# Optimal Consumption Choice with Intertemporal Substitution[†]

By Peter Bank and Frank Riedel[‡]

*Humboldt University of Berlin*

**Abstract**

We analyze the intertemporal utility maximization problem under uncertainty for the preferences proposed by Hindy, Huang, and Kreps. Existence and uniqueness of optimal consumption plans are established under arbitrary convex portfolio constraints, including both complete and incomplete markets. For the complete market setting, we prove an infinite-dimensional version of the Kuhn-Tucker theorem which implies necessary and sufficient conditions for optimality. Using this characterization, we show that optimal plans prescribe to consume just enough to keep the induced level of satisfaction always above some stochastic lower bound. When uncertainty is generated by a Lévy process and agents exhibit constant relative risk aversion, we derive solutions in closed form. Depending on the structure of the underlying stochastics, optimal consumption occurs at rates, in gulps, or in a singular way.

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# Introduction

In the theory of intertemporal consumption and portfolio choice, one typically uses *time–additive* utility functionals as a mathematical model for preferences between consumption plans. Time–additivity means that the overall utility assigned to a consumption plan is the sum (in continuous time, the integral) over all period utilities, where the period utility depends only on the quantity consumed in this period. Specifically, denoting by $C(t) = \int_0^t \dot{C}(s)\,ds$ $(0 \leq t \leq T)$ the process of cumulated consumption, the time–additive expected utility functional takes the well–known form

$$\mathbb{E}\tilde{U}(C) = \mathbb{E}\int_0^T u(t, \dot{C}(t))\,dt\,.$$

Such functionals form the basis of the modern theory of intertemporal consumption and portfolio choice, which was initiated in the seminal papers by Merton (1969, 1971).

The time–additive utility functionals suffer, however, from some drawbacks. For instance, they are restricted to the set of absolutely continuous consumption plans, thus excluding possibly relevant phenomena such as consumption in gulps or in singular form. A further drawback of these functionals is the lack of a reasonable discrete–time analogue which could sustain their structure by a limiting procedure.

A very fundamental caveat was raised by Hindy, Huang, and Kreps (1992). Their critique focuses on the very basis of continuous–time preference theory and applies not only to the time–additive utility functional but actually to all utility functionals which directly depend on the rate of consumption. The central point of their critique is that, concerning slight shifts of consumption in time, the mentioned functionals are not as robust as one would expect preferences of economic agents to be.

This point can be illustrated easily in the standard time–additive setting by examining the induced intertemporal substitution properties. Assume, for instance, that having a good meal is modeled by a certain constant rate of consumption for one hour. Now, compare the consumption plan of having such a meal once every day with the plan to have seven such meals from morning to evening on one day and no more meals for the rest of the week. One can hardly doubt that, due to obvious substitution effects, real economic agents would prefer the first plan to the latter. In the time–additive setting, however, both consumption plans will yield essentially the same utility as every single meal contributes to total utility separately. In other words, the standard setting exhibits complementarity of consumption over time rather than local substitutability of consumption.

Hindy, Huang, and Kreps argue that agents are indifferent between slight alterations of a consumption plan in both the amounts consumed at every time and the timing of the whole plan. Mathematically, this economic kind of closeness between consumption plans is captured by the Prohorov–distance between nonnegative, finite measures on some time interval. Therefore, utility functionals should be continuous with respect to this distance. However, as Hindy, Huang, and Kreps prove, any utility functional which directly depends on consumption rates in a non–linear way cannot have this economically desirable continuity property. The intuitive reason is that the rate of consumption reacts too sensitively to small changes of the consumption plan.

As a remedy, Hindy, Huang, and Kreps (1992) propose a new kind of utility functional where period utilities are no longer derived from the current consumption rate but from the current *level of satisfaction*. This level $Y(C)$ is modeled as a weighted average of past consumption:

$$Y(C)(t) \stackrel{\Delta}{=} \eta(t) + \int_0^t \vartheta(t,s)\, dC(s) \quad (0 \leq t \leq T)\,.$$

The quantity $\vartheta(t,s)$ describes the weight assigned at time $t$ to consumption made at time $s \leq t$; $\eta(t)$ may be interpreted as an exogenously given level of satisfaction for time $t$. Based on this level, Hindy, Huang, and Kreps specify a utility functional of the form

$$\mathbb{E}U(C) \stackrel{\Delta}{=} \mathbb{E} \int_0^T u(t, Y(C)(t))\, dt\,.$$

In contrast to the time–additive functional, the Hindy–Huang–Kreps–utilty is not confined to absolutely continuous consumption plans and it exhibits the economically desirable substitution properties; cf. Hindy, Huang, and Kreps (1992).

Once this new approach to intertemporal choice theory has been accepted, it is important to understand the consumption behavior which is induced by such 'HHK–preferences'. The analysis of this question in a general stochastic framework is the aim of the present paper. For the considerably easier deterministic case, we refer the reader to Hindy, Huang, and Kreps (1992) and Bank and Riedel (2000).

First, we establish existence and uniqueness of optimal consumption plans in a general setting. Existence of a solution to the utility maximization problem becomes an issue in the stochastic framework, since budget sets are no longer compact as in the deterministic setting. Using a new method which is based on a theorem of Komlós (1967) and its infinite–dimensional extension by Kabanov (1999), we give a short existence proof for optimal policy under convex portfolio constraints. This includes complete as well as incomplete markets and contains, as a special case, a result of Jin and Deng (1997) who prove existence in a diffusion model under short–sale constraints.

Moving on, we study the characterization and construction of optimal consumption plans when markets are complete. In a Markovian context, Hindy and Huang (1993) and Benth, Karlsen, and Reikvam (1999) derive sufficient conditions for optimality based on the *Hamilton–Jacobi–Bellman equation*; in a special case, this allows them to compute an explicit solution. Instead of using the Bellman approach, we extend our infinite–dimensional version of the *Kuhn–Tucker Theorem* in Bank and Riedel (2000) from the deterministic to the stochastic framework. In this way, we obtain necessary and sufficient conditions for optimality in a general semimartingale framework. In the context of HHK–preferences, our approach is analogous to the Cox and Huang (1989) method in the classical time–additive case.

In the case of uncertainty, the explicit construction of optimal plans is more difficult than in the deterministic case, where the optimal level of satisfaction is a smooth time–dependent function of the current price for consumption. The present context allows for price processes of unbounded variation. But, since the optimal level of satisfaction is an average, it has bounded variation. Thus, it can no longer be just a function of the present price.

Based on our Kuhn–Tucker characterization of optimal plans, we derive an equation (cf. (17)) characterizing a stochastic process which we call the *minimal level of satisfaction*. The optimal policy consists in consuming 'just enough' to keep the level of satisfaction always above this minimal level. This allows us to reduce the utility optimization problem to a solution of the *minimal level equation*. Thus, the minimal level equation (17) plays in our method the same role as the Hamilton–Jacobi–Bellman equation in the dynamic programming approach. The main advantage of our approach is that it works in a general semimartingale setting without any Markov assumptions.

In a homogeneous setting where prices are driven by a Lévy process, we are able to compute the solution to the minimal level equation. This gives the explicit description of the optimal consumption plan and allows us to calculate the indirect utility in closed form.

We carry out several case studies which illustrate the flexibility of the HHK–framework and of our approach to the corresponding optimization problem. A whole variety of consumption patterns can arise, depending on the structure of the underlying stochastic processes. If state prices are driven by Brownian motion, optimal consumption is singular, as already pointed out by Hindy and Huang (1993). If prices are driven by a Poisson process, the occurring price shocks induce the investor to consume in gulps whenever there is a 'favorable' downward price shock. If prices jump upward, he refrains from consumption for a while, until he has become 'unsatisfied' or rich enough to consume again.

An outline of the paper is as follows. In Section 1 we describe the general framework and formulate the utility maximization problem. Section 2 proves existence and uniqueness of a solution. In Section 3 we give necessary and sufficient conditions for optimality when the financial market is complete. Furthermore, we investigate the general structure of optimal consumption plans and introduce our concept of the 'minimal level of satisfaction'. Finally, Section 4 provides some explicit case studies.

# 1 Formulation of the Utility Maximization Problem

Consider an investor who wishes to consume his initial wealth $w \geq 0$ over a fixed finite time period $[0, T]$. Assume he can invest in at least one risky security and in a money market account whose interest rate $r = (r(t),\ 0 \leq t \leq T)$ is given as a bounded, progressively measurable process. Uncertainty is described by a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t,\ 0 \leq t \leq T), \mathbb{P})$ satisfying the usual conditions of right continuity and completeness; $\mathcal{F}_0$ is $\mathbb{P}$–a.s. trivial. A priori, the consumption plans at the investor's deposit are given by

$$\mathcal{X} \stackrel{\Delta}{=} \{C \mid C \text{ is the distribution function of a nonnegative optional random measure}\}$$

while his budget–feasible set is

$$\mathcal{A}(w) \stackrel{\Delta}{=} \{C \in \mathcal{X} \mid \Psi(C) \le w\}\,.$$

Here, $\Psi(C) \in [0, +\infty]$ denotes the minimal initial capital needed to finance a given consumption plan $C \in \mathcal{X}$ by investing in the assets of the financial market. We assume this quantity can be expressed in the form

$$\Psi(C) \stackrel{\Delta}{=} \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \int_0^T \gamma(t)\, dC(t) \quad (C \in \mathcal{X}) \tag{1}$$

where $\gamma(t) \stackrel{\Delta}{=} \exp\left(-\int_0^t r(s)\, ds\right)$ and $\mathcal{P}$ is a fixed nonempty set of $\mathbb{P}$–equivalent probability measures on $(\Omega, \mathcal{F}_T)$. The specific choice of this set is determined by the risk-structure of the considered financial market.

**Remark 1.1** *Note that the above formulation allows for incomplete markets and, more generally, even for markets under convex constraints; see, e.g., Föllmer and Kabanov (1998), Föllmer and Kramkov (1997), Cvitanic and Karatzas (1992).*

*To illustrate this, let us consider a model of a security market consisting of a riskless bond and a stock, and let us assume that short selling of the stock is prohibited. Föllmer and Kramkov (1997) show that this economic setting may be captured by choosing*

$$\mathcal{P} \stackrel{\Delta}{=} \{\mathbb{P}^* \sim \mathbb{P} \mid \mathbb{P}^* \text{ is a supermartingale measure for each } S \in \mathcal{S}\},$$

*where $\mathcal{S}$ denotes the set of all gain processes which are attainable by some admissible strategy without short selling. More precisely, they prove that*

$$\sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[\gamma(T)H]$$

*is the minimal amount needed to hedge a given contingent claim $H \ge 0$ with maturity $T$. For a consumption plan $C \in \mathcal{X}$, this induces formula* (1) *for the minimal budget the investor needs to finance it.*

**Convention** *In this paper, integration over time intervals is carried out including the involved finite boundaries. We let any consumption stream start in $C(0-) \stackrel{\Delta}{=} 0$; a positive value at time* 0 *indicates an initial consumption gulp and corresponds to a point mass $C(0) > 0$ of the random measure $dC$ at time $t = 0$. Similarly, we assume that any other integrator $B$ starts from some initial value $B(0-)$, which is supposed to be zero unless otherwise stated.*

With a given consumption plan $C \in \mathcal{X}$ the investor associates the utility

$$U(C) \triangleq \int_0^T u(t, Y(C)(t))\, dt$$

where $u : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ denotes a continuous felicity function which is increasing and concave in its second argument, and where

$$Y(C)(t) \triangleq \eta(t) + \int_0^t \vartheta(t, s)\, dC(s)$$

is the investor's level of satisfaction obtained from his consumption up to time $t \in [0, T]$. We assume that the deterministic functions $\eta : [0, T] \to \mathbb{R}$ and $\vartheta : [0, T]^2 \to \mathbb{R}$ are continuous and nonnegative. The quantity $\vartheta(t, s)$ describes the weight assigned at time $t$ to consumption made at time $s \leq t$; $\eta(t)$ may be interpreted as an exogenously given level of satisfaction for time $t$.

**Remark 1.2** *A standard choice for $\vartheta(.,.)$ and $\eta(.)$ is $\vartheta(t, s) \triangleq \beta e^{-\beta(t-s)}$ and $\eta(t) \triangleq \eta e^{-\beta t}$ with constants $\beta, \eta > 0$; compare, e.g., Sundaresan (1989), Constantinides (1990) where this quantity appears in a different context.*

The investor's problem is to maximize his expected utility

$$V(C) \triangleq \mathbb{E}U(C) \quad (C \in \mathcal{X})$$

over all budget–feasible consumption plans, i.e.,

$$\max_{C \in \mathcal{A}(w)} V(C). \tag{2}$$

# 2 Existence and Uniqueness

This section is devoted to the proof of existence and uniqueness of a solution for the utility maximization problem (2) under

**Assumption 1** *The family of budget–feasible utilities $(U(C),\ C \in \mathcal{A}(w))$ is uniformly $\mathbb{P}$–integrable.*

This assumption is slightly stronger than the condition that problem (2) is well–posed because the latter assumption amounts to requiring merely $L^1(\mathbb{P})$–boundedness of the family $(U(C),\ C \in \mathcal{A}(w))$. In particular, Assumption 1 ensures that the value of problem (2) is finite.

A sufficient condition for Assumption 1 to be satisfied is given by the following

**Lemma 2.1** *The utility functional $U$ satisfies Assumption 1 if the following two conditions hold true:*

*(i) For some* $\alpha \in (0,1)$*, the felicity function* $u$ *satisfies the power growth–condition*

$$|u(t,y)| \leq \text{const.}\ (1+y^\alpha) \quad \textit{for all } y \geq 0 \textit{ uniformly in } t \in [0,T]\,. \tag{3}$$

*(ii) There is a measure* $\hat{\mathbb{P}} \in \mathcal{P}$ *with density* $\hat{Z} \stackrel{\Delta}{=} \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$ *satisfying*

$$\hat{Z}^{-1} \in L^{\hat{p}}(\mathbb{P}) \tag{4}$$

*for some* $\hat{p} > \frac{\alpha}{1-\alpha}$.

