsymbol{n}) - z(\boldsymbol{n}-\boldsymbol{e}_l) \end{pmatrix}. \end{aligned} \tag{8.36}$$

At any point $\boldsymbol{n} \in \widetilde{S}$, the discrete z^{2a} satisfies the following constraint:

$$\sum_{j=1}^{d} n_j \frac{(z(\boldsymbol{n}+\boldsymbol{e}_j) - z(\boldsymbol{n}))(z(\boldsymbol{n}) - z(\boldsymbol{n}-\boldsymbol{e}_j))}{z(\boldsymbol{n}+\boldsymbol{e}_j) - z(\boldsymbol{n}-\boldsymbol{e}_j)} = az(\boldsymbol{n}). \tag{8.37}$$

Theorem 8.16. (Discrete w^{2a-1} is isomonodromic) *Consider the solution of the Hirota system in $(\mathbb{Z}_+)^d$ with the initial data (8.31). For a proper choice of $A(\mathbf{0};\lambda)$, the matrices $A(\boldsymbol{n};\lambda)$ at any point $\boldsymbol{n}\in(\mathbb{Z}_+)^d$ have simple poles only:*

$$A(\boldsymbol{n};\lambda)=\frac{A^{(0)}(\boldsymbol{n})}{\lambda}+\sum_{l=1}^{d}\frac{B^{(l)}(\boldsymbol{n})}{\lambda-\theta_l^{-2}}\,, \tag{8.38}$$

with

$$A^{(0)}(\boldsymbol{n})=\begin{pmatrix}-a/2 & * \\ 0 & a/2\end{pmatrix}, \tag{8.39}$$

(8.40)
$$B^{(l)}(\boldsymbol{n})=\frac{n_l}{w(\boldsymbol{n}+\boldsymbol{e}_l)+w(\boldsymbol{n}-\boldsymbol{e}_l)}\begin{pmatrix}w(\boldsymbol{n}+\boldsymbol{e}_l) & \theta_l w(\boldsymbol{n}+\boldsymbol{e}_l)w(\boldsymbol{n}-\boldsymbol{e}_l) \\ 1/\theta_l & w(\boldsymbol{n}-\boldsymbol{e}_l)\end{pmatrix}.$$

The upper right entry of the matrix $A^{(0)}(\boldsymbol{n})$, denoted by the asterisk in (8.39), is given by $A^{(0)}_{12}(\boldsymbol{n})=-\sum_{l=1}^d B^{(l)}_{12}(\boldsymbol{n})$. At any point $\boldsymbol{n}\in\widetilde{S}$, the discrete w^{2a-1} satisfies the following constraint:

$$\sum_{l=1}^{d} n_l\,\frac{w(\boldsymbol{n}+\boldsymbol{e}_l)-w(\boldsymbol{n}-\boldsymbol{e}_l)}{w(\boldsymbol{n}+\boldsymbol{e}_l)+w(\boldsymbol{n}-\boldsymbol{e}_l)}=\big(a-\tfrac12\big)\big(1-(-1)^{n_1+\cdots+n_d}\big). \tag{8.41}$$

Proof. The proof of both theorems follows the same scheme as the proof of Theorem 7.14: one first shows that the poles of $A(n\boldsymbol{e}_r;\lambda)$ remain simple, due to the recurrence relations (8.26), resp. (8.28), and then shows that the order of poles does not increase at the points $\boldsymbol{n}$ away from the coordinate axes, due to the multidimensional consistency. □

The transition between z and w variables is a matter of straightforward computations. Actually, both theorems are dealing with the same matrices but written in different variables.

