No Free Lunch for Large Investors

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Abstract

We study an extension of the classical Black-Scholes model which accounts for feedback effects from trading in an imperfectly elastic market. The proposed semimartingale model may be viewed as a compromise between the diffusion approach in, e.g., (Cuoco and Cvitanic 1998), (Cvitanic and Ma 1996) and the reaction function framework used in, e.g., (Jarrow 1992), (Frey and Stremme 1997). We motivate our model by a discrete-time approximation and provide sufficient conditions which exclude arbitrage opportunities for large investors.

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1 Introduction

The classical Black-Scholes model takes a very stylized view of the dynamics of security prices and neglects many important features of real financial markets. Recent research has focused on extensions of this model which take into account, e.g., nonconstant interest rates, transaction costs, and stochastic volatility. Another important feature of real security markets is imperfect market elasticity due to bounded market liquidity. This is the aspect we would like to focus on in this note.

In a financial market model with perfect elasticity, orders of arbitrary size do not affect the price of the traded asset. This assumption is justified only as long as one considers 'small' investors whose trading volume is easily covered by market liquidity. If, however, one assumes that there are 'large' investors in the market such as a hedge funds, market prices will no longer evolve independently of the trading strategies chosen by the 'big players'.

There has been a number of suggestions how to deal with this feedback effect. (Jarrow 1992) and (Jarrow 1994) propose a discrete-time framework where prices depend on the large trader's activities via a reaction function of his instantaneous holdings. (Frey and Stremme 1997) develops a continuous time analogue of this framework; this is slightly generalized by (Bierbaum 1997). In contrast to these contributions, (Cuoco and Cvitanic 1998) and (Cvitanic and Ma 1996) study a diffusion model for the price dynamics where the drift and volatility coefficients depend on the large investor's trading strategy.

In this note, our key idea is to add the large investor's trading strategy as a driving semimartingale to the price dynamics used in the diffusion approach. On the one hand, this allows for a comparably simple dynamics (c.f. (4)). On the other hand, the additional factor models the price reaction in a more direct way and ensures an immediate price reaction to orders by the 'large' investor. Our model is described in Section 2, where we also discuss some of its qualitative features. Section 3 discusses the microeconomic justification for our approach in terms of temporary equilibria in a discrete-time approximation.

Section 4 contains the main result of this note. We formulate sufficient conditions on a given strategy which exclude an arbitrage opportunity for the large investor. By definition, such an arbitrage opportunity would allow the large investor to produce a sure return without bearing any risk. Conditions precluding arbitrage possibilities for *small* investors are by now very well understood, c.f. (Delbaen and Schachermayer 1994). In the diffusion models of (Cuoco and Cvitanic 1998), (Cvitanic and Ma 1996) the fundamental characterization of absence of arbitrage in (Delbaen and Schachermayer 1994) can still be used because the large trader affects prices only indirectly via the drift and volatility terms. In our framework and in the reaction function approach, the feedback effect is more pronounced. As a consequence, a direct application of the usual arguments based on the existence of an equivalent local martingale measure is no longer possible. In the reaction function approach of (Jarrow 1992) and (Bierbaum 1997), the proof of absence of arbitrage for large investors relies heavily on the assumption that prices depend on the large investor's strategy only via his *instantaneous* market position. Since this is not the case in our setting, we use a different method which exploits the separable structure of our model. Our main idea is to consider an auxiliary market where the large investor is the only driving force. We show that any tame strategy which induces a loss in this auxiliary market cannot be an arbitrage opportunity for the large investor in the 'real' market where prices are also driven by exogenous stochastic factors. In the special context of Section 4.3, we are able to show that any tame strategy without short sales does induce a technical loss in the above sense. Hence, in this special case, the prohibition of short sales excludes any arbitrage opportunity for the large investor.

2 A Semimartingale Model with Feedback Effects

We consider a frictionless financial market containing a risky stock and a riskless bond paying interest at some rate r. There is one large investor in the market whose trades may affect the price process of the risky asset, but *not* the riskless interest rate. Hence, it suffices for our purposes to consider the case $r \equiv 0$.

In the usual Black-Scholes type models stock prices are assumed to follow a geometric Brownian motion with possibly time varying drift and volatility coefficients. (Cuoco and Cvitanic 1998) and (Cvitanic and Ma 1996) extend this setup by allowing these coefficients to depend on the large investor's trading strategy. However, this kind of feedback effect is rather indirect because only the model coefficients react immediately to the strategy, but not the price itself. In order to model the feedback effect in a more direct way, we introduce the large trader's strategy as an additional driving semimartingale in the stock price dynamics of (Cuoco and Cvitanic 1998), (Cvitanic and Ma 1996).

Let us fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t;\ 0 \leq t \leq T)$ satisfying the usual conditions of right-continuity and completeness. The stock price process will be driven by two factors, both given by Doléans-Dade exponentials. We recall the definition of these exponentials in

Definition 2.1 *Let $X = (X_t)$ be a $(\mathbb{P}, \mathbb{F})$-semimartingale. The Doléans-Dade exponential of X is the unique solution to the linear SDE*

$$Z(0-) = 1, \quad dZ(t) = Z(t-)\, dX(t) \quad (t \geq 0).$$

Explicitly, it is given by

$$\mathcal{E}(X)_t \triangleq \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \prod_{0 \leq s \leq t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right) \quad (t \geq 0),$$

where $\Delta X \triangleq X - X_-$ denotes the jump process associated to X.

Remark 2.2 *Note that the above definition allows for a jump of X at time $t = 0$. This slight modification of the usual definition as in, e.g., (Protter 1990), will prove useful later, compare Remark 2.3.*

The first driving factor in our setting is given by an extended version of the model in (Cuoco and Cvitanic 1998), (Cvitanic and Ma 1996). We fix a continuous local $(\mathbb{P},\mathbb{F})$-martingale M, some bounded, $\mathbb{F}$-predictable process λ, and two D-adapted functionals $\sigma,\mu : D[0,T] \to D[0,T]$[1]. This allows us to consider, for any given $\mathbb{F}$-semimartingale θ, the Doléans-Dade exponential $\mathcal{E}(S^\theta)$ of the stochastic integral

$$S^\theta \triangleq \int_0^\cdot (\sigma_-^\theta \, dM + \mu_-^\theta \lambda \, d[M,M]) = \sigma_-^\theta \cdot M + \mu_-^\theta \lambda \cdot [M,M]. \tag{1}$$

As in (Cuoco and Cvitanic 1998), (Cvitanic and Ma 1996) we think of θ as the large investor's strategy. The strategy θ induces a volatility process $\sigma^\theta = \sigma(\theta)$ and a drift process $\mu^\theta = \mu(\theta)$. Both processes have right-continuous paths with left limits, and we denote by σ_-^θ and μ_-^θ the corresponding left-continuous versions.