Proof : We show that $(U(C),\ C \in \mathcal{A}(w))$ is bounded in $L^p(\mathbb{P})$ where $p \stackrel{\Delta}{=} \frac{\hat{p}}{\alpha(1+\hat{p})} > 1$. Due to our growth condition (3), we have

$$U(0) \leq U(C) \leq \text{const.} \int_0^T (1 + Y(C)(t)^\alpha)\, dt \leq \text{const.}\,(1 + C(T)^\alpha))\,.$$

Hence, it suffices to show uniform integrability of the family $(C(T)^\alpha,\ C \in \mathcal{A}(w))$. For this, note that $\alpha p < 1$, and apply Hölder's inequality to get

$$\mathbb{E}\left[C(T)^{\alpha p}\right] \leq \mathbb{E}\left[C(T)\hat{Z}\right]^{\alpha p} \mathbb{E}\left[\hat{Z}^{-\frac{\alpha p}{1-\alpha p}}\right]^{1-\alpha p} \leq \text{const.}\, w^{\alpha p} \mathbb{E}\left[\hat{Z}^{-\hat{p}}\right]^{1-\alpha p}\,.$$

Note that, in connection with condition (4), this yields the desired $L^p(\mathbb{P})$–boundedness. The last estimate holds true since

$$\mathbb{E}\left[C(T)\hat{Z}\right] = \widehat{\mathbb{E}}\left[C(T)\right] \leq \text{const.}\,\widehat{\mathbb{E}}\left[\int_0^T e^{-\int_0^t r(s)\,ds}\, dC(t)\right] \leq \text{const.}\, w$$

for all $C \in \mathcal{A}(w)$. □

**Remark 2.2** *Assumptions similar to those of Lemma 2.1 have been made for the case of time–additive functionals in Cox and Huang (1991) and Aumann and Perles (1965). The example in Kramkov and Schachermayer (1999) suggests that a growth condition like* (3) *may in fact be necessary. An integrability condition similar to* (4) *can be found in Cuoco (1997).*

The following is the main result of this section.

**Theorem 2.3** *Under Assumption 1, the utility maximization problem* (2) *has a solution. This solution is unique if, in addition,* $u(t,.)$ *is strictly concave for every* $t \in [0,T]$ *and* $C \mapsto Y(C)$ *is injective up to* $\mathbb{P}$*–indistinguishability.*

**Remark 2.4** *Injectivity of* $C \mapsto Y(C)$ *follows, e.g., if* $\vartheta(t,s) = \vartheta^1(t)\vartheta^2(s)$ *for some strictly positive, continuous functions* $\vartheta^1,\ \vartheta^2 : [0,T] \to \mathbb{R}$.

Let us prepare the proof of Theorem 2.3 by the following technical

**Lemma 2.5** *(i) There is a constant $B > 0$ such that*

$$Y(C)(t) \leq B(1 + C(t)) \quad (0 \leq t \leq T)$$

*for all $C \in \mathcal{X}$.*

*(ii) If $C^n \in \mathcal{X}$ $(n = 1, 2, \dots)$ converge almost surely to $C \in \mathcal{X}$ in the weak topology of measures on $[0, T]$ then we have almost surely*

$$Y(C^n)(t) \to Y(C)(t)$$

*for $t = T$ and for every point of continuity $t$ of $C$.*

*(iii) $\mathcal{A}(w)$ is norm–bounded in $L^1(\mathbb{P}^*)$ for every $\mathbb{P}^* \in \mathcal{P}$, i.e.,*

$$\sup_{C \in \mathcal{A}(w)} \mathbb{E}^* C(T) < +\infty.$$

Proof : (i) and (ii) follow immediately from our assumptions on $\eta(.)$ and $\vartheta(.,.)$. The boundedness of the interest rate process $r$ implies (iii). □

Now we can give the

**Proof of Theorem 2.3** Choose a maximizing sequence $C^n \in \mathcal{A}(w)$ $(n = 1, 2, \dots)$ for (2). By Lemma 2.5 (iii) and Kabanov's version of Komlós' theorem (Kabanov (1999), Lemma 3.5; Komlós (1967)), there exists a subsequence, again denoted by $(C^n)$, which is almost surely weakly Cesaro convergent to some $C^* \in \mathcal{X}$, i.e., almost surely we have

$$\tilde{C}^n(t) \triangleq \frac{1}{n} \sum_{k=1}^{n} C^k(t) \to C^*(t) \quad (n \uparrow +\infty)$$

for $t = T$ and also for every point of continuity $t$ of $C^*$.

We claim that $C^*$ is optimal for (2). Indeed, since $\gamma$ is continuous, we have

$$\int_0^T \gamma(t)\, dC^*(t) = \lim_n \int_0^T \gamma(t)\, d\tilde{C}^n(t) \quad \mathbb{P}\text{–a.s.}$$

Hence, by Fatou's Lemma,

$$\mathbb{E}^* \int_0^T \gamma(t)\, dC^*(t) \leq \liminf_n \mathbb{E}^* \int_0^T \gamma(t)\, d\tilde{C}^n(t) \leq w,$$

for every $\mathbb{P}^* \in \mathcal{P}$, i.e., $C^* \in \mathcal{A}(w)$.

Furthermore, Lemma 2.5 (i) and (ii) yield $U(\tilde{C}^n) \to U(C^*)$ for $n \uparrow +\infty$ $\mathbb{P}$–a.s. by dominated convergence. In conjunction with Assumption 1 this yields

$$V(\tilde{C}^n) = \mathbb{E}U(\tilde{C}^n) \to \mathbb{E}U(C^*) = V(C^*)$$

by Lebesgue's theorem. Like $(C^n)$ also $(\tilde{C}^n)$ is a maximizing sequence for (2) by concavity of $V$. Thus, $C^*$ is indeed a budget–feasible consumption plan with maximal utility.

Let us now prove uniqueness. If two solutions $\tilde{C}$ and $C^*$ are not indistinguishable, then, by assumption, neither are their respective levels of satisfaction $\tilde{Y} \stackrel{\Delta}{=} Y(\tilde{C})$ and $Y^* \stackrel{\Delta}{=} Y(C^*)$. Optimality excludes that these levels only differ at time $t = T$ because this would imply a (suboptimal) final jump by one of the policies. Thus, on a set with positive probability $\tilde{Y}$ and $Y^*$ differ on an open time interval. Hence, by strict concavity of $u(t,.)$ for any $t \in [0,T]$,

$$\begin{aligned} V\left(\tfrac{1}{2}\{\tilde{C}+C^*\}\right) &= \mathbb{E}\int_0^T u\left(t, \tfrac{1}{2}\{\tilde{Y}(t)+Y^*(t)\}\right)\, dt \\ &> \mathbb{E}\int_0^T \tfrac{1}{2}\left\{u(t,\tilde{Y}(t)) + u(t,Y^*(t))\right\}\, dt \\ &= \tfrac{1}{2}\{V(\tilde{C})+V(C^*)\} \\ &= \max_{C\in\mathcal{A}(w)} V(C) \end{aligned}$$

in contradiction to $\frac{1}{2}\{\tilde{C}+C^*\} \in \mathcal{A}(w)$ and to the optimality of $\tilde{C}$ and $C^*$ in this set. $\square$

# 3 Solutions in the Complete Case

From now on we work under Assumption 1. In addition we make

**Assumption 2** *The financial market is complete in the sense that $\mathcal{P}$ is a singleton.*

Thus, there is precisely one measure $\mathbb{P}^* \in \mathcal{P}$ and we let

$$\psi(t) \stackrel{\Delta}{=} \gamma(t) \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} \qquad (0 \le t \le T)$$

denote the RCLL–version of its associated state–price density.

Furthermore we require the following

**Assumption 3** *The felicity function $u = u(t,y)$ is strictly concave and differentiable in $y$.*

Since a strictly concave and increasing function is strictly increasing, the above assumption ensures that the investor's utility function is non–satiated. Hence, he will always exhaust his budget.

## 3.1 First Order Conditions for Optimality

In the complete setting described above, we can prove the following analogue of the Kuhn–Tucker Theorem for the utility maximization problem (2). It provides a characterization of the solution to (2) in terms of necessary and sufficient first–order conditions.

In order to formulate our result, we let $\nabla V(C)$ denote the optional version of

$$\nabla V(C)(t) \stackrel{\Delta}{=} \mathbb{E}\left[\int_t^T \partial_y u(s, Y(C)(s))\vartheta(s,t)\,ds \,\middle|\, \mathcal{F}_t\right]. \tag{5}$$

**Remark 3.1** *(i) The quantity $\nabla V(C)(t)$ may be interpreted as the marginal expected utility resulting from an additional infinitesimal consumption at time $t$, otherwise following the consumption plan $C \in \mathcal{X}$. Mathematically, $\nabla V(C)$ may be viewed as the Riesz representation of the utility gradient at $C$, as pointed out by Duffie and Skiadas (1994) in their Example 5.*

*(ii) More precisely, we define $\nabla V(C)$ as the optional projection of the nonnegative, product–measurable process*

$$\Phi(\omega,t) \stackrel{\Delta}{=} \int_t^T \partial_y u(s, Y(C(\omega))(s))\vartheta(s,t)\,ds \quad (\omega \in \Omega,\ t \in [0,T]).$$

*Hence, $\nabla V(C)$ is uniquely determined up to $\mathbb{P}$–indistinguishability. Moreover, we have the identity*

$$\mathbb{E}\int_0^T \nabla V(C)(t)\,dC'(t) = \mathbb{E}\int_0^T \Phi(t)\,dC'(t)$$

*for all $C' \in \mathcal{X}$; cf., e.g., Théorème (1.33) in Jacod (1979).*

With this notation at hand, we now can give

**Theorem 3.2** *Under Assumption 1, a consumption plan $C^* \in \mathcal{X}$ solves the utility optimization problem (2) if and only if the following conditions (i)–(iii) hold true for some finite Lagrange multiplier $M > 0$:*

*(i)* $\mathbb{E}\int_0^T \psi(t)\,dC^*(t) = w$,

*(ii)* $\nabla V(C^*)(t) \le M\psi(t)$ *for every* $t \in [0,T]$ $\mathbb{P}$*–a.s.,*

*(iii)* $\mathbb{E}\int_0^T \{\nabla V(C^*)(t) - M\psi(t)\}\,dC^*(t) = 0$, *i.e., for $\mathbb{P}$–a.e. $\omega \in \Omega$, $C^*(\omega)$ is flat off the set*

$$\{t \in [0,T] \mid \nabla V(C^*)(\omega,t) = M\psi(\omega,t)\}.$$

PROOF : Let us first prove sufficiency. Assume $C^* \in \mathcal{X}$ satisfies conditions (i)–(iii) and consider another budget–feasible consumption plan $C \in \mathcal{A}(w)$. Put $Y \triangleq Y(C)$, $Y^* \triangleq Y(C^*)$. By concavity of $u$ and by definition of $Y$ and $Y^*$, one has

$$\begin{aligned} V(C^*) - V(C) &= \mathbb{E} \int_0^T \{u(s, Y^*(s)) - u(s, Y(s))\}\, ds \\ &\geq \mathbb{E} \int_0^T \{\partial_y u(s, Y^*(s))(Y^*(s) - Y(s))\}\, ds \\ &= \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \vartheta(s,t)\, [dC^*(t) - dC(t)] \right\} ds\,. \end{aligned}$$

We split up the last expectation in two terms:

$$I^* \triangleq \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \vartheta(s,t)\, dC^*(t) \right\} ds$$

and

$$I \triangleq \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \vartheta(s,t)\, dC(t) \right\} ds\,.$$

For the second term, Fubini's theorem yields

$$I = \mathbb{E} \int_0^T \left\{ \int_t^T \partial_y u(s, Y^*(s)) \vartheta(s,t)\, ds \right\} dC(t)\,.$$

Since $dC$ is an optional random measure, we may replace the $\{\dots\}$–term in the above expectation by its optional projection which, by definition, is $\nabla V(C^*)(t)$ $(0 \leq t \leq T)$; cf. Remark 3.1 (ii). Hence,

$$I = \mathbb{E} \int_0^T \nabla V(C^*)(t)\, dC(t) \leq M \mathbb{E} \int_0^T \psi(t)\, dC(t) \leq Mw$$

where the first inequality follows from condition (ii) and the last inequality is due to the budget constraint. By conditions (i) and (iii), the above calculation carried out for $C^*$ instead of $C$ shows

$$I^* = \mathbb{E} \int_0^T \nabla V(C^*)(t)\, dC^*(t) = M \mathbb{E} \int_0^T \psi(t)\, dC(t) = Mw\,.$$

Summing up, we obtain

$$V(C^*) - V(C) \geq I^* - I \geq Mw - Mw = 0\,,$$

establishing sufficiency. Necessity follows from Lemma 3.3 and Lemma 3.4 below. □

Before we attack the necessity part of Theorem 3.2, let us briefly sketch the argument. The idea is to proceed along the same lines as in the proof of the finite–dimensional Kuhn–Tucker Theorem. Thus, in a first step, we show that the optimal policy $C^*$ solves the problem linearized around $C^*$. This is done in Lemma 3.3 below. Solutions of the linear problem are easily characterized (Lemma 3.4), and it follows that $C^*$ has to satisfy the conditions given in Theorem 3.2.

The following two Lemmata are needed to prove necessity of the first–order conditions.

