Differentialgeometrie III

Compact Riemann Surfaces

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1 Definition of a Riemann Surface and Basic Examples

Let $\mathcal{R}$ be a two-real dimensional manifold and $\{U_\alpha\}_{\alpha\in A}$ an open cover of $\mathcal{R}$, i. e. $\cup_{\alpha\in A} U_\alpha = \mathcal{R}$. A *local parameter (local coordinate, coordinate chart)* is a pair (U_α, z_α) of U_α with a homeomorphism $z_\alpha : U_\alpha \to V_\alpha$ to an open subset $V_\alpha \subset \mathbb{C}$. Two coordinate charts (U_α, z_α) and (U_β, z_β) are called *compatible* if the mapping

$$f_{\beta,\alpha} = z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta),$$

which is called a *transition function* is holomorphic. The local parameter (U_α, z_α) will be often identified with the mapping z_a if its domain is clear or irrelevant.

If all the local parameters $\{U_\alpha, z_\alpha\}_{\alpha\in A}$ are compatible, they form a *complex atlas* $\mathcal{A}$ of $\mathcal{R}$. Two complex atlases $\mathcal{A} = \{U_\alpha, z_\alpha\}$ and $\tilde{\mathcal{A}} = \{\tilde{U}_\beta, \tilde{z}_\beta\}$ are compatible if $\mathcal{A} \cup \tilde{\mathcal{A}}$ is a complex atlas. An equivalence class Σ of complex atlases is called a *complex structure*. It can be identified with a maximal atlas $\mathcal{A}^*$, which consists of all coordinate charts, compatible with an atlas $\mathcal{A} \subset \Sigma$.

Definition 1.1 *A Riemann surface is a connected one-complex-dimensional analytic manifold, that is, a two-real dimensional connected manifold $\mathcal{R}$ with a complex structure Σ on it.*

When it is clear, which complex structure is considered we use the notation $\mathcal{R}$ for the Riemann surface.

Remark If $\{U, z\}$ is a coordinate on $\mathcal{R}$ then for every open set $V \subset U$ and every function $f : \mathbb{C} \to \mathbb{C}$, which is holomorphic and injective on $z(V)$, $\{V, f \circ z\}$ is also a local parameter on $\mathcal{R}$.

Remark The coordinate charts establish homeomorphisms of domains in $\mathcal{R}$ with domains in $\mathbb{C}$. This means, that locally the Riemann surface is just a domain in $\mathbb{C}$. But for any point $P \in \mathcal{R}$ there are many possible choices of these homeomorphisms. Therefore one can associate to $\mathcal{R}$ only the notions from the theory of analytic functions in $\mathbb{C}$, which are invariant with respect to biholomorphic maps, i. e. for definition of which one should not specify a local parameter. For example one can talk about an angle between two smooth curves γ and $\tilde{\gamma}$ on $\mathcal{R}$, intersecting at some point $P \in \mathcal{R}$. This angle equals to the one between the curves $z(\gamma)$ and $z(\tilde{\gamma})$, which lie in $\mathbb{C}$ and intersect at the point $z(P)$, where z is some local parameter at P. This definition is invariant with respect to the choice of z.

Remark If $(\mathcal{R}, \Sigma)$ is a Riemann surface, then the manifold $\mathcal{R}$ is orientable. The transition function $f_{\beta,\alpha}$ written in terms of real coordinates $(z = x + iy)$

$$(x_\alpha, y_\alpha) \to (x_\beta, y_\beta)$$

preserves orientation

$$dx_\alpha \wedge dy_\alpha = \frac{i}{2} dz_\alpha \wedge d\bar{z}_\alpha = \frac{i}{2} \left|\frac{dz_\alpha}{dz_\beta}\right|^2 dz_\beta \wedge d\bar{z}_\beta = \left|\frac{dz_\alpha}{dz_\beta}\right|^2 dx_\beta \wedge dy_\beta.$$

The simplest examples of Riemann surfaces are any domain (connected open subset) $U \subset \mathbb{C}$ in a complex plane, the complex plane $\mathbb{C}$ itself and the extended complex plane (or *Riemann sphere*) $\hat{\mathbb{C}} = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The complex structures on U and $\mathbb{C}$ are defined by single coordinate charts (U, id) and $(\mathbb{C}, id)$. The extended complex plane is the simplest compact Riemann surface. To define the complex structure on it we use two charts $(U_1, z_2), (U_2, z_2)$ with

$$U_1 = \mathbb{C}, \qquad z_1 = z,$$
$$U_2 = (\mathbb{C}\backslash\{0\}) \cup \{\infty\}, \qquad z_2 = 1/z.$$

The transition functions

$$f_{1,2} = z_1 \circ z_2^{-1}, \quad f_{2,1} = z_2 \circ z_1^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$$

are holomorphic

$$f_{1,2}(z) = f_{2,1}(z) = 1/z.$$

In large extend the beauty of the theory of Riemann surfaces is due to the fact that Riemann surfaces can be described in many completely different ways. Interrelations between these descriptions comprise an essential part of the theory. The basic examples of Riemann surfaces we are going to discuss now are exactly these foundation stones the whole theory is based on.

1.1 Non-singular Algebraic Curves

Definition 1.2 *An algebraic curve C is a subset in $\mathbb{C}^2$*

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 \mid \mathcal{P}(\mu, \lambda) = 0\}, \tag{1}$$

where $\mathcal{P}$ is an irreducible polynominal in λ and μ

$$\mathcal{P}(\mu, \lambda) = \sum_{i=0}^{N} \sum_{j=0}^{M} p_{ij} \mu^i \lambda^j.$$

The curve C is called non-singular if

$$\mathrm{grad}_{\mathbb{C}} \mathcal{P}_{|\mathcal{P}=0} = \left(\frac{\partial \mathcal{P}}{\partial \mu}, \frac{\partial \mathcal{P}}{\partial \lambda} \right)_{|\mathcal{P}(\mu,\lambda)=0} \neq 0. \tag{2}$$

To introduce a complex structure on the non-singular curve (1, 2) one uses a complex version of the implicit function theorem.

Theorem 1.1 *Let $\mathcal{P}(\mu, \lambda)$ be an analytic function of μ and λ in a neighbourhood of a point $(\mu_0, \lambda_0) \in \mathbb{C}^2$ with $\mathcal{P}(\mu_0, \lambda_0) = 0$, and, in addition*

$$\frac{\partial \mathcal{P}}{\partial \mu}(\mu_0, \lambda_0) \neq 0.$$

Then in a neighbourhood of (μ_0, λ_0) *the set*

$$\{(\mu, \lambda) \in \mathbb{C}^2 \mid \mathcal{P}(\mu, \lambda) = 0\}$$

is described as

$$\{(\mu(\lambda), \lambda) \mid \lambda \in U\},$$

where $U \subset \mathbb{C}$ *is a neighbourhood of* $\lambda_0 \in U$ *and* $\mu(\lambda)$ *is an analytic function. The derivative of the function* $\mu(\lambda)$ *is equal*

$$\frac{d\mu}{d\lambda} = -\frac{\partial\mathcal{P}/\partial\lambda}{\partial\mathcal{P}/\partial\mu}.$$

The complex structure on C is introduced as follows: the variable λ is taken to be a local parameter in the neighbourhoods of the points where $\partial\mathcal{P}/\partial\mu \neq 0$, and the variable μ is a local parameter near the points where $\partial\mathcal{P}/\partial\lambda \neq 0$. The holomorphic compatibility of the introduced local parameters results from Theorem 1.1.

The surface C can be made a compact Riemann surface $\hat{C}$ by joining point(s) $\infty^{(1)}, \ldots, \infty^{(N)}$

$$\hat{C} = C \cup \{\infty^{(1)}\} \cup \ldots \cup \{\infty^N\}$$

at infinity $\lambda \to \infty, \mu \to \infty$, and introducing proper local parameters at this(ese) point(s). In oder to explain this compactification let us define Riemann surfaces with punctures.

Definition 1.3 *Let* $\mathcal{R}$ *be a Riemann surface such that there exists an open subset* U_∞

$$U_\infty^{(1)} \cup \ldots \cup U_\infty^{(N)} = U_\infty \subset \mathcal{R}$$

such that $\mathcal{R} \backslash U_\infty$ *is compact and* $U_\infty^{(n)}$ *are homeomorphic to punctured discs*

$$z_n : U_\infty^{(n)} \to D \backslash \{0\} = \{z \in \mathbb{C} \mid 0 < |z| < 1\},$$

where homomorphisms z_n *are holomorphically compatible with the complex structure of* $\mathcal{R}$*. Then* $\mathcal{R}$ *is called a compact Riemann surface with punctures.*

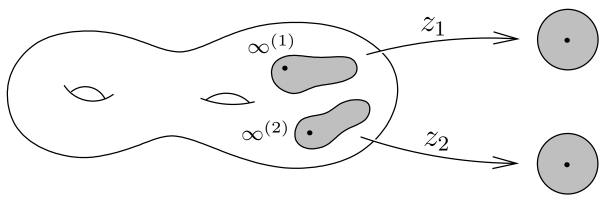

Figure 1: A compact Riemann surface with punctures.

Let us extend the homeomorphisms z_n to D

$$z_n : \hat{U}_\infty^{(n)} = U_\infty^{(n)} \cup \infty^{(n)} \to D = \{z \mid |z| < 1\}, \tag{3}$$

defining punctures $\infty^{(n)}$ by the condition $z_n(\infty^{(n)}) = 0, \quad n = 1, \dots, N$. A complex atlas for a new Riemann surface

$$\hat{\mathcal{R}} = \mathcal{R} \cup \{\infty^{(1)}\} \cup \ldots \cup \{\infty^{(n)}\}$$

is defined as a union of a complex atlas $\mathcal{A}$ of $\mathcal{R}$ with the coordinate charts (3) compatible with $\mathcal{A}$ due to Definition 1.3. We call $\hat{\mathcal{R}}$ a *compactification* of $\mathcal{R}$.

Hyperelleptic Curves.

Let us consider the important special case of hyperelleptic curves [1]

$$\mu^2 = \prod_{j=1}^{N} (\lambda - \lambda_j), \quad N \geq 3, \quad \lambda_j \in \mathbb{C}. \tag{4}$$

The curve is non-singular if all the points λ_j are different

$$\lambda_j \neq \lambda_i, \quad i, j = 1, \dots, N.$$

In this case the choice of local parameters can be additionally specified. Namely, in the neighbourhood of the points (μ_0, λ_0) with $\lambda_0 \neq \lambda_j \quad \forall j$, the local parameter is the homeomorphism

$$(\mu, \lambda) \to \lambda. \tag{5}$$

In the neighbourhood of each point $(0, \lambda_j)$ it is defined by the homeomorphism

$$(\mu, \lambda) \to \sqrt{\lambda - \lambda_j}. \tag{6}$$

Indeed, near $(0, \lambda_i)$

$$\mu = \sqrt{\lambda - \lambda_i} \left(\sqrt{\prod_{j=1}^{N} (\lambda_i - \lambda_j)} + o(1) \right), \qquad \lambda \to \lambda_i,$$

and the local parameter $\sqrt{\lambda - \lambda_j}$ is equivalent to μ.

The hyperelleptic curve (4) is a compact Riemann surface with a puncture (or punctures) at $\lambda \to \infty$. To show this one should consider the cases of even $N = 2g + 2$ and odd $N = 2g + 1$ separately. The formulas

$$m = \frac{\mu}{\lambda^{g+1}}, \qquad l = \frac{1}{\lambda}$$

describe a biholomorphic map $(\mu, \lambda) \mapsto (m, l)$ of a neighbourhood of infinity

$$U_\infty = \{(\mu, \lambda) \in C \mid |\lambda| > c > |\lambda_i|, \quad i = 1, \dots, N\}$$

onto the punctured neighbourhood

$$V_0 = \{(m, l) \in C' \mid 0 < |l| < c^{-1}\}$$

[1] When $N = 3$ or 4 the curve (4) is called elliptic

of the point $(m,l)=(0,0)$ of the curve

$$m^2 = l \prod_{i=1}^{2g+1}(1-l\lambda_i) \tag{7}$$

for $N=2g+1$, or onto punctured neighbourhoods of the points $(m,l)=(\pm 1,0)$ of the curve

$$m^2 = \prod_{i=1}^{2g+2}(1-l\lambda_i) \tag{8}$$

for $N=2g+2$. Formulas (5), (6) show that at the point $(0,0)$ of the curve (7) the local parameter is $\sqrt{l}$ and at the points $(\pm 1,0)$ of the curve (8) the local parameters are l.

Finally, for odd $N=2g+1$ the curve (4) has one puncture ∞

$$P \equiv (\mu,\lambda) \to \infty \Longleftrightarrow \lambda \to \infty,$$

and the local parameter in its neighbourhood is given by the homeomorphism

$$z_\infty : (\mu,\lambda) \to \frac{1}{\sqrt{\lambda}}. \tag{9}$$

For even $N=2g+2$ there are two punctures $\infty^\pm$ distinguished by the condition

$$P \equiv (\mu,\lambda) \to \infty^\pm \Longleftrightarrow \frac{\mu}{\lambda^{g+1}} \to \pm 1, \qquad \lambda \to \infty,$$

and the local parameters in the neighbourhood of both points are given by the homeomorphism

$$z_{\infty^\pm} : (\mu,\lambda) \to \lambda^{-1}. \tag{10}$$

Theorem 1.2 *The local parameters (5, 6, 9, 10) describe a compact Riemann surface*

$$\begin{aligned} \hat{C} &= C \cup \{\infty\} \qquad \text{if } N \text{ is odd}, \\ \hat{C} &= C \cup \{\infty^\pm\} \qquad \text{if } N \text{ is even}, \end{aligned}$$

of the hyperelleptic curve (4).

Later on we consider basically compact Riemann surfaces and call $\hat{C}$ shortly the Riemann surface of the curve C.

It turnes out that all compact Riemann surfaces can be described as compactifications of algebraic curves.

1.2 Quotients under Group Actions

Definition 1.4 *Let Δ be a domain*[2] *in $\mathbb{C}$. A group $G : \Delta \to \Delta$ of holomorphic transformations acts discontinously on Δ if for any $P \in \Delta$ there exists a neighbourhood $V \ni P$ such that*

$$gV \cap V = \emptyset, \qquad \forall g \in G, \qquad g \neq I. \tag{11}$$

[2] Similarly one can consider action of groups of holomorphic transformations on $\hat{\mathbb{C}}$.

One can introduce the equivalence relation between the points of Δ :

$$P \sim P' \Leftrightarrow \exists g \in G, \qquad P' = gP,$$

and the quotient space Δ/G of the equivalence classes.

Theorem 1.3 *Δ/G is a Riemann surface.*

Proof. Let us denote by

$$\pi : \Delta \to \Delta/G$$

the canonical projection, which associate to each point of Δ its equivalence class. We define the factor topology on Δ/G: a subset $U \subset \Delta/G$ is called open if $\pi^{-1}(U) \subset \Delta$ is open. Both Δ and Δ/G are connected. Every finite point $P \in \Delta$ has a neighbourhood V satisfying (11). Then $U = \pi(V)$ is open and $\pi_{|_V} : V \to U$ is a homeomorphism. Its inversion $z : U \to V \subset \Delta \subset \mathbb{C}$ is a local parameter. One can cover Δ/G by domains of this type. Let us consider two local parameters $z : U \to V$ and $\tilde{z} : U \to \tilde{V}$. The transition function $f = \tilde{z} \circ z : V \to \tilde{V} satisfies$

$$\pi(z) = \pi(f(z)).$$

For each point $z \in V$ there is a group element $g \in G$ such that

$$f(z) = g(z). \tag{12}$$

Since $f : V \to \tilde{V}$ is a homeomorphism and G acts discontinuously, the group element $g \in G$ in (12) is the same for all $z \in V$. This proves that the transition functions are holomorphic and $\mathcal{R}$ is a Riemann surface.

Tori

Let us consider the case $\Delta = \mathbb{C}$ and the group G generated by two shifts

$$z \to z + w, \qquad z \to z + w',$$

where $w, w' \in \mathbb{C}$ are two non-parallel vectors, Im $w'/w \neq 0$. The group G is commutative and consists of the elements

$$g_{n,m}(z) = z + nw + mw', \qquad n, m \in \mathbb{Z}. \tag{13}$$

The factor $\mathbb{C}/G$ has a nice geometrical realization as the parallelogram

$$T = \{z \in \mathbb{C} \mid z = aw + bw',\ a, b \in [0, 1)\}.$$

There are no G-equivalent points in T and on the other hand every point in $\mathbb{C}$ is equivalent to some point in T. Since the edges of the parallelogram T are G-equivalent $z \sim z + w,\ z \sim z + w'$, $\mathcal{R}$ is a compact Riemann surface, which is topologically a torus. We discuss this case in more detail in Section 6.

In frames of the uniformization theory it is proven that all compact Riemann surfaces can be described as factors Δ/G.

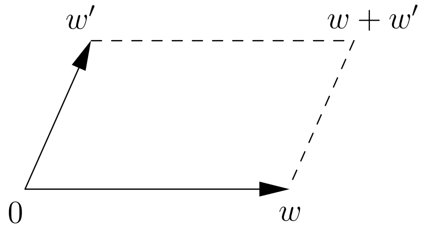

Figure 2: A complex torus

1.3 Euclidean Polyhedral Surfaces as Riemann Surfaces

It is not difficult to build a Riemann surface glueing together pieces of the complex plane $\mathbb{C}$.

Consider a finite set of disjoint Euclidean triangles F_i and identify their elements (vertices and edges) is such a way that they comprise a compact oriented Euclidean polyhedral surface. A polyheder in 3-dimensional Euclidean space is an example of such a surface. A required identification of edges and vertices is shown in Fig. 3. It is characterized by the following properties.

(i) If two triangles have common elements then these may be either a common vertex or a common edge.

(ii) Every edge of the surface belongs exactly to two triangles.

(iii) Triangles with a common vertex P are successively glued along edges passing through P (as in Fig. 3), i.e. the triangles with a common vertex P are arranged in a cyclic sequence $F_1, F_2, \ldots, F_n$ such that each pair F_i, F_{i+1} as well as F_n, F_1 has a common edge containing P.

(iv) All triangles can be oriented so that their orientations correspond.

In order to define a complex structure on an Euclidean polyhedral surface let us distinguish three kinds of points:

1. inner points of triangles,
2. inner points of edges,
3. vertices.

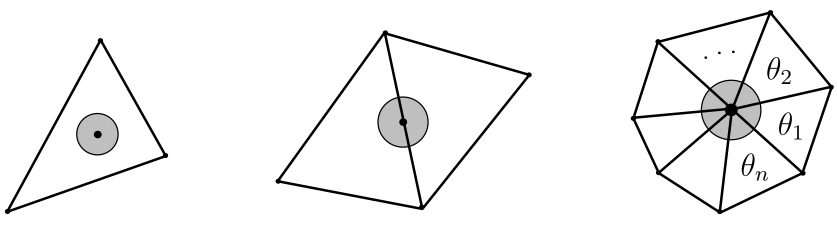

Figure 3: Three kinds of points on an Euclidean polyhedral surface

It is clear how to define local parameters for the points of the first and the second kind. By an Euclidean isometry one can map the corresponding triangles (or pairs of neighbouring triangles) into $\mathbb{C}$. This provides us with local parameters at the points of the first and the second kind. Next let P be a vertex and $F_i, \ldots, F_n$ the sequence of successive triangles with this vertex (see the point (iii) above). Denote by θ_i the angle of F_i at P. Then define

$$\gamma = \frac{2\pi}{\sum_{i=1}^{n} \theta_i}.$$

Consider a suitably small ball neighbourhood of P, which is the union $U^r = \cup_i F_i^r$, where $F_i^r = \{Q \in F_i \mid \mid Q - P \mid < r\}$. Each F_i^r is a sector with angle θ_i at P. We map it as above into $\mathbb{C}$ with P mapped to the origin and then apply $z \mapsto z^\gamma$, which produces a sector with the angle $\gamma\theta_i$. The mappings corresponding to different triangles F_i can be adjusted to provide a homeomorphism of U^r onto a disc in $\mathbb{C}$.

All transition functions of the constructed charts are holomorphic since they are compositions of maps of the form $z \mapsto az + b$ and $z \mapsto z^\gamma$ (away from the origin).

Using the algebraic curve representation of compact Riemann surfaces it is not difficult to show that any compact Riemann surface can be recovered from some Euclidean polyhedral surface [Bost].

1.4 Complex Structure Generated by Metric

There is a smooth version of the previous construction. Let $(\mathcal{R}, g)$ be a two-real dimensional orientable differential manifold with a metric g. In local coordinate $(x, y) : U \subset \mathcal{R} \to \mathbb{R}^2$ one has

$$g = a\, dx^2 + 2b\, dxdy + c\, dy^2, \qquad a > 0,\ c > 0,\ ac - b^2 > 0. \tag{14}$$

Definition 1.5 *Two metrics g and $\tilde{g}$ are called conformally equivalent if they differ by a function on $\mathcal{R}$*

$$g \sim \tilde{g} \Leftrightarrow g = f\tilde{g}, \qquad f : \mathcal{R} \to \mathbb{R}_+. \tag{15}$$

The relation (15) defines the classes of conformally equivalent metrics.

Remark The angles between tangent vectors are the same for conformally equivalent metrics.

We show that there is one to one correspondence between the conformal equivalence classes of metrics on an orientable two-manifold $\mathcal{R}$ and the complex structures on $\mathcal{R}$. In terms of the complex variable[3] $z = x + iy$ one rewrites the metric as

$$g = Adz^2 + 2Bdzd\bar{z} + \bar{A}d\bar{z}^2, \qquad A \in \mathbb{C}, B \in \mathbb{R}, B > |A|, \tag{16}$$

with

$$a = 2B + A + \bar{A}, \qquad b = i(A - \bar{A}), \qquad c = 2B - A - \bar{A}. \tag{17}$$

[3]Note that the complex coordinate z is *not* compatible with the complex structure we will define on $\mathcal{R}$ with the help of g.

Definition 1.6 *A coordinate $w : U \to \mathbb{C}$ is called conformal if the metric in this coordinate is of the form*

$$g = e^{\phi} dw d\bar{w}, \tag{18}$$

i.e. it is conformally equivalent to the standard metric of $\mathbb{R}^2 = \mathbb{C}$

$$dw d\bar{w} = du^2 + dv^2, \qquad w = u + iv.$$

Remark If $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is an immersed surface in $\mathbb{R}^3$ then the first fundamental form $< dF, dF >$ induces a metric on U. When the standard coordinate (x, y) of $\mathbb{R}^2 \supset U$ is conformal, the parameter lines

$$F(x, \Delta m), \qquad F(\Delta n, y), \qquad x, y \in \mathbb{R}, \qquad n, m \in \mathbb{Z}, \qquad \Delta \to 0$$

comprise an infinitesimal square net on the surface. The problem of conformal coordinates was studied already by Gauss, who proved their existence in the real-analytic case.

We start with a simple

Theorem 1.4 *Every compact Riemann surface admits a conformal Riemannian metric.*

Proof. Each point $P \in \mathcal{R}$ possesses a local parameter $z_P : U_P \to D_P \subset \mathbb{C}$, where D_P is a small open disc. Since $\mathcal{R}$ is compact there exists a finite covering $\cup_{i=1}^n U_{P_i} = \mathcal{R}$. For each i choose a smooth function $m_i : D_{P_i} \to \mathbb{R}$ with

$$m_i > 0 \quad \text{on } D_i, \qquad m_i = 0 \quad \text{on } \mathbb{C} \setminus D_i.$$

$m_i(z_{P_i}) dz_{P_i} d\bar{z}_{P_i}$ is a conformal metric on U_{P_i}. The sum of these metrics over $i = 1, \ldots, n$ yields a conformal metric on $\mathcal{R}$.

Let us show how one finds conformal coordinates. The metric (16) can be written as follows (we suppose $A \neq 0$)

$$g = s(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz), \qquad s > 0, \tag{19}$$

where

$$\mu = \frac{\bar{A}}{2B}(1 + |\mu|^2), \qquad s = \frac{2B}{1 + |\mu|^2}.$$

Here $|\mu|$ is a solution of the quadratic equation

$$|\mu| + \frac{1}{|\mu|} = \frac{2B}{|A|},$$

which can be chosen $|\mu| < 1$

$$|\mu| = \frac{1}{|A|}(B - \sqrt{B^2 - |A|^2}). \tag{20}$$

Comparing (19) and (18) we get

$$dw = \lambda(dz + \mu d\bar{z})$$

or

$$dw = \lambda(d\bar{z} + \bar{\mu} dz).$$

In the first case the map $w(z, \bar{z})$ satisfies the equation

$$w_{\bar{z}} = \mu w_z \tag{21}$$

and preserves the orientation $w : U \subset \mathbb{C} \to V \subset \mathbb{C}$ since $|\mu| < 1$: for the map $z \to w$ written in terms of the real coordinates

$$z = x + iy, \qquad w = u + iv$$

one has

$$du \wedge dv = |w_z|^2(1 - |\mu|^2) dx \wedge dy.$$

In the second case $w : U \to V$ inverses the orientation.

Definition 1.7 *Equation (21) is called the Beltrami equation and $\mu(z, \bar{z})$ is called the Beltrami coefficient.*

Let us postpone for a moment the discussion of the proof of existence of solutions to the Beltrami equation and let us assume that this equation can be solved in a small neighbourhood of any point of $\mathcal{R}$.

Theorem 1.5 *Let $\mathcal{R}$ be a two-dimensional orientable manifold with a metric g and an oriented atlas $((x_\alpha, y_\alpha) : U_\alpha \to \mathbb{R}^2)_{\alpha \in A}$ on $\mathcal{R}$. Let $(x, y) : U \subset \mathcal{R} \to \mathbb{R}^2$ be one of these coordinate charts with a point $P \in U$, $z = x + iy$, $\mu(z, \bar{z})$ - the Beltrami coefficient (20) and $w_\beta(z, \bar{z})$ be a solution to the Beltrami equation (21) in a neighbourhood $V_\beta \subset V = z(U)$ with $P \in U_\beta = z^{-1}(V_\beta)$. Then the coordinate w_β is conformal and the atlas $(w_\beta : U_\beta \to \mathbb{C})_{\beta \in B}$ defines a complex structure on $\mathcal{R}$.*

Proof. To prove the holomorphicity of the transition function let us consider two local parameters $w : U \to \mathbb{C}, \tilde{w} : \tilde{U} \to \mathbb{C}$ with a non-empty intersection $U \cap \tilde{U} \neq \emptyset$. Both coordinates are conformal

$$g = e^{\phi} dw d\bar{w} = e^{\tilde{\phi}} d\tilde{w} d\bar{\tilde{w}},$$

which happens in one of the two cases

$$\frac{\partial \tilde{w}}{\partial \bar{w}} = 0 \text{ or } \frac{\partial \tilde{w}}{\partial w} = 0 \tag{22}$$

only. The transition function $\tilde{w}(w)$ is holomorphic and not antiholomorphic since the map $w \to \tilde{w}$ preserves orientation.