Now we introduce a second driving factor which ensures an immediate price reaction to the transactions made by the large investor. This immediate price change will be determined by the size of the large investor's instantaneous transaction. This suggests to restrict the large investor to use only strategies θ which are semimartingales with respect to $(\mathbb{P},\mathbb{F})$. Then θ can be used as a stochastic integrator and, in this sense, the instantaneous transaction $d\theta$ has a clear mathematical interpretation. Furthermore, we can scale $d\theta$ by another D-adapted nonnegative functional $\eta : D[0,T] \to D[0,T]^+$. This allows us to introduce the feedback process

$$F^\theta \triangleq \eta_-^\theta \cdot \theta \tag{2}$$

defined as the stochastic integral of η_-^θ with respect to θ. The process $\eta^\theta \geq 0$ may be interpreted as a description of the market elasticity: the larger η_-^θ the less elastic is the market and the stronger is the feedback effect from the large investor's strategy.

Now we are in a position to define the stock price evolution. For a large investor strategy given by a semimartingale θ, we define the price process as

$$P^\theta \triangleq p_0 \mathcal{E}(S^\theta)\mathcal{E}(F^\theta), \tag{3}$$

i.e., as the solution of the stochastic differential equation

$$P_{0-}^\theta = p_0, \quad dP^\theta = P_-^\theta \left(dS^\theta + dF^\theta + d[S^\theta, F^\theta]\right) = P_-^\theta \left(\sigma_-^\theta dM + \mu_-^\theta \lambda \, d[M,M] + \eta_-^\theta d\theta + \sigma_-^\theta \eta_-^\theta d[M,\theta]\right), \tag{4}$$

where $p_0 > 0$ denotes the stock price before trading begins. The bracket term $[.,.]$ denotes the quadratic covariation between the respective semimartingales; see, e.g., (Protter 1990).

To ensure that the price process P^θ remains positive we must assume that any permissible large investor strategy θ satisfies

$$\eta_-^\theta \Delta\theta > -1 \,. \tag{5}$$

[1] $D[0,T]$ denotes the Skorohod space of all right-continuous functions with left limits $[0,T] \to \mathbb{R}$. We call a functional $F : D[0,T] \to D[0,T]$ D-adapted if we have $F(\vartheta)_t = F(\tilde{\vartheta})_t$ whenever $\vartheta, \tilde{\vartheta} \in D[0,T]$ coincide on $[0,t]$.

The natural class of large investor strategies is therefore given by the set

$$\Theta \triangleq \{\theta \,|\, \theta \text{ is an } \mathbb{F}\text{-semimartingale}, \ \eta^\theta_- \Delta\theta > -1\}.$$

Let us also introduce the set

$$\Theta_0 \triangleq \{\theta \in \Theta \,|\, \theta_T = 0\}$$

of all strategies which end with a zero position in the risky security.

The large investor starts with an initial zero position in both the bank account and the stock. Given a strategy $\theta \in \Theta$ the evolution of the large investor's bank account is described by

$$\beta^\theta_{0-} = 0, \quad d\beta^\theta = \text{interest } - \text{investments} = -(P^\theta_- \, d\theta + d[\theta, P^\theta]). \tag{6}$$

Note that the bracket term ensures that the large investor's orders are served only *after* the market has reacted on them. Indeed, suppose there is a jump in the large investor's strategy at time t, say $\Delta\theta_t > 0$. Then his cash position will jump, too, namely by

$$\Delta\beta^\theta_t = -P^\theta_{t-}\Delta\theta_t - P^\theta_{t-}\eta^\theta_{t-}\Delta\theta_t^2 = -P^\theta_{t-}(1 + \eta^\theta_{t-}\Delta\theta_t)\Delta\theta_t = -P^\theta_t \Delta\theta_t,$$

i.e., the large investor's order causes the price to rise up to $P^\theta_t = P^\theta_{t-}(1 + \eta^\theta_{t-}\Delta\theta_t)$, and it is this price he has to pay for the ordered $\Delta\theta_t$ shares.

Remark 2.3 *In order to allow for trading at time $t = 0$ we define $\theta_{0-} = 0$ for any strategy $\theta \in \Theta$. Thus, a value $\theta_0 > 0$ (< 0) corresponds to an initial purchase (sale) of $\Delta\theta_0 = \theta_0$ shares by the large investor.*

Having fixed our model, let us discuss some of its qualitative features. First of all we note that an increasing demand adds the positive term $\eta^\theta_- \, d\theta$ to the stochastic price increment in (4), while a decreasing demand puts its price under pressure. Moreover, the higher the traded volume the more pronounced is the immediate price reaction. In particular, our model allows to make the important distinction between the large investor's paper wealth and his real wealth. The paper wealth at a given time t is computed in terms of the market prices at that time, while the computation of the real wealth takes into consideration the feedback effect resulting from portfolio liquidation, c.f. Remark 4.5. This distinction is the main economic difference between our model and the diffusion models of (Cuoco and Cvitanic 1998) and (Cvitanic and Ma 1996). In their setting, these two descriptions for the investor's wealth coincide because only the drift and volatility react immediately to the liquidation orders, not the price itself. In the reaction function approach by (Jarrow 1994) or (Frey and Stremme 1997), the difference between these two notions does appear. However, the resulting price dynamics seems to be rather complicated as compared to the SDE description in a diffusion setup or to our model (4). In this sense one may view our model as a compromise between the somewhat static approach via reaction functions and the diffusion description.