**Lemma 3.3** *Let $C^* \in \mathcal{A}(w)$ be optimal for (2) and let $\phi^* \triangleq \nabla V(C^*)$. Then $C^*$ solves the linear problem*

$$\max_{C \in \mathcal{A}(w)} \mathbb{E} \int_0^T \phi^*(t)\, dC(t)\,, \tag{6}$$

*and the value of this problem is finite.*

PROOF : Consider $C \in \mathcal{A}(w)$ and let $C^\varepsilon \triangleq \varepsilon C + (1-\varepsilon) C^*$ $(0 \le \varepsilon \le 1)$. By optimality of $C^*$ and concavity of $u(t,.)$ $(0 \le t \le T)$, we have for $Y^\varepsilon \triangleq Y(C^\varepsilon)$, $Y \triangleq Y(C)$, $Y^* \triangleq Y(C^*)$:

$$\begin{aligned}
0 &\ge \frac{1}{\varepsilon}\{V(C^\varepsilon) - V(C^*)\} \\
&= \mathbb{E} \int_0^T \frac{1}{\varepsilon}\{u(s, Y^*(s) + \varepsilon(Y(s) - Y^*(s))) - u(s, Y^*(s))\}\, ds \\
&\ge \mathbb{E} \int_0^T \{\partial_y u(s, Y^\varepsilon(s))(Y(s) - Y^*(s))\}\, ds \\
&= \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^\varepsilon(s)) \int_0^s \vartheta(s,t)\, [dC(t) - dC^*(t)] \right\} ds \\
&= \mathbb{E} \int_0^T \left\{ \int_t^T \partial_y u(s, Y^\varepsilon(s)) \vartheta(s,t)\, ds \right\} [dC(t) - dC^*(t)] \\
&= \mathbb{E} \int_0^T \Phi^\varepsilon(t)\, [dC(t) - dC^*(t)]\,,
\end{aligned}$$

where $\Phi^\varepsilon(t) \triangleq \int_t^T \partial_y u(s, Y^\varepsilon(s)) \vartheta(s,t)\, ds$ $(0 \le t \le T)$. Let furthermore $\Phi^* \triangleq \Phi^0$. By Fatou's Lemma, we have

$$\liminf_{\varepsilon \downarrow 0} \mathbb{E} \int_0^T \Phi^\varepsilon(t)\, dC(t) \ge \mathbb{E} \int_0^T \Phi^*(t)\, dC(t)\,. \tag{7}$$

We claim that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \int_0^T \Phi^\varepsilon(t)\, dC^*(t) = \mathbb{E} \int_0^T \Phi^*(t)\, dC^*(t)\,. \tag{8}$$

Given (8) and using (7), we may let $\varepsilon \downarrow 0$ in the above series of estimations to infer

$$\mathbb{E}\int_0^T \Phi^*(t)\,dC(t) \le \mathbb{E}\int_0^T \Phi^*(t)\,dC^*(t)\,.$$

By Théorème (1.33) in Jacod (1979) (cf. Remark 3.1), we may replace $\Phi^*$ in the above inequality by its optional projection which, by definition, coincides with $\nabla V(C^*) = \phi^*$. This establishes optimality of $C^*$ for the linear problem (6).

It remains to prove (8). For this, it suffices to show that the family

$$I^\varepsilon = \int_0^T \Phi^\varepsilon(t)\,dC^*(t) \quad (0 \le \varepsilon \le \tfrac{1}{2})$$

has a $\mathbb{P}$–integrable upper bound. To this end, write

$$I^\varepsilon = \int_0^T \left\{\partial_y u(s, Y^\varepsilon(s)) \int_0^s \vartheta(s,t)\,dC^*(t)\right\} ds \le \int_0^T \partial_y u(s, Y^\varepsilon(s)) Y^*(s)\,ds\,.$$

As $Y^\varepsilon \ge \frac{1}{2}Y^*(s)$ for $\varepsilon \le \frac{1}{2}$, concavity of $u(s,.)$ implies that this is

$$\le \int_0^T \partial_y u(s, \tfrac{1}{2}Y^*(s))Y^*(s)\,ds = 2\int_0^T \partial_y u(s, \tfrac{1}{2}Y^*(s))\tfrac{1}{2}Y^*(s)\,ds$$

which, using the concavity–estimate $\partial_y u(s,y)(y-0) \le u(s,y) - u(s,0)$, is in turn

$$\le 2\int_0^T \{u(s, \tfrac{1}{2}Y^*(s)) - u(s,0)\}\,ds \le 2\left\{U(C^*) - \int_0^T u(s,0)\,ds\right\}.$$

Since $U(C^*)$ is $\mathbb{P}$–integrable by Assumption 1, we have found the required upper bound for $I^\varepsilon$ $(0 \le \varepsilon \le 1/2)$. □

Let us now discuss the linear problem (6).

**Lemma 3.4** *Let $\phi, \psi$ be two strictly positive, rightcontinuous, and adapted processes. Then every solution $C^*$ to the linear optimization problem*

$$\max_{C \in \mathcal{X}} \mathbb{E}\int_0^T \phi(t)\,dC(t) \quad s.t. \quad \mathbb{E}\int_0^T \psi(t)\,dC(t) \le w \tag{9}$$

*satisfies*

$$\mathbb{E}\int_0^T |\phi(t) - M\psi(t)|\,dC^*(t) = 0, \tag{10}$$

*where*

$$M \triangleq \operatorname*{ess\,sup}_{\Omega} \sup_{t\in[0,T]} \frac{\phi(t)}{\psi(t)}\,.$$

Proof :

1. We first show that the value $v$ of the linear problem (9) is given by $Mw$. Indeed, it is easy to see that $v \leq Mw$. On the other hand, note that for every $K < M$ the set
$$\left\{ \omega \in \Omega \;\middle|\; \left( \sup_{t \in [0,T]} \frac{\phi(t)}{\psi(t)} \right)(\omega) > K \right\}$$
has positive probability. Therefore, letting
$$\tau^K \triangleq \inf \left\{ t \in [0,T] \;\middle|\; \frac{\phi(t)}{\psi(t)} > K \right\}$$
we can find $k \geq 0$ such that $C^K \triangleq k 1_{[\tau^K, T]} \in \mathcal{X}$ satisfies $\mathbb{E} \int_0^T \psi \, dC^K = w$. We have
$$\begin{aligned} Mw \geq v \geq \mathbb{E} \int_0^T \phi \, dC^K &= \mathbb{E}\left[ k \phi(\tau^K) 1_{\{\tau^K < +\infty\}} \right] \\ &\geq \mathbb{E}\left[ k K \psi(\tau^K) 1_{\{\tau^K < +\infty\}} \right] = K \mathbb{E} \int_0^T \psi \, dC^K = Kw. \end{aligned}$$
Letting $K \uparrow M$ in the above inequality yields $v = Mw$.

2. Suppose that $C^*$ is a solution to (9). Then by Step 1 and the definition of $M$
$$Mw = \mathbb{E} \int_0^T \phi \, dC^* \leq M \mathbb{E} \int_0^T \psi \, dC^* \leq Mw$$
implying (10).

□

## 3.2 The Structure of Optimal Consumption Plans

As the finite–dimensional Kuhn–Tucker Theorem, our infinite–dimensional version does not yield immediately an explicit description of the optimum. However, we can use the characterization in Theorem 3.2 to analyze the general structure of the solution, as we will show in this section. The main result of this analysis will be Theorem 3.13. This theorem provides an equation characterizing what we call the 'minimal level of satisfaction'. This is a progressively measurable process $L = (L(t),\ 0 \leq t \leq T)$ which gives us a canonical lower bound for the investor's optimal level of satisfaction. As we will see, this property allows us to express the optimal consumption plan in terms of the minimal level process $L$. Thus, in our non–Markovian setup, the equation characterizing this level plays the same role as the Hamilton–Jacobi–Bellman equation does in Dynamic Programming.

As a first application of Theorem 3.2, let us now prove a version of the Dynamic Programming Principle:

**Proposition 3.5** *Let $S \leq T$ be a stopping time. If $C^* \in \mathcal{X}$ is a solution to (2) then, $\mathbb{P}$–a.s., it also solves the problem*

$$\textit{Maximize } \mathbb{E}[U(C) \,|\, \mathcal{F}_S] \;\; \textit{subject to } C \equiv C^* \textit{ on } [0, S) \textit{ and } \Psi_S(C) \leq \Psi_S(C^*)$$

*where*

$$\Psi_S(C) \triangleq \frac{1}{\psi(S)} \mathbb{E}\left[ \int_S^T \psi(t)\, dC(t) \,\middle|\, \mathcal{F}_S \right] \quad (C \in \mathcal{X})$$

*is the price–functional at time $S$. Thus, a consumption plan which is optimal at time zero is its best continuation at any time afterwards.*

Proof : Using the first–order conditions satisfied by $C^*$, this can be shown by the same calculation as for the sufficiency part of Theorem 3.2, now carried out using conditional expectations instead of ordinary expectations. □

Let us now study the dependence of the optimal consumption plan on the exogenous level of satisfaction $\eta(.)$. To make this precise, let us specify the following dynamics for the level of satisfaction:

**Assumption 4** *The exogenous level of satisfaction $\eta : [0, T] \to \mathbb{R}$ and the consumption weights $\vartheta : [0, T]^2 \to \mathbb{R}$ are given by*

$$\eta(t) \triangleq \eta e^{-\int_0^t \beta(s)\, ds}, \quad \vartheta(t, s) \triangleq \beta(s) e^{-\int_s^t \beta(v)\, dv} \quad (0 \leq s \leq t \leq T)\,,$$

*where $\beta(.)$ is a strictly positive, continuous function $[0, T] \to \mathbb{R}$ and $\eta \geq 0$ is a constant.*

**Remark 3.6** *Under the preceding assumption, the corresponding level of satisfaction*

$$Y(C)(t) = \eta e^{-\int_0^t \beta(s)\, ds} + \int_0^t \beta(s) e^{-\int_s^t \beta(v)\, dv}\, dC(s) \quad (0 \leq t \leq T)\,.$$

*induced by a consumption plan $C \in \mathcal{X}$ evolves according to the ODE*

$$(11) \qquad Y(C)(0-) = \eta, \quad dY(C)(t) = \beta(t)\, (dC(t) - Y(C)(t-)\, dt) \quad (0 \leq t \leq T)\,.$$

*Hence, this particular specification ensures that past consumption affects future levels of satisfaction only through the induced* current *level of satisfaction.*

Under Assumption 4, $C \mapsto Y(C)$ is injective, see Remark 2.4, and, therefore, we may apply Theorem 2.3 to obtain existence and uniqueness of an optimal consumption plan for every choice of the initial level of satisfaction $\eta \geq 0$. In order to stress its dependence on this parameter, let us denote this plan by $C^{M,\eta}$; $M > 0$ is the Lagrange multiplier induced by our Kuhn–Tucker Theorem 3.2.

The following lemma shows how the optimal plan $C^{M,\eta}$ depends on the initial level of satisfaction $\eta$.

**Lemma 3.7** *Let $Y(.)$ and $\tilde{Y}(.)$ denote the functionals for the level of satisfaction with initial value $\eta$ and $\tilde{\eta}$, respectively. Suppose $0 \le \eta \le \tilde{\eta}$.*

*Then the respective optimal levels of satisfaction $Y^* \triangleq Y(C^{M,\eta})$, $\tilde{Y}^* \triangleq \tilde{Y}(C^{M,\tilde{\eta}})$ with the* same *Lagrange multiplier $M > 0$ are related by*

$$\tilde{Y}^*(t) = \tilde{\eta} e^{-\int_0^t \beta(s)\,ds} \vee Y^*(t) \quad (0 \le t \le T)\,. \tag{12}$$

*In particular, we have*

$$dC^{M,\tilde{\eta}}(t) = 1_{\{\tau < t \le T\}}\, dC^{M,\eta}(t) + \tilde{\Delta}\, \delta_{\{\tau\}}(dt) \tag{13}$$

*where the second summand is the Dirac measure with point mass*

$$\tilde{\Delta} \triangleq \frac{1}{\beta(\tau)} \left( Y^*(\tau) - \tilde{\eta} e^{-\int_0^\tau \beta(s)\,ds} \right)$$

*at time*

$$\tau \triangleq \inf \left\{ t \ge 0 \,\middle|\, \tilde{\eta} e^{-\int_0^t \beta(s)\,ds} \le Y^*(t) \right\}.$$

Proof : Let $\tilde{C} \in \mathcal{X}$ be the consumption plan defined by the right side of (13). From the dynamics for the level of satisfaction specified in Assumption 4, it may easily be deduced that $\tilde{Y}(\tilde{C})$ coincides with the right side of (12). Moreover, we see that $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, T]$. We will show that $\tilde{C}$ is optimal for the problem with initial level of satisfaction $\tilde{\eta}$ and that it has Lagrange multiplier $M > 0$. By uniqueness of this plan, we then obtain equations (12) and (13).

To prove the claimed optimality of $\tilde{C}$, let us verify that it satisfies the appropriate first–order conditions. Denote the utility functional corresponding to initial level of satisfaction $\tilde{\eta}$ by $\tilde{V} : \mathcal{X} \to \mathbb{R}$. For any stopping time $S \le T$, we have

$$\begin{aligned} \nabla \tilde{V}(\tilde{C})(S) &= \mathbb{E}\left[ \int_S^T \partial_y u\left(t, \tilde{Y}(\tilde{C})(t)\right) \vartheta(t,S)\,dt \,\middle|\, \mathcal{F}_S \right] \\ &\le \mathbb{E}\left[ \int_S^T \partial_y u\left(t, Y^*(t)\right) \vartheta(t,S)\,dt \,\middle|\, \mathcal{F}_S \right] = \nabla V(C^{M,\eta})(S) \qquad (14) \\ &\le M\psi(S)\,, \qquad (15) \end{aligned}$$

where inequality (14) follows from $\tilde{Y}(\tilde{C}) \ge Y^*$; inequality (15) is due to the first–order conditions satisfied by $C^{M,\eta}$. Since the stopping time $S$ is arbitrary, the above estimate shows in conjunction with Meyer's optional section theorem that $\tilde{C}$ satisfies the first–order inequality constraint $\nabla \tilde{V}(\tilde{C}) \le M\psi$.