Repearting the arguments of the proof of Theorem 1.5 one immeadeately observes that conformaly equivalent metrics generate the same complex structure. Finally, we obtain the following

Theorem 1.6 *Conformal equivalence classes of metrics on an orientable two-manifold $\mathcal{R}$ are in one to one correspondence with the complex structures on $\mathcal{R}$.*

On Solution to the Beltrami Equation

For the real-analytic case $\mu \in C^\omega$ the existence of the solution to the Beltrami equation was known already to Gauss. It can be proven using the Cauchy-Kowalewski theorem.

Theorem 1.7 (Cauchy-Kowalewski)
Let

$$\frac{\partial^m u_i}{\partial x_0^m} = F_i(x_0, x, u, \frac{\partial^{m_0+\ldots+m_n}}{\partial x_0^{m_0} \ldots \partial x_n^{m_n}} u),$$

$$i = 1, \ldots, k, \quad x \in \mathbb{R}^n, \quad \sum_{j=0}^{n} m_j \leq m, \quad m_0 < m, \quad m \geq 1,$$

be a system of k partial differential equations for k functions $u_1(x, x_0), \ldots, u_k(x, x_0)$. The Cauchy problem

$$\left.\frac{\partial^j u_i}{\partial x_0^j}\right|_\sigma = \phi_{ij}(x), \qquad i = 1, \ldots, k; \qquad j = 0, \ldots, m-1,$$

where $\sigma = \{(x, x_0),\ x_0 = 0,\ x \in \Omega_0,\ \Omega_0$ is a domain in $\mathbb{R}^n\}$ with real-analytic data (all F_i, ϕ_{ij} are real-analytic functions of all their arguments), has a unique real-analytic solution $u(x, x_0)$ in some domain $\Omega \subset \mathbb{R}^{n+1}$ of variables (x, x_0) with $\Omega_0 \subset \Omega$.

In terms of real variables

$$z = x + iy, \qquad w = u + iv, \qquad \mu = p + iq$$

the Beltrami equation reads as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = \frac{1}{(1+p)^2 + q^2} \begin{pmatrix} 2q & p^2 + q^2 - 1 \\ 1 - p^2 - q^2 & 2q \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x . \tag{23}$$

If μ is real-analytic and $|\mu| < 1$ all the coefficients in (23) are real-analytic, which implies the existence of a real-analytic solution to the equation.

Solutions to the Beltrami equation exist in much more general case but the proof is much more involved.

Recall that a function is of Hölder class of order α ($0 < \alpha < 1$) on W, $f \in C^\alpha(W)$ if there exists a constant K such that

$$|f(p) - f(q)| \leq K|p - q|^\alpha, \forall p, q \in W.$$

If all mixed n-th order derivatives of f exist and are C^α then $f \in C^{n+\alpha}(W)$.

Theorem 1.8 *Let $z : U \to V \subset \mathbb{C}$ be a coordinate chart at some point $P \in U$ and $\mu \in C^\alpha(V)$ be the Beltrami coefficient. There is a solution $w(z, \bar{z})$ to the Beltrami equation of the class $w \in C^{\alpha+1}(W)$ in some neighbourhood W of the point $z(P) \in W \subset V$.*

Sketch of the proof of Theorem 1.8.

The Beltrami equation can be rewritten as an integral equation using

Lemma 1.9 ($\bar{\partial}$-Lemma)
Given $g \in C^\alpha(V)$, the formula

$$f(z) = \frac{1}{2\pi i} \int_V \frac{g(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}$$

defines a $C^{\alpha+1}(V)$ solution to the equation

$$f_{\bar{z}}(z) = g(z).$$

In case $g \in C^\infty$ or $g \in C^1$ this lemma is a standard result in complex analysis. For the proof in the case formulated above see [Bers] and [Spivak], v.4.

The $\bar{\partial}$-Lemma implies that the solution of

$$w(z) = h(z) + \frac{1}{2\pi i} \int_V \frac{\mu(\xi) w_\xi(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}, \tag{24}$$

where h is holomorph, satisfies the Beltrami equation. The proof of the existence of the solution to the integral equation (24) is standard: it is solved by iterations. Let us rewrite the equation to be solved as

$$w = Tw, \tag{25}$$

where Tw is the right-hand side of (24). Let us suppose that there complete metric space $\mathcal{H}$ such that

i) $T\mathcal{H} \subset \mathcal{H}$

ii) T is a contraction in $\mathcal{H}$, i. e. $\|Tw - Tw'\| < c\|w - w'\|$ for any $w, w' \in \mathcal{H}$ with some $c < 1$.

Then there exists a unique solution $w^* \in \mathcal{H}$ of (25) and this solution can be obtained from any starting point $w_0 \in \mathcal{H}$ by iteration

$$w^* = \lim_{n\to\infty} T^n w_0.$$

For the choice of the function space $\mathcal{H}$ and details of the proof see [Bers] and [Spivak], v.4.

The theorem above holds true also after replacing $\alpha \to \alpha + n, n \in \mathbb{N}$.

2 Holomorphic Mappings

Definition 2.1 *A mapping*

$$f : M \to N$$

between Riemann surfaces is called holomorphic (or analytic) if for every local parameter (U, z) on M and every local parameter (V, w) on N with $U \cap f^{-1}(V) \neq \emptyset$, the mapping

$$w \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \to w(V)$$

is holomorphic.

A holomorphic mapping into $\mathbb{C}$ is called a *holomorphic function*, a holomorphic mapping into $\hat{\mathbb{C}}$ is called a *meromorphic function.*

The following lemma characterizes a local behaviour of holomorphic mappings.

Lemma 2.1 *Let $f : M \to N$ be a holomorphic mapping. Then for any $a \in M$ there exist local parameters $(U, z), (V, w)$ such that $a \in U, f(a) \in V$ and $F = w \circ f \circ z^{-1} : z(U) \to w(V)$ equals*

$$F(z) = z^k, \quad k \in \mathbb{N}. \tag{26}$$

Proof Let us normalize local parameters $\tilde{z}$ near a and w near $f(a)$ to vanish at these points: $\tilde{z}(a) = w(f(a)) = 0$. Since $F(\tilde{z})$ is holomorphic and $F(0) = 0$ it can be represenred as $F(\tilde{z}) = \tilde{z}^k g(\tilde{z})$, where $g(\tilde{z})$ is holomorphic and $g(0) \neq 0$. The map $\tilde{z} \to z$ with

$$z = \tilde{z} h(\tilde{z}), \ h^k(\tilde{z}) = g(\tilde{z})$$

is biholomorphic and in terms of the local parameter z the mapping $w \circ f \circ z^{-1}$ is given by (26). $\square$

Corollary 2.2 *Let $f : M \to N$ be a non-constant holomorphic mapping, then f is open, i.e. an image of any open set is open.*

Corollary 2.3 *Let $f : M \to N$ be a non-constant holomorphic mapping and M compact. Then f is surjective $f(M) = N$ and N is also compact.*

Proof The previous corollary implies that $f(M)$ is open. On the other hand, $f(M)$ is compact since it is a continuous image of a compact set. $f(M)$ is open, closed and non-empty, therefore $f(M) = N$ and N compact. $\square$

Theorem 2.4 (Liouville)
There are no non-constant holomorphic functions on compact Riemann surfaces.

Proof An existence of a non-constant holomorphic mapping $f : M \to \mathbb{C}$ contradicts to the previous corollary since $\mathbb{C}$ is not compact. $\square$

Non-constant holomorphic mappings of Riemann surfaces $f : M \to N$ are *discrete*: for any point $P \in N$ the set $S_P = f^{-1}(P)$ is discrete, i.e. for any point $a \in M$ there is a neighbourhood $V \subset M$ intersecting with S_P in at most one point, $|V \cap S_P| \leq 1$. Non-discreteness of S for a holomorphic mapping would imply the existence of a limiting point in S_P and finally $f = \text{const}$, $f : M \to P \in N$. Non-constant holomorphic mappings of Riemann surfaces are also called *holomorphic coverings*.

Definition 2.2 *Let $f : M \to N$ be a holomorphic covering. A point $P \in M$ is called a branch point of f if it has no neighbourhood $V \ni P$ such that $f\big|_V$ is injective. A covering without branch points is called unramified (ramified or branched covering in the opposite case).*[4]

The number $k \in \mathbb{N}$ in Lemma 2.1 can be described in topological terms. There exist neighbourhoods $U \ni a, V \ni f(a)$ such that for any $Q \in V \backslash \{f(a)\}$ the set $f^{-1}(Q) \cap U$ consists of k points. One says that f has the *multiplicity* k at a. Lemma 2.1 allows us to characterize the branch points of a holomorphic covering $f : M \to N$ as the points with the multiplicity $k > 1$. Equivalently, P is a branch point of the covering $f : M \to N$ if

$$\left. \frac{\partial (w \circ f \circ z^{-1})}{\partial z} \right|_{z(P)} = 0, \tag{27}$$

where z and w are local parameters at P and $f(P)$ respectively (due to the chain rule this condition is independent of the choice of the local parameters). The number $b_f(P) = k-1$ is called the *branch number* of f at $P \in M$. The next lemma also immediately follows from Lemma 2.1.

Lemma 2.5 *Let $f : M \to N$ be a holomorphic covering. Then the set of branch points*

$$B = \{P \in M \mid b_f(P) > 0\}$$

is discrete. If M is compact, then B is finite.

Proof Let $P \in M$. Then exists $U \ni P$, such that $F(z) := (w \circ f \circ z^{-1})(z) = z^k$ on U, where $k = b_f + 1$. Since the map z^k is locally injective for all $z \neq 0$, the point P is the only possible branch point in U.

Now, if you suppose B to be infinite, this contradicts the discreteness of B, because an infinite subset in a compact set M has a limiting point $P \in M$ and therefore there would be infinitely many points with $b_f > 0$ in any neighbourhood of P. $\square$

---

[4]Note that there are various definitions of a covering of manifolds used in the literature (see for example [Bers, Jost, Beardon]). In particular often the term "covering" is used for unramified coverings of our definition. Ramified coverings are important in the theory of Riemann surfaces.

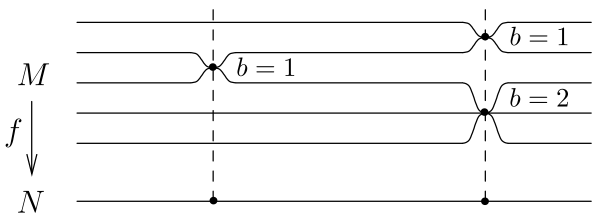

Figure 4: Covering

Theorem 2.6 *Let $f : M \to N$ be a non-constant holomorphic mapping between two compact Riemann surfaces. Then there exists $m \in \mathbb{N}$ such that every $Q \in N$ is assumed by f precisely m times - counting multiplicities; that is for all $Q \in N$*

$$\sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m. \tag{28}$$

Proof The set of branch points B is finite, therefore its projection $A = f(B)$ is also finite. Any two points $Q_1, Q_2 \in N\backslash A$ can be connected by a curve $l \subset N\backslash A$. Since $f^{-1}(l) \cap B = \emptyset$, the map f is a homeomorphism near each component of $f^{-1}(l)$, and $f^{-1}(l)$ consists of m non-intersecting curves $l_1, \ldots, l_m$ (m is finite, otherwise the set $f^{-1}(Q_1)$ has a limiting point and f is constant). This shows that the number of preimages for any points in $N\backslash A$ is the same.

Generally (see Fig. 4), for a point $Q \in N$ there are n preimages $P_1, \ldots, P_n$ with $f(P_i) = Q$ and the corresponding branch numbers $b(P_i)$. These points have non-intersecting neighbourhoods $U_1, \ldots, U_n$, $P_i \in U_i$, $\pi(U_i) = U \quad \forall i$, $U_i \cap U_j = \emptyset$ such that for any $\tilde{Q} \in U\backslash\{Q\}$ there are exactly $b(P_i) + 1$ points of $f^{-1}(\tilde{Q})$ lying in U_i. Since $\tilde{Q} \in N\backslash A$ the previous consideration implies (28). $\square$

Definition 2.3 *The number m above is called the degree of f. The covering $f : M \to N$ is called m-sheeted.*

Applying Theorem 2.6 to holomorphic mappings $f : \mathcal{R} \to \hat{\mathbb{C}}$ we get

Corollary 2.7 *A non-constant meromorphic function on a compact Riemann surface assumes every its value in $\hat{\mathbb{C}}$ m times, where m is the number of its poles (counting multiplicities).*

Remark A single non-constant meromorphic function $f : \mathcal{R} \to \hat{\mathbb{C}}$ completely determines the complex structure of the Riemann surface. A local parameter vanishing at $P_0 \in \mathcal{R}$ is given by

$$(f(P) - f(P_0))^{1/k(P_0)} \text{ for } f(P_0) \neq \infty,$$

where $k(P_0) = b_f(P_0) + 1$. For $f(P_0) = \infty$ one uses the local coordinate $1/z$ for a neighbourhood of ∞ in $\hat{\mathbb{C}}$, and a local parameter is given by

$$(f(P))^{-1/k(P_0)} \text{ for } f(P_0) = \infty.$$

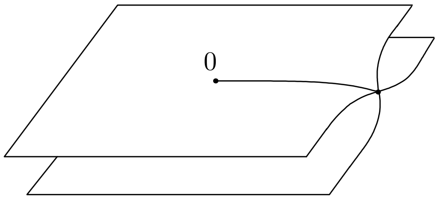

Figure 5: Riemann surface of $\sqrt{\lambda}$

2.1 Algebraic curves as coverings

Let C be a non-singular algebraic curve (1) and $\hat{C}$ its compatification. The mapping

$$(\mu, \lambda) \to \lambda \tag{29}$$

defines a holomorphic covering $\hat{C} \to \hat{\mathbb{C}}$. If N is the degree of the polynomial $\mathcal{P}(\mu, \lambda)$ in μ

$$\mathcal{P}(\mu, \lambda) = \mu^N p_N(\lambda) + \mu^{N-1} p_{N-1}(\lambda) + \ldots + p_0(\lambda),$$

where all $p_i(\lambda)$ are polynomials, then $\lambda : \hat{C} \to \hat{\mathbb{C}}$ is an N-sheeted covering.

The points with $\partial\mathcal{P}/\partial\mu = 0$ are the branch points of the covering $\lambda : C \to \mathbb{C}$. Indeed, at these points $\partial\mathcal{P}/\partial\lambda \neq 0$, and μ is a local parameter. The derivative of λ with respect to the local parameter vanishes

$$\frac{\partial\lambda}{\partial\mu} = -\frac{\partial\mathcal{P}/\partial\mu}{\partial\mathcal{P}/\partial\lambda} = 0,$$

which characterizes (27) the branch points of the covering (29). In the same way C covers $(\mu, \lambda) \to \mu$ the complex plane of μ. The branch points of this covering are the points with $\partial\mathcal{P}/\partial\lambda = 0$.

Hyperelliptic curves

Considering the hyperelliptic case let us remind a conventional description of the Riemann surface of the function $\mu = \sqrt{\lambda}$ from the basic course of complex analysis. One imagines oneself two copies of the complex plane $\mathbb{C}$ with a cut $[0, \infty]$ glued together crosswise along this cut (see Fig. 5). The image in Fig. 5 is in one to one correspondence with the points of the curve

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 \mid \mu^2 = \lambda\},$$

and the point $\lambda = 0$ gives an idea of a branch point.

The compactification $\hat{C}$ of the hyperelliptic curve

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 \mid \mu^2 = \prod_{i=1}^{N} (\lambda - \lambda_i)\} \tag{30}$$

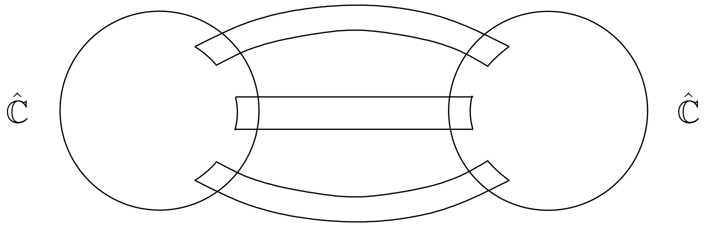

Figure 6: Topological image of a hyperelliptic surface

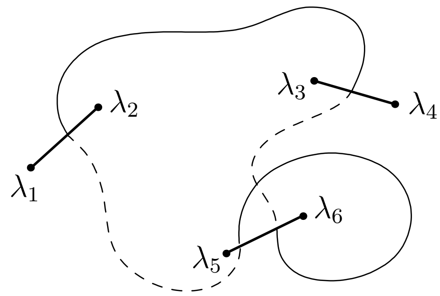

Figure 7: Hyperelliptic surface C as a two-sheeted cover. The parts of the curves on C that lie on the second sheet are indicated by dotted lines.

is a two sheeted covering of the extended complex plane $\lambda : \hat{C} \to \hat{\mathbb{C}}$. The branch points of this covering are

$$\begin{array}{llll}(0,\lambda_i), & i = 1,\ldots,N & \text{and } \infty & \text{for } N = 2g+1, \\ (0,\lambda_i), & i = 1,\ldots,N & & \text{for } N = 2g+2,\end{array}$$

with the branch numbers $b_\lambda = 1$ at these points. Only the branching at $\lambda = \infty$ possibly needs some clarification. The local parameter at $\infty \in \hat{\mathbb{C}}$ is $1/\lambda$, whereas the local parameter at the point $\infty \in \hat{C}$ of the curve $\hat{C}$ with $N = 2g+1$ is $1/\sqrt{\lambda}$ due to (9). In these coordinates the covering mapping reads as (compare with (26))

$$\frac{1}{\lambda} = \left(\frac{1}{\sqrt{\lambda}}\right)^2,$$

which shows that $b_\lambda(\infty) = 1$.

One can imagine oneself the Riemann surface $\hat{C}$ with $N = 2g+2$ as two Riemann spheres with the cuts

$$[\lambda_1, \lambda_2], [\lambda_3, \lambda_4], \ldots, [\lambda_{2g+1}, \lambda_{2g+2}]$$

glued together crosswise along the cuts. Fig. 6 presents a topological image of this Riemann surface. Later on we will use the image shown in Fig. 7, where we see the Riemann surface "from above" or "the first" sheet on the covering $\lambda : C \to \mathbb{C}$ and should add the points at infinity to this image. In the case $N = 2g+1$ one should move the branch point λ_{2g+2} to infinity.

The hyperelliptic curves obey a holomorphic involution

$$h : (\mu, \lambda) \to (-\mu, \lambda), \tag{31}$$

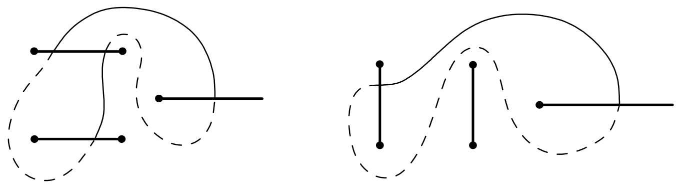

Figure 8: Two equivalent images of a hyperelliptic Riemann surface

which interchanges the sheets of the covering $\lambda : \hat{C} \to \hat{\mathbb{C}}$ and is called *hyperelliptic*. The branch points of the covering are the fixed points of h.

Remark The cuts in Fig. 7 are conventional and belong to the image shown in Fig. 7 and not to the hyperelliptic Riemann surface itself, which is determined by its branch points. In particular, the images shown in Fig.8 correspond to the same Riemann surface and to the same covering $(\mu, \lambda) \to \lambda$.

2.2 Quotients of Riemann Surfaces as Coverings

In Section 1.2 we defined the complex structure on the factor Δ/G, where Δ is a domain in $\mathbb{C}$, so that the canonical projection

$$\pi : \Delta \to \Delta/G$$

is holomorphic. This construction can be also applied to Riemann surfaces.

Theorem 2.8 *Let $\mathcal{R}$ be a (compact) Riemann surface and G a finite group of its holomorphic automorphisms*[5] *of order* ord$G$. *Then $\mathcal{R}/G$ is a Riemann surface with the complex structure determined by the condition that the canonical projection*

$$\pi : \mathcal{R} \to \mathcal{R}/G$$

is holomorphic. This is an ord$G$*-sheeted covering, ramified at fixed points of G.*

Proof The consideration for the case when $P \in \mathcal{R}$ is not a fixed point of G (there are finitely many fixed points of G) is the same as for Δ/G above. The canonical projection π defines an ord$G$-sheeted covering unramified at these points. Let P_0 be a fixed point and denote by

$$G_{P_0} = \{g \in G \mid gP_0 = P_0\}$$

the *stabilizer* of P_0. It is always possible to choose a neighborhood U of P_0 with no other fixed point than P_0 in U, which is invariant with respect to all elements of G_{P_0} and such that $U \cap gU = \emptyset$ for all $g \in G \setminus G_{P_0}$. Let us normalize the local parameter z on U by $z(P_0) = 0$. The local parameter w in $\pi(U)$, which is ordG_{P_0}-sheetedly covered by U is defined by the product of the values of the local parameter z at all equivalent points lying in U. In terms of the local parameter z all the elements of the stabilizer are represented by the functions $\tilde{g} = z \circ g \circ z^{-1} : z(U) \to z(U)$, which vanish at $z = 0$.

[5]We will see later that this group is always finite if the genus ≥ 2.

Since $\tilde{g}(z)$ are also invertible they can be represented as $\tilde{g}(z) = zh_g(z)$ with $h_g(0) \neq 0$. Finally the $w - z$ coordinate charts representation of π

$$w \circ \pi \circ z^{-1} : z \to z^{\mathrm{ord} G_{P_0}} \prod_{g \in G_{P_0}} h_g(z)$$

shows that the branch number of P_0 is $\mathrm{ord} G_{P_0}$. □

The compact Riemann surface $\hat{C}$ of the hyperelliptic curve

$$\mu^2 = \prod_{n=1}^{2N} (\lambda^2 - \lambda_n^2), \qquad \lambda_i^2 \neq \lambda_j^2,\ \lambda_k \neq 0 \tag{32}$$

has the following group of holomorphic automorphisms

$$\begin{aligned} h &: (\mu, \lambda) \to (-\mu, \lambda) \\ i_1 &: (\mu, \lambda) \to (\mu, -\lambda) \\ i_2 = hi_1 &: (\mu, \lambda) \to (-\mu, -\lambda). \end{aligned}$$

The hyperelliptic involution h interchanges the sheets of the covering $\lambda : \hat{C} \to \hat{\mathbb{C}}$, therefore the factor $\hat{C}/h$ is the Riemann sphere. The covering

$$\hat{C} \to \hat{C}/h = \hat{\mathbb{C}}$$

is ramified at all the points $\lambda = \pm\lambda_n$.

The involution i_1 has four fixed points on $\hat{C}$: two points with $\lambda = 0$ and two points with $\lambda = \infty$. The covering

$$\hat{C} \to \hat{C}_1 = \hat{C}/i_1 \tag{33}$$

is ramified at these points. The mapping (33) is given by

$$(\mu, \lambda) \to (\mu, \Lambda), \qquad \Lambda = \lambda^2,$$

and $\hat{C}_1$ is the Riemann surface of the curve

$$\mu^2 = \prod_{n=1}^{2N} (\Lambda - \lambda_n^2).$$

The involution i_2 has no fixed points. The covering

$$\hat{C} \to \hat{C}_2 = \hat{C}/i_2 \tag{34}$$

is unramified. The mapping (34) is given by

$$(\mu, \lambda) \to (M, \Lambda), \qquad M = \mu\lambda,\ \Lambda = \lambda^2,$$

and $\hat{C}_2$ is the Riemann surface of the curve

$$M^2 = \Lambda \prod_{n=1}^{2N} (\Lambda - \lambda_n^2).$$

3 Topology of Riemann Surfaces

3.1 Spheres with Handles

We have seen in Section 1 that any Riemann surface is a two-real-dimensional orientable smooth manifold. In this section we present basic facts about topology of these manifolds focusing on the compact case. We start with an intuitively natural fundamental classification theorem and comment its proof later on.

Theorem 3.1 (and Definition) *Any compact Riemann surface is homeomorphic to a sphere with handles* [6]. *The number $g \in \mathbb{N}$ of handles is called the genus of $\mathcal{R}$. Two manifolds with different genera are not homeomorphic.*

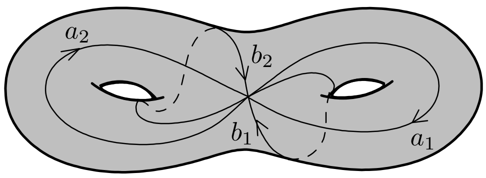

Figure 9: Sphere with 2 handles

The genus of the compactification $\hat{C}$ of the hyperelliptic curve (30) with $N = 2g + 1$ or $N = 2g + 2$ is equal to g.

For many purposes it is convenient to use planar images of spheres with handles.

Proposition 3.2 *Let Π_g be an extended plane*[7] *with $2g$ holes bounded by the non-intersecting curves*

$$\gamma_1, \gamma_1', \ldots, \gamma_g, \gamma_g'. \tag{35}$$

and the curves $\gamma_i \approx \gamma_i'$, $i = 1, \ldots, g$ are topologically identified in such a way that the orientations of these curves with respect to Π_g are opposite (see Fig. 10). Then Π_g is homeomorphic to a sphere with g handles.