3 A Microeconomic Motivation

This section provides a possible microeconomic justification for our model (3) by motivating its discrete-time analogue

$$P_t^\theta = p_0 \prod_{s=0}^{t} (1 + \Delta S_s^\theta)(1 + \Delta F_s^\theta) \quad (t = 0, 1, \ldots, T). \tag{7}$$

Assume that, given a proposed equilibrium price p, the small investors' aggregate excess demand is defined by

$$e_t^\theta(p) \triangleq \frac{1}{\eta_{t-1}^\theta} \left(1 - \frac{p}{\hat{P}_t^\theta}\right).$$

Here η_{t-1}^θ is the predictable market elasticity for time t resulting from the large investor's past positions. The small investors use a reference level $\hat{P}_t^\theta$ for prices proposed at time t. Specifically, we assume that this reference level takes the form

$$\hat{P}_t^\theta = P_{t-1}^\theta \left(1 + \Delta S_t^\theta\right),$$

where ΔS_t^θ is an aggregated small traders' estimate for the next asset return which may depend on an exogenous noisy signal ΔM_t and on the large investor's past activities.

Under these assumptions the market clearing condition

$$e_t^\theta(p) + \Delta\theta_t = 0$$

implies a price dynamics of the form (7) where

$$\Delta F_t^\theta \triangleq \eta_{t-1}^\theta \Delta\theta_t.$$

Remark 3.1 *Obviously, $e_t^\theta(p) > 0$ iff $\hat{P}_t^\theta > p$, i.e., small investors buy iff their reference level lies above the proposed equilibrium price p. Note furthermore that the absolute value of the small investors' excess demand decreases in η_{t-1}^θ. This means that a lower market elasticity (higher value of η) corresponds to a less liquid market.*

The methods in, e.g., (Duffie and Protter 1992) may now be used to approximate the continuous time version (3). To illustrate this point let us consider a sequence of discrete-time large investor strategies $\theta^{(n)}$, elasticity functionals $\eta^{(n),\theta^{(n)}}$, and cumulative return estimates $S^{(n),\theta^{(n)}}$ $(n = 1, 2, \ldots)$. Identify each of these discrete-time processes with its continuous-time counterpart, i.e., identify any given discrete-time process $Z_k^{(n)}$ $(k = 0, 1, \ldots, n)$ with the associated continuous-time process $Z_t^{(n)} \triangleq Z_{[nt/T]}^{(n)}$ $(0 \leq t \leq T)$. Assume that the resulting multidimensional sequence of semimartingales is 'good' in the sense of Definition 4.1 in (Duffie and Protter 1992) and that it converges in law to some limiting semimartingale $(\theta, \eta^\theta, S^\theta)$. As an additional component in this sequence, consider the discrete feedback processes

$$F_t^{(n),\theta^{(n)}} = \int_0^t \eta_{s-}^{(n),\theta^{(n)}} \, d\theta_s^{(n)} = \sum_{k=0}^{[nt/T]} \eta_{k-1}^{(n),\theta^{(n)}} \, \Delta\theta_k^{(n)} \quad (0 \leq t \leq T) .$$

By Proposition 4.1 in (Duffie and Protter 1992), this extended sequence is again 'good' and converges to $(\theta, \eta^\theta, S^\theta, F^\theta)$ where F^θ is defined by (2).

We go one step further and add the exponentials

$$\mathcal{E}(S^{(n),\theta^{(n)}}) = \prod_{0 \leq s \leq .} (1 + \Delta S_s^{(n),\theta^{(n)}}) \quad \text{and} \quad \mathcal{E}(F^{(n),\theta^{(n)}}) = \prod_{0 \leq s \leq .} (1 + \Delta F_s^{(n),\theta^{(n)}}) .$$

By Theorem 4.4 in (Duffie and Protter 1992), this further extension defines again a 'good' sequence which converges weakly to the limiting semimartingale $(\theta, \eta^\theta, S^\theta, F^\theta, \mathcal{E}(S^\theta), \mathcal{E}(F^\theta))$. In particular, the discrete-time asset prices (7) converge in law to the continuous-time asset prices defined by (3).

4 Absence of Arbitrage

In this section we are going to focus on conditions under which the financial market described by (3) excludes arbitrage. First we do so from the point of view of a small investor, and then we adopt the perspective of a large trader.

4.1 No Arbitrage for Small Investors

(Jarrow 1992) introduces the assumption that the stock price process excludes arbitrage by small investors in periods where the large investor does not trade. In our model we can verify this consistency requirement under the following

Assumption 4.1 *(i) σ and μ take values in $[\underline{\sigma}, \overline{\sigma}]$ and $[\underline{\mu}, \overline{\mu}]$ where $\underline{\sigma}, \overline{\sigma}, \underline{\mu}, \overline{\mu}$ are constants and $\underline{\sigma} > 0$.*

(ii) M satisfies the Novikov condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\frac{\overline{\mu}^2\|\lambda\|_\infty^2}{\underline{\sigma}^2}[M, M]_T\right)\right] < \infty.$$

Theorem 4.2 *Under Assumption 4.1 there is no free lunch with vanishing risk for small investors in periods where the large investor does not trade.*

Proof: Let $T_0 \leq T_1$ be two stopping times between which the large investor does not trade, i.e., $\theta_t = \theta_{T_0}$ for $t \in [T_0, T_1]$. Over this time period the dynamics (4) takes the form

$$dP^\theta = P_-^\theta(\sigma_-^\theta \, dM + \mu_-^\theta \lambda \, d[M, M])_T = P_-^\theta \, dS^\theta \quad \text{on } [T_0, T_1]. \tag{8}$$

Assumption 4.1 guarantees that the $\mathbb{P}$-equivalent measure $\mathbb{P}^\theta$ associated to the large investor strategy via

$$d\mathbb{P}^\theta \triangleq \mathcal{E}\left(-\frac{\mu_-^\theta}{\sigma_-^\theta}\lambda \cdot M\right)_T d\mathbb{P} \tag{9}$$

is indeed a probability measure. By Girsanov's Theorem S^θ becomes a local martingale under $\mathbb{P}^\theta$. Thus $\mathbb{P}^\theta$ is an equivalent local martingale for the price process considered over the period $[T_0, T_1]$. By (Delbaen and Schachermayer 1994) this is equivalent to 'no free lunch with vanishing risk' for small investors over $[T_0, T_1]$. □

Remark 4.3 1. *More generally, our model satisfies Jarrow's Assumption of*

> *(i) no free lunch with vanishing risk for small investors over periods where the large investor does not trade*

if and only if

> *(ii) there is an equivalent local martingale measure for S^θ considered over any period where θ is constant.*

Again, this follows from (Delbaen and Schachermayer 1994) and the special form (8) *of our price dynamics in periods where the large investor refrains from trading.*

2. *Jarrow's assumption (i) implies that the bounded variation part of the continuous semimartingale S^θ is absolutely continuous with respect to the variation process of its martingale part. This follows from Girsanov's Theorem as pointed out by, e.g., (Schweizer 1992). Thus, assumption (i) explains the special form we chose for S^θ in its definition* (1).