Hence, it remains to check the flat–off condition. Note first that $\operatorname{supp} d\tilde{C} \subset [\tau, T]$. Moreover, we have $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, T]$ and, therefore, also $\nabla \tilde{V}(\tilde{C}) = \nabla V(C^{M,\eta})$ on this interval. Hence,

$$\mathbb{E} \int_0^T \{\nabla \tilde{V}(\tilde{C})(t) - M\psi(t)\}\, d\tilde{C}(t) = \mathbb{E} \int_\tau^T \{\nabla \tilde{V}(C^{M,\eta})(t) - M\psi(t)\}\, d\tilde{C}(t) = 0$$

where the last equality is due to the absolute continuity of $d\tilde{C}$ with respect to $dC^{M,\eta}$ and to the flat–off condition satisfied by the latter consumption plan. □

**Remark 3.8** *The preceding lemma shows in particular that it suffices to find the optimal consumption plan for $\eta = 0$. All other cases may be recovered from this one by equations* (12) *and* (13).

We are now going to introduce our key concept of a 'minimal level of satisfaction'. Let us first motivate its definition by some heuristics.

For every stopping time $S < T$, consider an agent, called $S$–Adam, who is born at time $S$. $S$–Adam starts with an initial level of satisfaction of zero. Taking the history $\mathcal{F}_S$ as given, he solves

$$\text{Maximize } V_S(C) \triangleq \mathbb{E}\left[\int_S^T u\left(t, Y_S(C)(t)\right)\, dt \,\middle|\, \mathcal{F}_S\right] \text{ subject to } \Psi_S(C) \leq w_S^M \ ,$$

where

$$Y_S(C)(t) \triangleq \int_S^t \beta(s) e^{-\int_S^s \beta(v)\,dv}\, dC(s) \quad (S \leq t \leq T)$$

denotes the evolution of $S$–Adam's level of satisfaction if, from his birth on, he follows the consumption plan $C$. We assume that, at his time of birth, $S$–Adam is endowed with the initial capital $w_S = w_S^M$ needed to buy the optimal consumption plan $C_S^M$ which has Lagrange multiplier $M > 0$. This Lagrange multiplier is also shared by all his brothers.

Now imagine that $\tilde{S}$–Adam, with $\tilde{S} \leq S$, thinks about his consumption from time $S$ on. We claim that he can deduce his optimal behavior by observing his younger brother $S$–Adam. In fact, as long as $\tilde{S}$–Adam's level of satisfaction $Y_{\tilde{S}}(.) \triangleq Y_{\tilde{S}}(C_{\tilde{S}}^M)(.)$ is strictly higher than $S$–Adam's, he should not consume. Once $\tilde{S}$–Adam's level of satisfaction has dropped to $S$–Adam's level, it is optimal to mimic $S$–Adam's behavior. In particular, at whatever time before $S$ an agent is 'born', his optimal level of satisfaction at time $S$ will be above $S$–Adam's level $Y_S(S)$.

Heuristically, we argue therefore that

$$L(S) = Y_S(S) \quad \text{for every stopping time} \quad S < T$$

defines a *universal* lower bound from which we may recover *all* optimal consumption plans $C_S^M$ with the same Lagrange multiplier $M > 0$. Indeed, every $S$–Adam should optimally consume 'just enough' to ensure that his level of satisfaction never falls below this lower bound. Lemma 3.9 below makes precise what we mean by 'consuming just enough' in this sense. We state this result only for time of birth being equal to zero, the general case can be treated analogously. Figure 4.1 in Section 4.1 below illustrates the way a consumption plan may be defined by this property.

**Lemma 3.9** *Let $L = (L(t), 0 \leq t \leq T)$ be a real valued, progressively measurable process with upper–rightcontinuous paths. Set*

$$Y^L(t) \triangleq e^{-\int_0^t \beta(s)\,ds} \left(\eta \vee \sup_{0 \leq v \leq t} \left\{ L(v) e^{\int_0^v \beta(s)\,ds} \right\}\right) \quad (0 \leq t \leq T)\,.$$

*(i)* $Y^L$ *is an adapted RCLL–process of bounded variation with* $Y^L \geq L$.

*(ii)* *Consider the rightcontinuous process of bounded variation* $C^L$ *defined by*

$$C^L(0-) \triangleq 0, \quad C^L(t) \triangleq \int_0^t Y^L(s)\,ds + \int_0^t \beta(s)^{-1}\,dY^L(s) \quad (0 \leq t \leq T)\,.$$

*This process is nondecreasing and adapted, and defines, therefore, a consumption plan, i.e.,* $C^L \in \mathcal{X}$.

*(iii)* *The level of satisfaction induced by* $C^L$, $Y(C^L)$, *coincides with* $Y^L$ *and is minimal above* $L$ *in the following sense:*

$$Y(C^L)(t) = Y^L(t) = \min_{C \in \mathcal{X}, Y(C) \geq L} Y(C)(t) \quad \textit{for all} \quad 0 \leq t \leq T.$$

*In addition, if, for fixed* $\omega \in \Omega$, $t \in [0,T]$ *is a point of increase of* $C^L(\omega,.)$ *then* $Y(C^L)(\omega,t) = L(\omega,t)$.

The above lemma allows us to give

**Definition 3.10** *We say, an investor following the plan* $C^L$ *of the preceding lemma consumes just enough to keep his level of satisfaction always above* $L$. *Equivalently, we will say that the consumption plan* $C^L$ *tracks the level process* $L$.

**Proof of Lemma 3.9** Consider a consumption plan $C \in \mathcal{X}$. By Assumption 4, the process $A(C)$ defined by

$$A(C)(0-) \triangleq \eta, \qquad A(C)(t) \triangleq e^{\int_0^t \beta(s)\,ds} Y(C)(t) \quad (0 \leq t \leq T)$$

is increasing and adapted. In terms of $A(C)$, the restriction $Y(C) \geq L$ may be rewritten as

$$A(C)(t) \geq e^{\int_0^t \beta(s)\,ds} L(t) \quad \text{for all} \quad 0 \leq t \leq T\,.$$

Obviously, the minimal increasing process $A^L$ which starts in $A^L(0-) \triangleq \eta$ and dominates the right side of this inequality is the running supremum

$$A^L(t) \triangleq \sup_{0 \leq v \leq t} \{\eta \vee e^{\int_0^v \beta(s)\,ds} L(v)\} \quad (0 \leq t \leq T).$$

Since $L$ is progressively measurable, we may deduce from Théorème IV.2.33 in Dellacherie and Meyer (1975) that $A^L$ is progressively measurable, too. Due to the upper–rightcontinuity of $L$, $A^L$ is in fact an adapted RCLL–process. Consequently, this also holds true for $Y^L(t) = e^{-\int_0^t \beta(s)\,ds} A^L(t)$ $(t \geq 0)$. In addition, we obtain that

$$dC^L(t) = \frac{1}{\beta(t)} e^{-\int_0^t \beta(s)\,ds}\, dA^L(t) \quad (0 \leq t \leq T)$$

defines an optional random measure with $C^L \in \mathcal{X}$ and $Y(C^L) = Y^L$. Furthermore, minimality of $Y^L$ is inherited from the minimality of $A^L$. Finally, $t$ is a point of increase of $C^L(\omega,.)$ iff it is a point of increase of $A^L(\omega,.)$ . The latter implies $A^L(\omega,t) = e^{\int_0^t \beta(s)\,ds} L(\omega,t)$ which is equivalent to $Y(C^L)(\omega,t) = L(\omega,t)$. □

The above arguments suggest that, for a given Lagrange multiplier $M > 0$, there exists a canonical lower bound $L = L^M$ for the investor's level of satisfaction from which the optimal consumption behavior may be recovered as described in Lemma 3.9. However, the heuristic way to construct this minimal level sketched above is far from being constructive. Therefore, we would like to derive additional properties of this process that allow us to characterize it more explicitly.

To this end, let us continue our heuristics and suppose that the felicity function $u$ satisfies the Inada–condition

$$\partial_y u(t, 0+) = +\infty \quad \text{for all} \quad t \in [0, T] \,.$$

Then our Kuhn–Tucker conditions imply that every $S$–Adam immediately starts consuming at his time of birth $S$: otherwise his optimal level of satisfaction $Y_S(.) = Y_S(C_S^M)(.)$ would remain zero over an open time interval, contradicting the inequality restriction

$$\nabla V_S(C_S^M)(s) \triangleq \mathbb{E}\left[\int_s^T \partial_y u\left(t, Y_S(C_S^M)(t)\right) \vartheta(t,s)\, dt \,\middle|\, \mathcal{F}_t\right] \leq M\psi(s) \quad (S \leq s \leq T)$$

for optimal plans. Hence, at time $s = S$, the first–order condition is binding for $S$–Adam and, therefore, we obtain the following equality:

$$\nabla V_S(C_S^M)(S) = \mathbb{E}\left[\int_S^T \partial_y u(t, Y_S(t))\vartheta(t,S)\, dt \,\middle|\, \mathcal{F}_S\right] = M\psi(S) \,. \tag{16}$$

As pointed out above, we conjecture that $S$–Adam's optimal consumption plan tracks some level process $L$. Thus, Lemma 3.9 (adapted for initial time $S$ and initial satisfaction zero) allows us to rewrite equation (16) in terms of this process $L$:

$$\mathbb{E}\left[\int_S^T \partial_y u\left(t, \sup_{S \leq v \leq t}\left\{L(v)e^{-\int_v^t \beta(w)\,dw}\right\}\right) \vartheta(t,S)\, dt \,\middle|\, \mathcal{F}_S\right] = M\psi(S) \,. \tag{17}$$

Since $L$ is a universal lower bound for *every* $S$–Adam's level of satisfaction, this equality should hold true for *every* stopping time $S < T$. In fact, together with the preceding heuristics and the following assumption it justifies the formal Definition 3.12 of the minimal level of satisfaction given below.

**Assumption 5** *For every $M > 0$, there is a unique progressively measurable process $L = L^M$ with upper–rightcontinuous paths and $L(T) = 0$ such that the 'minimal level equation'* (17) *is satisfied for every stopping time $S < T$.*

**Remark 3.11** *In a discrete time setting, it is easy to construct a solution to (the discrete–time analogue of) equation* (17) *via backwards induction, provided*

$$\partial_y u(t, 0+) = +\infty \quad \textit{and} \quad \partial_y u(t, +\infty) = 0$$

*for all $t \in [0, T]$. In the present continuous–time framework, this construction is more involved. We refer the reader to Bank (2000) for a proof of existence and uniqueness.*

We now can give the following

**Definition 3.12** *The process $L = L^M$ of Assumption 5 which is associated with $M > 0$ will be called the minimal level of satisfaction for Lagrange parameter $M$.*

The following theorem establishes the usefulness of this concept.

**Theorem 3.13** *Under Assumptions 2–5, the consumption plan $C^L$ which tracks the minimal level of satisfaction $L = L^M$ is optimal for the utility maximization problem (2) given initial capital $w = \Psi(C^L)$; the constant $M > 0$ is its associated Lagrange multiplier.*

PROOF : We show that $C^L$ satisfies the first–order conditions for Lagrange parameter $M$. Note first that, by definition of $C^L$,

$$Y(C^L)(t) = \left\{ Y(C^L)(s) e^{-\int_s^t \beta(v)\,dv} \right\} \vee \sup_{s \le v \le t} \left\{ L(v) e^{\int_v^t \beta(w)\,dw} \right\}$$

for all $0 \le t \le s \le T$. Hence, we have for any $C \in \mathcal{X}$,

$$\begin{aligned} &\mathbb{E} \int_0^T \nabla V(C^L)(s)\,dC(s) \\ &\quad = \mathbb{E} \int_0^T \left\{ \int_s^T \partial_y u \left( t, \left\{ Y(C^L)(s) e^{-\int_s^t \beta(w)\,dw} \right\} \vee \sup_{s \le v \le t} \{ L(v) e^{\int_v^t \beta(w)\,dw} \} \right) \right. \\ &\qquad\qquad \left. \cdot\ \vartheta(t,s)\,dt \right\} dC(s) \\ &\quad \le \mathbb{E} \int_0^T \left\{ \int_s^T \partial_y u \left( t, \sup_{s \le v \le t} \{ L(v) e^{\int_v^t \beta(w)\,dw} \} \right) \vartheta(t,s)\,dt \right\} dC(s)\,. \qquad (18) \end{aligned}$$

As the measure $dC$ is optional, we may replace the $\{\dots\}$–term in the last expression by its optional projection which, due to our minimal level equation (17), is given by $M\psi 1_{[0,T)}$. This yields

$$\mathbb{E} \int_0^T \nabla V(C^L)(t)\,dC(t) \le \mathbb{E} \int_0^T M\psi(t)\,dC(t)\,.$$

As this estimate holds true for any $C \in \mathcal{X}$, we obtain the inequality condition $\nabla V(C^L) \le M\psi$ from Meyer's optional section theorem. To prove the flat–off condition, recall from Lemma 3.9 that, for any $\omega \in \Omega$, the measure $dC^L(\omega, .)$ charges only the set $\{t \in [0,T] \mid Y(C^L(\omega))(t) = L(\omega,t)\}$. Thus, for $C = C^L$, estimate (18) is tight and, therefore, we have equality in the above estimate for $C = C^L$. This is the desired flat–off condition. $\square$

The preceding theorem suggests the following method to construct explicit solutions to the utility maximization problem (2):

1. For every $M > 0$, find the progressively measurable process $L = L^M$ which solves the minimal level equation (17).

2. For each $M > 0$, compute the price $\Psi(C^M)$ of the consumption plan $C^M \stackrel{\Delta}{=} C^{L^M}$ which tracks the minimal level of satisfaction $L^M$.