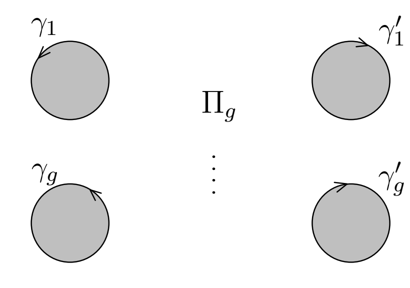

Figure 10: Planar image of a sphere with g handles

[6]By a sphere with handles we mean a topological manifold homeomorphic to a sphere with handles in Euclidean 3-space.

[7]By an extended plane we mean $\mathbb{R}^2 \cup \{\infty\}$, which is homeomorphic to S^2.

To prove this proposition one should cut up all the handles of a sphere with g handles.

A normalized simply-connected image of a sphere with g handles is described by the following proposition.

Proposition 3.3 *Let F_g be a 4g-gon with the edges*

$$a_1, b_1, a_1', b_1', \ldots, a_g, b_g, a_g', b_g', \tag{36}$$

listed in the order of traversing the boundary of F_g and the curves

$$a_i \approx a_i', b_i \approx b_i', \; i = 1, \ldots, g$$

are topologically identified in such a way that the orientations of the edges a_i and a_i' as well as b_i and b_i' with respect to F_g are opposite (see Fig. 11). Then F_g is homeomorphic to a sphere with g handles. The sphere without handles ($g = 0$) is homeomorphic to the 2-gon with the edges

$$a, a', \tag{37}$$

identified as above.

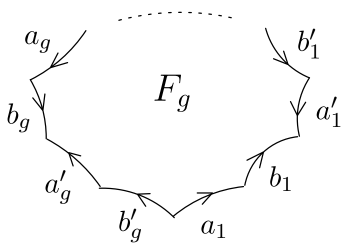

Figure 11: Simply-connected image of a sphere with g handles

Proof is given in Figs. 12, 13. One choice of closed curves $a_1, b_1, \ldots, a_g, b_g$ on a sphere with handles is shown in Fig. 9. □

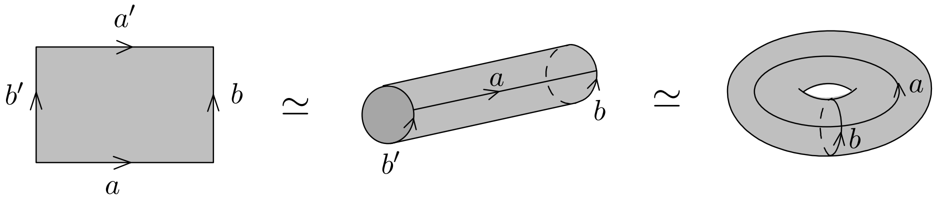

Figure 12: Gluing a torus

Let us consider a *triangulation* $\mathcal{T}$ of $\mathcal{R}$, i.e. a set $\{T_i\}$ of topological triangles on $\mathcal{R}$, which cover $\mathcal{R}$

$$\cup T_i = \mathcal{R}$$

and the intersection $T_i \cap T_j$ for any T_i, T_j is either empty or consists of one common edge or of one common vertex (compare with Section 1.3). Obviously, compact Riemann surfaces are triangularizable by finite triangulations[8].

[8]Due to Rado's theorem (see for example [AlforsSario]) any Riemann surface is triangularizable.

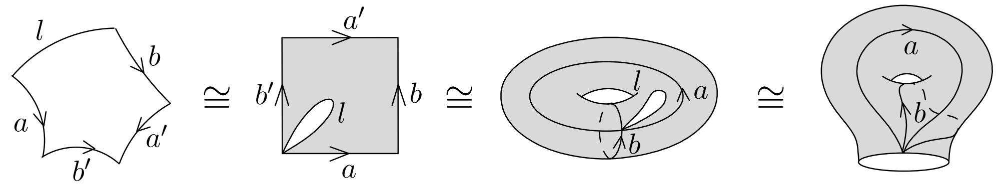

Figure 13: Gluing a handle

Definition 3.1 *Let $\mathcal{T}$ be a triangulation of a compact two-real dimensional manifold $\mathcal{R}$ and F be the number of triangles, E - the number of edges, V - the number of vertices of $\mathcal{T}$. The number*

$$\chi = F - E + V \tag{38}$$

is called the Euler characteristics of $\mathcal{R}$.

Proposition 3.4 *The Euler characteristic $\chi(\mathcal{R})$ of a compact Riemann surface*[9] *$\mathcal{R}$ is independent of the triangulation of $\mathcal{R}$.*

Proof. Introduce a conformal metric $e^u\, dzd\bar{z}$ on a Riemann surface (Theorem 1.4). The Gauss–Bonnet theorem provides us with the following formula for the Euler characteristic

$$\chi(\mathcal{R}) = \frac{1}{2\pi}\int_{\mathcal{R}} K, \tag{39}$$

where

$$K = -2u_{z\bar{z}}e^{-u}$$

is the curvature of the metric. The right hand side in (39) is independent of the triangulation, the left hand side is independent of the metric we introduced on $\mathcal{R}$. This proves that the Euler characteristics is a topological invariant of $\mathcal{R}$.

Corollary 3.5 *The Euler characteristics $\chi(\mathcal{R})$ of a compact Riemann surface $\mathcal{R}$ of genus g is equal*

$$\chi(\mathcal{R}) = 2 - 2g. \tag{40}$$

For the proof of this corollary it is convenient to consider the simply-connected model F_g of Proposition 3.3.

Sketch of the proof of Theorem 3.1. Let $\mathcal{R}$ be a compact Riemann surface and $\mathcal{T}$ a triangulation of $\mathcal{R}$ oriented in accordance with the orientation of $\mathcal{R}$. Each triangle T_i can be mapped onto an Euclidean triangle. Successively mapping neighboring triangles we finally obtain a $n+2$-gon, where n is the number of triangles in $\mathcal{T}$. Since each side of this polygon is identified with precisely one other side, the polygon has an even number

[9]The statement of the Proposition holds true also for general two-real dimensional manifolds. The proof is combinatorial.

of edges. Let us label the edges of this polygon, labelling one of the identified edges by c and the other by c'. We call the word obtained by writing the letters in order of traversing the boundary the *symbol* of the polygon. By cutting up the polygon and pasting it after that in another way one can simplify the symbol. The simplification to the normal form (35) ($g > 0$) or (36) ($g = 0$) can be described explicitly. All the details of this process can be found for example in [Springer, Bers]. We see that $\mathcal{R}$ is homeomorphic to F_g with some g. In its turn, due to Proposition 3.3 F_g is obviously homeomorphic to a sphere with g handles.

Directly from Definition 3.1 one gets that the Euler characteristics of two homeomorphic manifolds coincide. This implies that $\tilde{F}_{\tilde{g}}$ and F_g are homeomorphic if and only if $g = \tilde{g}$, which completes the proof.

Theorem 3.6 (Riemann-Hurwitz)
Let $f : \hat{\mathcal{R}} \to \mathcal{R}$ be an N-sheeted covering of compact Riemann surfaces and $\mathcal{R}$ is of genus g. Then the genus $\hat{g}$ of $\hat{\mathcal{R}}$ is given by

$$\hat{g} = N(g-1) + 1 + \frac{b}{2}, \tag{41}$$

where

$$b = \sum_{P \in \hat{\mathcal{R}}} b_f(P) \tag{42}$$

is the total branching number.

Proof As it was shown in Lemma 2.5 the set $B = \{P \in \hat{\mathcal{R}} \mid b_f(P) > 0\}$ is finite. We triangulate $\mathcal{R}$ so that every point of $A = f(B) \subset \mathcal{R}$ is a vertex of the triangulation. Let us assume that the triangulation has F faces, E edges and V vertices. Then the induced triangulation lifted to $\hat{\mathcal{R}}$ via the mapping f has NF faces, NE edges and $NV - b$ vertices, where b is given by (42). For the Euler characteristics of $\hat{\mathcal{R}}$ and $\mathcal{R}$ this implies

$$\chi(\hat{\mathcal{R}}) = N\chi(\mathcal{R}) - b,$$

which is equivalent to (41) because of (38). □

3.2 Fundamental group

Let P and Q be two points on $\mathcal{R}$ and γ_{PQ} a curve, i.e. a continuous map $\gamma : [0,1] \to \mathcal{R}$, connecting them $\gamma_{PQ}(0) = P,\ \gamma_{PQ}(1) = Q$.

Definition 3.2 *Two curves $\gamma^1_{PQ}, \gamma^2_{PQ}$ on $\mathcal{R}$ with the initial point P and the terminal point Q are called homotopic if they can be continuously deformed one to another, i.e. provided there is a continuous map $\gamma : [0,1] \times [0,1] \to \mathcal{R}$ such that $\gamma(t,0) = \gamma^1_{PQ}(t),\ \gamma(t,1) = \gamma^2_{PQ}(t),\ \gamma(0,\lambda) = P,\ \gamma(1,\lambda) = Q$. The set of homotopic curves forms a homotopic class, which we denote by $\Gamma_{PQ} = [\gamma_{PQ}]$.*

If the terminal point of γ_1 coincides with the initial point of γ_2 the curves can be multiplied:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

This multiplication is well-defined also for the corresponding homotopic classes

$$\Gamma_1 \cdot \Gamma_2 = [\gamma_1 \cdot \gamma_2].$$

Any two closed curves through P can be multiplied. The set of homotopic classes of these curves forms a group $\pi_1(\mathcal{R}, P)$ with the multiplication defined above. The curves, which can be contracted to a point correspond to the identity element of the group. It is easy to see that the groups $\pi_1(\mathcal{R}, P)$ and $\pi_1(\mathcal{R}, Q)$ based at different points are isomorphic as groups. Considering this group one can omit the second argument in the notation

$$\pi_1(\mathcal{R}, P) \approx \pi_1(\mathcal{R}, Q) \approx \pi_1(\mathcal{R}).$$

Definition 3.3 *The group $\pi_1(\mathcal{R})$ is called the fundamental group of $\mathcal{R}$.*

Examples

1. Sphere with N holes

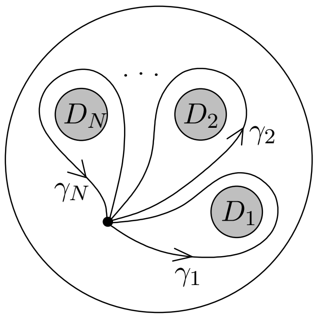

Figure 14: Fundamental group of a sphere with N holes

$\mathcal{R} = S \setminus \{\bigcup_{n=1}^{N} D_n\}$.

The fundamental group is generated by the homotopic classes of the closed curves $\gamma_1, \ldots, \gamma_N$ each going around one of the holes (Fig 14). The curve $\gamma_1\gamma_2 \ldots \gamma_N$ can be contracted to a point, which implies the relation

$$\Gamma_1\Gamma_2 \ldots \Gamma_N = 1 \tag{43}$$

in $\pi_1(S \setminus \{\bigcup_{n=1}^{N} D_n\})$.

2. Compact Riemann surface of genus g.

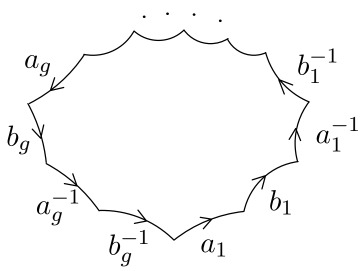

Figure 15: Fundamental group of a compact surface of genus g

It is convenient to consider the $4g$-gon model F_g (Fig. 15). The curves $a_1, b_1, \ldots, a_g, b_g$ are closed on $\mathcal{R}$. Their homotopic classes, which we denote by $A_1, B_1, \ldots, A_g, B_g$ generate $\pi_1(\mathcal{R})$.

The curve

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$$

comprises the oriented boundary of F_g. This implies the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} = 1 \tag{44}$$

in the fundamental group. There are no other independent relations. Indeed, such a relation would mean that some product p of the curves $a_1, \ldots, b_g$ can be contracted to a point. Since all the points of $\mathcal{R}$ are equivalent this point can be chosen inside F_g. This proves that $[p]$ is a multiple of (44).

3.3 First Homology Group of Riemann surfaces

Consider a Riemann surface $\mathcal{R}$ with an oriented triangulation $\mathcal{T}$. Formal sums of points $\sum n_i P_i$, oriented edges γ_i,

$$\gamma = \sum n_i \gamma_i \in C_1$$

and oriented triangles D_i,

$$D = \sum n_i D_i \in C_2$$

with integer coefficients $n_i \in \mathbb{Z}$ are called (simplicial) 0-chains, 1-chains and 2-chains respectively. We will denote these sets by C_0, C_1 and C_2. Define by $-\gamma_i$ (resp. $-D_i$) the curve γ_i (resp. the triangle D_i) with opposite orientation. It is clear that C_i form abelian groups under addition.

Denote by (P_1, P_2) the oriented edge from P_1 to P_2 and by $D_0 = (P_1, P_2, P_3)$ the oriented triangle bounded by the oriented edges $(P_1, P_2), (P_2, P_3)$ and (P_3, P_1). Define the boundary operator δ on the edge and triangle by

$$\delta(P_1, P_2) = P_2 - P_1, \qquad \delta D_0 = (P_1, P_2) + (P_2, P_3) + (P_3, P_1).$$

The boundary operator can be extended to whole C_1 and C_2 by linearity $\delta D = \sum k_i \delta D_i$, defining the group homomorphisms $\delta : C_1 \to C_0, \delta : C_2 \to C_1$.

C_1 contains two important subgroups - of cycles and of boundaries. A 1-chain γ with $\delta\gamma = 0$ is called a cycle, a 1-chain $\gamma = \delta D$ is called a boundary. We denote these subgroups by

$$Z = \{\gamma \in C_1 \mid \ \delta\gamma = 0\}, \qquad B = \delta C_2.$$

Due to $\delta^2 = 0$ every boundary is a cycle and we have $B \subset Z \subset C_1$.

One can introduce an equivalence relation between elements of C_1. Two 1-chains are called *homologous* if their difference is a boundary:

$$\gamma_1 \sim \gamma_2,\ \gamma_1, \gamma_2 \in C_1 \Leftrightarrow \gamma_1 - \gamma_2 \in B,\ \text{i.e. } \exists D \in C_2 : \delta D = \gamma_1 - \gamma_2.$$

Definition 3.4 *The factorgroup*

$$H_1(\mathcal{R}, \mathbb{Z}) = Z/B$$

is called the first homology group of $\mathcal{R}$*.*

All the groups we consider are abelian and the elements of $H_1(\mathcal{R}, \mathbb{Z})$ can be described as equivalence classes[10]

$$[\gamma] \in \frac{\{1-\text{cycles}\}}{\{1-\text{dimensional boundaries}\}}.$$

Any closed oriented continuous curve $\tilde{\gamma}$ (i.e. periodic continuous map $\tilde{\gamma} : [0,1] \to \mathcal{R}$) can be deformed homotopically into a 1-cycle in the triangulation $\mathcal{T}$. To show this one should consider the triangles of $\mathcal{T}$ close to $\tilde{\gamma}$ and construct the 1-cycle using the edges of their boundaries. Details of this construction can be found in [Springer]. Since homotopical simplicial 1-cycles are obviously homologous, this insight allows us to define the homology group as a homology group of cycles composed of arbitrary closed curves rather than symplicial 1-cycles on $\mathcal{R}$. We call such a curve $\tilde{\gamma}$ a simple cycle on $\mathcal{R}$.

This definition of homologous continuous cycles later will be shown to be independent of $\mathcal{T}$. Directly from the definition follows that freely homotopic closed curves are homologous. Note that the converse is however false in general as one can see from the example in Fig. 16.

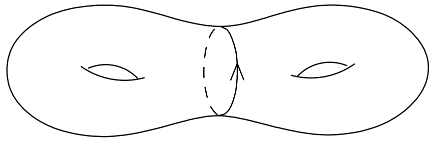

Figure 16: A cycle homologous to zero but not homotopic to a point.

[10]Considering n-chains on a triangulated manifold one can analogously define n-th homology group. Homology groups can be also introduced over arbitrary fields if one considers formal linear combinations with coefficients in these fields. For example so one can define $H_1(\mathcal{R}, \mathbb{Z}_2), H_n(\mathcal{R}, \mathbb{R})$ etc.

The first homology group is the fundamental group "made commutative"[11]. Indeed, let γ be a 1-cycle on $\mathcal{R}$ with a point $P_0 \in \gamma$ and $\Gamma_1, \ldots, \Gamma_n$ be generators of $\pi(\mathcal{R}, P_0)$. Denote by $[\gamma], [\Gamma_1], \ldots, [\Gamma_n] \in H_1(\mathcal{R}, \mathbb{Z})$ the corresponding homology classes. The cycle γ is homotopic to

$$\gamma = \Gamma_{i_1}^{j_1} \ldots \Gamma_{i_k}^{j_k}, \ i_1, \ldots, i_k \in \{1, \ldots, n\}, j_i \in \mathbb{Z},$$

which implies for the homology classes

$$[\gamma] = j_1[\Gamma_{i_1}] + \ldots j_k[\Gamma_{i_k}].$$

By linearity this representation can be extended to arbitrary combination of cycles in $H_1(\mathcal{R}, \mathbb{Z})$. As in Section 3.2 it is easy to see that $[\Gamma_i]$ are independent of P_0. Finally we see that the homology group is the abelian group generated by the elements $[\Gamma_i]$, $i = 1, \ldots, n$. This shows in particular that the whole construction is independent of the triangulation $\mathcal{T}$ we started with.

To introduce intersection numbers of elements of the first homology group it is convenient to represent them by smooth cycles. Every element of $H_1(\mathcal{R}, \mathbb{Z})$ can be represented by a C^∞-cycle. Moreover given two elements of $H_1(\mathcal{R}, \mathbb{Z})$ one can represent them by smooth cycles intersecting transversally in finite number of points.

Let γ_1 and γ_2 be two curves intersecting transversally at the point P. One associates to this point a number $(\gamma_1 \circ \gamma_2)_P = \pm 1$, where the sign is determined by the orientation of the basis $\gamma_1'(P), \gamma_2'(P)$ as it is shown in Fig. 17.

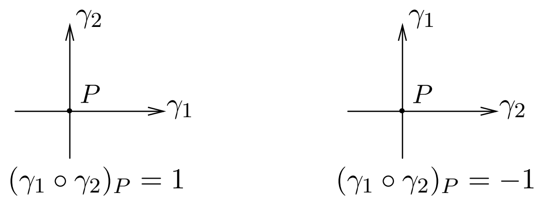

Figure 17: Intersection number at a point.

Definition 3.5 *Let γ_1, γ_2 be two smooth cycles intersecting transversally at the finite set of their intersection points. The intersection number of γ_1 and γ_2 is defined by*

$$\gamma_1 \circ \gamma_2 = \sum_{P \in \text{Intersection set}} (\gamma_1 \circ \gamma_2)_P. \tag{45}$$

Lemma 3.7 *The intersection number of any boundary β with any cycle γ vanishes, i.e. $\gamma \circ \beta = 0$.*

[11]Precisely

$$H_1(\mathcal{R}, \mathbb{Z}) = \frac{\pi(\mathcal{R})}{[\pi(\mathcal{R}), \pi(\mathcal{R})]},$$

where the denominator is the commutator subgroup, i.e. the subgroup of $\pi(\mathcal{R})$ generated by all elements of the form $ABA^{-1}B^{-1}, \ A, B \in \pi(\mathcal{R})$.

Proof. Since (45) is bilinear it is enough to prove the statement for a boundary of a domain $\beta = \delta D$ and a simple cycle γ. In this case the statement follows from the simple fact that the cycle γ goes as many times inside D as outside (see Fig. 18).

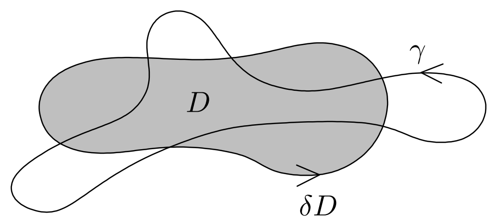

Figure 18: $\gamma \circ \delta D = 0$.

To define the intersection number on homologies represent $\gamma, \gamma' \in H_1(\mathcal{R}, \mathbb{Z})$ by C^∞-cycles

$$\gamma = \sum_i n_i \gamma_i, \qquad \gamma' = \sum_j m_j \gamma'_i,$$

where γ_i, γ'_j are smooth curves intersecting transversally. Define $\gamma \circ \gamma' = \sum_{ij} n_i m_j \gamma_i \circ \gamma'_j$. Due to Lemma 3.7 the intersection number is well defined on homologies.

Theorem 3.8 *The intersection number is a bilinear skew-symmetric map*

$$\circ : H_1(\mathcal{R}, \mathbb{Z}) \times H_1(\mathcal{R}, \mathbb{Z}) \to \mathbb{Z}.$$

Examples

1. Homology group of a sphere with N holes.

The homology group is generated by the loops $\gamma_1, \dots, \gamma_{N-1}$ (see Fig. 14). For the homology class of the loop γ_N one has

$$\gamma_N = -\sum_{i=1}^{N-1} \gamma_i,$$

since $\sum_{i=1}^{N} \gamma_i$ is a boundary.

2. Homology group of a compact Riemann surface of genus g.

Since the homotopy group is generated by the cycles $a_1, b_1, \dots, a_g, b_g$ shown in Fig. 15 it is also true for the homology group. The intersection numbers of these cycles are as follows

$$a_i \circ b_j = \delta_{ij}, \qquad a_i \circ a_j = b_i \circ b_j = 0. \tag{46}$$

The cycles $a_1, b_1, \dots, a_g, b_g$ build a basis of the homology group. They are distinguished by their intersection numbers and as a consequence are linearly independent.

Definition 3.6 *A homology basis $a_1, b_1, \ldots, a_g, b_g$ of a compact Riemann surface of genus g with the intersection numbers (46) is called canonical basis of cycles.*

Remark Canonical basis of cycles is by no means unique. Let (a, b) be a canonical basis of cycles. We represent it by a $2g$-dimensional vector

$$\begin{pmatrix} a \\ b \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_g \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix}.$$

Any other basis $(\tilde{a}, \tilde{b})$ of $H_1(\mathcal{R}, \mathbb{Z})$ is then given by the transformation

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}, \qquad A \in SL(2g, \mathbb{Z}). \tag{47}$$

Substituting (47) into

$$J = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \circ (\tilde{a}, \tilde{b}), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

we obtain that the basis $(\tilde{a}, \tilde{b})$ is canonical if and only if A is symplectic $A \in Sp(g, \mathbb{Z})$, i.e.

$$J = AJA^T. \tag{48}$$

Two examples of canonical basis of cycles are presented in Figs. 19, 20. The curves b_i in Fig. 19 connect identified points of the boundary curves and therefore are closed. In Fig. 20 the parts of the cycles lying on the "lower" sheet of the covering are marked by dotted lines.

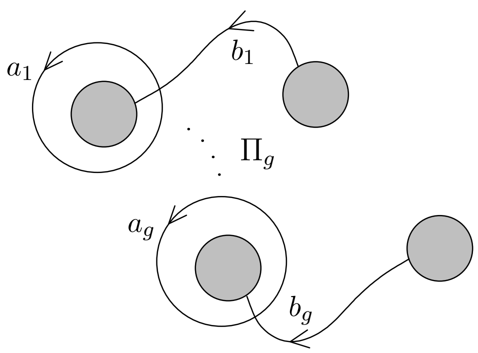

Figure 19: Canonical basis of cycles on the planar model Π_g of a compact Riemann surface.

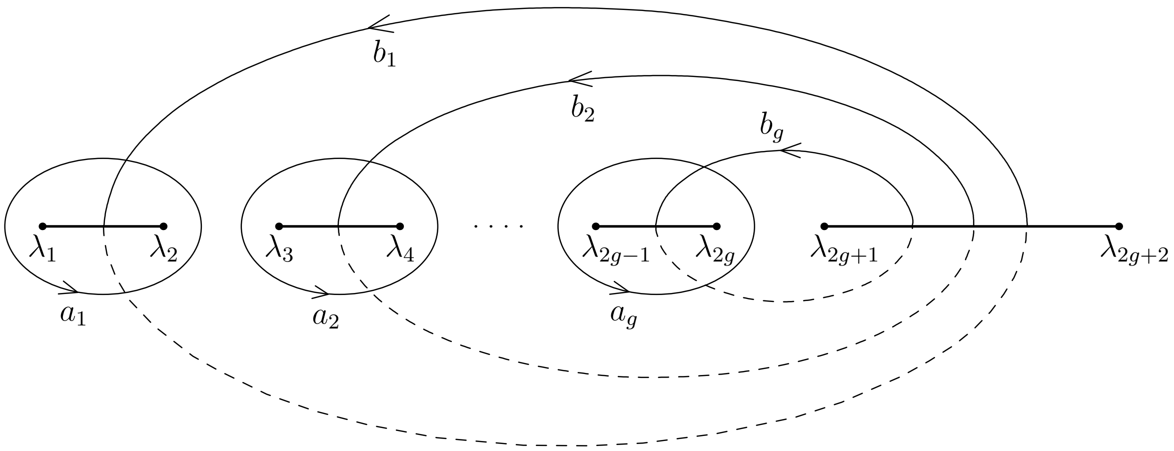

Figure 20: Canonical basis of cycles of a hyperelliptic Riemann surface.

4 Abelian differentials

Our main goal is to construct functions on compact Riemann surfaces with prescribed analytical properties (for example, meromorphic functions with prescribed singularities). This and next sections are devoted to this problem. We start with a description of meromorphic differentials, which are much simpler to handle than the functions and which are the basic tool to investigate and to construct functions.

4.1 Differential forms and integration formulas

We recall the theory of integration on 2-dimensional C^∞-manifolds using complex notations. Let $\mathcal{R}$ be such a manifold and

$$z : U \subset \mathcal{R} \to V \subset \mathbb{C}$$

be local parameters. The transition functions $\tilde{z}(z, \bar{z})$ defined for non-trivial intersections $U \cap \tilde{U}$

$$\tilde{z} \circ z^{-1} : z(U \cap \tilde{U}) \to \tilde{z}(U \cap \tilde{U}) \tag{49}$$

are C^∞.