4.2 No Arbitrage Opportunity for the Large Investor

A more delicate question is whether there exist arbitrage opportunities for the large investor. In our context, the fundamental characterization of absence of arbitrage (Delbaen and Schachermayer 1994) cannot be applied because of the direct feedback effect in the price dynamics. From an economic point of view, it is not immediately clear that the large investor does not have arbitrage opportunities. On the one hand, he typically has to buy at higher prices than a small investor. On the other hand, by buying additional shares he may increase the paper wealth of the shares he held before the transaction. This effect might allow him to create a market trend and then trade on it by liquidating the portfolio in a carefully chosen way.

(Jarrow 1992) and (Jarrow 1994) seem to be the first articles dealing with this problem in a rigorous way. In a discrete-time setting absence of arbitrage for the large investor is shown under the condition that prices are given by a reaction function, and that arbitrage by small investors is excluded as long as the large investor does not operate. Under analogous assumptions (Bierbaum 1997) extends these results to a general semimartingale framework. Since our model does not fit into a reaction function setup we have to use different arguments. This will be the topic of this and the following section.

Let us start by formalizing what we mean by a riskless profit for the large investor.

Definition 4.4 *A strategy $\theta \in \Theta_0$ is called an arbitrage opportunity for the large investor if*

$$\mathbb{P}[\beta_T^\theta \geq 0] = 1 \quad \text{and} \quad \mathbb{P}[\beta_T^\theta > 0] > 0. \tag{10}$$

Remark 4.5 *Following (Jarrow 1992), we may define the large investor's paper wealth V^θ at time $0 \leq t \leq T$ by*

$$V_t^\theta \triangleq \beta_t^\theta + P_t^\theta \theta_t \,.$$

The corresponding real wealth may be described by the large investor's maximal cash position which is attainable from the portfolio held just before time t by a single transaction δ at time t which induces a neutral or long position in the stock:

$$W_t^\theta \triangleq \max_{\delta \leq \theta_{t-}} \left\{ \beta_{t-}^\theta + P_{t-}^\theta (1 - \eta_{t-}^\theta \delta)\delta \right\} \,.$$

It is easy to see that $W_t^\theta = \beta_{t-}^\theta + P_{t-}^\theta(1 - \eta_{t-}^\theta \delta^*)\delta^*$ where $\delta^* \triangleq \theta_{t-} \wedge 1/(2\eta_{t-}^\theta)$.

Clearly, for $\theta \in \Theta_0$ both real and paper wealth at time T are equal to β_T^θ. This is why we restrict θ to this class in our definition of an arbitrage opportunity for the large investor. Furthermore, recall that the large investor starts with zero initial wealth $(\theta_{0-}, \beta_{0-}) = (0, 0)$. Thus, a strategy of class Θ_0 satisfying (10) does indeed produce a profit without risk.

Consider a strategy $\theta \in \Theta_0$. Our aim is to derive conditions on θ under which the large investor's total gain β_T^θ has nonpositive expectation under the $\mathbb{P}$-equivalent measure $\mathbb{P}^\theta$ defined in (9) — this clearly excludes arbitrage by θ.

For notational convenience we let $p_0 \triangleq 1$. Then the evolution of the investor's bank account is given by

$$\begin{aligned}
\beta^\theta &= -\int_0^\cdot P_-^\theta \, d\theta - [P^\theta, \theta] \\
&= -\int_0^\cdot \mathcal{E}(S^\theta)\mathcal{E}(F^\theta)_- \, d\theta - [\mathcal{E}(S^\theta)\mathcal{E}(F^\theta), \theta] \\
&= -\int_0^\cdot \mathcal{E}(S^\theta) \, d\left(\mathcal{E}(F^\theta)_- \cdot \theta\right) - [\mathcal{E}(S^\theta)\mathcal{E}(F^\theta), \theta] \,.
\end{aligned}$$

Integration by parts yields for the first term

$$\begin{aligned}
&\int_0^\cdot \mathcal{E}(S^\theta) \, d\left(\mathcal{E}(F^\theta)_- \cdot \theta\right) = \\
&\qquad \mathcal{E}(S^\theta)\left(\mathcal{E}(F^\theta)_- \cdot \theta\right) - [\mathcal{E}(S^\theta), \mathcal{E}(F^\theta)_- \cdot \theta] - \int_0^\cdot \left(\mathcal{E}(F^\theta)_- \cdot \theta\right)_- \, d\mathcal{E}(S^\theta),
\end{aligned}$$

while for the second term we get

$$
\begin{aligned}
&[\mathcal{E}(S^\theta)\mathcal{E}(F^\theta),\theta] \\
&= \int_0^\cdot \mathcal{E}(S^\theta)\mathcal{E}(F^\theta)_-\, d[S^\theta + F^\theta + [S^\theta, F^\theta],\theta] \\
&= \int_0^\cdot \mathcal{E}(S^\theta)\mathcal{E}(F^\theta)_-\, (d[S^\theta,\theta] + d[F^\theta,\theta]) \\
&= [\mathcal{E}(S^\theta), \mathcal{E}(F^\theta)_- \cdot \theta] + \int_0^\cdot \mathcal{E}(S^\theta)\, d[\mathcal{E}(F^\theta),\theta] \\
&= [\mathcal{E}(S^\theta), \mathcal{E}(F^\theta)_- \cdot \theta] + \mathcal{E}(S^\theta)[\mathcal{E}(F^\theta),\theta] - \int_0^\cdot [\mathcal{E}(F^\theta),\theta]_-\, d\mathcal{E}(S^\theta).
\end{aligned}
$$