3. The consumption plan $C^{M(w)}$ with $\Psi(C^{M(w)}) = w$ is then the unique solution to the investor's utility maximization problem (2).

# 4 Explicit Solution in a Homogeneous Setting

In this section, we are going to derive some explicit solutions to the utility maximization problem (2) in a homogeneous setting by applying the method described at the end of the preceding section.

## 4.1 Some Heuristics

Let us try to find a plausible candidate for the minimal level of satisfaction.

To this end, we first recall the structure of optimal consumption plans as they are derived in the 'classical' theory based on time-additive von Neumann-Morgenstern utility functionals. In such a setting, utility is obtained from the current *rate* of consumption, rather than from the instantaneous level of satisfaction. Applying methods of convex duality (confer, e.g., Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)), one shows that the marginal felicity of an optimal consumption rate for this problem should equal some fixed multiple of the state-price density. This leads to the absolutely continuous optimal consumption plan $dC^{ac}(t) \equiv i(t, K\psi(t))\,dt$, where $i(t,.) \stackrel{\Delta}{=} (\partial_y u(t,.))^{-1}$ is the inverse of marginal felicity and $K$ is a strictly positive constant.

At least formally, the level of satisfaction $Y(C)$ plays the same role for our utility functional $U(C)$ as does the rate of consumption for the classic von Neumann–Morgenstern utilities. Thus, the above solution suggests to choose $C \in \mathcal{X}$ such that $Y(C)(t) \equiv i(t, K\psi(t))$. However, the right side of this equality will typically be of unbounded variation, while, under Assumption 4, the left side must have bounded variation for any choice of $C \in \mathcal{X}$. Hence, there might be no $C \in \mathcal{X}$ inducing a level of satisfaction of the form suggested above. But we can try to stay as close as possible to this desirable level. This suggest to choose the consumption plan $C^K \stackrel{\Delta}{=} C^{L^K}$ which tracks the level process

$$L^K(t) \stackrel{\Delta}{=} i(t, K\psi(t)) \quad (t \geq 0)\,.$$

This definition is illustrated by Figure 4.1. Any $L^K$ gives us a plausible candidate for the minimal level of satisfaction $L$ we are looking for. In fact, this turns out to be the

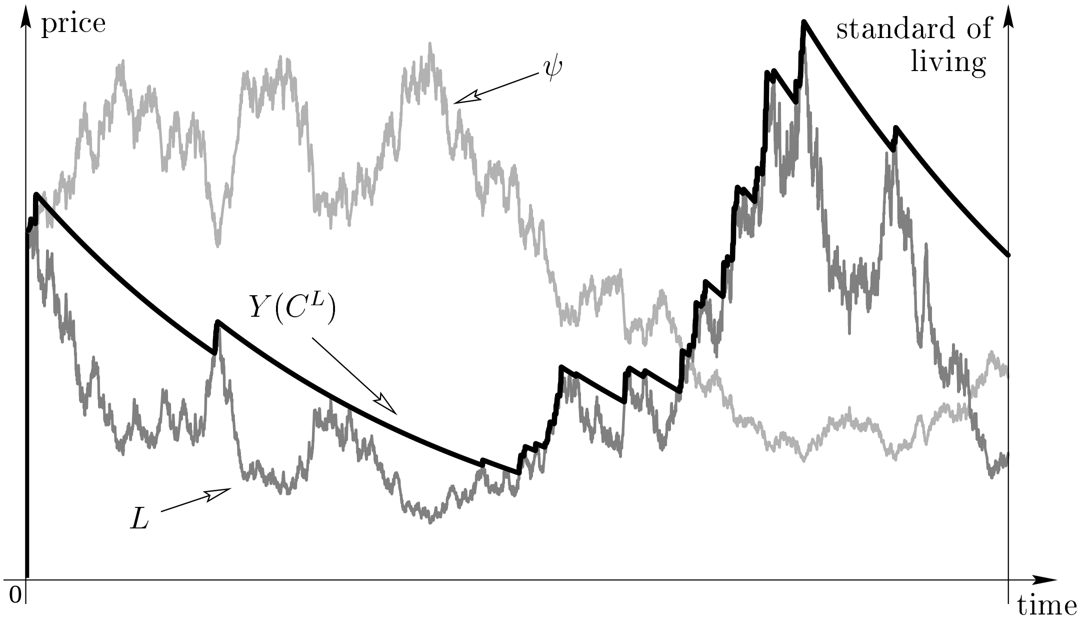


Figure 1: Typical paths for the state–price $\psi$ (light grey line), the level of satisfaction $Y(C^L)$ (black line) and its minimal level $L$ (grey line).

right choice in a homogeneous setting where we can give a rigorous argument; confer Corollary 4.6 below.

## 4.2 The Homogeneous Setting

For the remainder of this chapter, we fix the following setting.

The investor's time horizon is infinite: $T = +\infty$. We furthermore suppose his felicity function to have the separable, homogeneous form

$$u(t, y) = e^{-\delta t} \frac{1}{\alpha} y^{\alpha} \quad (t \geq 0, y \geq 0) \tag{19}$$

for some constant $\alpha \in (-\infty, 1)\backslash\{0\}$ and denote by

$$i(t, z) \stackrel{\Delta}{=} \left(e^{\delta t} z\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0, z > 0)$$

the inverse of its associated marginal felicity function $\partial_y u(t, .)$.

**Remark 4.1** *The case $\alpha = 0$, corresponding to 'log-felicity', can be treated with the same method as the 'power-felicities' above. For ease of exposition, we leave this case to the interested reader.*

We assume the function $\beta(.)$ of Assumption 4 to be a strictly positive constant $\beta(.) \equiv \beta > 0$. Hence, the level of satisfaction is a time–homogeneous, exponentially weighted average of past consumption:

$$Y(C)(t) = \eta e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)}\, dC(s)$$

with constants $\eta, \beta > 0$.

To ensure that $V(0)$ is finite, we have to make

**Assumption 6** $\delta + \alpha\beta > 0$.

For $\alpha \in (0,1)$ this condition is also necessary (not sufficient, see Theorem 4.7 below) to ensure that the problem is well–posed since otherwise $V \equiv +\infty$.

Furthermore, we assume that the unique state–price density $\psi$ is of the form

$$\psi(t) = \exp\left(-\theta X(t) - (r + \pi(-\theta))t\right) \quad (t \geq 0),$$

for some $(\mathbb{P}, \mathbb{F})$–Lévy process $X$ with finite Laplace–exponent $\pi(\xi)$ $(\xi \in \mathbb{R})$. Hence, interest rates are constant, $r(t) \equiv r \geq 0$, and uncertainty is introduced by a stochastic process $X$ with stationary and independent increments which possesses all exponential moments

$$\mathbb{E}\exp\left(\xi X(t)\right) < +\infty \quad (\xi \in \mathbb{R},\ t \geq 0)\,.$$

The Laplace–exponent $\pi(.)$ of $X$ is then defined via

$$\mathbb{E}\exp\left(\xi X(t)\right) = \exp(\pi(\xi)t) \quad \text{for all } \xi \in \mathbb{R},\, t \geq 0\,;$$

see, e.g., Bertoin (1996). The constant $\theta > 0$ can be viewed as the 'market price of risk'.

**Example 4.2** *1. For $X = (W(t),\ t \geq 0)$, a standard Brownian motion, we have $\pi(\xi) = \frac{1}{2}\xi^2$, and the state-price density*

$$\psi(t) = \exp\left(-\theta W(t) - \left(r + \tfrac{1}{2}\theta^2\right) t\right) \quad (t \geq 0)$$

*takes the well known form of a geometric Brownian motion. This specification of $\psi$ corresponds to the setup studied in Hindy and Huang (1993).*

*2. If $X = (\pm N(t),\ t \geq 0)$ is a Poisson process with upward (downward) jumps and intensity $\lambda$, then $\pi(\xi) = \lambda(e^{\pm\xi} - 1)$ and, therefore,*

$$\psi(t) = \exp\left(\mp\theta N(t) - \left(r + \lambda(e^{\mp\theta} - 1)\right) t\right) \quad (t \geq 0)$$

*is a geometric Poisson process.*

**Remark 4.3** *Note that the above examples describe complete financial markets if $\mathbb{F}$ is the augmented filtration generated by $X$.*

In the homogeneous setting of this section, the consumption plans $C^K$ $(K > 0)$ defined in the preceding section can be represented in the following form:

$$dC^K(t) = \frac{1}{\beta} e^{-\beta t}\, dA^K(t) \quad (t \geq 0)$$

where, for $t \geq 0$,

(20) $$A^K(0-) \triangleq \eta, \quad A^K(t) \triangleq \eta \vee \left\{ K^{-\frac{1}{1-\alpha}} S(t) \right\}$$

with

(21) $$S(t) \triangleq \sup_{0 \leq v \leq t} \left\{ \psi(v)^{-\frac{1}{1-\alpha}} e^{\left(\beta - \frac{\delta}{1-\alpha}\right) v} \right\}.$$

We have

$$Y(C^K)(t) = e^{-\beta t} A^K(t) \quad (t \geq 0).$$

## 4.3 A Solution to the Level Equation

Let us now prove that the consumption plans $C^K$ described above are indeed optimal in their respective class $\mathcal{A}(\Psi(C^K))$. Following the method proposed at the end of Section 3.2, we first show that the candidates $L^K$ $(K > 0)$ for the minimal level of satisfaction do indeed solve equation (17) for some $M = M(K) > 0$:

**Lemma 4.4** *In the homogeneous setting of Section 4.2, the process*

$$L^K(t) \triangleq i(t, K\psi(t)) = \left( K e^{\delta t} \psi(t) \right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0)$$

*solves the minimal level equation* (17) *for*

(22) $$M \triangleq \mathbb{E}\left[ \int_0^{+\infty} \beta e^{-(\delta + \alpha\beta)s} \inf_{0 \leq v \leq s} \{ e^{-(\beta(1-\alpha)-\delta)v} \psi(v) \}\, ds \right] K < +\infty.$$

PROOF : For any stopping time $S$, we have

$$\begin{aligned}
&\mathbb{E}\left[ \int_S^{+\infty} \partial_y u \left( t, e^{-\beta t} \sup_{S \leq v \leq t} \{ L^K(v) e^{\beta v} \} \right) \beta e^{-\beta(t-S)}\, dt \,\middle|\, \mathcal{F}_S \right] \\
&= \mathbb{E}\left[ \int_S^{+\infty} \beta e^{\beta S} e^{-(\delta+\alpha\beta)t} \inf_{S \leq v \leq t} \{ K e^{-(\beta(1-\alpha)-\delta)v} \psi(v) \}\, dt \,\middle|\, \mathcal{F}_S \right] \\
&= \mathbb{E}\left[ \int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)t} \inf_{0 \leq v \leq t} \left\{ K e^{-(\beta(1-\alpha)-\delta)v} \frac{\psi(S+v)}{\psi(S)} \right\} dt \,\middle|\, \mathcal{F}_S \right] \psi(S) \\
&= \mathbb{E}\left[ \int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)t} \inf_{0 \leq v \leq t} \{ K e^{-(\beta(1-\alpha)-\delta)v} \psi(v) \}\, dt \right] \psi(S)
\end{aligned}$$

where the last equation holds true because $X$ is a Lévy process. Thus, $L^K$ does indeed solve equation (17) for $M = M(K) > 0$ as defined in (22). Note that $M < +\infty$ because the infimum in its definition is always less than or equal to 1 and because $\delta + \alpha\beta > 0$ by Assumption 6. $\square$

By the same arguments as in the proof of Theorem 3.13, we now can show that the consumption plans $C^K$ satisfy the first–order conditions (ii) and (iii) with $T = +\infty$. Moreover, it is easy to see that these conditions are sufficient for optimality also in the infinite horizon case:

**Theorem 4.5** *A consumption plan $C^*$ solves the utility maximization problem* (2) *for $T = +\infty$ if the following conditions hold true for some Lagrange multiplier $M \geq 0$:*

*(i)* $\mathbb{E}\int_0^{+\infty} \psi(t)\, dC^*(t) = w$,

*(ii)* $\nabla V(C^*)(t) \leq M\psi(t)$ *for all* $t \geq 0$ $\mathbb{P}$*–a.s.,*

*(iii)* $\mathbb{E}\int_0^{+\infty} \{\nabla V(C^*)(t) - M\psi(t)\}\, dC^* = 0$, *i.e., for* $\mathbb{P}$*–a.e.* $\omega \in \Omega$, $C^*(\omega)$ *is flat off the set*

$$\{t \geq 0 \mid \nabla V(C^*)(\omega, t) = M\psi(\omega, t)\}$$

*where* $\nabla V(C)$ *is defined for* $C \in \mathcal{X}$ *by* (5) *with* $T = +\infty$.

Proof : Without loss of generality we may assume $V(C^*) < +\infty$. Then every expression in the argument for the sufficiency part of Theorem 3.2 is well–defined also when $T = +\infty$. Hence, we may use this argument to deduce that indeed $V(C^*) \geq V(C)$ for any other consumption plan $C \in \mathcal{A}(w)$. □

We finally obtain

**Corollary 4.6** *In the homogeneous setting described in Section 4.2 and under Assumption 6, the consumption plan $C^K$ is optimal given the initial capital $w = \Psi(C^K)$, provided this value is finite.*

## 4.4 Prices and Utilities

The preceding section shows that the consumption plans $C^K$ $(K > 0)$ are optimal in their respective class *provided their price is finite.* Hence, we still have to check for which parameter values of the problem this condition is satisfied. Furthermore, we have to calculate the exact prices for varying Lagrange multiplier $K > 0$ in order to find the plan whose price coincides with a given initial capital $w > 0$.