If to each local coordinate on $\mathcal{R}$ there are assigned complex valued functions[12] $f(z, \bar{z}),\ p(z, \bar{z}),\ q(z, \bar{z}),\ s(z, \bar{z})$ such that

$$\begin{aligned} f &= f(z, \bar{z}), \\ \omega &= p(z, \bar{z})dz + q(z, \bar{z})d\bar{z}, \\ S &= s(z, \bar{z})dz \wedge d\bar{z}. \end{aligned} \tag{50}$$

are invariant under coordinate changes (49) one says that the function (0-form) f, the differential (1-form) ω and the 2-form S are defined on $\mathcal{R}$. The identification

$$dz = dx + idy, \qquad d\bar{z} = dx - idy$$

[12] We will not treat the problems in the most general setup and assume that the functions are smooth. It will be enough for applications in the Riemann surface theory.

implies the standard description of ω, S in real coordinates x, y. The exterior product of two 1-forms ω_1 and ω_2 is the 2-form

$$\omega_1 \wedge \omega_2 = (p_1 q_2 - p_2 q_1) dz \wedge d\bar{z}.$$

If we let $\omega^{(1,0)} = p(z,\bar{z})dz, \omega^{(0,1)} = q(z,\bar{z})d\bar{z}$, the forms $\omega^{(1,0)}$ and $\omega^{(0,1)}$ are independent of the choice of the local holomorphic coordinate and therefore are differentials defined globally on $\mathcal{R}$. The 1-form ω is called a form of type (1,0) (resp. a form of type (0,1)) iff locally it may be written $\omega = p\,dz$ (resp. $\omega = q\,d\bar{z}$), i.e. its (0,1)-part (resp. (1,0)-part) vanish. The space of differentials is obviously a direct sum of the subspaces of (1,0) and (0,1) forms.

One can integrate:

1. 0-forms over 0-chains, which are finite sets $\{P_\alpha\}_\alpha$ of points $P_\alpha \in \mathcal{R}$:

$$\sum_\alpha f(P_\alpha),$$

2. 1-forms over 1-chains (paths, i.e. smooth oriented curves, and their finite unions):

$$\int_\gamma \omega,$$

3. 2-forms over 2-chains (finite unions of domains):

$$\int_D S.$$

Here if $\gamma : [0,1] \to U$ and $D \subset U$ are contained in a single coordinate disc, the integrals are defined by

$$\begin{aligned} \int_\gamma \omega &= \int_0^1 \left(p(z(\gamma(t)), \overline{z(\gamma(t))}) \frac{dz(\gamma)}{dt} + q(z(\gamma(t)), \overline{z(\gamma(t))}) \frac{d\overline{z(\gamma)}}{dt} \right) dt, \\ \int_U S &= \int_V s(z,\bar{z}) dz \wedge d\bar{z}. \end{aligned}$$

Due to invariance of (50) under coordinate changes the integrals are well-defined.

The differential operator d, which transforms k-forms into $(k+1)$-forms is defined by

$$\begin{aligned} df &= f_z dz + f_{\bar{z}} d\bar{z}, \\ d\omega &= (q_z - p_{\bar{z}}) dz \wedge d\bar{z}, \\ dS &= 0. \end{aligned} \tag{51}$$

Definition 4.1 *A differential df is called exact. A differential ω with $d\omega = 0$ is called closed.*

One can also easily check using (51), that

$$d^2 = 0$$

whenever d^2 is defined and

$$d(f\omega) = df \wedge \omega + f d\omega \tag{52}$$

for any function f and 1-form ω. This implies in particular that any exact form is closed. The most important property of d is contained in

Theorem 4.1 (Stokes)
Let D be a 2-chain with a piecewise smooth boundary ∂D. Then the Stokes formula

$$\int_D d\omega = \int_{\partial D} \omega \tag{53}$$

holds for any differential ω.

Our principal interest will be in 1-forms. Let γ_{PQ} be a curve connecting P and Q. When does the integral $\int_{\gamma_{PQ}} \omega$ depend on the points P, Q and not on the integration path?

Corollary 4.2 *A differential ω is closed, $d\omega = 0$, if and only if for any two homological paths γ and $\tilde{\gamma}$*

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$$

holds.

Proof The difference of two homological curves $\gamma - \tilde{\gamma}$ is a boundary for some D. Applying (53) we have

$$\int_\gamma \omega - \int_{\tilde{\gamma}} \omega = \int_{\partial D} \omega = \int_D d\omega = 0.$$

The differential ω is closed since D is arbitrary. □

Corollary 4.3 *Let ω be a closed differential, F_g be a simply connected model of Riemann surface of genus g (see Section 3) and P_0 be some point in F_g. Then the function*

$$f(P) = \int_{P_0}^{P} \omega, \qquad P \in F_g,$$

where the integration path lies in F_g is well-defined on F_g.

One can easily check the identity

$$d(\int_{P_0}^{P} \omega) = \omega(P). \tag{54}$$

Let $\gamma_1, \ldots, \gamma_n$ be a homology basis of $\mathcal{R}$ and ω a closed differential. Periods of ω are defined by

$$\Lambda_i = \int_{\gamma_i} \omega.$$

Any closed curve γ on $\mathcal{R}$ is homological to $\sum n_i \gamma_i$ with some $n_i \in \mathbb{Z}$, which implies

$$\int_{\gamma} \omega = \sum n_i \Lambda_i,$$

i.e. Λ_i generate the lattice of periods of ω. In particular, if $\mathcal{R}$ is a Riemann surface of genus g with the canonical homology basis $a_1, b_1, \ldots, a_g, b_g$, we denote the corresponding periods by

$$A_i = \int_{a_i} \omega, \qquad B_i = \int_{b_i} \omega.$$

Theorem 4.4 (Riemann's bilinear identity)
Let $\mathcal{R}$ be a Riemann surface of genus g with a canonical basis $a_i, b_i,\ i = 1, \ldots, g$ and F_g be its simply-connected model. Also let ω and ω' be two closed differentials on $\mathcal{R}$ and $A_i, B_i, A'_i, B'_i,\ i = 1, \ldots, g$ be their periods. Then

$$\int_{\mathcal{R}} \omega \wedge \omega' = \int_{\partial F_g} \omega'(P) \int_{P_0}^{P} \omega = \sum_{j=1}^{g} (A_j B'_j - A'_j B_j), \tag{55}$$

where P_0 is some point in F_g and the integration path $[P_0, P]$ lies in F_g.

Proof The Riemann surface $\mathcal{R}$ cut along all the cycles $a_i, b_i,\ i = 1, \ldots, g$ of the fundamental group is the simply connected domain F_g with the boundary (see Figs. 11, 15)

$$\partial F_g = \sum_{i=1}^{g} a_i + a_i^{-1} + b_i + b_i^{-1}. \tag{56}$$

The first identity in (55) follows directly from the Stokes theorem with $D = F_g$, Corollary 4.3, (52) and (54).

The curves a_j and a_j^{-1} of the boundary of F_g are identical on $\mathcal{R}$ but have opposite orientation. For the points P_j and P'_j lying on a_j and a_j^{-1} respectively and coinciding on $\mathcal{R}$ we have (see Fig. 21)

$$\omega'(P_j) = \omega'(P'_j),$$
$$\int_{P_0}^{P_j} \omega - \int_{P_0}^{P'_j} \omega = \int_{P'_j}^{P_j} \omega = -B_j. \tag{57}$$

In the same way for the points $Q_j \in b_j$ and $Q'_j \in b_j^{-1}$ coinciding on $\mathcal{R}$ one gets

$$\omega'(Q_j) = \omega'(Q'_j),$$
$$\int_{P_0}^{Q_j} \omega - \int_{P_0}^{Q'_j} \omega = \int_{Q'_j}^{Q_j} \omega = A_j. \tag{58}$$

Substituting, we obtain

$$\begin{aligned} \int_{\partial F_g} \omega'(P) \int_{P_0}^{P} \omega &= \sum_{j=1}^{g} \Big(- B_j \int_{a_j} \omega' + A_j \int_{b_j} \omega' \Big) = \\ &= \sum_{j=1}^{g} (A_j B'_j - A'_j B_j). \end{aligned}$$

Finally, to prove Riemann's bilinear identity for an arbitrary canonical basis of $H_1(\mathcal{R}, \mathbb{C})$ one can directly check that the right hand side of (55) is invariant with respect to the transformation (47, 48).

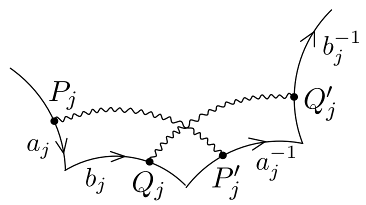

Figure 21: To the proof of the Riemann bilinear relations.

□

4.2 Abelian differentials of the first, second and third kind

Let now $\mathcal{R}$ be a Riemann surface. The transition functions (49) are holomorphic and one can define more special differentials on $\mathcal{R}$.

Definition 4.2 *A differential ω on a Riemann surface $\mathcal{R}$ is called holomorphic (or an Abelian differential of the first kind) if in any local chart it is represented as*

$$\omega = h(z)dz$$

where $h(z)$ is holomorphic. The differential $\bar{\omega}$ is called anti-holomorphic.

Holomorphic and anti-holomorphic differentials are closed.

Holomorphic differentials form a complex vector space, which is denoted by $H^1(\mathcal{R}, \mathbb{C})$. What is its dimension?

Lemma 4.5 *Let ω be a non-zero ($\omega \not\equiv 0$) holomorphic differential on $\mathcal{R}$. Then its periods A_j, B_j satisfy*

$$\mathrm{Im} \sum_{j=1}^{g} A_j \bar{B}_j < 0.$$

Proof The periods of $\bar{\omega}$ are $\bar{A}_j, \bar{B}_j$. Apply Theorem 4.4 to ω and $\bar{\omega}$ and use

$$i\omega \wedge \bar{\omega} = i|h|^2 dz \wedge d\bar{z} = 2|h|^2 dx \wedge dy > 0.$$

□

Corollary 4.6 *If all a-periods of the holomorphic differential ω are zero*

$$\int_{a_j} \omega = 0, \qquad j = 1, \ldots, g,$$

then $\omega \equiv 0$.

Corollary 4.7 *If all periods of a holomorphic differential ω are real, then $\omega \equiv 0$.*

Corollary 4.8 $\dim\, H^1(\mathcal{R}, \mathbb{C}) \leq g$.

Proof If $\omega_1, \ldots, \omega_{g+1}$ are holomorphic, then there exists a linear combination of them $\sum_{i=1}^{g+1} \alpha_i \omega_i$ with all zero a-periods. Corollary 4.6 implies $\sum_{i=1}^{g+1} \alpha_i \omega_i \equiv 0$, i.e. the differentials are linearly dependent. □

Theorem 4.9 *The dimension of the space of holomorphic differentials of a compact Riemann surface is equal to its genus*

$$\dim\, H^1(\mathcal{R}, \mathbb{C}) = g(\mathcal{R}).$$

We give a proof of this theorem in Section 4.4. When the Riemann surface $\mathcal{R}$ is concretely described, one can usually present the basis $\omega_1, \ldots, \omega_g$ of holomorphic differentials explicitly.

Theorem 4.10 *The differentials*

$$\omega_j = \frac{\lambda^{j-1} d\lambda}{\mu}, \qquad j = 1, \ldots, g \tag{59}$$

form a basis of holomorphic differentials of the hyperelliptic Riemann surface

$$\mu^2 = \prod_{i=1}^{N} (\lambda - \lambda_i) \qquad \lambda_i \neq \lambda_j, \tag{60}$$

where $N = 2g + 2$ or $N = 2g + 1$.

Proof The differentials (57) are obviously linearly independent. Their holomorphicity at all the points (μ, λ) with $\lambda \neq \lambda_k,\ \lambda \neq \infty$ is evident. Local parameters at the branch points $\lambda = \lambda_k$ are $z_k = \sqrt{\lambda - \lambda_k}$. In terms of z_k the differentials ω_j are holomorphic

$$\omega_j \approx \frac{\lambda_k^{j-1} d\lambda}{\sqrt{\prod_{i=1, i\neq k}^{N} (\lambda_k - \lambda_i)} \sqrt{\lambda - \lambda_k}} = \frac{2\lambda_k^{j-1}}{\sqrt{\prod_{i=1, i\neq k}^{N} (\lambda_k - \lambda_i)}} dz_k,\ \lambda \to \lambda_k.$$

If $N = 2g+2$ there are two infinity points $\infty^{\pm}$, and $z_\infty = 1/\lambda$ is a local parameter at these points. The differentials ω_j are holomorphic at these points

$$\omega_j \approx \pm \frac{\lambda^{j-1}}{\lambda^{g+1}} d\lambda = \pm z_\infty^{g-j} dz_\infty, \qquad \lambda \to \infty^{\pm}.$$

If $N = 2g+1$ there is one ∞ point and $z_\infty = 1/\sqrt{\lambda}$. At the point ∞ the differentials are holomorphic

$$\omega_i \approx \frac{\lambda^{j-1}}{\lambda^{g+1/2}} d\lambda = z_\infty^{2(g-j)} dz_\infty, \qquad \lambda \to \infty.$$

$\square$

One more example is the holomorphic differential

$$\omega = dz$$

on the torus $\mathbb{C}/G$ of Section 2. Here z is the coordinate of $\mathbb{C}$.

Corollary 4.6 implies that the matrix of a-periods

$$A_{ij} = \int_{a_i} \omega_j$$

of any basis $\omega_j,\ j = 1, \ldots, g$ of $H^1(\mathcal{R}, \mathbb{C})$ is invertible. Therefore the basis can be normalized as in the following

Definition 4.3 *Let $a_j, b_j\ j = 1, \ldots, g$ be a canonical basis of $H_1(\mathcal{R}, \mathbb{Z})$. The dual basis of holomorphic differentials $\omega_k,\ k = 1, \ldots, g$ normalized by*

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}$$

is called canonical.

We consider also differentials with singularities.

Definition 4.4 *A differential Ω is called meromorphic or Abelian differential if in any local chart $z : U \to \mathbb{C}$ it is of the form*

$$\Omega = g(z) dz,$$

where $g(z)$ is meromorphic. The integral

$$\int_{P_0}^{P} \Omega$$

of a meromorphic differential is called the Abelian integral.

Let z be a local parameter at the point P, $z(P) = 0$ and

$$\Omega = \sum_{k=N(P)}^{\infty} g_k z^k dz, \qquad N \in \mathbb{Z} \tag{61}$$

be the representation of the differential Ω at P. The numbers $N(P)$ and g_{-1} do not depend on the choice of the local parameter and are characteristics of Ω only. $N(P)$ is called the *order of the point* P. If $N(P)$ is negative $-N(P)$ is called the *order of the pole* of Ω at P. g_{-1} is called the *residue* of Ω at P. It also can be defined by

$$\mathrm{res}_P \Omega \equiv g_{-1} = \frac{1}{2\pi i} \int_{\gamma} \Omega, \tag{62}$$

where γ is a small closed simple loop going around P in the positive direction.
Let S be the set of singularities of Ω

$$S = \{P \in \mathcal{R} \mid N(P) < 0\}.$$

S is discrete and if $\mathcal{R}$ is compact then S is also finite.

Lemma 4.11 *Let Ω be an Abelian differential on a compact Riemann surface $\mathcal{R}$. Then*

$$\sum_{P_j \in S} \mathrm{res}_{P_j} \Omega = 0,$$

where S is the singular set of Ω.

Proof Use the simply connected model F_g of $\mathcal{R}$ and the equivalent definition of $\mathrm{res}_{P_j}\Omega$ via the integral

$$\sum_{P_j \in S} \mathrm{res}_{P_j} \Omega = \frac{1}{2\pi i} \sum_j \int_{\gamma_j} \Omega = \frac{1}{2\pi i} \int_{\partial F} \Omega = 0.$$

Here we used that Ω is holomorphic on $\mathcal{R} \setminus S$ and (56). □

Definition 4.5 *A meromorphic differential with singularities is called an Abelian differential of the second kind if the residues are equal to zero at all singular points. A meromorphic differential with non-zero residues is called an Abelian differential of the third kind.*

Lemma 4.11 motivates the following choice of basic meromorphic differentials. The differential of the second kind $\Omega_R^{(N)}$ has only one singularity. It is at the point $R \in \mathcal{R}$ and is of the form

$$\Omega_R^{(N)} = \left(\frac{1}{z^{N+1}} + O(1)\right) dz, \tag{63}$$

where z is the local parameter at R with $z(R) = 0$. The Abelian differential of the third kind Ω_{RQ} has two singularities at the points R and Q with

$$\mathrm{res}_R \Omega_{RQ} = -\mathrm{res}_Q \Omega_{RQ} = 1,$$

$$\begin{aligned} \Omega_{RQ} &= \left(\frac{1}{z_R} + O(1)\right) dz_R \qquad \text{near } R, \\ \Omega_{RQ} &= \left(-\frac{1}{z_Q} + O(1)\right) dz_Q \qquad \text{near } Q, \end{aligned} \tag{64}$$

where z_R and z_Q are local parameters at R and Q with $z_R(R) = z_Q(Q) = 0$. For the corresponding Abelian integrals this implies

$$\int^P \Omega_R^{(N)} = -\frac{1}{N z^N} + O(1) \qquad P \to R, \tag{65}$$

$$\begin{aligned} \int^P \Omega_{RQ} &= \log z_R + O(1) \qquad P \to R, \\ \int^P \Omega_{RQ} &= -\log z_Q + O(1) \qquad P \to Q. \end{aligned} \tag{66}$$

Remark The Abelian integrals of the first and second kind are single-valued on F_g. The Abelian integral of the third kind Ω_{RQ} is single-valued on $F_g \setminus [R, Q]$, where $[R, Q]$ is a cut from R to Q lying inside F_g.

Remark The Abelian differential of the second kind $\Omega_R^{(N)}$ depends on the choice of the local parameter z.

One can add Abelian differentials of the first kind to $\Omega_R^{(N)}$, Ω_{RQ} preserving the form of the singularities. By addition of a proper linear combination $\sum_{i=1}^g \alpha_i \omega_i$ the differential can be normalized as follows:

$$\int_{a_j} \Omega_R^{(N)} = 0, \qquad \int_{a_j} \Omega_{RQ} = 0 \tag{67}$$

for all a-cycles $j = 1, \dots, g$.

Definition 4.6 *The differentials $\Omega_R^{(N)}$, Ω_{RQ} with the singularities (63), (64) and all zero a-periods (67) are called the normalized Abelian differentials of the second and third kind.*

Theorem 4.12 *Given a compact Riemann surface $\mathcal{R}$ with a canonical basis of cycles $a_1, b_1, \ldots, a_g, b_g$, points $R, Q \in \mathcal{R}$, a local parameter z at R and $N \in \mathbb{N}$ there exist unique normalized Abelian differentials of the second $\Omega_R^{(N)}$ and of the third Ω_{RQ} kind.*

The existence will be proven in Section 4.4. The proof of the uniqueness is simple. The holomorphic difference of two normalized differentials with the same singularities has all zero a-periods and vanishes identically due to Corollary 4.6.

Remark Due to Corollary 4.7 Abelian differentials of the second and third kind can be normalized by a more symmetric then (67) condition. Namely all the periods can be normalized to be pure imaginary

$$\mathrm{Re} \int_\gamma \Omega = 0, \qquad \forall \gamma \in H_1(\mathcal{R}, \mathbb{Z}).$$

Corollary 4.13 *The normalized Abelian differentials form a basis in the space of Abelian differentials on $\mathcal{R}$.*

Again, as in the case of holomorphic differentials, we present the basis of Abelian differentials of the second and third kind in the hyperelliptic case

$$\mu^2 = \prod_{k=1}^{M} (\lambda - \lambda_k).$$

Denote the coordinates of the points R and Q by

$$R = (\mu_R, \lambda_R), \qquad Q = (\mu_Q, \lambda_Q).$$

We consider the case when both points R and Q are finite $\lambda_R \neq \infty$, $\lambda_Q \neq \infty$. The case $\lambda_R = \infty$ or $\lambda_Q = \infty$ is reduced to the case we consider by a fractional linear transformation. If R is not a branch point, then to get a proper singularity we multiply $d\lambda/\mu$ by $1/(\lambda - \lambda_R)^n$ and cancel the singularity at the point $\pi R = (-\mu_R, \lambda_R)$ by multiplication by a linear function of μ.

The following differentials are of the third kind with the singularities (64)

$$\begin{aligned}
\hat{\Omega}_{RQ} &= \left(\frac{\mu + \mu_R}{\lambda - \lambda_R} - \frac{\mu + \mu_Q}{\lambda - \lambda_Q} \right) \frac{d\lambda}{2\mu} \quad \text{if} \quad \mu_R \neq 0, \quad \mu_Q \neq 0, \\
\hat{\Omega}_{RQ} &= \left(\frac{\mu + \mu_R}{\mu(\lambda - \lambda_R)} - \frac{1}{\lambda - \lambda_Q} \right) \frac{d\lambda}{2} \quad \text{if} \quad \mu_R \neq 0, \quad \mu_Q = 0, \\
\hat{\Omega}_{RQ} &= \left(\frac{1}{\lambda - \lambda_R} - \frac{1}{\lambda - \lambda_Q} \right) \frac{d\lambda}{2} \quad \text{if} \quad \mu_R = \mu_Q = 0.
\end{aligned}$$

The differentials

$$\hat{\Omega}_R^{(N)} = \frac{\mu + \mu_R^{[N]}}{(\lambda - \lambda_R)^{N+1}} \frac{d\lambda}{2\mu} \quad \text{if} \quad \mu_R \neq 0,$$

where $\mu_R^{[N]}$ is the Taylor series at R up to the term of order N

$$\mu_R^{[N]} = \mu_R + \left.\frac{\partial\mu}{\partial\lambda}\right|_R (\lambda - \lambda_R) + \ldots + \frac{1}{N!}\left.\frac{\partial^N\mu}{\partial\lambda^N}\right|_R (\lambda - \lambda_R)^N$$

have the singularities at R of the form

$$\big(z^{-N-1} + o(z^{-N-1})\big)dz \tag{68}$$

with $z = \lambda - \lambda_R$. If R is a branch point $\mu_R = 0$ the following differentials have the singularities (68) with $z = \sqrt{\lambda - \lambda_R}$

$$\hat{\Omega}_R^{(N)} = \frac{d\lambda}{2(\lambda - \lambda_R)^n \mu}\sqrt{\prod_{\substack{i=1\\ i\neq R}}^{N}(\lambda_R - \lambda_i)} \quad \text{for} \quad N = 2n-1,$$

$$\hat{\Omega}_R^{(N)} = \frac{d\lambda}{2(\lambda - \lambda_R)^n} \quad \text{for} \quad N = 2n-2.$$

Taking proper linear combinations of these differentials with different N's we obtain the singularity (63). The normalization (67) is obtained by addition of holomorphic differentials (57)

4.3 Periods of Abelian differentials. Jacobi variety

Definition 4.7 *Let $a_j, b_j,\ j = 1,\ldots,g$ be a canonical homology basis of $\mathcal{R}$ and $\omega_k,\ k = 1,\ldots,g$ the dual basis of $H^1(\mathcal{R},\mathbb{C})$. The matrix*

$$B_{ij} = \int_{b_i} \omega_j \tag{69}$$

is called the period matrix of $\mathcal{R}$.

Theorem 4.14 *The period matrix is symmetric and its real part is negative definite*

$$B_{ij} = B_{ji}, \tag{70}$$

$$\mathrm{Re}(B\alpha,\alpha) < 0, \qquad \forall\alpha \in \mathbb{R}^g \setminus \{0\}. \tag{71}$$

Proof For the proof of (70) substitute two normalized holomorphic differentials $\omega = \omega_i$ and $\omega' = \omega_j$ into the Riemann bilinear identity (55). The vanishing of the left hand side $\omega_i \wedge \omega_j \equiv 0$ implies (70). Lemma 4.5 with $\omega = \sum \alpha_k\omega_k$ yields

$$0 > \mathrm{Im}\sum_{j=1}^{g} A_j\bar{B}_j = \mathrm{Im}\left(\sum_{j=1}^{g} 2\pi i\alpha_j \sum_{k=1}^{g}\bar{B}_{jk}\alpha_k\right) = 2\pi\mathrm{Re}(B\alpha,\alpha).$$

□

The period matrix depends on the homology basis. Let us use the column notations

$$\begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix} = \begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}, \qquad \begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix} \in Sp(g,\mathbb{Z}). \tag{72}$$

Lemma 4.15 *The period matrices B and $\tilde{B}$ of the Riemann surface $\mathcal{R}$ corresponding to the homology basis (a,b) and $(\tilde{a},\tilde{b})$ respectively are related by*

$$\tilde{B} = 2\pi i(\mathsf{D}B + 2\pi i\mathsf{C})(\mathsf{B}B + 2\pi i\mathsf{A})^{-1},$$

where $\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}$ are the coefficients of the symplectic matrix (72).

Proof Let $\omega = (\omega_1, \ldots, \omega_g)$ be the canonical basis of holomorphic differentials dual to (a,b). Labeling columns of the matrices by differentials and rows by cycles we get

$$\int\limits_{\tilde{a}} \omega = 2\pi i\mathsf{A} + \mathsf{B}B, \qquad \int\limits_{\tilde{b}} \omega = 2\pi i\mathsf{C} + \mathsf{D}B.$$

The canonical basis of $H^1(\mathcal{R},\mathbb{C})$ dual to the basis $(\tilde{a},\tilde{b})$ is given by the right multiplication

$$\tilde{\omega} = 2\pi i\omega(2\pi i\mathsf{A} + \mathsf{B}B)^{-1}.$$

For the period matrics this implies

$$\tilde{B} = \int\limits_{\tilde{b}} \tilde{\omega} = (2\pi i\mathsf{C} + \mathsf{D}B)2\pi i(2\pi i\mathsf{A} + \mathsf{B}B)^{-1}$$

$\square$

Using the Riemann bilinear identity the periods of the normalized Abelian differentials of the second and third kind can be expressed in terms of the normalized holomorphic differentials.

Lemma 4.16 *Let $\omega_j, \Omega_R^{(N)}, \Omega_{RQ}$ be the normalized Abelian differentials from Definition 4.6. Let also z be a local parameter at R with $z(R) = 0$ and*

$$\omega_j = \sum_{k=0}^{\infty} \alpha_{k,j} z^k dz \qquad P \sim R \tag{73}$$

the representation of the normalized holomorphic differentials at R. The periods of $\Omega_R^{(N)}$, Ω_{RQ} are equal to:

$$\int\limits_{b_j} \Omega_R^{(N)} = \frac{1}{N}\alpha_{N-1,j} \tag{74}$$

$$\int\limits_{b_j} \Omega_{RQ} = \int\limits_{Q}^{R} \omega_j, \tag{75}$$

where the integration path $[R,Q]$ in (75) does not cross the cycles a,b.