Combining the above expressions we conclude

$$
\begin{aligned}
\beta^\theta &= \int_0^\cdot \Big(\mathcal{E}(F^\theta)_- \cdot \theta + [\mathcal{E}(F^\theta),\theta]\Big)_-\, d\mathcal{E}(S^\theta) \\
&\quad - \mathcal{E}(S^\theta)\left\{\Big(\mathcal{E}(F^\theta)_- \cdot \theta\Big) + [\mathcal{E}(F^\theta),\theta]\right\}.
\end{aligned}
\tag{11}
$$

How should we interpret this equation? Set

$$\xi^\theta \triangleq \mathcal{E}(F^\theta)_- \cdot \theta + [\mathcal{E}(F^\theta),\theta] = \theta\mathcal{E}(F^\theta) - \theta_- \cdot \mathcal{E}(F^\theta)\,.$$

Let us introduce an auxiliary 'fictitious asset' with price process $\mathcal{E}(S^\theta)$. This process reflects the price evolution of the real asset in a fictitious world without direct feedback effects. Now, we may rewrite (11) in the form

$$\beta^\theta + \xi^\theta\mathcal{E}(S^\theta) = \int_0^\cdot \xi^\theta_-\, d\mathcal{E}(S^\theta). \tag{12}$$

The right side in this equation may be viewed as the fictitious gain resulting from strategy ξ^θ in the fictitious asset $\mathcal{E}(S^\theta)$. The left side is composed by the costs for strategy θ in the real asset P^θ and by the value of the current stock position, the latter being expressed in terms of ξ^θ 'units' in the fictitious asset $\mathcal{E}(S^\theta)$. Hence, we may interpret ξ^θ as a fictitious strategy describing the large investor's real position adjusted for feedback effects. This point of view is also sustained by the observation that $\xi^\theta \equiv \theta$ if $\eta^\theta_t \equiv 0$, i.e., if there is no direct feedback.

Definition 4.6 *A strategy $\theta \in \Theta_0$ is called tame if the fictitious gain defined by* (12) *is a $\mathbb{P}^\theta$-supermartingale.*

Remark 4.7 *A large investor strategy $\theta \in \Theta_0$ is tame iff the associated 'feedback adapted' strategy ξ^θ is admissible in the usual sense with respect to the fictitious price process $\mathcal{E}(S^\theta)$. Note that, in any case, the stochastic integral in* (12) *is a local $\mathbb{P}^\theta$-martingale. Hence, for θ to be tame, it suffices to assume that this integral is bounded from below by some constant.*

In order to exclude arbitrage by a tame strategy $\theta \in \Theta_0$ we add the restriction that the large investor ends up in a fictitious long position $\xi_T^\theta \geq 0$. In this case, his real gain β_T^θ will be dominated by the fictitious gain from trading in the artificial asset $\mathcal{E}(S^\theta)$. For a tame strategy, this gain has nonpositive $\mathbb{P}^\theta$-expectation, and so arbitrage by θ will be excluded:

Theorem 4.8 *Under Assumption 4.1 there is no arbitrage opportunity for the large investor among all tame strategies $\theta \in \Theta_0$ such that*

$$\xi_T^\theta \geq 0 \ a.s. \tag{13}$$

PROOF: For any tame strategy $\theta \in \Theta_0$, the stochastic integral appearing in (11) has nonpositive $\mathbb{P}^\theta$-expectation. This allows us to conclude

$$\mathbb{E}^\theta \beta_T^\theta \leq \mathbb{E}^\theta \left[-\mathcal{E}(S^\theta)_T \left\{ \left(\mathcal{E}(F^\theta)_- \cdot \theta \right)_T + [\mathcal{E}(F^\theta), \theta]_T \right\} \right] = \mathbb{E}^\theta \left[-\mathcal{E}(S^\theta)_T \xi_T^\theta \right] .$$

Together with (13) this implies that β_T^θ has nonpositive $\mathbb{P}^\theta$-expectation. Thus, a tame strategy θ satisfying (13) cannot be an arbitrage opportunity for the large investor. □

Let us give another interpretation of (13), which will be useful for the remainder of this note. Write

$$L^\theta \triangleq - \int_0^T \mathcal{E}(F^\theta)_- \, d\theta - [\mathcal{E}(F^\theta), \theta]_T = -\xi_T^\theta \tag{14}$$

so that condition (13) takes the form

$$L^\theta \leq 0 \, . \tag{15}$$

Note that the price P^θ reduces to the process $\mathcal{E}(F^\theta)$ if the large investor's strategy is the *only* driving semimartingale for the price process, i.e., if $M \equiv 0$. In this case, L^θ coincides with the final profit β_T^θ resulting from $\theta \in \Theta_0$. From this point of view, condition (14) means that the feedback mechanism in the market causes the large investor to suffer a 'technical loss', provided there are no other influences on the stock price.

4.3 A Case Study: Constant Market Elasticity

Let us now focus on sufficient conditions for (15) under the *assumption of constant market elasticity.* For simplicity we assume $\eta_t^\theta \equiv 1$ so that $F^\theta = \theta$ and

$$L^\theta = 1 - \mathcal{E}(\theta)_T - [\mathcal{E}(\theta), \theta]_T \, . \tag{16}$$

This simple description of L^θ allows us to conclude

Lemma 4.9 $L^\theta = 0$ *for any continuous $\theta \in \Theta_0$ of bounded variation.*

PROOF: The assumed properties cause the bracket term in (16) to vanish while the Doléans-Dade exponential equals 1 because $\theta_{0-} = \theta_T = 0$. □

Remark 4.10 *The lemma shows that the large investor's technical profit or loss due to feedback effects is zero if he trades 'slowly', i.e., if he uses a continuous strategy of bounded variation.*

The following is an analogue of Lemma 4.9 for continuous strategies of possibly unbounded variation.