### 4.4.1 Well–Posedness of the Utility Maximization Problem

We show that, in our homogeneous framework, the optimization problem (2) is well–posed if and only if all prices of our candidate policies $C^K$ $(K > 0)$ are finite. Thus our method yields the complete solution to the utility maximization problem provided it is well–posed.

**Theorem 4.7** *We have equivalence between* *(i) Finite prices:* $\mathbb{E}\int_0^{+\infty} \psi(t)\, dC^K(t) < +\infty$ *for some (all)* $K > 0$.

*and*

*(ii) The investor's rate of time preference* $\delta$ *satisfies*

$$\delta > \hat{\delta} \triangleq \alpha r + (1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta)\,. \tag{23}$$

*For* $\alpha < 0$, *these two assertions are always satisfied, and the utility–maximization problem is always well–posed. For* $\alpha \in (0,1)$, *assertions (i) and (ii) are equivalent to*

*(iii) Finite utilities:* $V(C^K) < +\infty$ *for some (all)* $K > 0$,

*and the (joint) violation of these conditions entails that, for any initial wealth* $w > 0$, *there is a budget–feasible plan* $C$ *with infinite expected utility* $V(C) = +\infty$, *i.e., the utility maximization problem is ill–posed.*

**Remark 4.8** *Note that there is a slight gap in Theorem 4.7, since it leaves open whether or not the optimization problem is well–posed in case* $\alpha > 0$ *and* $\delta = \hat{\delta}$. *In Proposition 4.13 below, this case is treated under some additional assumption.*

The proof of Theorem 4.7 will be prepared by the following Lemmata 4.9–4.11.

**Lemma 4.9** *(i) In terms of the increasing process* $A^K$, *we may express the price of the consumption plan* $C^K$ *as*

$$\Psi^K \triangleq \mathbb{E}\int_0^T \psi(t)\, dC^K(t) = \tfrac{1}{\beta}\left(\mathbb{E}^* A^K(\tau^*) - \eta\right) \quad (K > 0)\,, \tag{24}$$

*where* $\tau^*$ *is an independent exponential random time with parameter* $r + \beta > 0$.

*(ii) We have* $\Psi^K < +\infty$ *for some* $K > 0$ *iff*

$$\mathbb{E}^* S(\tau^*) < +\infty\,, \tag{25}$$

*where* $\tau^*$ *is as in (i). In particular, the price of* every *policy* $C^K$ $(K > 0)$ *is finite if just one of these prices is finite.*

*(iii) The mapping* $K \mapsto \Psi^K$ *is nonnegative, nondecreasing, and convex. If prices are finite, we have* $\Psi^K \uparrow +\infty$ *as* $K \downarrow 0$ *and* $\Psi^K \downarrow 0$ *as* $K \uparrow +\infty$. *In particular, for every initial capital* $w > 0$ *there is a consumption policy* $C^K$ *with price* $\Psi^K = w$ *in this case.*

PROOF : From $dC^K(t) = \frac{1}{\beta}e^{-\beta t}\,dA^K(t)$ and partial integration of the price functional, we deduce that for all $K > 0$

$$\begin{aligned}\Psi^K &= \mathbb{E}^* \int_0^{+\infty} e^{-rt}\,dC^K(t) \\ &= \frac{1}{\beta}\mathbb{E}^* \lim_{T\uparrow+\infty} \left( A^K(T)e^{-(r+\beta)T} - \eta + \int_0^T A^K(t)(r+\beta)e^{-(r+\beta)t}\,dt \right) . \end{aligned} \tag{26}$$

Hence, condition

$$\mathbb{E}^* A^K(\tau^*) = \mathbb{E}^* \int_0^{+\infty} A^K(t)(r+\beta)e^{-(r+\beta)t}\,dt < +\infty \tag{27}$$

is necessary for $\Psi^K < +\infty$. It is also sufficient since it implies

$$\lim_{T\uparrow+\infty} A^K(T)e^{-(r+\beta)T} = 0 \quad \mathbb{P}^*\text{–a.s.} \tag{28}$$

Indeed, otherwise we have $\limsup_{T\uparrow+\infty} A^K(T)e^{-(r+\beta)T} > 0$ with positive $\mathbb{P}^*$–probability. Thus, on a set with positive $\mathbb{P}^*$–measure, there is a random $\varepsilon > 0$ such that

$$A^K(\sigma_n)e^{-(r+\beta)\sigma_n} \geq \varepsilon$$

along a sequence of random times $\sigma_n$ tending to $+\infty$ as $n \uparrow +\infty$. Without loss of generality we may assume that $\sigma_{n+1} - \sigma_n \geq 1$ for all $n$. Since $A^K$ is nondecreasing we have $A^K(t)e^{-(r+\beta)t} \geq \varepsilon e^{-(r+\beta)} > 0$ whenever $t \in [\sigma_n, \sigma_n + 1)$ for some $n$. This implies $\int_0^{+\infty} A^K(t)(r+\beta)e^{-(r+\beta)t}\,dt = +\infty$ with positive $\mathbb{P}^*$–probability. Hence, (27) implies (28). Furthermore the preceding considerations yield that (i) is implied by (26).

For (ii) it remains to note that $\mathbb{E}^* A^K(\tau^*) < +\infty$ is equivalent to $\mathbb{E}^* S(\tau^*) < +\infty$. This follows from

$$K^{-\frac{1}{1-\alpha}} S(\tau^*) \leq A^K(\tau^*) \leq \eta + K^{-\frac{1}{1-\alpha}} S(\tau^*) .$$

From (i) we deduce that $K \mapsto \Psi^K$ is nonnegative, nondecreasing, and convex, since so is $K \mapsto A^K$. If prices are finite, $A^{K_0}(\tau^*)$ is $\mathbb{P}^*$–integrable. Thus, $\Psi^K \downarrow 0$ for $K \uparrow +\infty$ by dominated convergence. For $K \downarrow 0$, we have $\Psi^K \geq \Delta C^K(0) \uparrow +\infty$. This yields (iii). $\square$

The following is an analogue of Lemma 4.9 for utilities instead of prices:

**Lemma 4.10** *(i) In terms of the increasing process $A^K$, we may express the expected utility of plan $C^K$ as*

$$V(C^K) = \frac{1}{\alpha(\delta+\alpha\beta)}\mathbb{E}\left(A^K(\tau)\right)^\alpha \quad (K > 0) , \tag{29}$$

*where $\tau$ is an independent exponential random time with parameter $\delta + \alpha\beta$.*

*(ii) In case $\alpha \in (0,1)$, we have $V(C^K) < +\infty$ for some $K > 0$ iff*

$$\mathbb{E}\left(S(\tau)\right)^\alpha < +\infty , \tag{30}$$

*where $\tau$ is as in (i). In particular, the expected utility of* every *policy $C^K$ is finite if just one of these utilities is finite.*

PROOF : Note first that, because of Assumption 6, we have $\delta+\alpha\beta>0$, and, therefore, $\tau$ is well–defined. Now, (i) follows from $Y(C^K)(t)=e^{-\beta t}A^K(t)$ and the definition of the utility functional $U(.)$. For (ii) we note that, in case $\alpha\in(0,1)$,

$$K^{-\frac{\alpha}{1-\alpha}}\left(S(\tau)\right)^{\alpha}\le A^K(\tau)^{\alpha}\le\eta^{\alpha}+K^{-\frac{\alpha}{1-\alpha}}\left(S(\tau)\right)^{\alpha}\,.$$

□

Lemma 4.9 and Lemma 4.10 are valid for any semimartingale state–price density which induces a constant interest rate. For the following lemma we need the special Lévy–structure of $\psi$.

**Lemma 4.11** *Let $\sigma$ be an exponential random time independent of $X$.*

*(i) We have the Wiener–Hopf factorization*

$$\mathbb{E}\exp\left(\xi\sup_{0\le s\le\sigma}X(s)\right)\mathbb{E}\exp\left(\xi\inf_{0\le s\le\sigma}X(s)\right)=\mathbb{E}\exp\left(\xi X(\sigma)\right)\tag{31}$$

*for all $\xi\in\mathbb{R}$.*

*(ii) If $X$ has no positive jumps and is neither a deterministic drift nor the negative of a subordinator, then $\sup_{0\le s\le\sigma}X(s)$ is exponentially distributed. The parameter $\zeta$ of its distribution is uniquely determined by $\pi(\zeta)=\xi$, where $\xi$ is the parameter of the exponential distribution of $\sigma$.*

*(iii) Under the risk–neutral measure $\mathbb{P}^*$ induced by $\psi$, $X$ is again a Lévy process with finite exponential moments. Its $\mathbb{P}^*$–Laplace exponent is given by*

$$\pi^*(\xi)=\pi(\xi-\theta)-\pi(-\theta)\quad(\xi\in\mathbb{R})\,.\tag{32}$$

PROOF :

(i) For $t\ge0$, let $\tilde{X}(t)\stackrel{\Delta}{=}\sup_{0\le s\le t}X(s)$. By Theorem VI.5 (i) in Bertoin (1996), the random variables $\tilde{X}(\sigma)$ and $\tilde{X}(\sigma)-X(\sigma)$ are independent. Hence,

$$\begin{aligned}\mathbb{E}\exp(\xi X(\sigma))&=\mathbb{E}\left[\exp(\xi\tilde{X}(\sigma))\exp(-\xi(\tilde{X}(\sigma)-X(\sigma)))\right]\\&=\mathbb{E}\exp(\xi\tilde{X}(\sigma))\mathbb{E}\exp(-\xi(\tilde{X}(\sigma)-X(\sigma)))\,.\end{aligned}\tag{33}$$

Using the Duality Lemma II.2 in Bertoin (1996) and the independence of $X$ and $\sigma$, we see that

$$\tilde{X}(\sigma)-X(\sigma)=\sup_{0\le s\le\sigma}\{X((\sigma-s)-)-X(\sigma)\}$$

has the same law as

$$\sup_{0\le s\le\sigma}\{-X(s)\}=-\inf_{0\le s\le\sigma}X(s)\,.$$

In connection with equation (33), this yields (i).

(ii) This is Corollary VII.1.2 in Bertoin (1996).

(iii) By definition of $\psi$, the density process $Z$ for $\mathbb{P}$ and $\mathbb{P}^*$ is given by

$$Z(t) \triangleq \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(-\theta X(t) - \pi(-\theta)t) \quad (t \geq 0)\,.$$

Hence, for $s, t \geq 0$, we may calculate the conditional $\mathbb{P}^*$–Laplace transform of the increment $X(t+s) - X(t)$ given $\mathcal{F}_t$ as follows:

$$\begin{aligned}
&\mathbb{E}^*\left[\exp(\xi(X(t+s) - X(t)))\,|\,\mathcal{F}_t\right] \\
&= \frac{1}{Z(t)}\mathbb{E}\left[\exp(\xi(X(t+s) - X(t)))Z(t+s)\,|\,\mathcal{F}_t\right] \\
&= \frac{1}{Z(t)}\mathbb{E}\left[\exp((\xi-\theta)(X(t+s) - X(t)))\,|\,\mathcal{F}_t\right]\exp(-\theta X(t) - \pi(-\theta)(t+s)) \\
&= \exp(s(\pi(\xi-\theta) - \pi(-\theta)))\,.
\end{aligned}$$

Since the last quantity is deterministic and does not depend on $t$, the above calculation shows that, also under $\mathbb{P}^*$, the process $X$ has independent and stationary increments. Furthermore, we can easily read off the transformation rule (32) for the $\mathbb{P}^*$–Laplace exponent $\pi^*(.)$.

□

Now, we are in a position to give the

**Proof of Theorem 4.7**

1. We first prove equivalence between (i) and (ii).

   By Lemma 4.9 (ii), we know that (i) is equivalent to

$$\begin{aligned}
+\infty > \mathbb{E}^* S(\tau^*) &= \mathbb{E}^* \sup_{0 \leq s \leq \tau^*} \exp\left(\frac{\theta}{1-\alpha}X(s) + \left(\frac{\pi(-\theta) + r + \beta(1-\alpha) - \delta}{1-\alpha}\right)s\right) \\
&= \mathbb{E}^* \exp\left(\frac{\theta}{1-\alpha}\sup_{0 \leq s \leq \tau^*}\{X(s) + \mu s\}\right)
\end{aligned}$$

   where $\tau^*$ is an independent exponential random time with parameter $r + \beta > 0$ and

$$\mu \triangleq (\pi(-\theta) + r + \beta(1-\alpha) - \delta)/\theta\,.$$

   In turn, the Wiener–Hopf factorization of Lemma 4.11 (i) entails equivalence of this condition and

$$\mathbb{E}^* \exp\left(\frac{\theta}{1-\alpha}\{X(\tau^*) + \mu\tau^*\}\right) < +\infty\,.$$

Since $\tau^*$ is independent of $X$ and exponentially distributed with parameter $r+\beta$, we may use Fubini's theorem to obtain equivalence of (i) and

$$r+\beta > \pi^*\left(\frac{\theta}{1-\alpha}\right) + \frac{\theta\mu}{1-\alpha}.$$

Using the transformation rule (32), it is finally easy to see that this condition is indeed equivalent to (ii).

2. We next prove $(ii) \Leftrightarrow (iii)$ for $\alpha \in (0,1)$.

   For these values of $\alpha$, we may use Lemma 4.10 (ii) and follow a similar line of arguments as in Step 1. This yields that (iii) is equivalent to

   $$\mathbb{E}\exp\left(\frac{\alpha\theta}{1-\alpha}\left\{X(\tau)+\mu\tau\right\}\right) < +\infty$$

   where $\tau$ is an independent exponential random time with parameter $\delta+\alpha\beta > 0$, and where $\mu$ is defined as above.