It is interesting to study the limiting behavior of the function z^{2a} as $a\to 0$. It is not difficult to see that for all $\boldsymbol{n}\neq\mathbf{0}$ one has $z^{2a}(\boldsymbol{n})\to 1$. Denote

$$\mathrm{L}(\boldsymbol{n})=\lim_{a\to 0}\frac{z^{2a}(\boldsymbol{n})-1}{2a}. \tag{8.42}$$

This function is called the *discrete logarithmic function*; it should not be confused with the namesake function $\ell(\boldsymbol{n})$ in the linear theory (Section 7.5). From (8.42) the following characterization is found: the discrete logarithmic function L is the solution of the discrete cross-ratio system on $\widetilde{S}$ defined by the values on the coordinate semiaxes $\mathrm{L}_n^{(r)} = \mathrm{L}(n\boldsymbol{e}_r)$, $r \in [m, m+d-1]$, which solve the recurrence relation

$$n\,\frac{(\mathrm{L}_{n+1}-\mathrm{L}_n)(\mathrm{L}_n-\mathrm{L}_{n-1})}{\mathrm{L}_{n+1}-\mathrm{L}_{n-1}} = \frac{1}{2} \tag{8.43}$$

with the initial conditions

$$\mathrm{L}_0^{(r)} = \mathrm{L}(\boldsymbol{0}) = \infty, \qquad \mathrm{L}_1^{(r)} = \mathrm{L}(\boldsymbol{e}_r) = \log\theta_r, \tag{8.44}$$

where $\log\theta_r$ is chosen in the interval (7.26). Explicit expressions:

$$\mathrm{L}_{2n}^{(r)} = \log\theta_r + \sum_{k=1}^{n-1}\frac{1}{k} + \frac{1}{2n}\,, \qquad \mathrm{L}_{2n+1}^{(r)} = \log\theta_r + \sum_{k=1}^{n}\frac{1}{k}\,. \tag{8.45}$$

Theorem 8.17. (Circle pattern logarithm is isomonodromic) *The discrete logarithm is isomonodromic and satisfies, at any point $\boldsymbol{n} \in \widetilde{S}$, the following constraint:*

$$\sum_{j=1}^{d} n_j\,\frac{(\mathrm{L}(\boldsymbol{n}+\boldsymbol{e}_j)-\mathrm{L}(\boldsymbol{n}))(\mathrm{L}(\boldsymbol{n})-\mathrm{L}(\boldsymbol{n}-\boldsymbol{e}_j))}{\mathrm{L}(\boldsymbol{n}+\boldsymbol{e}_j)-\mathrm{L}(\boldsymbol{n}-\boldsymbol{e}_j)} = \frac{1}{2}. \tag{8.46}$$

By restriction to quad-surfaces $\Omega_{\mathcal{D}}$, we come to the discrete logarithmic function on arbitrary quasicrystallic quad-graphs $\mathcal{D}$. By construction, they all correspond to circle patterns. A conjecture that these circle patterns are embedded seems plausible (see Figure 8.5).

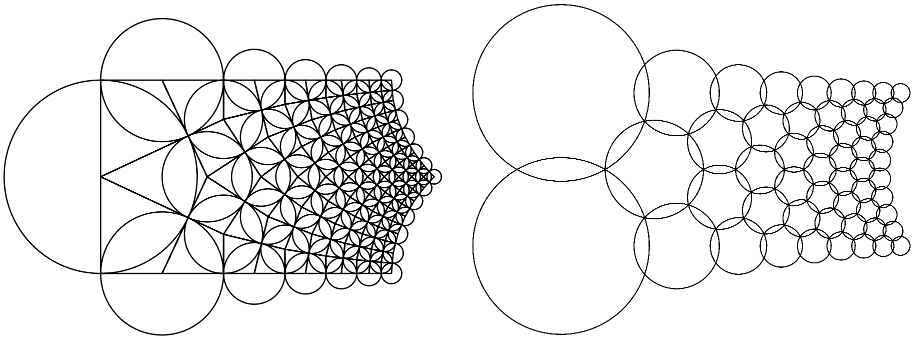


Figure 8.5. Discrete logarithm circle patterns with combinatorics of the regular square and hexagonal lattices.