Lemma 4.11 *If $\theta \in \Theta_0$ is continuous and $\theta \geq -\log 2$ then $L(\theta) \leq 0$.*

PROOF: Consider a strategy $\theta \in \Theta_0$ satisfying $\theta \geq -l$ for some $l \geq 0$. Using $\theta_{0-} = \theta_T = 0$ and the explicit description of the Doléans-Dade exponential $\mathcal{E}(\theta)$ in Definition 2.1, we get

$$\begin{aligned} L^\theta &= 1 - e^{-\frac{1}{2}[\theta,\theta]_T} - \int_0^T e^{\theta - \frac{1}{2}[\theta,\theta]}\, d[\theta,\theta] \\ &= 1 - e^{-\frac{1}{2}[\theta,\theta]_T} - 2\int_0^T e^{\theta}\, d\left(-\exp(-\tfrac{1}{2}[\theta,\theta])\right) \\ &\leq (1-2e^{-l})(1-\exp(-\tfrac{1}{2}[\theta,\theta]_T)). \end{aligned}$$

Letting $l \triangleq \log 2$ yields $L^\theta \leq 0$. □

Remark 4.12 *In economic terms, the preceding lemma shows that it does not pay for the large investor to trade 'too much' provided he is not 'too short' the asset. The short selling restriction is indeed necessary as can be seen by the following strategy:*

- *first, sell short $\log 2 + \epsilon$ assets 'slowly' (in the sense of Remark 4.10), where $\epsilon > 0$;*
- *then, keeping a short position of at least $\log 2 + \epsilon/2$ assets, use a continuous semimartingale strategy which produces a positive quadratic variation;*
- *finally, clear the short position 'slowly'.*

Applying the same arguments as in the proof of Lemma 4.11 we see that such a strategy induces a 'technical gain' $L^\theta > 0$.

An analogue of Lemma 4.11 for possibly discontinuous strategies is more involved. Intuitively, it should not pay for the large investor to trade in jumps because prices rise *before* he can buy. This intuition is sustained by the proof of the following

Lemma 4.13 *A simple strategy $\theta \in \Theta_0$ of the form*

$$\theta = \sum_{i=0}^{n} \theta_{t_i} 1_{[t_i, t_{i+1})} \quad \textit{for some } 0 = t_0 \leq t_1 < \ldots < t_{n+1} = T$$

satisfies $L^\theta \leq 0$ if $\theta \geq 0$, i.e., if there are no short sales.

Proof: Without loss of generality we may assume that the portfolio is already liquidated at time $t_n < T$, i.e., $\theta_{t_n} = 0$. We will show that the continuous linear interpolation of θ with $\tilde{\theta}^*_{t_i} \triangleq \theta_{t_{i-1}}$ $(i = 0, \dots, n+1)$ [2] yields a strategy $\tilde{\theta}^*$ with

$$L^\theta \leq L^{\tilde{\theta}^*} . \tag{17}$$

By construction, $\tilde{\theta}^*$ is continuous and of bounded variation with $\tilde{\theta}^*_T = 0$, and therefore Lemma 4.9 allows us to conclude $L^{\tilde{\theta}^*} = 0$; in connection with (17) this proves our assertion.

In order to establish the estimate (17), we consider an arbitrary nonnegative strategy $\hat{\theta} \in \Theta_0$ which has a jump at time $t_i < T$, is constant over $[t_i, t_{i+1})$, and which is continuous and of bounded variation afterwards. We show below that the partial interpolation

$$\tilde{\theta} \triangleq \begin{cases} \hat{\theta} \text{ on } [0, t_i) \cup (t_{i+1}, T] \\ \text{linearly interpolating between } \hat{\theta}_{t_i-} \text{ and } \hat{\theta}_{t_i} = \hat{\theta}_{t_{i+1}} \text{ on } [t_i, t_{i+1}] \end{cases}$$

satisfies $L^{\hat{\theta}} \leq L^{\tilde{\theta}}$. Beginning with $\hat{\theta} \triangleq \theta$ and iterating this construction we then get a sequence of nonnegative strategies $\tilde{\theta}^{(i)}$ $(i = 0, \dots, n)$ with

$$L^\theta \leq L^{\tilde{\theta}^{(0)}} \leq L^{\tilde{\theta}^{(1)}} \leq \dots \leq L^{\tilde{\theta}^{(n)}}.$$

Obviously, $\tilde{\theta}^* \equiv \tilde{\theta}^{(n)}$ and the estimate (17) follows.

So let us compute $L^{\hat{\theta}}$ with $\hat{\theta} \in \Theta_0$ and $\tau \triangleq t_i$ as above:

$$\begin{aligned} L^{\hat{\theta}} &= -\int_0^T \mathcal{E}(\hat{\theta})_- \, d\hat{\theta} - \int_0^T \mathcal{E}(\hat{\theta})_- \, d[\hat{\theta}, \hat{\theta}] \\ &= -\int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d\hat{\theta} - \int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d[\hat{\theta}, \hat{\theta}] - \mathcal{E}(\hat{\theta})_{\tau-}(\Delta\hat{\theta}_\tau + \Delta\hat{\theta}^2_\tau) - \int_{\tau+}^T \mathcal{E}(\hat{\theta})_- \, d\hat{\theta} \\ &= -\int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d\hat{\theta} - \int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d[\hat{\theta}, \hat{\theta}] \\ &\quad - \mathcal{E}(\hat{\theta})_{\tau-} \left(\Delta\hat{\theta}_\tau + \Delta\hat{\theta}^2_\tau + (1 + \Delta\hat{\theta}_\tau)(e^{-\hat{\theta}_\tau} - 1) \right), \end{aligned}$$

where the last equality follows because $\hat{\theta}$ is continuous and of bounded variation on $(\tau, T]$. On the other hand, $\hat{\theta} = \tilde{\theta}$ on $[0, \tau)$ yields

$$\begin{aligned} L^{\tilde{\theta}} &= -\int_0^T \mathcal{E}(\tilde{\theta})_- \, d\tilde{\theta} - \int_0^T \mathcal{E}(\tilde{\theta})_- \, d[\tilde{\theta}, \tilde{\theta}] \\ &= -\int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d\hat{\theta} - \int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d[\hat{\theta}, \hat{\theta}] - \int_\tau^T \mathcal{E}(\tilde{\theta})_- \, d\tilde{\theta} \\ &= -\int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d\hat{\theta} - \int_0^{\tau-} \mathcal{E}(\hat{\theta})_- \, d[\hat{\theta}, \hat{\theta}] - \mathcal{E}(\hat{\theta})_{\tau-} \left(e^{-\hat{\theta}_{\tau-}} - 1 \right) , \end{aligned}$$