Proof Substitute $\omega = \Omega_R^{(N)}, \omega' = \omega_j$ into (55). The integral

$$\int\limits_{\partial F_g} \omega_j(P) \int^P \Omega_R^{(N)}$$

can be calculated by residues. The integrand is a meromorphic function on F_g with only one singularity, which is at the point R. Multiplying (65) and (73) we have

$$\mathrm{res}_R\ \omega_j(P) \int^P \Omega_R^{(N)} = -\frac{1}{N}\alpha_{N-1,j}.$$

On the right hand side of (55) only the term with $A'_j = 2\pi i$ does not vanish, which yields (74). The same calculation with $\omega = \omega_j,\ \omega' = \Omega_{RQ}$ proves (75)

$$\int\limits_{\partial F_g} \Omega_{RQ}(P) \int\limits_{P_0}^{P} \omega_j = 2\pi i \left(\int\limits_{P_0}^{R} \omega_j - \int\limits_{P_0}^{Q} \omega_j \right) = 2\pi i \int\limits_{Q}^{R} \omega_j = 2\pi i \int\limits_{b_j} \Omega_{RQ}.$$

At the end of this section we introduce two notions, which play a central role in the studies of functions on compact Riemann surfaces. □

Let Λ be the lattice

$$\Lambda = \{2\pi i N + BM, \qquad N, M \in \mathbb{Z}^g\}$$

generated by the periods of $\mathcal{R}$. It defines an equivalence relation in $\mathbb{C}^g$: two points of $\mathbb{C}^g$ are equivalent if they differ by an element of Λ.

Definition 4.8 *The complex torus*

$$\mathrm{Jac}(\mathcal{R}) = \mathbb{C}^g/\Lambda$$

is called the Jacobi variety (or Jacobian) of $\mathcal{R}$.

Definition 4.9 *The map*

$$\mathcal{A} : \mathcal{R} \to \mathrm{Jac}(\mathcal{R}), \qquad \mathcal{A}(P) = \int\limits_{P_0}^{P} \omega, \tag{76}$$

where $\omega = (\omega_1, \ldots, \omega_g)$ is the canonical basis of holomorphic differentials and $P_0 \in \mathcal{R}$, is called the Abel map.

4.4 Harmonic differentials and proof of existence theorems

As we mentioned in Section 1 angles between tangent vectors are well defined on Riemann surfaces. In particular one can introduce rotation of tangent spaces on angle $\pi/2$. The induced transformation of the differentials[13] is called the *conjugation operator*

$$\omega = f\,dz + g\,d\bar{z} \mapsto *\omega = -if\,dz + ig\,d\bar{z}.$$

[13]For $X + iY = Z \in T_P\mathcal{R}$ we defined $*\omega(Z) = \omega(-iZ)$ or equivalently $*\omega(X,Y) = \omega(Y, -X)$.

Clearly $** = -1$. In terms of the conjugation operator the differentials of type $(1,0)$ (resp. of type $(0,1)$) can be characterized by the property $*\omega = -i\omega$ (resp. $*\omega = i\omega$).

Let $\mathcal{R}$ be a Riemann surface (not necessarily compact !). Consider the Hilbert space $L_2(\mathcal{R})$ of square integrable differentials with the scalar product

$$(\omega_1, \omega_2) = \int_{\mathcal{R}} \omega_1 \wedge *\bar{\omega}_2. \tag{77}$$

In local coordinate $z : U \subset \mathcal{R} \to V \subset \mathbb{C}$ one has

$$\int_U \omega_1 \wedge *\bar{\omega}_2 = 2 \int_V (f_1 \bar{f}_2 + g_1 \bar{g}_2) dx \wedge dy.$$

One can easily see that formula (77) defines a Hermitian scalar product, i.e.

$$(\omega_2, \omega_1) = \overline{(\omega_1, \omega_2)},$$
$$(\omega, \omega) \geq 0 \text{ and } (\omega, \omega) = 0 \Leftrightarrow \omega = 0.$$

Introduce the subspaces E and E^* of *exact* and *co-exact* differentials

$$\begin{aligned} E &= \overline{\{df \mid f \in C_0^\infty(\mathcal{R})\}}, \\ E^* &= \overline{\{*df \mid f \in C_0^\infty(\mathcal{R})\}}, \end{aligned}$$

where $C_0^\infty(\mathcal{R})$ is the space of smooth functions on $\mathcal{R}$ with compact support and the bar denotes the closure in $L_2(\mathcal{R})$. Consider the orthogonal complements $E^\perp$ and $E^{*\perp}$ and their intersection

$$H := E^\perp \cap E^{*\perp}.$$

Let us note that E and E^* are orthogonal. It is enough to check this statement for exact and co-exact C^∞-differentials

$$(df, *dg) = \int_{\mathcal{R}} df \wedge d\bar{g} = \int_{\mathcal{R}} \bar{g}\, d(df) = 0.$$

Here we used the Stokes theorem for functions with compact support and $d^2 = 0$. We obtain the orthogonal decomposition

$$L_2(\mathcal{R}) = E \oplus E^* \oplus H$$

shown in Fig. 22.

To get an idea of interpretation of these subspaces one should consider smooth differentials. A C^1-differential α is said to be *closed* (resp. *co-closed*) iff $d\alpha = 0$ (resp. $d{*}\alpha = 0$).

Lemma 4.17 *Let $\alpha \in L_2(\mathcal{R})$ be of class C^1. Then $\alpha \in E^\perp$ (resp. $\alpha \in E^{*\perp}$) iff α is co-closed (resp. closed).*

Proof follows directly from the Stokes theorem: $\alpha \in E^{*\perp}$ is equivalent

$$0 = (\alpha, *df) = \int_{\mathcal{R}} \alpha \wedge d\bar{f} = \int_{\mathcal{R}} \bar{f} d\alpha$$

for arbitrary $f \in C_0^\infty(\mathcal{R})$. This implies $d\alpha = 0$. $\square$

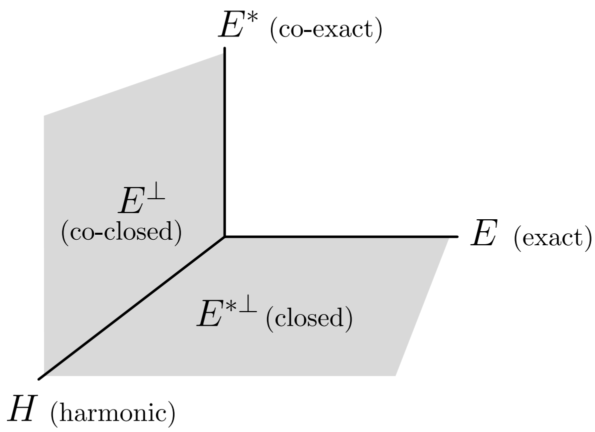

Figure 22: Orthogonal decomposition of $L_2(\mathcal{R})$.

Corollary 4.18 *Let $\alpha \in H$ be of class C^1. Then locally $\alpha = f\,dz + g\,d\bar{z}$, where f is holomorphic and g is antiholomorphic functions.*

Definition 4.10 *A differential h is called* harmonic *if it is locally $(z : U \subset \mathcal{R} \to V \subset \mathbb{C})$ of the form*

$$h = dH$$

with $H \in C^\infty(V)$ a harmonic function, i.e. $\frac{\partial^2}{\partial z \partial \bar{z}} H = 0$.

Harmonic and holomorphic differentials are closely related.

Lemma 4.19 *A differential h is harmonic iff it is of the form*

$$h = \omega_1 + \bar{\omega}_2, \qquad \omega_1, \omega_2 \text{ -holomorphic.} \tag{78}$$

A differential ω is holomorphic iff it is of the form

$$\omega = h + i * h, \qquad h \text{ -harmonic.} \tag{79}$$

Proof Let h be harmonic and locally $h = dH$. Since $H_{z\bar{z}} = 0$ the differential $H_z dz$ is holomorphic and the differential $H_{\bar{z}} d\bar{z}$ is antiholomorphic. Conversely, $h = f\,dz + g\,d\bar{z}$ with holomorphic f and antiholomorphic g can be rewritten as $h = d(F + G)$ with holomorphic F and antiholomorphic G defined by $F_z = f, G_{\bar{z}} = g$. The function $F + G$ is obviously harmonic. To prove the second part of the lemma note that for h given by (78) the sum

$$h + i * h = 2\omega_1$$

is always holomorphic. Conversely, given holomorphic ω,

$$h = \frac{\omega - \bar{\omega}}{2}$$

is a harmonic differential satisfying (79). □

To prove the next theorem we need an L_2-characterization of holomorphic functions.

Lemma 4.20 *(Weil's lemma). Let f be a square integrable function on the unit disc D. Then f is holomorphic iff*

$$\int_D f\eta_{\bar z}\, dz \wedge d\bar z = 0$$

for every $\eta \in C_0^\infty(D)$ (with compact support).

Proof See [FarkasKra, Jost]. □

Theorem 4.21 *The space H is the space of harmonic differentals.*

Proof A harmonic differential h is closed, co-closed and of class C^1. Lemma 4.17 implies $h \in H$.

Conversely, suppose $\alpha \in H$. For any $\eta \in C_0^\infty(\mathcal{R})$ we have

$$(\alpha, d\eta) = (\alpha, *d\eta) = 0. \tag{80}$$

Take local coordinate $z : U \to V$. For $\alpha = f\, dz + g\, d\bar z$ formulas (80) imply

$$\int_V f\eta_{\bar z} dz \wedge d\bar z = \int_V g\eta_z dz \wedge d\bar z = 0.$$

for every $\eta \in C_0^\infty(V)$. Holomorphicity of f and $\bar g$ follows from Weil's lemma. Lemma 4.19 completes the proof. □

Corollary 4.22 *Every square integrable differential α on $\mathcal{R}$ can be uniquely represented as an orthogonal sum of its exact df, co-exact $*dg$ and harmonic h parts:*

$$\alpha = df + *dg + h. \tag{81}$$

Now let us show how to construct $2g$ linearly independent harmonic differentials on a compact Riemann surface $\mathcal{R}$. Take a simple (without self-intersections) loop γ on $\mathcal{R}$. Consider a small strip Γ containing γ. It is an annulus and γ splits it into two annuli Γ^+ and Γ^-. Take a smaller strip Γ_0 (with corresponding one-sided strips $\Gamma_0^\pm$) around γ in Γ (see Fig. 23). Construct a real-valued function F on $\mathcal{R}$ satisfying

$$F_{|\Gamma_0^-} = 1, \quad F_{|\mathcal{R}\setminus\Gamma^-} = 0, \quad F \in C^\infty(\mathcal{R} \setminus \gamma).$$

Define a smooth differential

$$\alpha_\gamma = \begin{cases} dF & \text{on } \Gamma \setminus \gamma \\ 0 & \text{on } (\mathcal{R} \setminus \Gamma) \cup \gamma. \end{cases}$$

Consider now a simply connected model F_g of $\mathcal{R}$ and take one of the basic cycles $a_1, b_1, \ldots, a_g, b_g$, say a_1 as γ. The differential α_γ we constructed has a non-vanishing period along the cycle b_1. Chosing properly the orientation we obtain

$$\int_{b_1} \alpha_\gamma = 1$$

whereas all other periods of α_γ vanish. The differential α_γ is closed and non-exact. It can be decomposed into its exact df_γ and harmonic h_γ components

$$\alpha_\gamma = df_\gamma + h_\gamma.$$

Note that both parts are automatically smooth. The harmonic differential h_γ has the same periods as the original differential α_γ.

Chosing different cylces from $a_1, b_1, \ldots, a_g, b_g$ as γ one constructs $2g$ linearly independent harmonic differentials. For the dimension we obtain

$$\dim\, H \geq 2g. \tag{82}$$

Consider again holomorphic and antiholomorphic differentials and denote their spaces by $\mathcal{H} = H^1(\mathcal{R}, \mathbb{C})$ and $\bar{\mathcal{H}}$ respectively. These spaces are obviously orthogonal $\mathcal{H} \perp \bar{\mathcal{H}}$.

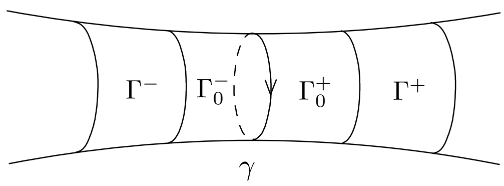

Figure 23: Consruction of a closed non-exact form.

Proposition 4.23 *Let $\mathcal{R}$ be a compact Riemann surface of genus g. Then*

$$\dim\, H^1(\mathcal{R}, \mathbb{C}) \geq g.$$

Proof The spaces $\mathcal{H}$ and $\bar{\mathcal{H}}$ are orthogonal and have the same dimension. On the other hand due to Lemma 4.19

$$H \subset \mathcal{H} \oplus \bar{\mathcal{H}},$$

which implies $\dim\, H \leq 2\, \dim\, \mathcal{H}$. The inequality (82) completes the proof. □

Theorem 4.9 follows from Proposition 4.23 and Corollary 4.8.

As a corollary of Theorem 4.9 we obtain $\dim\, H \leq 2g$, and finally $\dim\, H = 2g$. This observation combined with the construction of harmonic differentials h_γ above implies the following

Proposition 4.24 *Given a compact Riemann surface with a canonical basis of cycles $a_1, b_1, \ldots, a_g, b_g$ there exist unique $2g$ harmonic differentials $h_1, \ldots, h_{2g}$ with the periods*

$$\int_{a_j} h_i = \int_{b_j} h_{g+i} = \delta_{ij}, \quad \int_{a_j} h_{g+i} = \int_{b_j} h_i = 0, \quad i = 1, \ldots, g.$$

Let us now construct Abelian differentials of the second kind $\Omega_R^{(N)}$. Consider nested neighborhoods $R \in U_0 \subset U_1 \subset \mathcal{R}$ of the point R and a smooth function $\rho \in C^\infty(\mathcal{R})$ satisfying

$$\rho = \begin{cases} 1 & \text{on } U_0 \\ 0 & \text{on } \mathcal{R} \setminus U_1. \end{cases}$$

Let z be be a local parameter in U_1 with $z(R) = 0$. Take a differential

$$\psi := d\left(-\frac{\rho}{Nz^N}\right) = \left(-\frac{\rho_z}{Nz^N} + \frac{\rho}{z^{N+1}}\right) dz - \left(\frac{\rho_{\bar{z}}}{Nz^N}\right) d\bar{z}$$

with the same kind of singularity as the one of $\Omega_R^{(N)}$. The $(0,1)$-part of ψ is smooth on $\mathcal{R}$ and can be decomposed into its closed, co-closed and harmonic components[14]

$$\psi - i * \psi = df + *dg + h \in E(\mathcal{R}) \oplus E^*(\mathcal{R}) \oplus H(\mathcal{R}).$$

Consider

$$\alpha := \psi - df.$$

Lemma 4.25 *The differential α is harmonic on $\mathcal{R} \setminus R$ and the differential $\alpha - \frac{dz}{z^{N+1}}$ is harmonic on U_0.*

Proof Chose a closed $\bar{U} \subset U_0$. For α we have

$$\alpha = d\left(-\frac{\rho}{Nz^{N+1}}\right) - df,$$

which implies $\alpha \perp E^*(\mathcal{R} \setminus \bar{U})$. On the other hand

$$\alpha = i * \psi + *dg + h,$$

which implies $\alpha \perp E(\mathcal{R} \setminus \bar{U})$. Combining these two observations we obtain $\alpha \in H(\mathcal{R} \setminus \bar{U})$ for arbitrary $U \ni R$. Concerning the representation of α in U_0 let us observe that $\psi - \frac{dz}{z^{N+1}} \mid_{U_0} \equiv 0$. On U_0 this implies:

$$\alpha - \frac{dz}{z^{N+1}} = -df = *dg + h.$$

As above $\alpha - \frac{dz}{z^{N+1}}$ must be orthogonal to both $E(U_0)$ and $E^*(U_0)$ and therefore belongs to $H(U_0)$. $\square$

As a direct corollary of Lemmas 4.19, 4.25 we obtain the following

Proposition 4.26 *The differential*

$$\Omega := \frac{1}{2}(\alpha + i * \alpha)$$

is holomorphic on $\mathcal{R} \setminus R$ and the differential $\Omega - \frac{dz}{z^{N+1}}$ is holomorphic on U_0.

[14] We have incorporated $\mathcal{R}$ into the notations of the spaces $E(\mathcal{R}), E^*(\mathcal{R})$ and $H(\mathcal{R})$ since we will consider spaces corresponding to various Riemann surfaces.

The existence of the normalized differential of the second kind $\Omega_R^{(N)}$ claimed in Theorem 4.12 follows from Proposition 4.26.

To prove existence of differentials of the third kind one should start with the differential

$$\psi_{P_1P_2} = d\left(\rho \log \frac{z - z_1}{z - z_2}\right),$$

where $z_1 = z(P_1)$ and $z_2 = z(P_2)$ are local coordinates of two points $P_1, P_2 \in U_0$. Applying the same technique as above one obtains an Abelian differential of the third kind $\Omega_{P_1P_2}$ with

$$\mathrm{res}_{P_1}\Omega_{P_1P_2} = -\mathrm{res}_{P_2}\Omega_{P_1P_2} = 1.$$

Finally, any Abelian differential of the third kind Ω_{RQ} on a compact Riemann surface can be obtained as a finite sum of these basic differentials $\Omega_{P_1P_2}$.

5 Meromorphic functions on compact Riemann surfaces

5.1 Divisors and the Abel theorem

Analyzing functions and differentials on Riemann surfaces one characterizes them in terms of their zeros and poles. It is convenient to consider formal sums of points on $\mathcal{R}$. (Later these points will become zeros and poles of functions and differentials).

Definition 5.1 *The formal linear combination*

$$D = \sum_{j=1}^{N} n_j P_j, \qquad n_j \in \mathbb{Z},\ P_j \in \mathcal{R} \tag{83}$$

is called a divisor on the Riemann surface $\mathcal{R}$. The sum

$$\deg\ D = \sum_{j=1}^{N} n_j$$

is called the degree of D.

The set of all divisors with the obviously defined group operations

$$n_1 P + n_2 P = (n_1 + n_2) P, \qquad -D = \sum_{j=1}^{N} (-n_j) P_j$$

forms an Abelian group $\mathrm{Div}(\mathcal{R})$. A divisor (83) with all $n_j \geq 0$ is called *positive* (or integral, or effective). This notion allows us to define a partial ordering in $\mathrm{Div}(\mathcal{R})$

$$D \leq D' \iff D' - D \geq 0.$$

Definition 5.2 *Let f be a meromorphic function on $\mathcal{R}$ and $P_1, \ldots, P_M$ be its zeros with the multiplicities $p_1, \ldots, p_M > 0$ and $Q_1, \ldots, Q_N$ be its poles with the multiplicities $q_1, \ldots, q_N > 0$. The divisor*

$$D = p_1 P_1 + \ldots + p_M P_M - q_1 Q_1 - \ldots - q_N Q_N = (f)$$

is called the divisor of f and is denoted by (f) . A divisor D is called principal if there exists a function with $(f) = D$.

Obviously we have

$$(fg) = (f) + (g), \qquad (\mathrm{const} \neq 0) = 0,$$

where f and g are two meromorphic functions on $\mathcal{R}$.

Definition 5.3 *Two divisors D and D' are called linearly equivalent if the divisor $D - D'$ is principal. The corresponding equivalence class is called the divisor class.*

We denote linearly equivalent divisors by $D \equiv D'$. Divisors of Abelian differentials are also well-defined. We have seen already, that the order of the point $N(P)$ defined by (61) is independent of the choice of a local parameter and is a characteristic of the Abelian differential. The set of points $P \in \mathcal{R}$ with $N(P) \neq 0$ is finite.

Definition 5.4 *The divisor of an Abelian differential* Ω *is*

$$(\Omega) = \sum_{P \in \mathcal{R}} N(P)P,$$

where $N(P)$ *is the order of the point* P *of* Ω.

Since the quotient of two Abelian differentials

$$\Omega_1/\Omega_2$$

is a meromorphic function any two divisors of Abelian differentials are linearly equivalent. The corresponding class is called *canonical*. We will denote it by $\mathcal{C}$.

Any principal divisor can be represented as the difference of two positive linearly equivalent divisors

$$(f) = D_0 - D_\infty, \qquad D_0 \equiv D_\infty,$$

where D_0 is the zero divisor and D_∞ is the pole divisor of f. Corollary 2.7 implies that

$$\deg(f) = 0,$$

i.e. all principal divisors have zero degree. Also all canonical divisors have equal degrees.

The Abel map is defined for divisors in a natural way

$$\mathcal{A}(D) = \sum_{j=1}^{N} n_j \int_{P_0}^{P_j} \omega. \tag{84}$$

If the divisor D is of degree zero, then $\mathcal{A}(D)$ is independent of P_0

$$D = P_1 + \ldots + P_N - Q_1 - \ldots - Q_N,$$
$$\mathcal{A}(D) = \sum_{i=1}^{N} \int_{Q_i}^{P_i} \omega. \tag{85}$$

Theorem 5.1 (Abel)
The divisor $D \in \mathrm{Div}\ (\mathcal{R})$ *is principal if and only if:*

1) $\deg D = 0$,

2) $\mathcal{A}(D) \equiv 0$.

Proof The necessity of the first condition is already proven. Let f be a meromorphic function with the divisor

$$(f) = P_1 + \ldots + P_N - Q_1 - \ldots - Q_N$$

(these points are not necessarily assumed to be different). Then

$$\Omega = \frac{df}{f} = d(\log f)$$

is an Abelian differential of the third kind. All periods of Ω are integer multiples of $2\pi i$:

$$\int_{a_k} \Omega = 2\pi i\, n_k, \quad \int_{b_k} \Omega = 2\pi i\, m_k; \quad n_k, m_k \in \mathbb{Z}.$$

Applying the Riemann bilinear identity (55) with $\omega = \omega_j, \omega' = \Omega$ (compare with the proof of formula (75)) one obtains

$$\begin{aligned} \sum_{k=1}^{N} \int_{P_k}^{Q_k} \omega_j &= \sum_{P} \operatorname{res} \Omega(\mathrm{P}) \int_{\mathrm{P}_0}^{\mathrm{P}} \omega_{\mathrm{j}} = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{F_g}} \Omega(\mathrm{P}) \int_{\mathrm{P}_0}^{\mathrm{P}} \omega_{\mathrm{j}} \\ &= 2\pi i\, m_j - \sum_{k=1}^{N} n_k \int_{b_k} \omega_j \equiv 0 \end{aligned}$$

and finally

$$\mathcal{A}(D) \equiv 0. \tag{86}$$

Conversely, if (86) is fulfilled, let us choose $[P_i, Q_i]$, which do not intersect the cycles, and consider the normalized Abelian differentials of the third kind $\Omega_{P_iQ_i}$. The differential

$$\hat{\Omega} = \sum_{i=1}^{N} \Omega_{P_iQ_i}$$

has all zero a-periods, and its b-periods belong to the Jacobian lattice (because of (75))

$$\int_b \hat{\Omega} = \sum_{i=1}^{N} \int_b \Omega_{P_iQ_i} = \sum_{i=1}^{N} \int_{Q_i}^{P_i} \omega = 2\pi i N + BM, \quad N, M \in \mathbb{Z}^g.$$

Then all the periods of the differential

$$\hat{\Omega} - \sum_{j=1}^{g} \omega_j M_j, \qquad M = (M_1, \ldots, M_g)$$

are multiples of $2\pi i$. Finally, the meromorphic function

$$f(P) = \exp\left(\int^{P} \left(\sum_{i=1}^{N} \Omega_{P_iQ_i} - \sum_{j=1}^{g} \omega_j M_j \right) \right)$$

has the divisor D. $\square$

Corollary 5.2 *All linearly equivalent divisors are mapped by the Abel map to the same point of the Jacobian.*

Proof

$$\mathcal{A}((f) + D) = \mathcal{A}((f)) + \mathcal{A}(D) = \mathcal{A}(D).$$

□

Remark The Abel theorem can be formulated in terms of any basis $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_g)$ of holomorphic differentials. In this case the second condition of the theorem reads

$$\sum_{i=1}^{N} \int_{Q_i}^{P_i} \tilde{\omega} \equiv 0 \quad (\text{mod periods of } \tilde{\omega}).$$

5.2 The Riemann-Roch theorem

Let D_∞ be a positive divisor on $\mathcal{R}$. A natural problem is to describe the vector space of meromorphic functions with poles at D_∞ only. More generally, let D be a divisor on $\mathcal{R}$. Let us consider the vector space

$$L(D) = \{f \text{ meromorphic functions on } \mathcal{R} \mid (f) \geq D \text{ or } f \equiv 0\}.$$

Let us split

$$D = D_0 - D_\infty$$

into negative and positive parts

$$D_0 = \sum n_i P_i, \qquad D_\infty = \sum m_k Q_k,$$

where both D_0 and D_∞ are positive. The space $L(D)$ of dimension

$$l(D) = \dim L(D)$$

is comprised by the meromorphic functions with zeros of order at least n_i at P_i and with poles of order at most m_k at Q_k.

Similarly, let us denote by

$$H(D) = \{\Omega \text{ Abelian differential on } \mathcal{R} \mid (\Omega) \geq D \text{ or } \Omega \equiv 0\}$$

the corresponding vector space of differentials, and by

$$i(D) = \dim H(D)$$

its dimension, which is called the *index of speciality* of D.

Remark The following properties are obvious:

1. $D_1 \geq D_2$ implies $L(D_1) \subset L(D_2)$ and $l(D_1) \leq l(D_2)$

2. The space $L(0)$ consists of constants, $l(0) = 1$
3. $\deg D \geq 0,\ D \neq 0$ implies $l(D) = 0$.
4. $i(0) = g$ since $H(0)$ is the space of holomorphic differentials.

Lemma 5.3 *$l(D)$ and $i(D)$ depend only on the divisor class of D, and*

$$i(D) = l(D - C), \tag{87}$$

where C is the canonical divisor class.