8.5. Linearization

Let $\theta : \vec{E}(\mathcal{D}) \to \mathbb{C}$ be an edge labelling, and let $p : V(\mathcal{D}) \to \mathbb{C}$ be the corresponding parallelogram realization of $\mathcal{D}$ defined by $p(y)-p(x) = \theta(x,y)$. Consider the trivial solutions

$$z_0(x) = p(x), \qquad w_0(x) = 1, \qquad \forall x \in V(\mathcal{D})$$

of the cross-ratio system (8.9) and the corresponding Hirota system (8.11). Suppose that $z_0 : V(\mathcal{D}) \to \mathbb{C}$ belongs to a differentiable one-parameter family of solutions $z_\epsilon : V(\mathcal{D}) \to \mathbb{C}$, $\epsilon \in (-\epsilon_0, \epsilon_0)$, of the same cross-ratio system, and denote by $w_\epsilon : V(\mathcal{D}) \to \mathbb{C}$ the corresponding solutions of the Hirota system. Denote

$$g = \frac{dz_\epsilon}{d\epsilon}\bigg|_{\epsilon=0}, \qquad f = \left(w_\epsilon^{-1}\frac{dw_\epsilon}{d\epsilon}\right)_{\epsilon=0}. \tag{8.47}$$

Theorem 8.18. (Discrete derivative for discrete holomorphic functions) *Both functions $f, g : V(\mathcal{D}) \to \mathbb{C}$ solve discrete Cauchy-Riemann equations* (7.16)*.*

Proof. By differentiating (8.12), we obtain a relation between the functions $f, g : V(\mathcal{D}) \to \mathbb{C}$:

$$g(y) - g(x) = \big(f(x) + f(y)\big)\big(p(y) - p(x)\big), \qquad \forall (x, y) \in \vec{E}(\mathcal{D}). \tag{8.48}$$

The proof of the theorem is based on this relation solely. Indeed, the exactness condition for the form on the right-hand side on an elementary quadrilateral reads

$$\begin{aligned}\big(f(x_0) + f(y_0)\big)\big(p(y_0) - p(x_0)\big) + \big(f(y_0) + f(x_1)\big)\big(p(x_1) - p(y_0)\big)\\ +\big(f(x_1) + f(y_1)\big)\big(p(y_1) - p(x_1)\big) + \big(f(y_1) + f(x_0)\big)\big(p(x_0) - p(y_1)\big) = 0,\end{aligned}$$

which is equivalent to (7.16) for the function f. Similarly, the exactness condition for f, that is,

$$\big(f(x_0) + f(y_0)\big) - \big(f(y_0) + f(x_1)\big) + \big(f(x_1) + f(y_1)\big) - \big(f(y_1) + f(x_0)\big) = 0,$$

yields

$$\frac{g(y_0) - g(x_0)}{p(y_0) - p(x_0)} - \frac{g(x_1) - g(y_0)}{p(x_1) - p(y_0)} + \frac{g(y_1) - g(x_1)}{p(y_1) - p(x_1)} - \frac{g(x_0) - g(y_1)}{p(x_0) - p(y_1)} = 0.$$

Under the condition $p(y_0) - p(x_0) = p(x_1) - p(y_1)$, this is equivalent to (7.16) for g. ∎

Remark. This proof shows that, given a discrete holomorphic function $f : V(\mathcal{D}) \to \mathbb{C}$, relation (8.48) correctly defines a unique, up to an additive constant, function $g : V(\mathcal{D}) \to \mathbb{C}$, which is also discrete holomorphic. Conversely, for any g satisfying the discrete Cauchy-Riemann equations (7.16), relation (8.48) defines a function f uniquely (up to an additive black-white constant); this function f also solves the discrete Cauchy-Riemann equations (7.16). Actually, formula (8.48) expresses that the discrete holomorphic function f is the *discrete derivative* of g, and so g is obtained from f by *discrete integration.*

Summarizing, we have the following statement.