[2] Let $\theta_{t_{-1}} \triangleq 0$.

where we used that $\tilde{\theta}$ is continuous and of bounded variation on the closed interval $[\tau, T]$. From the above calculations we infer

$$L^{\hat{\theta}} - L^{\tilde{\theta}} = -\mathcal{E}(\hat{\theta})_{\tau-}\left(\Delta\hat{\theta}_\tau^2 + \Delta\hat{\theta}_\tau e^{-\hat{\theta}_\tau} + e^{-\hat{\theta}_\tau} - e^{-\hat{\theta}_{\tau-}}\right).$$

Letting $x \triangleq \hat{\theta}_{\tau-}$ and $\Delta \triangleq \Delta\hat{\theta}_\tau$, we may rewrite this as $L^{\hat{\theta}} - L^{\tilde{\theta}} = -\mathcal{E}(\hat{\theta})_{\tau-} f(x, \Delta)$ where

$$f(x, \Delta) \triangleq \Delta^2 + e^{-x}((1+\Delta)e^{-\Delta} - 1).$$

Thus, in order to establish our claim $L^{\hat{\theta}} \leq L^{\tilde{\theta}}$, it suffices to prove $f(x, \Delta) \geq 0$.

The estimate $e^{-\Delta} \geq 1 - \Delta$ yields

$$f(x, \Delta) \geq \Delta^2 + e^{-x}((1+\Delta)(1-\Delta) - 1) = (1 - e^{-x})\Delta^2.$$

Because $x \geq 0$ due to the prohibition of short sales, the last term in this expression is nonnegative. Hence, we have indeed $f(x, \Delta) \geq 0$. $\square$

In order to extend the last result from simple to general strategies $\theta \in \Theta_0$ we only need to show

Lemma 4.14 *For $\theta \in \Theta$ let*

$$\theta^{(n)} \triangleq \sum_{t_i \in \tau^{(n)}} \theta_{t_i} 1_{[t_i, t_{i+1})} + \theta_T 1_{[T]} \tag{18}$$

where $\tau^{(n)} = \{0 = t_0 < t_1 < \ldots < t_{n+1} = T\}$ is a partition of $[0, T]$ such that $\sup_{t_i \in \tau^{(n)}} |t_{i+1} - t_i| \to 0$ as $n \uparrow +\infty$. Then $L^\theta = \lim_n L^{\theta^{(n)}}$.

Proof: By Theorem V.17 in (Protter 1990) the process $R^{(n)} \triangleq \mathcal{E}(\theta) - \mathcal{E}(\theta^{(n)})$ converges to zero uniformly in probability. Using (Protter 1990), Theorem II.30, this allows us to conclude

$$\begin{aligned}
[\mathcal{E}(\theta), \theta]_T &= \int_0^T \mathcal{E}(\theta)_- \, d[\theta, \theta] \\
&= \lim_n \sum_{t_i \in \tau^{(n)}} \mathcal{E}(\theta)_{t_i} (\theta_{t_{i+1}} - \theta_{t_i})^2 \\
&= \lim_n \sum_{t_i \in \tau^{(n)}} \mathcal{E}(\theta^{(n)})_{t_i} (\theta^{(n)}_{t_{i+1}} - \theta^{(n)}_{t_i})^2 + \lim_n \sum_{t_i \in \tau^{(n)}} R^{(n)}_{t_i} (\theta^{(n)}_{t_{i+1}} - \theta^{(n)}_{t_i})^2 \\
&= \lim_n \int_0^T \mathcal{E}(\theta^{(n)})_- \, d[\theta^{(n)}, \theta^{(n)}] \\
&= \lim_n [\mathcal{E}(\theta^{(n)}), \theta^{(n)}]_T.
\end{aligned}$$

$\square$

Combining the above results we end up with

Theorem 4.15 *Under Assumption 4.1 and constant market elasticity there is no arbitrage opportunity for the large investor among all tame strategies $\theta \in \Theta_0$ without short sales.*

PROOF: If $\theta \in \Theta_0$ does not sell short then the discretized strategies $\theta^{(n)}$ defined in (18) have the same property, i.e., $\theta^{(n)} \geq 0$. Combining Lemma 4.13 and Lemma 4.14 we see that our condition $L^\theta \leq 0$ is satisfied, and so we can apply Theorem 4.8. $\square$

Let us now show that the prohibition of short sales is 'almost necessary' for the preceding theorem. To this end, we consider strategies of the form

$$\theta_t^\epsilon \triangleq \frac{x}{\epsilon} t 1_{[0,\epsilon)}(t) + \frac{x+\Delta}{\epsilon}(2\epsilon - t) 1_{[\epsilon,2\epsilon)}(t) \quad (0 \leq t \leq T) \tag{19}$$

with $x < 0$, $-1 < \Delta < 0$ and $0 < \epsilon \leq T/2$. An investor applying strategy θ^ϵ starts by short selling shares at constant rate $\dot{\theta}^\epsilon = -|x|/\epsilon$ until time $\epsilon > 0$. At this time, he sells short another $|\Delta|$ shares en bloc. The obtained short position $\theta_\epsilon^\epsilon = x + \Delta < 0$ is then cleared over the period $(\epsilon, 2\epsilon]$ by buying back shares at rate $\dot{\theta}^\epsilon = (|x| + |\Delta|)/\epsilon$.