Proof The existence of h with $(h) = D_1 - D_2$ is equivalent to $D_1 \equiv D_2$. The map $L(D_2) \to L(D_1)$ defined by the multiplication

$$L(D_2) \ni f \longrightarrow hf \in L(D_1)$$

is an isomorphism, which proves $l(D_2) = l(D_1)$.

Let Ω_0 be a non-zero Abelian differential and $C = (\Omega_0)$ be its divisor. The map $H(D) \to L(D - C)$ defined by

$$H(D) \ni \Omega \longrightarrow \frac{\Omega}{\Omega_0} \in L(D - C)$$

is an isomorphism of linear spaces, which proves $i(D) = l(D - C)$. □

Theorem 5.4 (Riemann-Roch)
Let $\mathcal{R}$ be a compact Riemann surface of genus g and D a divisor on $\mathcal{R}$. Then

$$l(-D) = \deg\, D - g + 1 + i(D). \tag{88}$$

We prove the Riemann-Roch theorem in several steps.

Lemma 5.5 *The Riemann-Roch theorem holds for positive divisors D.*

Proof Due to the Remark, formula (88) holds for $D = 0$. Let D be positive and $D \neq 0$. We give a proof for the case when all points of the divisor have multiplicity one

$$D = P_1 + \ldots + P_k.$$

Treatment of the general case requires no essential additional work, but complicates notations. If $f \in L(-D)$ then its differential df lies in the space of differentials

$$df \in H(-D^{(+1)}),$$

where

$$D^{(+1)} = 2D = 2P_1 + \ldots + 2P_k.$$

Moreover, df lies in the subspace $H_0(-D^{(+1)}) \subset H(-D^{(+1)})$

$$H_0(-D^{(+1)}) = \{\Omega \text{ Abelian differentials on } \mathcal{R} \mid (\Omega) \geq -D^{(+1)}; \\ \operatorname{res}_{P_j}\Omega = 0\ \forall j\ ;\ \int_{a_i} \Omega = 0\ \forall i \text{ or } \Omega \equiv 0\}.$$

The normalized differentials of the second kind $\Omega^{(1)}_{P_j}$, $j = 1, \ldots, k$ form a basis for $H_0(-D^{(+1)})$,

$$\dim\, H_0(-D^{(+1)}) = k = \deg D.$$

Let us denote the linear operator $f \to df$ by

$$d : L(-D) \longrightarrow H_0(-D^{(+1)}).$$

Since only constant functions lie in the kernel of d

$$l(-D) = 1 + \dim \text{ Image } d. \tag{89}$$

The image of d can be described explicitly

$$df = \sum_{j=1}^{k} f_j \Omega^{(1)}_{P_j}, \tag{90}$$

where f_j are constants such that all the b-periods of df vanish

$$\int_{b_j} df = 0, \qquad i = 1, \ldots, g. \tag{91}$$

The conditions (91) is a system of g linear equations for $\deg D$ variables f_j. This observation immediately implies

$$\dim \text{ Image } d \geq \deg\, D - g.$$

Theorem 5.6 (Riemann's inequality)
For any positive divisor D

$$l(-D) \geq \deg\, D + 1 - g.$$

We interrupt the proof of Lemma 5.5 for two simple corollaries of Riemann's inequality.

Corollary 5.7 *For any positive divisor D with* $\deg\, D = g + 1$ *there exists a non-trivial meromorphic function in $L(-D)$.*

Corollary 5.8 *Any Riemann surface of genus* 0 *is conformally equivalent to the complex sphere $\hat{\mathbb{C}}$.*

Proof Let us consider a divisor which consists of one point $D = P$. Riemann's inequality implies $l(-P) \geq 2$. There exists a non-trivial function f with 1 pole on $\mathcal{R}$. It is a holomorphic covering $f : \mathcal{R} \to \hat{\mathbb{C}}$. Since f has only one pole, every value is assumed once (Corollary 2.7), therefore $\mathcal{R}$ and $\hat{\mathbb{C}}$ are conformally equivalent. □

Due to (74) the system (90), (91) can be rewritten as

$$\sum_{j=1}^{k} f_j \alpha_{0,i}(P_j) = 0, \qquad i = 1, \ldots, g.$$

In the matrix form this reads as

$$(f_1, \ldots, f_k)H = 0, \tag{92}$$

where H is the matrix

$$H = \begin{pmatrix} \alpha_{0,1}(P_1) & \ldots & \alpha_{0,g}(P_1) \\ \vdots & & \vdots \\ \alpha_{0,1}(P_k) & \ldots & \alpha_{0,g}(P_k) \end{pmatrix}$$

This is a linear map $H : \mathbb{C}^g \to \mathbb{C}^{\deg D}$, and due to (92)

$$\dim \text{ Image } d = \dim \ker \ H^T = \deg D - \text{rank} H. \tag{93}$$

Near the points P_j the normalized holomorphic differentials ω_i have the following asymptotics

$$\omega_i = (\alpha_{0,i}(P_j) + o(1))dz_j.$$

This shows that the linear spaces ker H and $H(D)$ are isomorphic

$$(\beta_1, \ldots, \beta_g) \in \ker \ H \Longleftrightarrow \sum_{i=1}^{g} \beta_i \omega_i \in H(D).$$

This observation implies

$$i(D) = \dim \ H(D) = \dim \ker \ H = g - \text{rank} H,$$

which combined with (89, 93) completes the proof of Lemma 5.5. □

Corollary 5.9 *The degree of the canonical class is*

$$\deg \ C = 2g - 2.$$

Proof The differential dz on the complex sphere has a double pole at $z = \infty$

$$dz = -\frac{1}{\tau^2} d\tau, \qquad \tau = \frac{1}{z}.$$

Since the degree is a characteristics of a divisor class, this proves the statement for $g = 0$. If $g > 0$ then there exists a non-trivial holomorphic differential ω. Its divisor $(\omega) = C$ is positive. Lemma 5.5 yields

$$l(-C) = \deg C - g + 1 + i(C).$$

Remarks 5.2 and Lemma 5.3 imply

$$l(-C) = i(0) = g, \qquad i(C) = l(0) = 1,$$

which completes the proof of the corollary. □

Corollary 5.10 *On a compact Riemann surface there is no point where all holomorphic differentials vanish simultaneously.*

Proof Suppose there exists a point $P \in \mathcal{R}$ where all holomorphic differentials vanish, i.e. $i(P) = g$. Applying the Riemann-Roch theorem for the divisor $D = P$ one obtains $l(-P) = 2$, i.e. there exists a non-constant meromorphic function f with the only pole. Due to Corollary 2.7 $f : \mathcal{R} \to \hat{\mathbb{C}}$ is bi-holomorphic, which implies $g = 0$. Due to Corollary 5.9 there are no holomorphic differentials on a Riemann surface of genus $g = 0$. □

Lemma 5.11 *The Riemann-Roch theorem holds for the divisors D, if D or $C - D$ are linearly equivalent to a positive divisor.*

Proof If D is linearly equivalent to a positive divisor the statement is trivial, since both $l(-D)$ and $i(D)$ depend on the divisor class only. Applying Lemma 5.5 to the positive divisor $C - D$ one gets

$$l(D - C) = \deg\,(C - D) - g + 1 + i(C - D)$$

or using Lemma 5.3, Corollary 5.9 and formula (88) for D

$$i(D) = 2g - 2 - \deg\, D - g + 1 + l(-D).$$

□

Lemma 5.12

$$\begin{aligned} l(-D) > 0 \quad &\Longleftrightarrow \quad D \equiv D_+ \geq 0, \\ i(D) > 0 \quad &\Longleftrightarrow \quad C - D \equiv D_+ \geq 0. \end{aligned}$$

Proof $l(-D) > 0$ implies the existence of $f \in L(-D)$. Since $(f) \geq -D$ we get that the divisor $(f) + D \geq 0$ is positive. Similarly $i(D) > 0$ is equivalent to $l(D - C) > 0$. This implies $(f) + C - D \geq 0$, where $f \in L(D - C)$. □

Finishing of the proof of Theorem 5.4. Due to Lemma 5.11 and Lemma 5.12 only one case remains to consider. We should prove that $i(D) = l(-D) = 0$ implies $\deg\, D = g - 1$. Represent D as a difference of two positive divisors

$$D = D_1 - D_2, \; D_2 \neq 0.$$

Then Riemann's inequality implies

$$l(-D_1) \geq \deg\, D_1 - g + 1 = \deg\, D + \deg\, D_2 - g + 1.$$

Let us suppose that deg $D \geq g$. Then

$$l(-D_1) \geq \deg D_2 + 1$$

and there exists a function in $L(-D_1)$ with the zero divisor $\geq D_2$. This yields $l(-D) > 0$, which contradicts our assumption. We have proven that deg $D \leq g - 1$.

In the same way using $i(D) = l(D - C) = 0$ one gets

$$\deg\ (C - D) \leq g - 1.$$

Combined with Corollary 5.9 this implies deg $D \geq g - 1$, and finally

$$\deg D = g - 1,$$

which completes the proof of the Riemann-Roch theorem.

5.3 Special divisors and Weierstrass points

Definition 5.5 *A positive divisor D of degree* deg $D = g$ *is called special if $i(D) > 0$, i. e. there exists a holomorphic differential ω with*

$$(\omega) \geq D. \tag{94}$$

The Riemann-Roch theorem implies that (94) is equivalent to the existence of a non-constant function f with $(f) \geq -D$. Since the space of holomorphic differentials is g-dimensional, (94) is a homogeneous linear system of g equations in g variables. This shows that most of the positive divisors of degree g are non-special.

Proposition 5.13 *Let the divisor*

$$D = P_1 + \ldots + P_g$$

be non-special. There exist neighborhoods $U_1, \ldots, U_g$ of the points of the divisor $P_j \in U_j,\ j = 1, \ldots, g$ such that any divisor

$$D' = P_1' + \ldots + P_{g'}$$

with $P_j' \in U_j,\ j = 1, \ldots, g$ is non-special. Arbitrary close to any special divisor D there exists a non-special positive divisor of degree g.

This proposition will be proved later (see Lemma 5.14) for divisors which are multiples of a point $D = gP$. The proof of the general case is analogous. Note that special divisors may be "non-rigid". In particular, if $l(-D_1) \geq 2$ for some $D_1 > 0, \deg D_1 < g$ then the divisor $D = D_1 + D_2$ is special with arbitrary $D_2 > 0, \deg D_2 = g - \deg D_1$.

Definition 5.6 *A point $P \in \mathcal{R}$ is called the Weierstrass point if the divisor $D = gP$ is special.*

The Weierstrass points are special points of $\mathcal{R}$. We prove that these points exist and estimate their number.

Remark There are no Weierstrass points on Riemann surfaces of genus $g = 0$ or $g = 1$.

Lemma 5.14 *Let $\omega_k = h_k(z)dz,\ k = 1, \dots, g$ be the local representation of a basis of holomorphic differentials in a neighborhood of P_0. The point P_0 is a Weierstrass point if and only if*

$$\Delta[h_1, \dots, h_g] \equiv \det \begin{pmatrix} h_1 & \dots & h_g \\ h_1' & \dots & h_g' \\ \vdots & & \vdots \\ h_1^{(g-1)} & \dots & h_g^{(g-1)} \end{pmatrix} \tag{95}$$

vanishes at P_0.

Proof Δ vanishes at P_0 iff the matrix in (95) has a non-trivial kernel vector $(\alpha_1, \dots, \alpha_g)^T$. In this case the differential $\sum_{k=1}^{g} \alpha_k h_k$ has a zero of order g at P_0, which implies $i(gP_0) > 0$. □

Since Δ is holomorphic in a neighborhood of P_0 the Weierstrass points are isolated. Moreover their number is finite due to compactness of $\mathcal{R}$.

Definition 5.7 *Let P_0 be a Weierstrass point on $\mathcal{R}$ and z a local parameter at P_0, with $z(P_0) = 0$. The order $\tau(P_0)$ of the zero of Δ at P_0*

$$\Delta = z^{\tau(P_0)} O(1) \tag{96}$$

is called the weight of the Weierstrass point P_0.

It turns out that Δ is well defined on $\mathcal{R}$ globally.

Definition 5.8 *If to every local coordinate $z : U \subset \mathcal{R} \to V \subset \mathbb{C}$ there assigned a holomorphic function $r(z)$ such that*

$$r = r(z)dz^q, \qquad q \in \mathbb{Z} \tag{97}$$

is invariant under holomorphic coordinate changes (49) one says that the holomorphic q-differential r is defined on $\mathcal{R}$.

In the same way as for the Abelian differentials one defines the divisor (r) of the q-differentials.

Lemma 5.15 deg $(r) = (2g - 2)q$

Proof Let ω be an Abelian differential. Then

$$f = \frac{r}{\omega^q}$$

is a meromorphic function on $\mathcal{R}$, which implies deg $(f) = 0$ and

$$\deg\,(r) = \deg\,(\omega^q) = q\deg\,(\omega) = q(2g-2).$$

□

Theorem 5.16 *$\Delta[h_1, \ldots, h_g]$ defined by (95) is a (non-trivial) holomorphic q-differential on $\mathcal{R}$ with*

$$q = \frac{g(g+1)}{2}.$$

Proof We have to check that $h_k(z)dz = \tilde{h}_k(\tilde{z})d\tilde{z}$ implies $\Delta dz^q = \tilde{\Delta} d\tilde{z}^q$. It is easy to verify that

$$\begin{aligned}
\tilde{\Delta} &= \det \begin{pmatrix} \tilde{h}_1 & \ldots & \tilde{h}_g \\ \frac{d}{d\tilde{z}}\tilde{h}_1 & \ldots & \frac{d}{d\tilde{z}}\tilde{h}_g \\ \vdots & & \vdots \\ \frac{d^{g-1}}{d\tilde{z}^{g-1}}\tilde{h}_1 & \ldots & \frac{d^{g-1}}{d\tilde{z}^{g-1}}\tilde{h}_g \end{pmatrix} \\
&= \left(\frac{dz}{d\tilde{z}}\right)^{g(g-1)/2} \det \begin{pmatrix} \tilde{h}_1 & \ldots & \tilde{h}_g \\ \frac{d}{dz}\tilde{h}_1 & \ldots & \frac{d}{dz}\tilde{h}_g \\ \vdots & & \vdots \\ \frac{d^{g-1}}{dz^{g-1}}\tilde{h}_1 & \ldots & \frac{d^{g-1}}{dz^{g-1}}\tilde{h}_g \end{pmatrix} \\
&= \left(\frac{dz}{d\tilde{z}}\right)^{g(g-1)/2} \Delta\left[h_1\frac{dz}{d\tilde{z}}, \ldots, h_g\frac{dz}{d\tilde{z}}\right]. \qquad (98)
\end{aligned}$$

On the other hand algebraic properties of determinant imply also

$$\Delta[fh_1, \ldots, fh_g] = f^g \Delta[h_1, \ldots, h_g], \qquad (99)$$

where f is an arbitrary holomorphic function. Combined with (98) for $f = \frac{dz}{d\tilde{z}}$ this yields

$$\tilde{\Delta} = \left(\frac{dz}{d\tilde{z}}\right)^{g(g+1)/2} \Delta.$$

Since the differentials ω_i are linearly independent $\Delta \not\equiv 0$. □

Lemma 5.15 and Theorem 5.16 imply

Corollary 5.17 *The number N of the Weierstrass points on a Riemann surface $\mathcal{R}$ of genus g is less or equal then*

$$N_W \leq g^3 - g.$$

Moreover

$$\sum \tau(P_k) = g^3 - g \qquad (100)$$

holds, where the sum is taken over all the Weierstrass points of $\mathcal{R}$.

5.4 Jacobi inversion problem

Now we are in a position to prove more complicated properties of the Abel map. Let us fix a point $P_0 \in \mathcal{R}$.

Proposition 5.18 *The Abel map*

$$\begin{aligned} \mathcal{A} : \mathcal{R} &\rightarrow Jac(\mathcal{R}) \\ P &\mapsto \int_{P_0}^{P} \omega \end{aligned} \tag{101}$$

is an embedding, i.e. the mapping (101) is an injective immersion (the differential vanishes nowhere on $\mathcal{R}$).

Proof From corollary 5.10 of the Riemann-Roch theorem we know, that not all holomorphic differentials can vanish simultaniously at a point P. Therefore $d\mathcal{A}(\mathcal{P}) = \omega(\mathcal{P}) \neq 0$, which shows that the Abel map is an immersion.
Now suppose there exist $P_1, P_2 \in \mathcal{R}$ with $\mathcal{A}(P_1) = \mathcal{A}(P_2)$. According to the Abel theorem the divisor $P_1 - P_2$ is principal. A function with one pole does not exist for Riemann surfaces of genus $g > 0$, thus the points must coincide $P_1 = P_2$. □

Although the next theorem looks technical it is an important result often used in the theory of Riemann surfaces and its applications.

Theorem 5.19 (Jacobi inversion)
Let $\mathcal{D}_g$ be the set of positive divisors of degree g. The Abel map on this set

$$\mathcal{A} : \mathcal{D}_g \rightarrow Jac(\mathcal{R})$$

is surjective, i.e. for any $\xi \in Jac(\mathcal{R})$ there exist a degree g positive divisor $P_1 + \ldots + P_g \in \mathcal{D}_g$ (P_i are not necessarily different) satisfying

$$\sum_{i=1}^{g} \int_{P_0}^{P_i} \omega = \xi. \tag{102}$$

Proof Start with a non-special divisor $D_R = R_1 + \ldots + R_g$. In a neighborhood $\mathcal{U}$ of D_R the differential of the Abel map does not vanish and all divisors are non-special (Proposition 5.13). Choosing sufficiently large $N \in \mathbb{N}$ one can achieve that $\mathcal{A}(D_R) + \xi/N$ lies in $\mathcal{A}(\mathcal{U})$ and therefore can be represented as

$$\mathcal{A}(D_Q) = \mathcal{A}(D_R) + \xi/N, \qquad D_Q = Q_1 + \ldots + Q_g \in \mathcal{U}.$$

The problem (102) is equivalent to

$$\mathcal{A}(P_1 + \ldots + P_g) = N(\mathcal{A}(D_Q) - \mathcal{A}(D_R)).$$

Applying the Riemann inequality to the divisor $N(D_Q - D_R) + gP_0$ we get

$$l(-N(D_Q - D_R) - gP_0) \geq 1,$$

i.e. there exists a function f with $(f) \geq N(-D_Q + D_R) - gP_0$. Applying the Abel theorem one obtains for the rest g zeros $P_1, \ldots, P_g$ of this function

$$\mathcal{A}(P_1 + \ldots + P_g) = N\mathcal{A}(D_Q - D_R) = \xi,$$

which coincides with (102). □

6 Hyperelliptic Riemann surfaces

6.1 Classification of hyperelliptic Riemann surfaces

Let us investigate in more detail hyperelliptic Riemann surfaces, which are the simplest Riemann surfaces existing for arbitrary genus. We give a new definition of these surfaces. The equivalence of this definition with the one of Section 1.1 will be proven.

Definition 6.1 *A compact Riemann surface $\mathcal{R}$ of genus $g \geq 2$ is called hyperellyptic provided there exists a positive divisor D on $\mathcal{R}$ with*

$$\deg D = 2, \qquad l(-D) \geq 2.$$

Equivalently, $\mathcal{R}$ is hyperellyptic if and only if there exists a non-constant meromorphic function Λ on $\mathcal{R}$ with precisely 2 poles counting multiplicities. If $\mathcal{R}$ carries such a function, it defines a two-sheeted covering of the complex sphere

$$\Lambda : \mathcal{R} \to \hat{\mathbb{C}}. \tag{103}$$

All the ramification points of this covering have branch numbers 1. The Riemann-Hutwitz formula (41) gives the number of these points

$$N_B = 2g + 2.$$

Let P_k be one the branch points of the covering (103). $\Lambda(P) - \Lambda(P_k)$ has a zero of order 2 at P_k and no other zeros. This implies $\Lambda(P_k) \neq \Lambda(P_m)$ for $k \neq m$. The function

$$W(P) = \frac{1}{\Lambda(P) - \Lambda(P_k)} \tag{104}$$

has the only pole at the point P_k and this pole is of order 2. This proves that all the branch points of (103) are the Weierstrass points of $\mathcal{R}$.

Lemma 6.1 *The Weierstrass points of the hyperellyptic surface $\mathcal{R}$ are of the weight $g(g-1)/2$ and coincide with the branch points of the covering (103).*

Proof Let P_k be one of the branch points of the covering (103). The functions $1, W(P), W^2(P), \ldots, W^{g-1}(P)$ have the pole divisors $0, 2P_k, 4P_k, \ldots, 2(g-1)P_k$. respectively. For the vector spaces $L(-2nP_k)$ we have

$$1, W(P), \ldots, W^n(P) \in L(-2nP_k),$$

which implies for their dimensions

$$l(-2nP_k) \geq n + 1.$$

For the dimensions of the corresponding spaces of holomorphic differentials this yields due to the Riemann-Roch theorem

$$i(2nP_k) \geq g - n. \tag{105}$$

One can choose a basis $\omega_1, \ldots, \omega_g$ of holomorphic differentials $\omega_n = h_n(z)dz,\ z(P_k) = 0$ such that

$$h_n = z^{m_n} g_n(z), \qquad g_n(z) \neq 0$$

with

$$0 \leq m_1 < m_2 < \ldots < m_g, \qquad m_n \in \mathbb{Z}.$$

and

$$m_n \geq 2(n-1) \tag{106}$$

because of (105). This observation allows us to estimate the weight of the Weierstrass point P_k. Using (99) we get

$$\begin{array}{rcr} & \Delta[h_1, \ldots, h_g] = \Delta[z^{m_1} g_1, \ldots, z^{m_g} g_g] & = \\ = & (z^{m_1} g_1)^g \Delta[1, z^{m_2 - m_1} \frac{g_2}{g_1}, \ldots, z^{m_g - m_1} \frac{g_g}{g_1}] & = \\ = & (z^{m_1} g_1)^g \Delta_{g-1}[(z^{m_2 - m_1} \frac{g_2}{g_1})', \ldots, (z^{m_g - m_1} \frac{g_g}{g_1})'] & = \\ = & (z^{m_1} g_1)^g \Delta_{g-1}[z^{m_2 - m_1 - 1} \tilde{g}_1, \ldots, z^{m_g - m_1 - 1} \tilde{g}_{g-1}], & \end{array}$$

where $\tilde{g}_k(z)$ defined by

$$z^{m_k - m_1 - 1} \tilde{g}_k = \left(z^{m_{k+1} - m_1} \frac{g_{k+1}}{g_1} \right)'$$

are holomorphic near $z = 0$ and $\tilde{g}_k(z) \neq 0$. Proceeding futher we get for the order of the zero of Δ at P_k

$$\begin{array}{rl} \text{ord} \geq & g m_1 + (g-1)(m_2 - m_1 - 1) + (g-2)(m_3 - m_2 - 1) + \ldots \\ & +(m_g - m_{g-1} - 1) = \sum_{n=1}^{g} (m_n - n + 1). \end{array}$$

Combined with (106) this yields

$$\tau(P_k) \geq \sum_{n=1}^{g} (n-1) = \frac{g(g-1)}{2}.$$

But there are $2g+2$ branch points of the covering (103) and the sum of their weights is $\leq g(g-1)(g+1)$. Applying identity (100) we obtain

$$\tau(P_k) = \frac{g(g-1)}{2}.$$

Moreover the points $P_k,\ k = 1, \ldots, 2g+2$ are the only Weierstrass points of $\mathcal{R}$. □

Lemma 6.2 *Let $\mathcal{R}$ be a hyperellyptic Riemann surface in the sence of Definition 6.1. Then the above mentioned (103) function $\Lambda : \mathcal{R} \to \hat{\mathbb{C}}$ is unique up to fractional linear transformations.*

Proof Let $\Lambda : \mathcal{R} \to \hat{\mathbb{C}}$ and $\tilde{\Lambda} : \mathcal{R} \to \hat{\mathbb{C}}$ be two hyperelliptic covering as in Definition 6.1. We know that their branch points coincide and are the Weierstrass points of $\mathcal{R}$. Consider the functions Λ and $\tilde{\Lambda}$ on $\mathcal{R}$. Their polar divisors are $Q_1 + Q_2$ and $\tilde{Q}_1 + \tilde{Q}_2$ respectively. Let P_k be one of the Weierstrass points with $\Lambda(P_k) \neq \infty$, $\tilde{\Lambda}(P_k) \neq \infty$ (one can always find such a point from $2g + 2$ Weierstrass points). The existence of the functions

$$\frac{1}{\Lambda(P) - \Lambda(P_k)}, \qquad \frac{1}{\tilde{\Lambda}(P) - \tilde{\Lambda}(P_k)}$$

shows that the divisors $Q_1 + Q_2$ and $\tilde{Q}_1 + \tilde{Q}_2$ are equivalent

$$Q_1 + Q_2 \sim 2P_k \sim \tilde{Q}_1 + \tilde{Q}_2.$$

There exists a meromorphic function ξ with the divisor $(\xi) = Q_1 + Q_2 - \tilde{Q}_1 - \tilde{Q}_2$, establishing the isomorphism of $L(-Q_1 - Q_2)$ and $L(-\tilde{Q}_1 - \tilde{Q}_2)$

$$\xi L(-Q_1 - Q_2) = L(-\tilde{Q}_1 - \tilde{Q}_2).$$

Since $\{1, \Lambda\}$ and $\{1, \tilde{\Lambda}\}$ form the basises of $L(-Q_1 - Q_2)$ and $L(-\tilde{Q}_1 - \tilde{Q}_2)$ respectively we get

$$\begin{aligned} \tilde{\Lambda} &= \alpha\xi\Lambda + \beta\xi 1 \\ 1 &= \gamma\xi\Lambda + \delta\xi 1, \end{aligned}$$

and finally eliminating ξ

$$\tilde{\Lambda} = \frac{\alpha\Lambda + \beta}{\gamma\Lambda + \delta}.$$

□

Remark It is not difficult to prove [FarkasKra] that the hyperelliptic surfaces give the lower bound for the number of the Weierstrass points

$$2g + 2 \leq N_W \leq g^3 - g.$$

Theorem 6.3 *Definition 6.1 is equivalent to the definition of the compact Riemann surface of hyperelliptic curve in Section 1.1.*

Proof Let $\hat{C}$ be a compact Riemann surface of hyperelliptic curve as in Theorem 1.2. For any λ_0 the pole divisor of the function

$$\Lambda = \frac{1}{\lambda - \lambda_0}$$

provides us the divisor D of Definition 6.1. On the other hand, let $\lambda : \mathcal{R} \to \hat{\mathbb{C}}$ be a meromorphic function with 2 poles as in Definition 6.1. Let $\lambda_k = \lambda(P_k), k = 1, \ldots, 2g-2$ be the values of λ at the Weierstrass points. We have seen above that all of them are different $\lambda_k \neq \lambda_m$ for $k \neq m$. At this point it is easy to check that the complex structure of $\mathcal{R}$ coincides with the complex structure of the compactification $\hat{C}$ of the hyperellyptic curve

$$\mu^2 = \prod_{k=1}^{2g+2} (\lambda - \lambda_k),$$

described in Section 1.1 $\square$

Theorem 6.3 and Lemma 6.2 imply the following

Corollary 6.4 *Two hyperelliptic Riemann surfaces are conformally equivalent if and only if their branch points differ by fractional linear transformation.*

Proposition 6.5 *Let $\mathcal{R}$ be a hyperelliptic Riemann surface and $\lambda : \mathcal{R} \to \hat{\mathbb{C}}$ the corresponding two-sheeted covering. A positive divisor D of degree g is singular if and only if it contains a paar of points*

$$(\mu_0, \lambda_0), \qquad (-\mu_0, \lambda_0)$$

with the same λ-coordinate or a double branch point

$$2(0, \lambda_k).$$

Proof $i(D) > 0$ implies that there exists a differential ω with $(\omega) \geq D$. The differential ω is holomorphic and due to Theorem 4.10 can be represented as

$$\omega = \frac{P_{g-1}(\lambda)}{\mu} d\lambda,$$

where $P_{g-1}(\lambda)$ is a polynominal of degree $g-1$. The differential ω has $g-1$ pairs of zeros

$$(\mu_n, \lambda_n),\ (-\mu_n, \lambda_n),\ n = 1, \ldots, g-1,\ P_{g-1}(\lambda_n) = 0.$$

Since D is of degree g it must contain at least one of these pairs. $\square$

6.2 Riemann surfaces of genus one and two

As it was proven in Corollary 5.8 there exists only one Riemann surface of genus zero, it is the Riemann sphere $\hat{\mathbb{C}}$. In this section we classify Riemann surfaces of genus one and two.