   Using Fubini's theorem allows us to conclude the equivalence of (ii) and

   $$\delta+\alpha\beta > \pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\mu}{1-\alpha},$$

   which, by an easy calculation, can be shown to be equivalent to (ii).

3. We now verify that (ii) holds true when $\alpha < 0$.

   Indeed, convexity of the Laplace exponent $\pi(.)$ and $\alpha < 0$ imply

   $$(1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta) \leq \pi(0) = 0\,.$$

   This yields $\hat{\delta} \leq \alpha r$. In turn, Assumption 6 in conjunction with $r+\beta > 0$ entails $\alpha r < -\alpha\beta < \delta$ for $\alpha < 0$. Thus, (ii) is satisfied.

4. To prove that problem (2) is ill–posed for $\alpha \in (0,1)$ in case $\delta < \hat{\delta}$, consider the consumption plan $\bar{C}^K$ obtained from tracking the level process

   $$\bar{L}^K(t) \triangleq \left(Ke^{\bar{\delta}t}\psi(t)\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0)$$

   where $\bar{\delta} > \hat{\delta}$ is some constant. The corresponding increasing process $\bar{A}^K$ is given by

   $$\bar{A}^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha}\sup_{0\leq v\leq t}\left\{X(v)+\bar{\mu}v\right\}\right) \quad (t \geq 0)$$

   where

   $$\bar{\mu} \triangleq (\pi(-\theta) + r + \beta(1-\alpha) - \bar{\delta})/\theta\,.$$

From $(i) \Leftrightarrow (iii)$ we know that the price of every policy $\bar{C}^K$ is finite because $\bar{\delta} > \hat{\delta}$. Hence, for any initial wealth $w > 0$, we can find $K = K(w)$ such that $\bar{C}^{K(w)}$ is budget–feasible.

By the same arguments as in the proof of Lemma 4.10, one can now show that the expected utility of the plan $\bar{C}^{K(w)}$ is finite iff

$$\mathbb{E}\exp\left(\frac{\alpha\theta}{1-\alpha}\sup_{0\le v\le\tau}\{X(v)+\bar{\mu}(v)\}\right) < +\infty$$

where $\tau$ is, as before, an independent exponential random time with parameter $\delta + \alpha\beta > 0$. From the Wiener–Hopf factorization (31), we deduce that the above relation holds true iff

$$\mathbb{E}\exp\left(\frac{\alpha\theta}{1-\alpha}\{X(\tau)+\bar{\mu}\tau\}\right) < +\infty\,.$$

Since $\tau$ is independent of $X$ and exponentially distributed, this is equivalent to

$$\delta + \alpha\beta > \pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\theta\bar{\mu}}{1-\alpha}\,. \tag{34}$$

Now, note that, for $\bar{\delta} \downarrow \hat{\delta}$, the right side of this inequality increases to

$$\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\theta\hat{\mu}}{1-\alpha} = \hat{\delta} + \alpha\beta > \delta + \alpha\beta\,, \tag{35}$$

where

$$\hat{\mu} \triangleq \lim_{\bar{\delta}\downarrow\hat{\delta}}\bar{\mu} = \left(\pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta}\right)/\theta\,.$$

The equation in (35) follows by definition of $\hat{\delta}$. Hence, there are $\bar{\delta} > \hat{\delta}$ for which inequality (34) is violated and for which, therefore, the associated plans $\bar{C}^K$ have infinite expected utility, even though their price is finite.

□

### 4.4.2 Some Explicit Computations

In order to obtain closed–form solutions for the optimization problem (2), it still remains to calculate all prices $\Psi^K = \mathbb{E}\int_0^T \psi(t)\,dC^K(t)$ $(K > 0)$ and to identify the parameter $K(w)$ for which $\Psi^{K(w)} = w$. To this end, let us introduce the Lévy process

$$Z(t) \triangleq X(t) + \mu t \quad (t \ge 0),$$

where

$$\mu \triangleq (\pi(-\theta) + r + \beta(1-\alpha) - \delta)/\theta\,. \tag{36}$$

This allows us to rewrite $A^K$ in the form

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha}\tilde{Z}(t)\right) \quad (t \geq 0)$$

where

$$\tilde{Z}(t) \stackrel{\Delta}{=} \sup_{0\leq v\leq t} Z(s) = \sup_{0\leq v\leq t} \{X(v) + \mu v\} \quad (t \geq 0)\,.$$

Now, we are able to compute the prices $\Psi^K$ and utilities $V(C^K)$ $(K > 0)$ explicitly in the following two cases:

**Assumption 7** *Either*

*(i) $Z$ is a decreasing process.*

*or*

*(ii) $Z$ is neither a decreasing nor a deterministic process, and all its jumps are non-positive ($\Delta Z \leq 0$).*

**Remark 4.12** *(i) Recall that a Lévy process is decreasing iff it is the negative of a subordinator.*

*(ii) The process $Z$ is deterministic iff the prices for consumption are deterministic. This case has already been treated in Hindy, Huang, and Kreps (1992) and Bank and Riedel (2000).*

Let $\tau$ and $\tau^*$ be exponential random times, independent of $X$ with parameter $\delta+\alpha\beta > 0$ and $r + \beta > 0$, respectively. Then, Assumption 7 ensures that $\tilde{Z}(\tau)$ and $\tilde{Z}(\tau^*)$ are exponentially distributed under $\mathbb{P}$ and $\mathbb{P}^*$ respectively. In fact, if Assumption 7 (i) holds true, we evidently have $\tilde{Z}(t) \equiv 0$ which corresponds to the parameter values $\zeta = \zeta^* = 0$ for the respective exponential distributions. Under Assumption 7 (ii), we may apply Lemma 4.11 (ii) with $Z$ instead of $X$ to identify the exponential parameters as the unique positive solutions to

$$\pi(\zeta) + \mu\zeta = \delta + \alpha\beta \quad \text{and} \quad \pi^*(\zeta^*) + \mu\zeta^* = r + \beta \tag{37}$$

respectively.

Thus, proceeding from Equations (24) and (29), we now can compute

$$\Psi(C^K) = \frac{1}{\beta} \cdot \begin{cases} \left(K^{-\frac{1}{1-\alpha}} - \eta\right)^+ & \text{if } \zeta^* = 0 \\ \dfrac{(1-\alpha)\zeta^*}{(1-\alpha)\zeta^* - \theta} K^{-\frac{1}{1-\alpha}} - \eta & \text{if } \eta \leq K^{-\frac{1}{1-\alpha}},\ \zeta^* > 0 \\ \dfrac{\theta}{(1-\alpha)\zeta^* - \theta} \eta^{-\frac{(1-\alpha)\zeta^* - \theta}{\theta}} K^{-\frac{\zeta^*}{\theta}} & \text{else} \end{cases} \tag{38}$$

and

$$V(C^K) = \frac{1}{\alpha(\delta+\alpha\beta)} \cdot \begin{cases} \eta^\alpha \vee K^{-\frac{\alpha}{1-\alpha}} & \text{if } \zeta = 0 \\ \dfrac{(1-\alpha)\zeta}{(1-\alpha)\zeta - \alpha\theta} K^{-\frac{\alpha}{1-\alpha}} & \text{if } \eta \leq K^{-\frac{1}{1-\alpha}},\ \zeta > 0 \\ \eta^\alpha + \dfrac{\alpha\theta}{(1-\alpha)\zeta - \alpha\theta} \eta^{-\frac{(1-\alpha)\zeta - \alpha\theta}{\theta}} K^{-\frac{\zeta}{\theta}} & \text{else.} \end{cases}$$

Hence, an agent with initial wealth $w > 0$ optimally follows the consumption plan $C^{K(w)}$ with

$$K(w) \triangleq \begin{cases} (\beta w + \eta)^{-(1-\alpha)} & \text{if } \zeta^* = 0 \\ \left( \dfrac{(1-\alpha)\zeta^* - \theta}{(1-\alpha)\zeta^*} (\beta w + \eta) \right)^{-(1-\alpha)} & \text{if } w \geq \hat{w},\ \zeta^* > 0 \\ \left( \dfrac{(1-\alpha)\zeta^* - \theta}{\theta} \eta^{\frac{(1-\alpha)\zeta^* - \theta}{\theta}} \beta w \right)^{-\frac{\theta}{\zeta^*}} & \text{else} \end{cases}$$

where $\hat{w} \triangleq \frac{1}{\beta} \frac{\theta}{(1-\alpha)\zeta^* - \theta} \eta$.

Furthermore, using Lemma 4.11 (iii), one can show that $\zeta^* = \zeta + \theta$ by a straightforward calculation. This allows us to represent the agent's maximal utility (the value $v(w)$ of the program (2)) by

$$v(w) = \frac{1}{\alpha(\delta+\alpha\beta)} \cdot \begin{cases} (\beta w + \eta)^\alpha & \text{if } \zeta^* = 0 \\ \zeta \left( \dfrac{1-\alpha}{(1-\alpha)\zeta - \alpha\theta} \right)^{1-\alpha} \left( \dfrac{\beta w + \eta}{\zeta + \theta} \right)^\alpha & \text{if } w \geq \hat{w},\ \zeta^* > 0 \\ \eta^\alpha + \alpha \eta^{-\frac{(1-\alpha)\zeta - \alpha\theta}{\zeta+\theta}} \left( \dfrac{\theta\beta w}{(1-\alpha)\zeta - \alpha\theta} \right)^{\frac{\zeta}{\zeta+\theta}} & \text{else.} \end{cases}$$

The above formulae give us the desired explicit solution to the investor's utility maximization problem (2) in the homogeneous setting of Section 4.2.

As pointed out in Remark 4.8, Theorem 4.7 does not characterize completely the parameter values for which problem (2) is well–posed in the present context. Under Assumption 7, this problem can be solved:

**Proposition 4.13** *Under Assumption 7, the parameter restriction $\delta > \hat{\delta}$ of Theorem 4.7 (iii) is also necessary for problem (2) to be well–posed if $\alpha \in (0,1)$. More precisely, suppose that Assumption 7 is satisfied and that the parameters of the problem are such that*

$$\delta \leq \hat{\delta} = \alpha r + (1-\alpha)\pi \left( \frac{\alpha\theta}{1-\alpha} \right) + \alpha\pi(-\theta), \quad \alpha \in (0,1). \tag{39}$$

*Then we have*

$$\sup_{C \in \mathcal{A}(w)} V(C) = +\infty$$

*for any initial capital $w > 0$.*

PROOF : As in the proof of Theorem 4.7, choose some $\bar{\delta} > \hat{\delta}$ and consider, for every $K > 0$, the lower bound $\bar{L}^K$ obtained from $L^K$ by replacing $\delta$ with $\bar{\delta}$. Again, the corresponding consumption plans will be denoted by $\bar{C}^K$, and we will write $\bar{S}$ for the analogue of the supremum process $S$. For simplicity, we assume that $\eta = 0$.

We have

$$\Psi(\bar{C}^K) = \frac{K^{-\frac{1}{1-\alpha}}}{\beta} \mathbb{E}^* \bar{S}(\tau^*)$$

and, since $\alpha \in (0,1)$,

$$V(\bar{C}^K) \geq \mathbb{E} \int_0^\infty e^{-\hat{\delta} t} \frac{1}{\alpha} \left( e^{-\beta t} K^{-\frac{1}{1-\alpha}} \bar{S}(t) \right)^\alpha \, dt = \frac{K^{-\frac{1}{1-\alpha}}}{\alpha(\delta + \alpha\beta)} \mathbb{E} \bar{S}(\tau)^\alpha$$

where $\tau^*$ and $\tau$ are independent exponential random times with parameters $r + \beta > 0$ and $\hat{\delta} + \alpha\beta > 0$ respectively.

In order to meet the budget–constraint, we choose $K$ such that $\Psi(\bar{C}^K) = w$. Note that this is indeed possible because $\bar{\delta} > \hat{\delta}$. By the above calculations, this gives us

$$v(w) \triangleq \sup_{C \in \mathcal{A}(w)} V(C) \geq \frac{(\beta w)^\alpha}{\alpha(\delta + \alpha\beta)} \frac{\mathbb{E} \bar{S}(\tau)^\alpha}{(\mathbb{E}^* \bar{S}(\tau^*))^\alpha}.$$

Hence, to show that $v(w) \equiv +\infty$, it suffices to prove

$$\frac{\mathbb{E} \bar{S}(\tau)^\alpha}{(\mathbb{E}^* \bar{S}(\tau^*))^\alpha} \to +\infty \quad \text{as} \quad \bar{\delta} \downarrow \hat{\delta}\,. \tag{40}$$

Using Lemma 4.11 (ii), it is easy to see that

$$\mathbb{E} \bar{S}(\tau)^\alpha = \frac{\bar{\zeta}}{\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}} \quad \text{and} \quad \mathbb{E}^* \bar{S}(\tau^*) = \frac{\bar{\zeta}^*}{\frac{\theta}{1-\alpha} - \bar{\zeta}^*}$$

where $\bar{\zeta}$ and $\bar{\zeta}^*$ are determined by

$$\pi(\bar{\zeta}) + \bar{\mu}\bar{\zeta} = \hat{\delta} + \alpha\beta \quad \text{and} \quad \pi^*(\bar{\zeta}^*) + \bar{\mu}\bar{\zeta}^* = r + \beta \tag{41}$$

with $\bar{\mu} \triangleq (\pi(-\theta) + r + \beta(1-\alpha) - \bar{\delta})/\theta$. A straightforward calculation based on Lemma 4.11 (iii) shows that $\bar{\zeta}^* = \bar{\zeta} + \theta$.