Lemma 4.16 *For a suitable choice of x, Δ and $\epsilon_n \downarrow 0$, the strategies (θ^{ϵ_n}) yield an asymptotic arbitrage opportunity for the large investor, i.e.,*

$$\lim_{k \uparrow +\infty} \beta_T^{\theta^{\epsilon_k}} > 0 \quad \mathbb{P}\text{-}a.s. \tag{20}$$

Remark 4.17 *For any strategy θ^ϵ we easily compute*

$$L^{\theta^\epsilon} = 1 - (1 - |\Delta|)e^{|\Delta|} - |\Delta|^2 e^{-|x|} \triangleq L(x, \Delta)\,. \tag{21}$$

Hence, L^{θ^ϵ} does not depend on ϵ, and for $|\Delta|$ sufficiently close to 1 it is strictly positive. Formula (21) shows that a market with constant elasticity and no other driving force but the large investor does contain arbitrage opportunities for the 'big player' if he is allowed to sell short.

If prices are also driven by the exogenous stochastic factor $\mathcal{E}(S^\theta)$, the strategies (19) still yield the asymptotic *arbitrage opportunity (20) for the large investor. The large investor's risk due to the additional factor vanishes as $\epsilon \downarrow 0$, because eventually he operates so fast that $\mathcal{E}(S^{\theta^\epsilon})$ may be viewed as constant over his trading period, and so we are back in the context where the large investor is the only driving force in the market. This heuristics will be made precise in the following proof.*

PROOF OF LEMMA 4.16: For notational convenience we let $S^\epsilon \triangleq S^{\theta^\epsilon}$, $\sigma^\epsilon \triangleq \sigma^{\theta^\epsilon}$, and $\beta^\epsilon \triangleq \beta^{\theta^\epsilon}$. Without loss of generality we can furthermore assume that $\mathbb{E}[M, M]_1 < +\infty$.

1. Let us first show

   $$\max_{0 \leq t \leq 2\epsilon_n} |\mathcal{E}(S^{\epsilon_n})_t - 1| \to 0 \quad \mathbb{P}\text{-a.s.}$$

   along a suitable sequence $\epsilon_n \downarrow 0$. For this it suffices to show $\mathbb{P}$-a.s. *(i)* $[S^{\epsilon_n}, S^{\epsilon_n}]_{2\epsilon_n} \to 0$ and *(ii)* $\max_{0 \leq t \leq 2\epsilon_n} |S_t^{\epsilon_n}| \to 0$. By Assumption 4.1, we have

   $$[S^\epsilon, S^\epsilon]_{2\epsilon} = \int_0^{2\epsilon} \left(\sigma_-^\epsilon\right)^2 d[M, M] \leq \overline{\sigma}^2 [M, M]_{2\epsilon}$$

which allows us to deduce *(i)*. To prove *(ii)*, we write

$$|S^\epsilon_t| \le |\int_0^t \sigma^\epsilon_- \, dM| + \overline{\mu}\|\lambda\|_\infty [M,M]_t$$

and note that it suffices to find $\epsilon_n \downarrow 0$ such that

$$\max_{0\le t\le 2\epsilon_n} |\int_0^t \sigma^{\epsilon_n}_- \, dM| \to 0 \quad \mathbb{P}\text{-a.s.} \tag{22}$$

By Doob's inequality, we have

$$\mathbb{E} \max_{0\le t\le 2\epsilon} |\int_0^t \sigma^\epsilon_- \, dM|^2 \le 4\mathbb{E}\int_0^{2\epsilon} \left(\sigma^\epsilon_-\right)^2 \, d[M,M] \le 4\overline{\sigma}^2 \mathbb{E}[M,M]_{2\epsilon}\,.$$

As $\epsilon \downarrow 0$ the last term converges to zero by dominated convergence. This implies (22) for a suitable sequence $\epsilon_n \downarrow 0$.

2. The uniform convergence established in 1. allows us to conclude

$$\beta^{\epsilon_n}_{\epsilon_n -} = \int_0^{\epsilon_n -} \mathcal{E}(S^{\epsilon_n})_t e^{-\frac{|x|}{\epsilon_n}t} \frac{|x|}{\epsilon_n}\, dt = \int_0^{|x|} \mathcal{E}(S^{\epsilon_n})_{\frac{\epsilon_n t}{|x|}} e^{-t}\, dt \longrightarrow 1 - e^{-|x|}.$$

Furthermore we deduce

$$\Delta\beta^{\epsilon_n}_{\epsilon_n} = \mathcal{E}(S^{\epsilon_n})_{\epsilon_n} e^{-|x|}(1-|\Delta|)|\Delta| \longrightarrow e^{-|x|}(1-|\Delta|)|\Delta|$$

and

$$\begin{aligned}
\beta^{\epsilon_n}_T - \beta^{\epsilon_n}_{\epsilon_n} &= -e^{-|x|}(1-|\Delta|)\int_{\epsilon_n +}^{2\epsilon_n} \mathcal{E}(S^{\epsilon_n})_t e^{\frac{|x|+|\Delta|}{\epsilon_n}(t-\epsilon_n)} \frac{|x|+|\Delta|}{\epsilon_n}\, dt \\
&= -e^{-|x|}(1-|\Delta|)\int_0^{|x|+|\Delta|} \mathcal{E}(S^{\theta^\epsilon_n})_{\epsilon_n + \frac{t}{|x|+|\Delta|}\epsilon_n} e^t\, dt \\
&\longrightarrow -(1-|\Delta|)(e^{|\Delta|} - e^{-|x|})\,.
\end{aligned}$$

Summing up we find

$$\beta^{\epsilon_n}_T \longrightarrow 1 - e^{-|x|} + e^{-|x|}(1-|\Delta|)|\Delta| - (1-|\Delta|)(e^{|\Delta|} - e^{-|x|}) = L(x,\Delta)\,.$$

By Remark 4.17, this is strictly positive for suitably chosen $x < 0$ and $-1 < \Delta < 0$.

□

Under the condition of constant market elasticity we have thus shown that 'no arbitrage for the large investor' is *essentially* equivalent to the prohibition of short sales. It would be interesting to construct other models where absence of arbitrage follows under less limiting assumptions.

If market elasticity is constant then all the strategies θ^ϵ in (19) have the same technical profit or loss $L^{\theta^\epsilon} = L(x,\Delta)$, no matter how fast the transactions take place. Clearly, this cannot be expected for a real market where market liquidity depends heavily on the volume of recent transactions. More flexible choices of the elasticity process η^θ will be discussed elsewhere.

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