Let $\mathcal{R}$ be a Riemann surface of genus one and ω a holomorphic differential on it. Take a point $P_0 \in \mathcal{R}$. Due to Corollary 5.9 ω does not vanish on $\mathcal{R}$, therefore by $\omega = dz$ it defines a local parameter $z : U \to \mathbb{C}, z(P_0) = 0$ in a neighbourhood of $P_0 \in U$. The Riemann-Roch theorem implies $l(-2P_0) = 2$, thus there exists a non-constant function g with a double pole in P_0. Normalizing we have the following asymptotics of g at $z = 0$:

$$g(z) = \frac{1}{z^2} + o(1), z \to 0.$$

This asymptotics can be further detalized using the fact that $g\omega$ and $g^2\omega$ are Abelian differentials on $\mathcal{R}$. Indeed, these differentials are singular at P_0 only and therefore must have vanishing residues at this point (Lemma 4.11)

$$\mathrm{res}_{P_0}\ g\omega = \mathrm{res}_{P_0}\ g^2\omega = \mathrm{res}_{P_0}\ g^3\omega = 0.$$

For the asymptotics of g this implies

$$g(z) = \frac{1}{z^2} + az^2 + bz^4 + o(z^4).$$

Define another function $h := dg/\omega$ on $\mathcal{R}$. It is holomorphic on $\mathcal{R} \setminus P_0$ with a pole at P_0

$$h(z) = -\frac{2}{z^3} + 2az + 4bz^3 + o(z^3).$$

A direct computation shows that the function $h^2 - 4g^3 + 20ag + 28b$ vanishes at P_0. On the other hand this function is holomorphic on $\mathcal{R}$ and therefore must vanish identically

$$h^2 = 4g^3 - 20ag - 28b. \tag{107}$$

Lemma 6.6 *The zeros of the cubic polynomial*

$$P_3(x) := 4x^3 - 20ax - 28b$$

are all different.

Proof Suppose $P_3(x)$ has a double zero at x_0, i.e.

$$h^2 = 4(g - x_0)^2(g + 2x_0)$$

or equivalently

$$4(g + 2x_0) = \left(\frac{h}{g - x_0}\right)^2.$$

Since $g + 2x_0$ is of degree 2 the meromorphic function $h/(g - x_0)$ has only one pole on $\mathcal{R}$ and must establish a holomorphic isomorphism $\mathcal{R} = \hat{\mathbb{C}}$. This contradiction proves the lemma. □

By an appropriate affine coordinate change $\mu = \alpha h, \lambda = \beta g + \gamma$ we can reduce (107) to

$$\mu^2 = \lambda(\lambda - 1)(\lambda - A)$$

with some $A \in \mathbb{C} \setminus \{0, 1\}$ which can be explicitly computed in terms of a and b.

Proposition 6.7 *Every compact Riemann surface of genus one is the compactification $\hat{C}$ of an elliptic curve C*

$$\mu^2 = \lambda(\lambda - 1)(\lambda - A), \qquad A \in \mathbb{C} \setminus \{0, 1\}.$$

Proof Consider the elliptic curve C and its compactification $\hat{C} = C \cup \{\infty\}$ (see Section 1.1). The holomorphic covering

$$f : \mathcal{R} \setminus P_0 \overset{(\mu,\lambda)}{\longrightarrow} C$$

can be extended to P_0 by $f(P_0) = \infty$. So defined holomorphic covering $f : \mathcal{R} \to \hat{C}$ is an isomorphism of Riemann surfaces. Indeed $f^{-1}(\infty) = P_0$ and f is unramified at P_0 (the local parameter λ/μ on $\hat{C}$ at ∞ is equivalent to z). $\square$

As we have shown in Section 6.1 the branch points are parameters in the module space of hyperelliptic curves. The complex dimension of this space is $2g - 1$. Indeed, there are $2g + 2$ branch points and three of them can be normalized to $0, 1, \infty$ by a fractional linear transformation. We see that for $g = 2$ this dimension coincides with the complex dimension $3g - 3$ of the space of Riemann surfaces of genus g.

This simple observation gives a hint that there exist non-hyperellyptic Riemann surfaces with $g \geq 3$ and that all Riemann surfaces of genus $g = 2$ are hyperelliptic.

Theorem 6.8 *Any Riemann surface of genus $g = 2$ is hyperelliptic.*

Proof Let ω be a holomorphic differential on $\mathcal{R}$ and $P_1 + P_2$ its zero divisor (of degree $2g - 2$). Since $i(P_1 + P_2) > 0$, the Riemann-Roch theorem implies

$$l(-P_1 - P_2) \geq 2.$$

There exists a non-constant function λ with the pole divisor $P_1 + P_2$ and $\mathcal{R}$ is hyperelliptic. $\square$

In Section 6.1 it was shown that the values λ_k of the function λ at the branch points of $\lambda : \mathcal{R} \to \hat{\mathbb{C}}$ are all different. Normalizing three of them by affine transformations of coordinates to $0, 1$ and ∞ we prove the following proposition.

Proposition 6.9 *Every compact Riemann surface of genus two is the compactification $\hat{C}$ of a hyperelliptic curve C*

$$\mu^2 = \lambda(\lambda - 1)(\lambda - A_1)(\lambda - A_2)(\lambda - A_3), \qquad A_i \in \mathbb{C} \setminus \{0, 1\}, A_i \neq A_j.$$

Riemann surfaces of genus one can be also classified using the Abel map. Let us fix a point $P_0 \in \mathcal{R}$. In Section 5.4 it was shown that the Abel map is an embedding.

Proposition 6.10 *A Riemann surface of genus one is conformally equivalent to its Jacobi variety.*

Proof The Jacobi variety of a Riemann surface of genus one is a one-dimensional complex torus, which is itself a Riemann surface of genus one (see Section 1.2). The Abel map (101) is obviously an unramified holomorphic covering (it is holomorphic with non-vanishing derivative). The surjectivity of (101) follows from the Jacobi inversion Theorem 5.19. The injectivity is a simple corollary of the Abel theorem proved in Proposition 5.18. $\square$

Theorem 6.11 *Every Riemann surface of genus one is conformally equivalent to a one-dimensional complex torus $\mathbb{C}/\Lambda_\tau$, where Λ_τ is the lattice*

$$\Lambda_\tau = \{n + \tau m \mid n, m \in \mathbb{Z}\}, \qquad \mathrm{Im}\ \tau > 0.$$

Every torus $\mathbb{C}/\Lambda_\tau$ is a Riemann surface of genus one. The tori corresponding to different τ are conformally equivalent $\mathbb{C}/\Lambda_\tau \cong \mathbb{C}/\Lambda_{\tilde\tau}$ iff τ and $\tilde\tau$ are related by a modular transformation

$$\tilde\tau = \frac{c + d\tau}{a + b\tau}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{108}$$

Proof The first statement follows from Proposition 6.10 if one uses another normalization of the Abel map

$$P \mapsto z = \frac{1}{2\pi i} \int_{P_0}^{P} \omega.$$

In this normalization the period lattice is generated by 1 and $\tau = B/2\pi i$, where B is the period of the Riemann surface. The conditions Im $\tau > 0$ and Re $B < 0$ are equivalent. Chosing another canonical homology basis of $\mathcal{R}$ one obtains a period which differs by the modular transformation (48) described in Lemma 4.15. In terms of τ this is equivalent to (108) since $Sp(1, \mathbb{Z}) = SL(2, \mathbb{Z})$.

On the other hand a bi-holomorphic map $f : \mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\tilde\tau}$ can be lifted to the corresponding (unramified) covering

$$\begin{array}{ccc} z \in \mathbb{C} & \stackrel{F}{\to} & \mathbb{C} \ni w \\ \downarrow & & \downarrow \\ \mathbb{C}/\Lambda_\tau & \stackrel{f}{\to} & \mathbb{C}/\Lambda_{\tilde\tau}. \end{array}$$

Any conformal automorphism $F : \mathbb{C} \to \mathbb{C}$ is of the form (see for example [Beardon])

$$w = \alpha z + \beta, \qquad \alpha, \beta \in \mathbb{C},\ \alpha \neq 0.$$

For the corresponding lattices this implies $\Lambda_{\tilde\tau} = \alpha\Lambda_\tau$. Basises $1, \tilde\tau$ and $\alpha, \alpha\tau$ of the lattice $\Lambda_{\tilde\tau}$ are related by a modular transformation

$$\begin{pmatrix} 1 \\ \tilde\tau \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha\tau \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

which proves (108). $\square$

We see that the theory of meromorphic functions on Riemann surfaces of genus one is equivalent to the theory of elliptic functions, i.e. of doubly periodic meromorphic functions.

7 Theta functions

7.1 Definition and simplest properties

We start with a notion of an Abelian torus which is a natural generalization of the Jacobi variety. Consider a g-dimensional complex torus $\mathbb{C}^g/\Lambda$ where Λ is a lattice of full rank:

$$\Lambda = AN + BM, \qquad A, B \in gl(g, \mathbb{C}),\ N, M \in \mathbb{Z}^g, \tag{109}$$

and all $2g$ columns of A, B are $\mathbb{R}$-linearly independent. Non-constant meromorphic functions on $\mathbb{C}^g/\Lambda$ exist only (see, for example, [Siegel]) if the complex torus is an *Abelian torus*, i.e. after an appropriate linear choice of coordinates in $\mathbb{C}^g/\Lambda$ it is as described in the following

Definition 7.1 *Let B be a symmetric $g \times g$ matrix with negative real part*[15] *and A a diagonal matrix of the form*

$$A = 2\pi i \ \mathrm{diag}(a_1 = 1, \ldots, a_g), \quad a_k \in \mathbb{N},\ a_k \mid a_{k+1}.$$

The complex torus $\mathbb{C}^g/\Lambda$ with the lattice (109) is called an Abelian torus.

An Abelian torus with $a_1 = \ldots = a_g = 1$ is called *principally polarized.* Jacobi varieties of Riemann surfaces are principally polarized Abelian tori. Meromorphic functions on Abelian tori are constructed in terms of theta functions, which are defined by their Fourier series.

Definition 7.2 *Let B be a symmetric $g \times g$ matrix with negative real part. The theta function is defined by the following series*

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp\{\frac{1}{2}(Bm, m) + (z, m)\}, \qquad z \in \mathbb{C}.$$

Here

$$(Bm, m) = \sum_{ij} B_{ij} m_i m_j, \quad (z, m) = \sum_j z_j m_j.$$

Since $\mathrm{Re}B < 0$ the series converge absolutely and defines an entire function on $\mathbb{C}^g$.

Proposition 7.1 *The theta function is even*

$$\theta(-z) = \theta(z)$$

and possesses the following periodicity property:

$$\theta(z + 2\pi i N + BM) = \exp\{-\frac{1}{2}(BM, M) - (z, M)\}\theta(z), \quad N, M \in \mathbb{Z}^g. \tag{110}$$

[15] Note that B is not necessarily a period matrix of a Riemann surface.

Proof is a direct computation

$$\theta(z + 2\pi i N + BM) = \theta(z + BM) =$$
$$\sum_{m\in\mathbb{Z}^g} \exp\{\tfrac{1}{2}(B(m+M),(m+M)) + (z, m+M) - (z,M) - \tfrac{1}{2}(BM,M)\} =$$
$$\sum_{m\in\mathbb{Z}^g} \exp\{-\tfrac{1}{2}(BM,M) - (z,M)\}\theta(z).$$

□

It is useful also to introduce the *theta functions with characteristics* $[\alpha, \beta]$

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z) = \sum_{m\in\mathbb{Z}^g} \exp\left\{\frac{1}{2}(B(m+\alpha), m+\alpha) + (z + 2\pi i\beta, m+\alpha)\right\} = \tag{111}$$
$$\theta(z + 2\pi i\beta + B\alpha)\exp\left\{\frac{1}{2}(B\alpha,\alpha) + (z+2\pi i\beta,\alpha)\right\}, \quad z\in\mathbb{C}^g,\ \alpha,\beta\in\mathbb{R}^g.$$

with the corresponding transformation laws

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z + 2\pi i N + BM) = \tag{112}$$
$$\exp\left\{-\tfrac{1}{2}(BM,M) - (z,M) + 2\pi i((\alpha,N) - (\beta,M))\right\}\ \theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z)$$

Theta functions with half-integer characteristics $\alpha_k, \beta_k \in \{0, 1/2\}$, $\forall k$ are most useful. A half-integer characteristic is called *even* (resp. *odd*) according to the parity of $4(\alpha,\beta) = 4\sum \alpha_k\beta_k$. The corresponding theta functions with these characteristics are even (resp. odd) with respect to z. There are 4^g half-integer characteristics, $2^{g-1}(2^g - 1)$ of which are odd and $2^{g-1}(2^g + 1)$ are even.

7.2 Theta functions of Riemann surfaces

From now on we consider the case of an Abelian torus being a Jacobi variety $\mathbb{C}/\Lambda = Jac(\mathcal{R})$ and theta functions generated by Riemann surfaces. In this case combining the theta function with the Abel map one obtains the following useful mapping on a Riemann surface

$$\Theta(P) := \theta(\mathcal{A}_{P_0}(P) - d), \qquad \mathcal{A}_{P_0}(P) = \int_{P_0}^{P}\omega. \tag{113}$$

Here we incorporated the based point $P_0 \in \mathcal{R}$ in the notation of the Abel map, and the parameter $d \in \mathbb{C}^g$ is arbitrary. The periodicity properties of the theta function (110) imply the following

Proposition 7.2 *$\Theta(P)$ is an entire function on the universal covering $\tilde{\mathcal{R}}$ of $\mathcal{R}$. Under analytical continuation along a- and b-cycles on the Riemann surface it is transformed as follows:*

$$\mathcal{M}_{a_k}\ \Theta(P) = \Theta(P),$$
$$\mathcal{M}_{b_k}\ \Theta(P) = \exp\{-\tfrac{1}{2}B_{kk} - \textstyle\int_{P_0}^{P}\omega_k + d_k\}\ \Theta(P). \tag{114}$$

The zero divisor (Θ) of $\Theta(P)$ on $\mathcal{R}$ is well defined.

Theorem 7.3 *The theta function $\Theta(P)$ either vanishes identically on $\mathcal{R}$ or has exactly g zeros (counting multiplicities):*

$$\deg(\Theta) = g.$$

Proof Suppose $\Theta \not\equiv 0$. As in Section 4 consider the simply connected model F_g of the Riemann surface. The differential $d\log\Theta$ is well defined on F_g and the number of zeros of Θ is equal

$$\deg(\Theta) = \frac{1}{2\pi i}\int_{\partial F_g} d\log\Theta(P).$$

using the periodicity properties of Θ we get[16] for the values of $d\log\Theta$ at the corresponding points

$$\begin{aligned} d\log\Theta(Q_j') &= d\log\Theta(Q_j), \\ d\log\Theta(P_j') &= d\log\Theta(P_j) - \omega_j(P_j). \end{aligned} \tag{115}$$

For the number of zeros of the theta function this implies

$$\deg(\Theta) = \frac{1}{2\pi i}\sum_{j=1}^{g}\int_{a_j}\omega_j = g.$$

□

The location of the zeros of Θ can be described by the following Jacobi inversion problem, which is important for further study of theta functions in Section 7.3.

Proposition 7.4 *Let $\Theta \not\equiv 0$. Then its g zeros $P_1, \ldots, P_g$ satisfy*[17]

$$\sum_{i=1}^{g}\int_{P_0}^{P_i}\omega = d - K, \tag{116}$$

where K is the vector of Riemann constants

$$K_k = \pi i + \frac{B_{kk}}{2} - \frac{1}{2\pi i}\sum_{j\neq k}\int_{a_j}\omega_j\int_{P_0}^{P}\omega_k. \tag{117}$$

Proof Consider the integral

$$I_k = \frac{1}{2\pi i}\int_{\partial F_g} d\log\Theta(P)\int_{P_0}^{P}\omega_k.$$

along the boundary of the simply connected model F_g of $\mathcal{R} \ni P_0$. Note that the Riemann bilinear identity can not be applied in this case since $d\log\Theta$ is not a differential on $\mathcal{R}$. The integral I_k can be computed by residues

$$I_k = \sum_{i=1}^{g}\int_{P_0}^{P}\omega_k.$$

[16]For notations see Section 4.1 and in particular Theorem 4.4.
[17]The identities are, of course, in Jac(R), i.e. modulo periods.

On the other hand, comparing again the integrand in the corresponding points $P_j \equiv P'_j$ and $Q_j \equiv Q'_j$ (which coincide on $\mathcal{R}$, see Fig. 21) one has

$$\int_{P_0}^{Q'_j} \omega_k = \int_{P_0}^{Q_j} \omega_k - 2\pi i \delta_{jk}, \qquad \int_{P_0}^{P'_j} \omega_k = \int_{P_0}^{P_j} \omega_k + B_{jk},$$

which combined with (115) implies

$$\frac{1}{2\pi i} \int_{a_j + a_j^{-1}} d \log \Theta(P) \int_{P_0}^{P} \omega_k = \frac{1}{2\pi i} \int_{a_j} \{ d \log \Theta(P) \int_{P_0}^{P} \omega_k -$$
$$(d \log \Theta(P) - \omega_j(P))(\int_{P_0}^{P} \omega_k + B_{jk}) \} = \frac{1}{2\pi i} \int_{a_j} \omega_j(P) \int_{P_0}^{P} \omega_k .$$

Note that we compute I_k modulo periods which allowed us to cancel the additional term

$$B_{jk} - B_{jk} \frac{1}{2\pi i} \int_{a_j} d \log \Theta(P)$$

in the last identity. The same computation for the b-periods is shorter

$$\frac{1}{2\pi i} \int_{b_j + b_j^{-1}} d \log \Theta(P) \int_{P_0}^{P} \omega_k = \delta_{jk} \int_{b_j} d \log \Theta(P).$$

For I_k this implies

$$I_k = \frac{1}{2\pi i} \sum_{j=1}^{g} \int_{a_j} \omega_j(P) \int_{P_0}^{P} \omega_k + \int_{b_k} d \log \Theta(P). \tag{118}$$

This expression can be simplified further. Let R_1, R_2, R_3 be the vertices of F_g (on $\mathcal{R}$ these three points correspond to the same point R) connected by the cycles a_k and b_k as in Fig. 24.

Using the periodicity (114) one obtains

$$\int_{b_k} d \log \Theta(P) = \log \Theta(R_3) - \log \Theta(R_2) = -\frac{1}{2} B_{kk} + d_k - \int_{P_0}^{R_2} \omega_k .$$

This integral should be combined with one of the integrals in the sum in (118)

$$\frac{1}{2\pi i} \int_{a_k} \omega_k(P) \int_{P_0}^{P} \omega_k = \frac{1}{4\pi i} \int_{a_k} d \left(\int_{P_0}^{P} \omega_k \right)^2 =$$
$$\frac{1}{4\pi i} \left(\left(\int_{P_0}^{R_2} \omega_k \right)^2 - \left(\int_{P_0}^{R_1} \omega_k \right)^2 \right) = \int_{P_0}^{R_2} \omega_k - \pi i,$$

where one uses that R_1 differs from R_2 by the period a_k. Finally comparing of the derived expressions for I_k completes the proof. □

One can easily check that $K \in Jac(\mathcal{R})$ is well defined by (117), i.e. is independent of the integration path. On the other hand K depends on the choice of the canonical homology basis and the base point P_0. To emphasize the last dependence we denote it by

$$K_{P_0}.$$

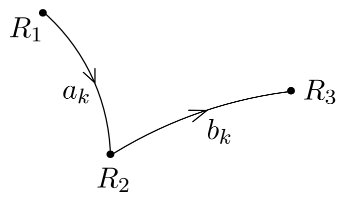

Figure 24: To the proof of Proposition 7.4.

7.3 Theta divisor

Let us denote by J_k the set of equivalence classes (of linear equivalent divisors, see Section 6.1) of divisors of degree k. The Abel theorem and the Jacobi inversion allow us to identify J_g with the Jacobi variety

$$D \in J_g \longleftrightarrow \mathcal{A}(D) \in Jac(\mathcal{R}).$$

The zero set of the theta function of a Riemann surface, which is called *theta divisor* can also be characterized in terms of divisors on $\mathcal{R}$.

Theorem 7.5 *The theta divisor is isomorphic to the set J_{g-1} of equivalence classes of positive divisors of degree $g-1$:*

$$\theta(e) = 0 \Leftrightarrow \exists D \in J_{g-1}, D \geq 0 : \; e = \mathcal{A}(D) + K.$$

Proof Suppose $\theta(e) = 0$. Then there exists $s \in \mathbb{N}$ and positive divisors $D_1, D_2 \in J_s$ such that

$$\theta(\mathcal{A}_{P_0}(D_1) - \mathcal{A}_{P_0}(D_2) - e) \neq 0$$

and for all positive divisors $\tilde{D}_1, \tilde{D}_2 \in J_k$ of lower degree $k = 0, \ldots, s-1$ the theta function

$$\theta(\mathcal{A}_{P_0}(\tilde{D}_1) - \mathcal{A}_{P_0}(\tilde{D}_2) - e) = 0$$

vanishes. The existence of such an $s \leq g$ follows from the Jacobi inversion (see Section 5.4). Take now two points P_1 in D_1 and P_2 in D_2

$$D_1 = P_1 + D_1', \; D_2 = P_2 + D_2', \; D_1', D_2' \geq 0, \; D_1', D_2' \in J_{s-1}$$

and consider the theta-function

$$f(P) = \theta\left(\int_{P_2}^{P} \omega + \mathcal{A}(D_1') - \mathcal{A}(D_2') - e\right)$$

on $\mathcal{R}$. Due to our assumption f vanishes at the divisor D_2

$$(f) \geq D_2$$

and does not vanish identically. Proposition 7.4 implies for the zero divisor $D_3 := (f)$

$$\mathcal{A}_{P_2}(D_3) = e - \mathcal{A}_{P_2}(D_1') + \mathcal{A}_{P_2}(D_2') - K_{P_2}. \tag{119}$$

Since D_3 can be decomposed into the sum ($\deg D_3 = g$)

$$D_3 = D_2 + D', \qquad D' \geq 0, \ \deg D' = g - s,$$

one obtains from (119)

$$e = \mathcal{A}_{P_2}(D_1' + D') + K_{P_2}.$$

The divisor $D_1' + D'$ is of degree $g - 1$.

Conversely, let $D = P_0 + D', \ \deg D' = g - 1, D' \geq 0$ be a non-special divisor of degree g. Take

$$e = \mathcal{A}_{P_0}(D) + K_{P_0}$$

and consider

$$\Theta(P) = \theta(\mathcal{A}_{P_0}(P) - e).$$

If $\Theta(P)$ does not vanish identically its zero divisor $D_\Theta := (\Theta)$ is of degree g. Proposition 7.4 implies

$$\mathcal{A}_{P_0}(D_\Theta) = e - K_{P_0} = \mathcal{A}_{P_0}(D).$$

Since the divisor D is non-special we get $D = D_\Theta$ and $\Theta(P_0) = 0$, i.e.

$$\theta(\mathcal{A}_{P_0}(D') + K_{P_0}) = 0. \tag{120}$$

On the other hand if $\Theta(P)$ vanishes identically it vanishes also at P_0 and thus again (120) holds. The claim is proven for the dense set and therefore for any positive divisor of degree $g - 1$. □

Remark For any $D \in J_{g-1}$ the expression $\mathcal{A}_{P_0}(D) + K_{P_0} \in Jac(\mathcal{R})$ is independent of the choice of P_0 and therefore P_0 can be omitted in the formulation of Theorem 7.5.