Theorem 8.19. (Linearization of circle patterns)

a) *A tangent space to the set of solutions of an integrable cross-ratio system, at a point corresponding to a rhombic embedding of a quad-graph, consists of discrete holomorphic functions on this embedding. This holds in both descriptions of the above set: in terms of variables z satisfying the cross-ratio equations, and in terms of variables w satisfying the Hirota equations. The corresponding two descriptions of the tangent space are related via the discrete derivative (resp. antiderivative) of discrete holomorphic functions.*

b) *A tangent space to the set of integrable circle patterns of a given combinatorics, at a point corresponding to an isoradial pattern, consists of discrete holomorphic functions on the rhombic embedding of the corresponding quad-graph, which take real values at white vertices and pure imaginary values at black ones. This holds in the description of circle patterns in terms of circle radii and rotation angles at intersection points (Hirota system).*

A spectacular example of this linearization property is delivered by the isomonodromic discrete logarithm studied in Section 7.5 and isomonodromic z^{2a} circle patterns of Section 8.4.

Theorem 8.20. (Linearization of w^{2a-1} circle patterns is the discrete logarithm) *The tangent vector to the space of integrable circle patterns along the curve consisting of patterns w^{2a-1}, at the isoradial point corresponding to $a = 1/2$, is the discrete logarithmic function ℓ defined in Section* 7.5.

Proof. We have to prove that the discrete logarithm ℓ and the discrete power function w^{2a-1} are related by

$$\ell(\boldsymbol{n}) = \Big(\frac{1}{2}\frac{d}{da}\,w^{2a-1}(\boldsymbol{n})\Big)_{a=1/2}.$$

Due to Theorem 8.18, it is enough to prove this for the initial data on the coordinate semiaxes. But this follows by differentiating with respect to a the initial values (8.31) at the point $a = 1/2$, where all $w = 1$: the result coincides with (7.31). □

8.6. Exercises

8.1. Check that formulas (8.30), (8.31) give solutions to the corresponding difference equations (8.26), (8.28).

8.2. Prove asymptotic relations (8.32), (8.33).

8.3. Fill in the details of the proofs of Theorems 8.15, 8.16.

8.4. For every solution $z : \mathbb{Z}^d \to \mathbb{C}$ of the cross-ratio system (8.21), define the dual solution $z^* : \mathbb{Z}^d \to \mathbb{C}$ by

$$z^*(\boldsymbol{n} + \boldsymbol{e}_j) - z^*(\boldsymbol{n}) = \frac{\theta_j^2}{z(\boldsymbol{n} + \boldsymbol{e}_j) - z(\boldsymbol{n})}.$$

The dual solution is defined uniquely up to translation, and this freedom can be fixed by prescribing $z^*(0)$. Show that for $a \in (0,1)$ the dual solution to the discrete z^{2a}, normalized to vanish at $\boldsymbol{n} = 0$, coincides with the discrete $z^{2(1-a)}$.

8.5. Show that the limit $a \to 1$ in Definition 8.12 leads to the discrete z^2 as a solution of the cross-ratio equation, satisfying the recurrence relations (8.26) with $a = 1$ on the coordinate semiaxes, and with the initial data

$$z(0) = 0, \quad z(\boldsymbol{e}_j) = 0, \quad z(2\boldsymbol{e}_j) = \theta_j^2, \quad z(\boldsymbol{e}_j + \boldsymbol{e}_k) = \frac{\theta_j^2 - \theta_k^2}{2(\log \theta_j - \log \theta_k)}.$$

In particular, one sector of the discrete z^2, defined on $(\mathbb{Z}_+)^2$, in the case of $\theta_1 = 1$, $\theta_2 = i$, is characterized by the initial data

$$z(0,0) = z(1,0) = z(0,1) = 0, \quad z(2,0) = 1, \quad z(0,2) = -1, \quad z(1,1) = i\,\frac{2}{\pi}.$$

8.6. Show that the dual solution to the discrete z^2 is the discrete logarithm L.

8.7. Show that for the cross-ratio system on $(\mathbb{Z}_+)^2$ with $\theta_1 = 1$, $\theta_2 = i$, the dual solution to $z(m,n) = 1/(m + in)$ is given by

$$z^*(m,n) = \frac{1}{3}\big((m+in)^3 - (m - in)\big).$$

This can be regarded as the discrete z^3.