This allows us to conclude that

$$\frac{\mathbb{E} \bar{S}(\tau)^\alpha}{(\mathbb{E}^* \bar{S}(\tau^*))^\alpha} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha} \frac{(\frac{\theta}{1-\alpha} - \bar{\zeta}^*)^\alpha}{\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha} \left( \frac{\alpha\theta}{1-\alpha} - \bar{\zeta} \right)^{-(1-\alpha)}. \tag{42}$$

Using the definition of $\hat{\delta}$, one can show that $\hat{\zeta} \triangleq \frac{\alpha\theta}{1-\alpha}$ is the unique solution to

$$\pi(\hat{\zeta}) + \hat{\mu}\hat{\zeta} = \hat{\delta} + \alpha\beta \tag{43}$$

with $\hat{\mu} \triangleq \frac{1}{\theta}(\pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta})$. Since, by definition, $\bar{\zeta}$ depends continuously on $\bar{\delta}$ and because equation (43) is the limit of equation (41) for $\bar{\delta} \downarrow \hat{\delta}$, this shows that $\bar{\zeta} \to \hat{\zeta} = \frac{\alpha\theta}{1-\alpha}$ as $\bar{\delta} \downarrow \hat{\delta}$. Now, the claimed convergence (40) can be read off equation (42).

□

Theorem 4.7 and Proposition 4.13 show that our method provides the complete solution to the utility maximization problem (2) in our homogeneous setting.

## 4.5 Case Studies

This section illustrates the preceding results by two case studies where $X$ is either a Brownian motion or a Poisson process. For the special case of Brownian motion, our results allow to recover and extend the results by Hindy and Huang (1993). In particular, we will recover the singularity of optimal consumption plans in this case. By contrast, in the Poisson case, optimal consumption may occur in gulps and at rates.

### 4.5.1 Geometric Brownian Motion

Let $X = (W(t),\ t \geq 0)$ be a Brownian motion. In this case, our optimization problem (2) is well–posed for $\alpha \in (0,1)$ if and only if

$$\delta > \hat{\delta} = \alpha r + \frac{1}{2}\frac{\alpha\theta^2}{1-\alpha}\,. \tag{44}$$

Note that this is exactly the regularity assumption needed in the context of the classical Merton portfolio problem; compare, e.g., Karatzas and Shreve (1998), Remark 3.9.23, Merton (1990), Section 4.6, or Korn (1997), Corollary 3.3.7.

Recall that the result in Hindy and Huang (1993) is obtained by use of the Bellman methodology under the additional parameter restriction

$$\delta < r + \beta(1-\alpha)\,; \tag{45}$$

confer their equation (41). Our approach shows that this assumption can be dispensed with. We only need the natural condition (44); compare Theorem 4.7 and Proposition 4.13.

Let us now focus on the economic interpretation of the results in the Brownian case. Recall that the agent consumes whenever the process

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha}\sup_{0\leq s\leq t}\{X(s)+\mu s\}\right)$$

increases. Since $X$ is Brownian motion, $\mu$ is given by

$$\mu = \left(\tfrac{1}{2}\theta^2 + r + \beta(1-\alpha) - \delta\right)/\theta\,.$$

From this we can immediately infer the following fundamental difference between the classic time–additive models and the Hindy–Huang–Kreps (HHK) approach: there is no open time interval during which the HHK–agent consumes all the time. Consumption occurs in a singular way, similar to the behavior of Brownian local time. This has already been pointed out by Hindy and Huang (1993). In their case, i.e., when (45) holds true, the process $A^L$ diverges to infinity. Hence, the agent never refrains from consumption completely. In fact, our analysis shows that this is the case iff $\mu \geq 0$, i.e., iff

$$\delta \leq r + \beta(1-\alpha) + \tfrac{1}{2}\theta^2 .$$

It is interesting to see what happens if this inequality does not hold true. In this case, the overall supremum of the Brownian motion with drift $(W(t) + \mu t,\ t \geq 0)$ is finite almost surely. Thus, there is an almost surely finite last time of consumption. However, since this is not a stopping time the agent will not consume all his wealth at that time because he does not know for sure that there will not be another opportunity for consumption! To illustrate this point further, let us calculate the optimal portfolio for an agent in a standard Samuelson–type model of the asset market.

**Portfolios** Consider a complete financial market with one risky asset whose price evolves according to

$$P(0) > 0, \quad dP(t) = P(t)\,(\sigma\, dW(t) + (r + \theta\sigma)\, dt) \quad (t \geq 0)$$

for some $\sigma > 0$. The agent uses the asset and the bond to finance his consumption plan $C^K$. Under $\mathbb{P}^*$,

$$W^*(t) \stackrel{\Delta}{=} W(t) + \theta t \quad (t \geq 0)$$

becomes a Brownian motion and the discounted asset price $\bar{P} = (e^{-rt}P(t),\ t \geq 0)$ is — as usual — a $\mathbb{P}^*$–martingale with

$$d\bar{P}(t) = \sigma \bar{P}(t)\, dW^*(t) \quad (t \geq 0)\,.$$

Denote by

$$V^K(t) \stackrel{\Delta}{=} E^*\left[\int_t^{+\infty} e^{-r(s-t)}\, dC^K(s) \middle| \mathcal{F}_t\right]$$

the present value of the remaining consumption at time $t \geq 0$. The portfolio strategy $\pi^K$ we are looking for has to satisfy

$$dV^K(t) = \pi^K(t)\, d\bar{P}(t) - e^{-rt}\, dC^K(t) \quad (t \geq 0)\,.$$

The following result has already been proved by Hindy and Huang (1993) using different methods based on their dynamic programming approach.

**Theorem 4.14** *The agent puts a constant fraction of his wealth in the risky asset:*

$$\frac{\pi^K(t)\bar{P}(t)}{V^K(t)} \equiv \frac{\zeta^*}{\sigma}\,,$$

*where* $\zeta^*$ *is as in* (37).

**Remark 4.15** *This similarity to the original Merton portfolio problem has already been observed by Hindy and Huang (1993).*

PROOF : We are interested in the representation of the martingale part of $V^K$ as a stochastic integral with respect to $W^*$. We will therefore compute $V^K$ explicitly.

We have $V^K(0-) = \Psi^K$, which has been computed in (38). For $t > 0$ we proceed along the same lines as in the proof of Theorem 4.7 and the calculation of (38) to obtain:

$$V^L(t) = \frac{e^{-\beta t}}{\beta}\left(\mathbb{E}^*\left[\int_t^{+\infty}(r+\beta)e^{-(r+\beta)(s-t)}A^K(s)\,ds\middle|\mathcal{F}_t\right] - A^K(t)\right).$$

The above expectation can be rewritten as

$$\mathbb{E}^*\left[A^L(t)\vee K^{-\frac{1}{1-\alpha}}e^{\frac{\theta}{1-\alpha}\left(W^*(t)+\mu^*t+\sup_{0\le s\le\tau^*}\{W^*(t+s)-W^*(t)+\mu^*s\}\right)}\middle|\mathcal{F}_t\right]$$

where $\tau^*$ is an independent exponential random variable with parameter $r+\beta$ and

$$\mu^* \triangleq \frac{r+\beta(1-\alpha)-\delta}{\theta} - \frac{1}{2}\theta\,.$$

The Markov property of Brownian motion and Lemma 4.11 (ii) allow us to conclude that this is equal to

$$A^K(t) + \frac{K^{-\zeta^*/\theta}}{\nu}e^{\zeta^*\mu^*t}A^K(t)^{-\nu}e^{\zeta^*W^*(t)}$$

where $\zeta^*$ is determined by (37) and $\nu \triangleq ((1-\alpha)\zeta^*-\theta)/\theta$, a strictly positive constant because of condition (44). The present value of the consumption policy $C^K$ is therefore given by

$$V^K(t) = \frac{K^{-\zeta^*/\theta}}{\beta\nu}e^{(\zeta^*\mu^*-\beta)t}A^K(t)^{-\nu}e^{\zeta^*W^*(t)}. \tag{46}$$

Hence,

$$dV^K(t) = V^K(t)\zeta^*\,dW^*(t) + \text{terms of bounded variation}$$

and we conclude that at each time $t \ge 0$ the investor must hold

$$\pi^K(t) \triangleq \frac{\zeta^*}{\sigma}\frac{V^K(t)}{\bar{P}(t)}$$

shares of the risky asset in his portfolio in order to finance the consumption policy $C^K$.

□

**Remark 4.16** *If $\sigma = \zeta^*$, the agent invests all his wealth in the risky asset. This case can be viewed as a single–agent equilibrium of the stock market for this type of investors.*

Consider again the case when there is an almost surely finite, yet imperceptible last time of consumption. This occurs, as we pointed out above, iff

$$\delta > r + \beta(1-\alpha) + \tfrac{1}{2}\theta^2.$$

In this case, the investor's level of satisfaction eventually decreases at rate $\beta$, inducing an ever increasing appetite. His wealth, however, decreases at the higher rate $|\zeta^*\mu^*| + \beta$, as can be read off (46). Thus, the investor's relative level of satisfaction — the fraction of his level of satisfaction and his wealth — remains large. This in turn drives him to wait for better times to come. He keeps being engaged in the risky asset although he knows that with positive probability he may never consume again. This illustrates that, as already noted by Hindy and Huang (1993), an HHK–investor is less risk averse than his time-additive counterpart, because he obtains utility from past consumption. He can afford to invest in the risky asset and to refrain from consumption for a while in order to speculate on a higher future level of satisfaction.

### 4.5.2 Geometric Poisson Process

Let us now study Poisson price processes, i.e., we let $X = (\pm N(t),\ t \geq 0)$. A jump of the process $N$ corresponds to an unpredictable 'price shock' or, in the terminology of Hindy and Huang (1993), an 'information surprise'. We distinguish the two cases where the shocks are upward (price increase) or downward (price decrease).

**Upward Price Shocks** First we consider the case of upward price shocks, i.e., $X = (-N(t),\ t \geq 0)$, a Poisson process with downward jumps and intensity $\lambda > 0$ under the objective probability $\mathbb{P}$.

For this choice of $X$, the optimization problem (2) is well–posed iff

$$\delta > \hat{\delta} = \alpha r + \lambda\left((1-\alpha)e^{-\frac{\alpha\theta}{1-\alpha}} + \alpha e^{\theta} - 1\right).$$

As in the Brownian case,

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0\leq v\leq t}\{X(v) + \mu v\}\right),$$

but here we have

$$\mu \stackrel{\Delta}{=} \left(\lambda(e^{\theta}-1) + r + \beta(1-\alpha) - \delta\right)/\theta\,.$$

In contrast to the Brownian case, it now may happen that $Z = (X(t) + \mu t,\ t \geq 0)$ is a decreasing process. Indeed, this is the case iff $\mu \leq 0$, i.e., iff

$$\delta \geq \lambda(e^{\theta}-1) + r + \beta(1-\alpha)\,.$$

Hence, a very impatient agent (characterized by a high rate of time preference $\delta$), optimally consumes his whole wealth by one single gulp at time $t = 0$. If the agent is not that impatient, then, apart from a possible initial gulp, he only consumes at rates

$$dC^K(t) = \frac{1}{\beta}e^{-\beta t}dA^K(t) = \frac{r + \beta(1-\alpha) - \delta + \lambda\theta}{\beta(1-\alpha)}e^{-\beta t}A^K(t)1_{\{\dot{A}^K(t)\neq 0\}}\,dt \quad (t > 0)$$

until an upward price shock makes him refrain from consumption. After a while, when his wealth and appetite have become large enough again, he restarts consumption until the next shock, etc.

**Downward Price Shocks** In the second Poisson example, there are downward price shocks, i.e., $X = (N(t),\ t \geq 0)$ with $N$ as before.

As before,

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0 \leq v \leq t} \{X(v) + \mu v\}\right)$$

where, in this case,

$$\mu \triangleq \left(\lambda(e^{-\theta} - 1) + r + \beta(1-\alpha) - \delta\right)/\theta\,.$$

Observe that now $X$ has positive jumps and, therefore, neither Assumption 7 (i) nor Assumption 7 (ii) holds true. Hence, the closed–form expressions for the prices of optimal consumption plans and their utilities as derived at the end of Section 4.4 are no longer valid here.

However, we still have that the utility maximization problem (2) is well–posed if

$$\delta > \hat{\delta} = \alpha r + \lambda\left((1-\alpha)e^{\frac{\alpha\theta}{1-\alpha}} + \alpha e^{-\theta} - 1\right)\,. \tag{47}$$

**Remark 4.17** *We conjecture, but cannot yet prove that condition* (47) *is also necessary for problem* (2) *to be well–posed in the case considered here. We know by Theorem 4.7 that the problem is ill–posed if $\delta < \hat{\delta}$. Thus, the only open case is $\delta = \hat{\delta}$.*

Depending on the parameter values, two types of (optimal) consumption behavior can emerge in the presence of downward price shocks:

- If we have $\mu \geq 0$, then, once the investor has started consumption, he consumes continually at rates

  $$\dot{C}^K(t) = \frac{\lambda(e^{-\theta} - 1) + r + \beta(1-\alpha)}{\beta(1-\alpha)} e^{-\beta t} A^K(t)$$

  and takes a gulp

  $$\Delta C^K(t) = \frac{e^{\frac{\theta}{1-\alpha}}}{\beta} e^{-\beta t} A^K(t-) \Delta N(t)$$

  whenever a price shock occurs. This is due to the fact that prices decline very fast and the relative wealth of the consumer increases.

- If the world is not such a comfortable one, i.e., if $\mu < 0$, then the agent consumes only in gulps, namely every time a 'favorable' price shock causes $A^K$ to reach a new maximum.

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Peter Bank
Institut für Mathematik
Humboldt–Universität zu Berlin
Unter den Linden 6
D–10099 Berlin, Germany
E–mail: pbank@mathematik.hu-berlin.de

Frank Riedel
Institut für Wirtschaftstheorie I
Humboldt–Universität zu Berlin
Spandauer Straße 1
D–10178 Berlin, Germany
E–mail: riedel@wiwi.hu-berlin.de