Using the characterization of the theta divisor one can complete the description of Proposition 7.4 of the divisor of the function Θ

Theorem 7.6 *Let $\Theta(P) = \theta(\mathcal{A}_{P_0}(P) - d)$ be the theta function (113) on a Riemann surface and the divisor $D \in J_g, \ D \geq 0$ a Jacobi inversion (102) of $d - K$*

$$d = \mathcal{A}(D) + K.$$

Then the following alternative holds:

(i) $\Theta \equiv 0$ iff $i(D) > 0$, i.e. the divisor D is special,

(ii) $\Theta \not\equiv 0$ iff $i(D) = 0$ i.e. the divisor D is non-special. In the last case D is precisely the zero divisor of Θ.

Proof Evenness of theta function and Theorem 7.5 imply that $\theta(d - \mathcal{A}(P)) \equiv 0$ is equivalent to existence (for any P) of a positive divisor D_P of degree $g - 1$ satisfying $\mathcal{A}(D) + K - \mathcal{A}(P) = \mathcal{A}(D_P) + K$. Due to the Abel theorem the last identity holds if and only if the divisors D and $D_P + P$ are linearly equivalent, i.e. there exists a function in $L(-D)$ vanishing at (arbitrary) point P. In terms of the dimension of $L(-D)$ the last property can be formulated as $l(-D) > 1$, which is equivalent to $i(D) > 0$.

Suppose now that D is non-special. Then as we have proven above $\Theta \not\equiv 0$ and Proposition 7.4 implies for the zero divisor of Θ

$$\mathcal{A}((\Theta)) = \mathcal{A}(D).$$

Non-speciality of D implies $D = (\Theta)$. $\square$

Although the vector of Riemann constants K appeared in Proposition 7.4 just as a result of computation K plays an important role in the theory of theta functions. The geometrical nature of K is partially clarified by the following

Proposition 7.7

$$2K = -\mathcal{A}(C),$$

where C is a canonical divisor.

The proof of this proposition is based on the following lemma

Lemma 7.8 *Let D be a positive divisor of degree $2g-2$ such that for any $D_1 \geq 0$, $\deg D_1 = g-1$ there exists $D_2 \geq 0$, $\deg D_2 = g-1$ such that $D \equiv D_1 + D_2$. Then $l(-D) \geq g$, or equivalently $i(D) > 0$.*

Proof Suppose $l(-D) = s < g$ and $f_1, \dots, f_s$ is a basis of $L(-D)$. Choose $P_s \in \mathcal{R}$ such that $f_s(P_s) \neq 0$. The functions

$$\phi_k(P) = f_k(P) f_s(P_s) - f_s(P) f_k(P_s), \qquad k = 1, \dots, s-1,$$

form a basis of $L(-D + P_s)$. Proceeding further this way we find $s \leq g-1$ points $P_1, \dots, P_s$ with $l(-D + P_1 + \dots + P_s) = 0$, which contradicts to the assumption of the lemma. $\square$

Proof of Proposition 7.7. Take an arbitrary $D_1 \in J_{g-1},\ D_1 \geq 0$. Due to Theorem 7.5 theta function vanishes at

$$e = \mathcal{A}(D_1) + K.$$

Theorem 7.5 applied to $\theta(-e) = 0$ implies the existence of a divisor $D_2 \in J_{g-1},\ D_2 \geq 0$ with

$$-e = \mathcal{A}(D_2) + K.$$

For $2K$ this gives

$$2K = \mathcal{A}(D_1 + D_2)$$

with an arbitrary $D_1 \in J_{g-1},\ D_1 \geq 0$. Applying Lemma 7.8 to the divisor $D_1 + D_2$ we get $i(D_1 + D_2) > 0$, i.e. $D_1 + D_2 = (\omega)$ for some holomorphic differential ω.

Vanishing of theta functions at some points follows from their algebraic properties.

Definition 7.3 *Half-periods of the period lattice*

$$\Delta = 2\pi i\alpha + B\beta, \qquad \alpha = (\alpha_1, \ldots, \alpha_g), \beta = (\beta_1, \ldots, \beta_g), \quad \alpha_k, \beta_k \in \{0, \frac{1}{2}\}.$$

are called half periods or theta characteristics. A half period is called even (resp. odd) according to the parity of $4(\alpha, \beta) = 4\sum \alpha_k\beta_k$.

We denote the theta characteristics by $\Delta = [\alpha, \beta]$. A simple calculation

$$\theta(\Delta) = \theta(-\Delta + 4\pi i\alpha + 2B\beta) = \theta(-\Delta)\exp(-4\pi i(\alpha, \beta))$$

shows that theta function $\theta(z)$ vanishes in all odd theta characteristics.

Corollary 7.9 *To any odd theta characteristic* Δ *there corresponds*

$$\Delta = \mathcal{A}(D_\Delta) + K \tag{121}$$

a positive divisor D_Δ *of degree* $g - 1$ *such that*

$$2D_\Delta \equiv C.$$

Proof The existence of D_Δ follows from $\theta(\Delta) = 0$. Since 2Δ belongs to the lattice of $Jac(\mathcal{R})$ doubling of (121) yields

$$\mathcal{A}(2D_\Delta) = -2K = \mathcal{A}(C).$$

The claim of the next corollary follows from the Abel theorem. □

Corollary 7.10 *For any odd theta characteristic* Δ *there exists a holomorphic differential* ω_Δ *with*[18]

$$(\omega_\Delta) = 2D_\Delta. \tag{122}$$

In particular all zeros of ω_Δ *are of even multiplicity.*

The differential ω_Δ of Corollary 7.10 can be described explicitly in theta functions.

To any point z of the Abelian torus on can associate a number $s(z)$ determined by the condition that all partial derivatives of θ up to order $s(z) - 1$ vanish at z and there exists a non-vanishing at z partial derivative of order $s(z)$. For most of the points $s = 0$. The points of the theta divisor are precisely those with $s > 0$, in particular $s(\Delta) > 0$ for any odd theta characteristics Δ. An odd theta characteristics Δ is called *non-singular* iff $s(\Delta) = 1$.

Proposition 7.11 *Let* Δ *be a non-singular odd theta characteristics and* D_Δ *the corresponding (121) positive divisor of degree* $g - 1$. *Then the holomorphic differential* ω_Δ *of Corollary 7.10 is given by the expression*

$$\omega_\Delta = \sum_{i=1}^{g} \frac{\partial\theta}{\partial z_i}(\Delta)\omega_i,$$

where ω_i *are normalized holomorphic differentials.*

[18] Note, that identity (122) is an identity on divisors and not only on equivalence classes of divisors.

Proof Let $D = P_1 + \ldots + P_{g-1}$ be a positive divisor of degree $g-1$. Consider the function $f(P_1, \ldots, P_{g-1}) = \theta(\mathcal{A}(D) + K)$ of $g-1$ variables. Since f vanishes identically differentiating it with respect to P_k one obtains

$$\sum_i \frac{\partial \theta}{\partial z_i}(\mathcal{A}(D) + K)\omega_i(P_k) = 0$$

for all points P_k. The holomorphic differential

$$h = \sum_i \frac{\partial \theta}{\partial z_i}(e)\omega_i$$

with e given by $e = \mathcal{A}(D) + K$ vanishes at all points P_k. Note that we have proven $(h) \geq D$ only in the case when all the points of D have multiplicity one.

Let Δ be an odd non-singular theta characteristics. Define $D_\Delta \in J_{g-1}$ by (121). Let us show that D_Δ is uniquely determined by the identity (121), i.e. $i(D_\Delta) = 1$. Suppose $i(D_\Delta) > 1$, i.e. there exists a non-constant function $f \in L(-D_\Delta)$. The divisor of $f - f(P_0)$ is $P_0 + D_{P_0} - D_\Delta$ with some $D_{P_0} \in J_{g-2}$, $D_{P_0} \geq 0$, and P_0 is arbitrary. Consider

$$h_\Delta := \sum_i \frac{\partial \theta}{\partial z_i}(\Delta)\omega_i.$$

As it was shown above h_Δ vanishes in all points of the divisor D_Δ and in the same way of the divisor $P_0 + D_{P_0}$. Thus we obtain $h_\Delta(P_0) = 0$ for arbitrary $P_0 \in \mathcal{R}$ which implies $h_\Delta(P_0) \equiv 0$ and contradicts to non-singularity of Δ. Assume[19] that all points of D_Δ are different. As we have shown above $(h_\Delta) \geq D_\Delta$. On the other hand the differential ω_Δ of Corollary 7.10 also vanishes at D_Δ. Since the space of holomorphic differentials vanishing at D_Δ is one-dimensional ($i(D_\Delta) = 1$) the differentials ω_Δ and h_Δ coincide up to a constant. □

We finish this Section with the complete description of the theta divisor by Riemann. The proof of this classical theorem can be found for example in [FarkasKra, Lewittes]. It is based on considerations similar to the ones in the present Section.

Theorem 7.12 *The following two characterizations of a point $e \in Jac(\mathcal{R})$ are equivalent:*

- *Theta function and all its partial derivatives up to order $s-1$ vanish in e and there exists a non-vanishing in e partial derivative of order s.*
- *$e = \mathcal{A}(D) + K$ where D is a positive divisor of degree g and $i(D) = s$.*

[19] Proof for the case of multiple points in D is more technically involved.

8 Holomorphic line bundles

In this section we reformulate results of the previous sections in the language of holomorphic line bundles. This language is useful for generalizations to manifolds of higher dimension, where one does not have so concrete tools as in the case of Riemann surfaces and should rely on more abstract geometric constructions.

8.1 Holomorphic line bundles and divisors

Let (U_α, z_α) be coordinate charts of an open cover $\cup_{\alpha\in A} U_\alpha = \mathcal{R}$ of a Riemann surface. The geometric idea behind the concept of the holomorphic line bundle is the following. One takes the union $U_\alpha\times\mathbb{C}$ over all $\alpha\in A$ and "glue" them together identifying $(P,\xi_\alpha)\in U_\alpha\times\mathbb{C}$ with $(P,\xi_\beta)\in U_\beta\times\mathbb{C}$ for $P\in U_\alpha\cap U_\beta$ linearly holomorphically, i.e. $\xi_\beta = g(P)\xi_\alpha$ where $g(P)$ is holomorphic.

Let us make this "constructive" definition rigorous. Denote by

$$\mathcal{O}^*(U)\subset\mathcal{O}(U)\subset\mathcal{M}(U)$$

the sets of nowhere vanishing holomorphic, holomorphic and meromorphic functions on $U\subset\mathcal{R}$ respectively. A holomorphic line bundle is given by its *transition functions*, which are holomorphic non-vanishing functions $g_{\alpha\beta}\in\mathcal{O}^*(U_\alpha\cap U_\beta)$ satisfying

$$g_{\alpha\beta}(P)g_{\beta\gamma}(P) = g_{\alpha\gamma}(P) \qquad \forall P\in U_\alpha\cap U_\beta\cap U_\gamma. \tag{123}$$

Remark Identity (123) implies in particular

$$g_{\alpha\alpha}=1, \qquad g_{\alpha\beta}g_{\beta\alpha}=1.$$

Introduce on triples $[P,U_\alpha,\xi],\ P\in U_\alpha, \alpha\in A, \xi\in\mathbb{C}$ the following equivalence relation[20]:

$$[P,U_\alpha,\xi]\sim[Q,U_\beta,\eta]\Leftrightarrow P=Q\in U_\alpha\cap U_\beta,\ \eta=g_{\beta\alpha}\xi. \tag{124}$$

Definition 8.1 *The union of $U_\alpha\times\mathbb{C}$ identified by the equivalence relation (124) is called a holomorphic line bundle $L=L(\mathcal{R})$. The mapping $\pi: L\to\mathcal{R}$ defined by $[P,U_\alpha,\xi]\mapsto P$ is called the canonical projection. The linear space $L_P := \pi^{-1}(P)\cong P\times\mathbb{C}$ is called a fibre of L.*

The line bundle with all $g_{\alpha\beta}=1$ is called *trivial*.

A set of meromorphic functions $\phi_\alpha\in\mathcal{M}(U_\alpha),\ \forall\alpha\in A$ such that $\phi_\alpha/\phi_\beta\in\mathcal{O}^*(U_\alpha\cap U_\beta)\ \forall\alpha,\beta$ is called a *meromorphic section* ϕ of a line bundle $L(\mathcal{R})$ defined by the transition functions[21]

$$g_{\alpha\beta}=\phi_\alpha/\phi_\beta.$$

[20] The condition (123) implies that the relation (124) is indeed an equivalence relation.

[21] The bundle condition (123) is automatically satisfied.

Note that the divisor (ϕ) of the meromorphic section ϕ is well defined by

$$(\phi)\Big|_{U_\alpha} = (\phi_\alpha)\Big|_{U_\alpha}.$$

In the same way one defines a line bundle $L(U)$ and its sections on an open subset $U \subset \mathcal{R}$. Bundles are locally trivializable, i.e. there always exist local sections: a local holomorphic section over U_α can be given simply by

$$U_\alpha \ni P \mapsto [P, U_\alpha, 1]. \qquad (125)$$

One immediately recognizes that holomorphic (Abelian) differentials (see Definitions 4.2, 4.4) are holomorphic (meromorphic) sections of a holomorphic line bundle. This line bundle given by the transition functions

$$g_{\alpha\beta}(P) = \frac{dz_\beta}{dz_\alpha}(P)$$

is called *canonical* and denoted by K.

Note that obviosly a line bundle is completely determined by its meromorphic section. In Sections 4,6 we deal with meromorphic sections directly and formulate results in terms of sections without using the bundle language.

The following proposition can be used as an alternative (”descriptive”) definition of holomorphic line bundles.

Proposition 8.1 *A holomorphic line bundle $\pi : L \to \mathcal{R}$ is holomorphic projection π of a two-dimensional complex manifold L with a $\mathbb{C}$-linear structure on each fibre $\pi^{-1}(P)$, such that for any point $P \in \mathcal{R}$ there exists an open $U \ni P$ with a bi-holomorphic trivialization $\phi_U : L(U) = \pi^{-1}(U) \to U \times \mathbb{C}$ preserving the linear structure of fibres. Holomorphic (meromorphic) sections of L are holomorphic (meromorphic) mappings $s : \mathcal{R} \to L$ with $\pi \circ s = id$.*

Proof Local coordinates on L can be introduced using local coordinates z_α on $\mathcal{R}$

$$Z_\alpha : U_\alpha \times \mathbb{C} \to z_\alpha(U_\alpha) \times \mathbb{C} \subset \mathbb{C}^2, \qquad [P, U_\alpha, \xi] \mapsto (z_\alpha(P), \xi).$$

The transition functions $Z_\beta \circ Z_\alpha^{-1}$ are obviously holomorphic. All other claims of the proposition can also be easily checked □

Let L be a holomorphic line bundle (124) with trivializations (125) on U_α. Local sections

$$U_\alpha \ni P \mapsto [P, U_\alpha, h_\alpha(P)],$$

where $h_\alpha \in \mathcal{O}^*(U_\alpha)$ define another holomorphic line bundle L' which is called (holomorphically) *isomorphic* to L. We see that fibres of isomorphic holomorphic line bundles can be holomorphically identified $h_\alpha : L(U_\alpha) \to L'(U_\alpha)$. This is equivalent to the following homological definition[22].

[22]Refining the coverings of L and L' if necessary one may assume that the line bundles are defined through the same open covering.

Definition 8.2 *Two holomorphic line bundles L and L' are isomorphic if their transition functions are related by*

$$g'_{\alpha\beta} = g_{\alpha\beta}\frac{h_\alpha}{h_\beta} \tag{126}$$

with some $h_\alpha \in \mathcal{O}^*(U_\alpha)$.

We have seen that holomorphic line bundles can be described through their meromorphic sections. Therefore it is not suprising that holomorphic line bundles and divisors are intimately related. To each divisor one can naturally associate a class of isomorphic holomorphic line bundles. Let D be a divisor on $\mathcal{R}$. Consider a covering $\{U_\alpha\}$ such that each point of the divisor belongs to only one U_α. Take $\phi_\alpha \in \mathcal{M}(U_\alpha)$ such that the divisor of ϕ_α is presicely the part of D lying in U_α

$$(\phi_\alpha) = D_\alpha := D\mid_{U_\alpha}.$$

One can take for example $\phi_\alpha = z_\alpha^{n_i}$, where z_α is a local parameter vanishing at the point $P_i \in U_\alpha$ of the divisor $D = \sum n_i P_i$. The meromorphic section ϕ determines a line bundle L associated with D. If $\phi'_\alpha \in \mathcal{M}(U_\alpha)$ are different local sections with the same divisor $D = (\phi')$, then $h_\alpha = \phi'_\alpha/\phi_\alpha \in \mathcal{O}^*(U_\alpha)$ and ϕ' determines a line bundle L' isomorphic to L. We see that a divisor D determines not a particular line bundle but a class of isomorphic line bundles together with corresponding meromorphic sections ϕ such that $(\phi) = D$. This relation is clearly an isomorphism. Let us denote by $L[D]$ isomorphic line bundles determined by D. The degree $\deg D$ is called the *degree* of the line bundle $L[D]$.

It is natural to get rid of sections in this relation and to describe line bundles in terms of divisors.

Lemma 8.2 *Divisors D and D' are linearly equivalent iff the holomorphic line bundles $L[D]$ and $L[D']$ are isomorphic.*

Proof Chose a covering $\{U_\alpha\}$ such that each point of D and D' belongs to only one U_α. Take $h \in \mathcal{M}(\mathcal{R})$ with $(h) = D - D'$. This function is holomorphic on each $U_\alpha \cap U_\beta$, $\alpha \neq \beta$. If ϕ is a meromorphic section of $L[D]$ then $h\phi$ is a meromorphic section of $L[D']$, which implies (126) for the transition functions. Conversely, let ϕ and ϕ' be meromorphic sections of isomorphic line bundles $L[D]$ and $L[D']$ respectively, $(\phi) = D, (\phi') = D'$. Identity (126) implies that $\phi_\alpha h_\alpha/\phi'_\alpha$ is a meromorphic fiction on $\mathcal{R}$. The divisor of this function is $D - D'$, which yields $D \equiv D'$. □

Lemma 8.2 clarifies in particular why equivalent divisors are called *linearly* equivalent.

It turnes out that Lemma 8.2 provides us a complete classification of holomorphic line bundles. Namely every holomorphic line bundle L comes as a bundle associated to the divisor $L = L[(\phi)]$ of its meromorphic section ϕ, provided the last one exists.

Lemma 8.3 *Every holomorphic line bundle possesses a meromorphic section.*

I do not know an analytic proof of this lemma. Proofs based on homological methods are rather involved [GriffithsHarris, Gunning, Springer].

The following fundamental classification theorem follows immediately from Lemmas 8.2,8.3.

Theorem 8.4 *There is a one to one correspondence between classes of isomorphic holomorphic line bundles and classes of linearly equivalent divisors.*

Thus, holomorphic line bundles are classified by elements of J_n (see Section 7.3), where n is the degree of the bundle $n = \deg L$. Due to the Abel theorem and the Jacobi inversion elements of J_n can be identified with the points of the Jacobi variety. Namely, chose some $D_0 \in J_n$ as a reference point. Then due to the Abel theorem the class of divisor $D \in J_n$ is given by the point

$$\mathcal{A}(D - D_0) = \int_{D_0}^{D} \omega \in Jac(\mathcal{R}).$$

Conversely, due to the Jacobi inversion, given some $D_0 \in J_n$ to any point $d \in Jac(\mathcal{R})$ there corresponds $D \in J_n$ satisfying $\mathcal{A}(D - D_0) = d$.

¿From now on we do not distinguish isomorphic line bundles and denote by $L[D]$ isomorphic line bundles associated with the divisor class D.

8.2 Picard group. Holomorphic spin bundle.

The set of line bundles can be equiped with an Abelian group structure. If L and L' are bundles with transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ respectively, then the line bundle[23] $L'L^{-1}$ is defined by the transition functions $g'_{\alpha\beta} g^{-1}_{\alpha\beta}$.

Definition 8.3 *The Abelian group of line bundles on $\mathcal{R}$ is called the Picard group of $\mathcal{R}$ and denoted by $Pic(\mathcal{R})$*

Using the classification of Section 8.1 of holomorphic line bundles in terms of divisors one immediately obtains the following result.

Theorem 8.5 *The Picard group $Pic(\mathcal{R})$ is isomorphic to the group of divisors $Div(\mathcal{R})$ modulo linear equivalence.*

Proof Take meromorphic sections ϕ and ϕ' of L and L' respectively. Then ϕ'/ϕ is a meromorphic section of $L'L^{-1}$. For the divisors of the sections one has $(\phi'/\phi) = (\phi') - (\phi)$. The claim of the theorem for bundles follows from passing to the corresponding equivalence classes of the divisors. □

Holomorphic q-differentials of Definition 5.8 are holomorphic sections of the bundle K^q.

[23] This is a special case of the tensor product $L' \otimes L^*$ defined for vector bundles.

Corollary 8.6 *The holomorphic line bundles L_1, L_2, L_3 satisfy*

$$L_3 = L_2 L_1^{-1}$$

if and only if

$$\deg L_3 = \deg L_2 - \deg L_1 \quad \text{and} \quad \mathcal{A}(D_3 - D_2 + D_1) = 0,$$

where D_i are the divisors corresponding to $L_i = L[D_i]$.

For the proof one uses the characterization of line bundles through their meromorphic sections ϕ_1, ϕ_2, ϕ_3 and applies the Abel theorem to the meromorphic function $\phi_3\phi_1/\phi_2$.

Since the canonical bundle K is of even degree one can define a "square root" of it.

Definition 8.4 *A holomorphic line bundle S satisfying*

$$SS = K$$

is called holomorphic spin bundle. Holomorphic (meromorphic) sections of S are called holomorphic (meromorphic) spinors.

Spinors are differentials of order 1/2 and their local description $s(z)\sqrt{dz}$ is not familiar from the standard course of complex analysis.

Proposition 8.7 *There exist exactly 4^g non-isomorphic spin bundles on a Riemann surface of genus g.*

Proof Fix a reference point $P_0 \in \mathcal{R}$. As it was already mentioned at the end of Section 8.1 the classes of linear equivalend divisors are isomorphic to points of the Jacobi variety

$$D \in J_n \leftrightarrow d = \mathcal{A}_{P_0}(D) = \mathcal{A}(D - nP_0) \in Jac(\mathcal{R}).$$

For the divisor class D_S of a holomorphic spin bundle Corollary 8.6 implies

$$\deg D_S = g - 1 \quad \text{and} \quad 2\mathcal{A}_{P_0}(D_S) = \mathcal{A}_{P_0}(C),$$

where C is the canonical divisor. Proposition 7.7 provides us with general solution to this problem

$$\mathcal{A}_{P_0}(D_S) = -K_{P_0} + \Delta,$$

where K_{P_0} is the vector of Riemann constants and Δ is one of 4^g half-periods of Definition 7.3. Due to the Jacobi inversion the last equation is solvable (the divisor $D_S \in J_{g-1}$ is not necessarily positive) for any Δ. We denote by $D_\Delta \in J_{g-1}$ the divisor class corresponding to the half-period Δ and by S_Δ the corresponding holomorphic spin bundle $S_\Delta := L[D_\Delta]$. The line bundles with different half-periods can not be isomorphic since the images of their divisors in the Jacobi variety are different. $\square$

Note that we obtained a geometrical interpretation for the vector of Riemann constants.

Corollary 8.8 *Up to a sign the vector of Riemann constants is the Abel map of the divisor of the holomorphic spin bundle with the zero theta characteristic*

$$K_{P_0} = -\mathcal{A}(D_{[0,0]} - (g-1)P_0).$$

This corollary clarifies the dependence of K_{P_0} on the base point and on the choice of canonical homology basis.

Remark In the same way one can show that for a given line bundle L which degree is a multiple of $n \in \mathbb{N}, \deg L = nm$ there exist exactly n^{2g} different "n-th roots" of L, i.e. line bindles $L^{1/n}$ satisfying $(L^{1/n})^n = L$.

Finally, let us give a geometric interpretation of the Riemann-Roch theorem. Denote by $h^0(L)$ the dimension of the space of holomorphic sections of the line bundle L.

Theorem 8.9 (Riemann-Roch)
For any holomorphic line bundle $\pi : L \to \mathcal{R}$ over a Riemann surface of genus g holds

$$h^0(L) = \deg L - g + 1 + h^0(KL^{-1}). \tag{127}$$

Proof This theorem is just a reformulation of Theorem 5.4. Indeed, let $D = (\phi)$ be the divisor of a meromorphic section of the line bundle $L = L[D]$ and let h be a holomorphic section of L. The quotient h/ϕ is a meromorphic function with the divisor $(h/\phi) \geq -D$. On the other hand, given $f \in \mathcal{M}(\mathcal{R})$ with $(f) \geq -D$ the product $f\phi$ is a holomorphic section of L. We see that the space of holomorphic sections of L can be identified with the space of meromorphic functions $L(-D)$ defined in Section 5.2. Similarly, holomorphic sections of KL^{-1} can be identified with Abelian differentials with divisors $(\Omega) \geq D$. This is the space $H(D)$ of Section 5.2 and its dimension is $i(D)$. Now the claim follows from (88). $\square$

The Riemann-Roch theorem does not help to compute the number of holomorphic sections of a spin bundle. The identity (127) implies only trivial $\deg S = g-1$. Computation of $h^0(S)$ is a rather delicate problem. It turnes out that the dimension of the space of holomorphic sections of S_Δ depends on the theta-characteristics Δ and is even for even theta-characteristics and odd for odd theta-characteristics [Atiah]. Spin bundles with non-singular theta-characteristics have no holomorphic sections if the characteristic is even and have a unique holomorphic section if the characteristic is odd.

Results of Section 7.3 allow us to prove this easily for odd theta-characteristics.

Proposition 8.10 *Spin bundles S_Δ with odd theta-characteristics Δ possess global holomorphic sections.*

Proof Take the differential ω_Δ of Corollary 7.10. The square root of it $\sqrt{\omega_\Delta}$ is a holomorphic section of S_Δ. $\square$

If Δ is a non-singular theta-characteristic then the corresponding positive divisor D_Δ of degree $g-1$ is unique (see the proof of Proposition 7.11). This implies the uniqueness of the differential with $(\omega) = D_\Delta$ and $h^0(S_\Delta) = 1$. This holomorphic section is given by

$$\sqrt{\sum_{i=1}^{g} \frac{\partial \theta}{\partial z_i}(\Delta)\omega_i}.$$

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