8.7. Bibliographical notes

Section 8.1: Circle patterns. The idea that circle packings and, more generally, circle patterns serve as a discrete counterpart of analytic functions is by now well established; see the monograph by Stephenson (2005). The origin of this idea is connected with the approach by Thurston (1985) to the Riemann mapping theorem via circle packings. Since then the theory bifurcated to several areas.

One of them is dealing mainly with approximation problems. The most popular are hexagonal packings, for which the convergence to the Riemann mapping was established in Rodin-Sullivan (1987). In He-Schramm (1998) it was shown that this convergence actually holds in the class C^∞, that is, all higher derivatives are approximated. Similar results are available also for circle patterns with combinatorics of the square grid introduced in Schramm (1997), and even for more general circle patterns; see Bücking (2007).

Another area concentrates around the uniformization theorem of Koebe-Andreev-Thurston, and is dealing with circle packing realizations of cell complexes of prescribed combinatorics, rigidity properties, constructing hyperbolic 3-manifolds, etc.; see Thurston (1997), He (1999), Stephenson (2005).

A variational description of circle packings was initiated by Colin de Verdière (1991). Further progress is due to Brägger (1992), Rivin (1994), and Bobenko-Springborn (2004). The extremals of the functional used in the last paper are described by equation (8.4). An application of this approach in discrete differential geometry is the construction of discrete minimal surfaces through circle patterns in Bobenko-Hoffmann-Springborn (2006).

The main topic of this chapter is interrelations of circle patterns with integrable systems. See the notes to Section 8.3.

Section 8.2: Integrable cross-ratio and Hirota systems. In this generality (for arbitrary quad-graphs) this material is due to Bobenko-Suris (2002a). On $\mathbb{Z}^2$ the relation between the cross-ratio and Hirota systems is considered in Capel-Nijhoff (1995). Our presentation follows Bobenko-Mercat-Suris (2005).

Section 8.3: Integrable circle patterns. Orthogonal circle patterns with combinatorics of the square grid were studied in Schramm (1997). Hexagonal circle patterns with fixed intersection angles were investigated in Bobenko-Hoffmann (2003), and with the multiratio property, in Bobenko-Hoffmann-Suris (2002). The general theory presented here is formulated in Bobenko-Mercat-Suris (2005).

Section 8.4: z^a and $\log z$ circle patterns. The circle patterns z^a on the square lattice were introduced in Bobenko (1999) and studied in Bobenko-Pinkall (1999) and Agafonov-Bobenko (2000). The conjecture that these patterns are embedded, i.e., the interiors of different kites are disjoint, was formulated in the first of these papers. The study was extended to the regular hexagonal grid in Bobenko-Hoffmann (2003). The fact that the circle patterns z^a are immersed, i.e., the neighboring kites do not overlap, was proven in Agafonov-Bobenko (2000) for the square grid and in Agafonov-Bobenko (2003) for the hexagonal grid combinatorics. The embeddedness was proven in Agafonov (2003) for the case of the square grid combinatorics. The isomonodromic constraint (8.37) was obtained first for $a = 1/2$ in Nijhoff (1996), with no geometric interpretation. For the Hirota system, the isomonodromic constraint (8.41) was derived in Nijhoff-Ramani-Grammaticos-Ohta (2001), also with no relation to geometry. Our presentation here follows Bobenko-Mercat-Suris (2005).

Section 8.5: Linearization. The operation of discrete integration for discrete holomorphic functions was considered in Duffin (1956, 1968), and Mercat (2001). Linearization of circle patterns was studied in Bobenko-Mercat-Suris (2005); in particular, the derivation of Green's function from the z^a circle pattern is taken from this paper.

Section 8.6: Exercises.

Ex. 8.3: See Bobenko-Mercat-Suris (2005).

Ex. 8.5: See Agafonov-Bobenko (2000).

Ex. 8.6: See Agafonov-Bobenko (2000) in the case of the regular square grid.

Ex. 8.7: See Bobenko-Pinkall (1999).